A CLASS OF UNARY SEMIGROUPS ADMITTING A REES MATRIX REPRESENTATION

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Abstract. In [1] the authors introduced the notion of an associate inverse subsemigroup of a regular semigroup. This is a subsemigroup T of a regular semigroup S containing a least associate of each $s \in S$, in relation to the natural partial order $\leq$. In [2] it is shown that the class of regular semigroups containing an associate inverse subsemigroup is a variety of unary semigroups. In this paper, we establish a structure theorem for the subvariety of such semigroups that satisfy two natural conditions. The structure described in this paper generalises the Rees matrix representation for completely simple semigroups.

1. Introduction

In a semigroup S, an element $t \in S$ is an associate of $s \in S$ if $s = sts$. The concept of an associate inverse subsemigroup of a regular semigroup S was introduced in [1] and extends the concept of an associate subgroup of a semigroup first presented in [3]. An associate inverse subsemigroup of a regular semigroup S is a subsemigroup S^* of S containing a least associate x^* of each $x \in S$, in relation to the natural partial order $\leq$. Due to a simple characterisation of inverse semigroups in terms of the natural partial order on an arbitrary semigroup, such a semigroup S^* is necessarily inverse:

Theorem 1.1. [[1], Theorem 2.1] *Let S be a regular semigroup. The following are equivalent:*

(i) *S is inverse;*
(ii) *for all $a \in S$, the set $\{x \in S : a = axa\}$ contains a least element with respect to the natural partial. order.*

Defining $x^{**} = (x^*)^*$ for every $x \in S$, $(x^*)^{**} = ((x^*)^*)^* = (x^{**})^*$ and so we can write as x^{***}. Since $S^* = \{s^* \mid s \in S\}$ is inverse we have clearly $s^{***} = s^*$.

2000 *Mathematics Subject Classification.* Primary 20M10.

Key words and phrases. Regular semigroup, associate inverse subsemigroup, variety, completely simple semigroups.

This research was financed by FEDER Funds through ”Programa Operacional Factores de Competitividade COMPETE” and by Portuguese Funds through FCT - ”Fundação para a Ciência e a Tecnologia”, within the project PEst-C/MAT/UI0013/2011.

This research was also financed by the Portuguese Government through the FCT Fundação para a Ciência e a Tecnologia under the project PEst-OE/MAT/UI4080/2011.

Semigroups S containing an associate inverse subsemigroup S^* can be regarded as unary semigroups, the unary operation being $x \mapsto x^*$. An axiomatic characterisation of this class of semigroups, consisting of three axioms, two of which are identities, is provided in [[1], Theorem 2.5]. In a recent paper, the authors proved the somewhat surprising result, [[2], Theorem 2.2], that the three axioms referred to above are equivalent to four identities, thus making the class of semigroups containing an associate inverse subsemigroup a variety of unary semigroups.

In Section 2, we establish a structure theorem for a certain class of regular semigroups with associate inverse subsemigroup in terms of the Rees matrix representation. The importance of this class is highlighted in Section 3, where we show that it contains all completely simple semigroups.

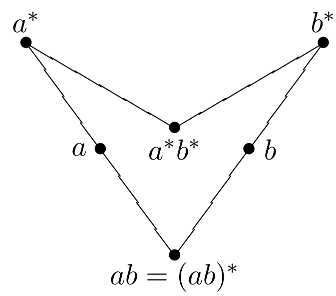
2. A class of regular semigroups with associate inverse subsemigroup

We aim at describing the structure of a regular semigroup S with associate inverse subsemigroup S^* satisfying:

(M1) $st(st)^* = ss^*tt^*$ and $(st)^*st = s^*st^*t$ for $s, t \in S$;
(M2) $(su^*t)^* = t^*u^{**}s^*$ for $s, t, u \in S$.

We observe first that, in general, these two conditions are mutually independent of each other.

Example 2.1. Let $S = \{a, b, ab, a^*, b^*, a^*b^*\}$ be the semilattice



As every semilattice with associate subsemilattice, S trivially satisfies (M1) since for all $s \in S$, $ss^* = s = s^*s$. However, it does not satisfy (M2) since $(aa^*b)^* = (ab)^* = ab \neq a^*b^*$.

Example 2.2. Let $S = \langle a, b \mid a^2 = aba = a, bab = b, b^2 = 0\rangle = \{a, b, ab, ba, 0\}$ be the 5-element non-orthodox 0-simple semigroup. The semigroup $S^* = \{a\}$ is an associate inverse subsemigroup of S that trivially satisfies (M2). However, since

$$b(ba)(b(ba))^* = b^2aa = 0$$

and

$$bb^*(ba)(ba)^* = babaa = ba \neq 0,$$

condition (M1) is not satisfied.

Lemma 2.1. *Let S be a regular semigroup with associate inverse subsemigroup S^* satisfying condition (M1). The inverse semigroup S^* is a Clifford semigroup.*

Proof. By [[6], 5.2 Theorem 12], S^* is a Clifford semigroup if, for all $x \in S^*$, $x^*x = xx^*$. Let $x \in S^*$. Since x^*x is an idempotent, we have, by (M1), that

$$\begin{aligned} x^*x &= (x^*x)^*x^*x \\ &= (x^*)^*x^*x^*x && [(\text{M1})] \\ &= xx^*x^*x \\ &= xx^*(xx^*)^* && [(\text{M1})] \\ &= xx^*. \end{aligned}$$

□

Lemma 2.2. *Let S be a regular semigroup with associate inverse subsemigroup S^* satisfying condition (M1). Then the following equalities hold:*

$$ss^*(ss^*)^* = ss^* \qquad \text{and} \qquad (s^*s)^*s^*s = s^*s.$$

Proof. From (M1) and since S^* is a Clifford semigroup, we obtain

$$ss^*(ss^*)^* = ss^*s^*s^{**} = ss^*s^{**}s^* = ss^*.$$

Similarly, $(s^*s)^*s^*s = s^*s$. □

Lemma 2.3. *Let S be a regular semigroup with associate inverse subsemigroup S^* satisfying condition (M1). The equalities $(ss^*)^* = s^{**}s^*$ and $(s^*s)^* = s^*s^{**}$ hold for all $s \in S$.*

Proof. Let $s \in S$. By Lemma 2.2

$$ss^*(ss^*)^*s = ss^*s = s$$

and so

$$s^* \leq s^*(ss^*)^*.$$

This implies

$$s^{**}s^* \leq s^{**}s^*(ss^*)^* \leq (ss^*)^*.$$

Also, by the proof of Lemma 2.2, we obtain

$$ss^*(s^{**}s^*)ss^* = ss^*$$

and then

$$(ss^*)^* \leq s^{**}s^*.$$

The equality follows. Dually, we have $(s^*s)^* = s^*s^{**}$. □

Lemma 2.4. *Let S be a regular semigroup with associate inverse subsemigroup S^*. For all $s, t \in S$, the equality $(st)^* = t^*(s^*stt^*)^*s^*$ holds.*

Proof. We have

$$st \cdot t^*(s^*stt^*)^*s^* \cdot st = ss^*stt^*(s^*stt^*)^*s^*stt^*t = ss^*stt^*t = st$$

and so

$$(st)^* \leq t^*(s^*stt^*)^*s^*,$$

that is,

$$(st)^* = e \cdot t^*(s^*stt^*)^*s^* = t^*(s^*stt^*)^*s^* \cdot f,$$

for some $e, f \in E(S)$. Then

$$t^*t^{**}(st)^* = (st)^* = (st)^*s^{**}s^* = t^*t^{**}(st)^*s^{**}s^*.$$

Thus,

$$s^*stt^*t^{**}(st)^*s^{**}s^*stt^* = s^*st(st)^*stt^* = s^*stt^*.$$

Consequently, we obtain

$$(s^*stt^*)^* \leq t^{**}(st)^*s^{**}$$

and so

$$t^*(s^*stt^*)^*s^* \leq t^*t^{**}(st)^*s^{**}s^* = (st)^*.$$

We have shown that $(st)^* = t^*(s^*stt^*)^*s^*$. □

Lemma 2.5. *Let S be a regular semigroup with associate inverse subsemigroup S^* satisfying condition (M2). For all $s, u \in S$,*

$$(su^*)^* = u^{**}s^* \text{ and } (u^*s)^* = s^*u^{**}.$$

Proof. We have, by (M2), for all $s, u \in S$,

$$(su^*)^* = (su^*(u^{**}u^*))^* = (u^{**}u^*)^*u^{**}s^* = u^{**}s^*.$$

Dually, $(u^*s)^* = s^*u^{**}$. □

We proceed to establish the structure of regular semigroups with associate inverse subsemigroup satisfying (M1) and (M2).

Construction 2.6. *Let G be a Clifford semigroup and let A and B be bands with the common associate inverse subsemilattice $E(G)$ satisfying (M1) and (M2). Assume that for all $a \in A$ and $x \in B$,*

$$aa^* = a, \qquad x^*x = x, \qquad a^*a \in A \cap B, \qquad xx^* \in A \cap B.$$

Let $P : B \times A \to G$, defined by $(x, a)P = p_{x,a}$ be a mapping which satisfies the following:

- (P1) $p_{yx,a} = y^*p_{x,a}$; $p_{x,ab} = p_{x,a}b^*$, *for all $a, b \in A, x, y \in B$;*
- (P2) $p_{x,a^*} = x^*a^* = p_{x^*,a}$, *for all $a \in A$ and $x \in B$;*
- (P3) $p_{x,a}p_{x,a}^{-1} = x^*a^*$, *for all $a \in A$ and $x \in B$.*

Define

$$\mathcal{M} = \{(a, g, x) \in A \times G \times B : a^* = x^* = gg^{-1}, a^*a = xx^*\}.$$

Lemma 2.7. *For every $a, b \in A$ and $x, y \in B$,*

$$(ab)^* = a^*b^*, \qquad (xy)^* = x^*y^*.$$

Proof. Since $a = aa^*$, it follows from (M2) that, for all $a, b \in A$,

$$(ab)^* = (aa^*b)^* = b^*a^{**}a^* = b^*a^*a^* = b^*a^* \in E(G).$$

Since $E(G)$ is a subsemilattice of A, $(ab)^* = a^*b^*$.

Similarly, we have, for all $x, y \in B$, $(xy)^* = x^*y^*$. □

Theorem 2.8. *With respect to the multiplication defined by*

$$(a, g, x)(b, h, y) := (ab, gp_{x,b}h, xy),$$

$\mathcal{M}$ forms a regular semigroup.

Proof. Let $(a, g, x), (b, h, y) \in \mathcal{M}$. Then $a^* = x^* = gg^{-1}$, $a^*a = xx^*$, $b^* = y^* = hh^{-1}$ and $b^*b = yy^*$.

We want to show that $(a, g, x)(b, h, y) \in \mathcal{M}$, i.e., that $(ab, gp_{x,b}h, xy)$ is such that

$$(ab)^* = (xy)^* = (gp_{x,b}h)(gp_{x,b}h)^{-1}$$

and

$$(ab)^*(ab) = (xy)(xy)^*.$$

We have

$$\begin{aligned}
(gp_{x,b}h)(gp_{x,b}h)^{-1} &= gp_{x,b}hh^{-1}p_{x,b}^{-1}g^{-1} \\
&= ghh^{-1}p_{x,b}p_{x,b}^{-1}g^{-1} && \text{[idempotents in } G \text{ are central]} \\
&= ghh^{-1}g^{-1}p_{x,b}p_{x,b}^{-1} && \text{[idempotents in } G \text{ are central]} \\
&= gg^{-1}hh^{-1}p_{x,b}p_{x,b}^{-1} && \text{[idempotents in } G \text{ are central]} \\
&= a^*b^*x^*b^* && \text{[def. of } \mathcal{M} \text{ and (P3)]} \\
&= a^*b^* = x^*y^* \\
&= (ab)^* = (xy)^*. && \text{[Lemma 2.7]}
\end{aligned}$$

Also, by (M1), we have

$$(ab)^*(ab) = a^*ab^*b = xx^*yy^* = (xy)(xy)^*.$$

Thus $(a, g, x)(b, h, y) \in \mathcal{M}$.

Next we show that the multiplication is associative

$$\begin{aligned}
[(a,g,x)(b,h,y)](c,k,z) &= (ab, gp_{x,b}h, xy)(c,k,z) \\
&= (abc, gp_{x,b}hp_{xy,c}k, xyz) \\
&= (abc, gp_{x,b}hx^*p_{y,c}k, xyz) && [(\text{P1})] \\
&= (abc, gp_{x,b}hgg^{-1}p_{y,c}k, xyz) && [gg^{-1} = x^*] \\
&= (abc, gg^{-1}gp_{x,b}hp_{y,c}k, xyz) && [S^* \text{ Clifford}] \\
&= (abc, gp_{x,b}hp_{y,c}k, xyz).
\end{aligned}$$

Similarly, $(a,g,x)[(b,h,y)(c,k,z)] = (abc, gp_{x,b}hp_{y,c}k, xyz)$ and so associativity holds.

We now show that (a^*, g^{-1}, x^*) is an associate of $(a,g,x) \in \mathcal{M}$. In fact, if $(a,g,x) \in \mathcal{M}$ then, by the definition of $\mathcal{M}$, $(a^*, g^{-1}, x^*) \in \mathcal{M}$ and

$$\begin{aligned}
(a,g,x)(a^*,g^{-1},x^*)(a,g,x) &= (aa^*a, gp_{x,a^*}g^{-1}p_{x^*,a}g, xx^*x) \\
&= (a, gx^*a^*g^{-1}x^*a^*g, x) && [(\text{P2})] \\
&= (a, gg^{-1}gx^*a^*, x) && [a^* = x^* \text{ is central}] \\
&= (a, gg^{-1}gg^{-1}g, x) \\
&= (a,g,x).
\end{aligned}$$

□

Lemma 2.9. $E(\mathcal{M}) = \{(a,g,x) \in \mathcal{M} \mid g = p_{x,a}^{-1}\}$.

Proof. Let $(a,g,x) \in \mathcal{M}$. Then

$$\begin{aligned}
(a,g,x)(a,g,x) = (a,g,x) &\Leftrightarrow (a, gp_{x,a}g, x) = (a,g,x) \\
&\Leftrightarrow gp_{x,a}g = g \\
&\Leftrightarrow g^{-1}gp_{x,a}gg^{-1} = g^{-1} \\
&\Leftrightarrow x^*p_{x,a}a^* = g^{-1} \\
&\Leftrightarrow p_{xx,aa} = g^{-1} && [(\text{P1})] \\
&\Leftrightarrow p_{x,a} = g^{-1} \\
&\Leftrightarrow g = p_{x,a}^{-1}.
\end{aligned}$$

□

Lemma 2.10. *Let* $(a,g,x), (b,h,y) \in \mathcal{M}$. *Then*

$$(a,g,x) \leq (b,h,y) \Leftrightarrow a \leq b,\ g \leq h,\ x \leq y.$$

Proof. Suppose that $a \leq b$, $g \leq h$ and $x \leq y$. Then

$$a = ab = ba,\ g = gg^{-1}h,\ x = xy = yx.$$

By Lemma 2.9, $(a, p_{x,a}^{-1}, x) \in E(\mathcal{M})$. Since

$$\begin{aligned}
(a, p_{x,a}^{-1}, x)(b, h, y) &= (ab, p_{x,a}^{-1} p_{x,b} h, xy) \\
&= (a, p_{xy,ba}^{-1} p_{xy,b} h, x) \\
&= (a, (x^* p_{y,b} a^*)^{-1} x^* p_{y,b} h, x) && [(P1)] \\
&= (a, a^* p_{y,b}^{-1} x^* x^* p_{y,b} h, x) \\
&= (a, a^* x^* p_{y,b}^{-1} p_{y,b} h, x) && [(P3)] \\
&= (a, a^* y^* b^* h, x) \\
&= (a, a^* y^* h, x) \\
&= (a, g g^{-1} h h^{-1} h, x) \\
&= (a, g g^{-1} h, x) = (a, g, x)
\end{aligned}$$

and

$$\begin{aligned}
(b, h, y)(a, p_{x,a}^{-1}, x) &= (ba, h p_{y,a} p_{x,a}^{-1}, yx) \\
&= (a, h p_{y,a} p_{xy,a}^{-1}, x) \\
&= (a, h p_{y,a} p_{y,a}^{-1} x^*, x) && [(P1)] \\
&= (a, h y^* a^* x^*, x) && [(P3)] \\
&= (a, h h^{-1} h g g^{-1}, x) \\
&= (a, h g g^{-1}, x) = (a, g, x)
\end{aligned}$$

we have that

$$(a, g, x) \leq (b, h, y).$$

Conversely, suppose that $(a, g, x) \leq (b, h, y)$. Then, by [[7]] there are $(c, k, z), (d, l, w)) \in \mathcal{M}$ such that

$$\begin{aligned}
(a, g, x) &= (b, h, y)(c, k, z) = (d, l, w)(b, h, y) \\
&= (a, g, x)(c, k, z).
\end{aligned}$$

Then

$$a = bc = db = ac,$$

$$x = yz = wy = xz,$$

$$g = h p_{y,c} k = l p_{w,b} h = g p_{x,c} k.$$

The first two give $a \leq b$ and $x \leq y$. The third one, together with

$$\begin{aligned}
g &= g p_{x,c} k \\
&= g p_{xy,c} k \\
&= g x^* p_{y,c} k && [(P1)] \\
&= g g^{-1} g p_{y,c} k \\
&= g p_{y,c} k,
\end{aligned}$$

gives that $g \leq h$. □

Theorem 2.11. *The semigroup $\mathcal{M}$ has an associate inverse subsemigroup*

$$\mathcal{M}^* = \{(a^*, g, a^*) : a^* = gg^{-1}\},$$

isomorphic to G, satisfying (M1) and (M2).

Proof. Clearly, $\mathcal{M}^* \subseteq \mathcal{M}$. Also, $\mathcal{M}^*$ is a subsemigroup of $\mathcal{M}$ since, for $(a^*, g, a^*), (b^*, h, b^*) \in \mathcal{M}^*$, we have

$$\begin{aligned}(a^*, g, a^*)(b^*, h, b^*) &= (a^*b^*, gp_{a^*,b^*}h, a^*b^*) \\ &= (a^*b^*, ga^*b^*h, a^*b^*) && \text{[(P2)]} \\ &= ((ab)^*, gg^{-1}ghh^{-1}h, (ab)^*) && \text{[Lemma 2.7 and definition of } \mathcal{M}] \\ &= ((ab)^*, gh, (ab)^*)\end{aligned}$$

with

$$(gh)(gh)^{-1} = ghh^{-1}g^{-1} = gg^{-1}hh^{-1} = a^*b^* = (ab)^*.$$

Also, from the proof of Theorem 2.8, $(a^*, g^{-1}, a^*) = (a^*, g^{-1}, x^*)$ is an associate of $(a, g, x) \in \mathcal{M}$.

We now show that this is the least such element in $\mathcal{M}^*$. In fact, if $(b^*, h, y^*) \in \mathcal{M}^*$, then

$$\begin{aligned}(a, g, x)(b^*, h, b^*)(a, g, x) = (a, g, x) &\Leftrightarrow (ab^*a, gp_{x,b^*}hp_{b^*,a}g, xb^*x) = (a, g, x) \\ &\Leftrightarrow ab^*a = a,\ xb^*x = x,\ gp_{x,b^*}hp_{y^*,a}g = g \\ &\Rightarrow a^* \leq b^*,\ x^* \leq b^*,\ gx^*b^*hb^*a^*g = g \\ &\Leftrightarrow a^* \leq b^*,\ x^* \leq b^*,\ gg^{-1}ghh^{-1}hh^{-1}hgg^{-1}g = g \\ &\Leftrightarrow a^* \leq b^*,\ x^* \leq b^*,\ ghg = g \\ &\Leftrightarrow a^* \leq b^*,\ x^* \leq b^*,\ g^{-1} \leq h \qquad [G \text{ inv and Thrm 1.1}] \\ &\Leftrightarrow (a^*, g^{-1}, x^*) \leq (b^*, h, b^*).\end{aligned}$$

The inverse subsemigroup $\mathcal{M}^*$ is obviously isomorphic to G.

We now prove that conditions (M1) and (M2) are satisfied.

We first observe that for $(a, g, x) \in \mathcal{M}$ we have

$$(\star) \qquad (a, g, x)(a, g, x)^* = (a, gg^{-1}, xx^*).$$

In fact,

$$\begin{aligned}(a, g, x)(a, g, x)^* &= (a, g, x)(a^*, g^{-1}, x^*) \\ &= (aa^*, gp_{x,a^*}g^{-1}, xx^*) \\ &= (a, gx^*a^*g^{-1}, xx^*) && \text{[(P2)]} \\ &= (a, gg^{-1}gg^{-1}gg^{-1}, xx^*) \\ &= (a, gg^{-1}, xx^*).\end{aligned}$$

Let $(a,g,x),(b,h,y)\in\mathcal{M}$. Then

$$\begin{aligned}
(a,g,x)(b,h,y)((a,g,x)(b,h,y))^* &= (ab,gp_{x,b}h,xy)(ab,gp_{x,b}h,xy)^* \\
&= (ab,gp_{x,b}h(gp_{x,b}h)^{-1},xy(xy)^*) && [(\star)] \\
&= (ab,gp_{x,b}hh^{-1}p_{x,b}^{-1}g^{-1},xx^*yy^*) && [B \text{ satisfies (M1)}] \\
&= (ab,ghh^{-1}p_{x,b}p_{x,b}^{-1}g^{-1},xx^*yy^*) && [hh^{-1} \text{ is central}] \\
&= (ab,ghh^{-1}x^*b^*g^{-1},xx^*yy^*) && [\text{(P3)}] \\
&= (ab,ghh^{-1}g^{-1}ghh^{-1}g^{-1},xx^*yy^*) \\
&= (ab,gg^{-1}hh^{-1},xx^*yy^*) && [hh^{-1},gg^{-1} \text{ are central}] \\
&= (ab,gg^{-1}x^*b^*hh^{-1},xx^*yy^*) \\
&= (ab,gg^{-1}p_{xx^*,b}hh^{-1},xx^*yy^*) && [\text{(P1) and (P2)}] \\
&= (a,gg^{-1},xx^*)(b,hh^{-1},yy^*) \\
&= (a,g,x)(a,g,x)^*(b,h,y)(b,h,y)^*. && [(\star)]
\end{aligned}$$

Similarly, $((a,g,x)(b,h,y))^*(a,g,x)(b,h,y) = (a,g,x)^*(a,g,x)(b,h,y)^*(b,h,y)$ and so (M1) holds.

Moreover, for $(a,g,x),(b,h,y)\in\mathcal{M}$ and $(c^*,k,c^*)\in\mathcal{M}^*$ we have

$$\begin{aligned}
((a,g,x)(c^*,k,c^*)(b,h,y))^* &= (ac^*b,gp_{x,c^*}kp_{xc^*,b}h,xc^*y)^* \\
&= (ac^*b,gx^*c^*kx^*c^*b^*h,xc^*y)^* && [\text{(P1) and (P2)}] \\
&= (ac^*b,gg^{-1}gkk^{-1}kg^{-1}gk^{-1}khh^{-1}h,xc^*y)^* \\
&= (ac^*b,gkh,xc^*y)^* && [\text{idempotents are central}] \\
&= (b^*c^*a^*,h^{-1}k^{-1}g^{-1},b^*c^*a^*) \\
&= (b^*,h^{-1},b^*)(c^*,k^{-1},c^*)(a^*,g^{-1},a^*) \\
&= (b,h,y)^*(c^*,k,c^*)^*(a,g,x)^*
\end{aligned}$$

and (M2) holds. □

Theorems 2.8 and 2.11 can be summarized in the following theorem:

Theorem 2.12. *With respect to the multiplication*

$$(a,g,x)(b,h,y) := (ab,gp_{x,b}h,xy),$$

the set $\mathcal{M}$ defined in Construction 2.6 forms a regular semigroup with an associate inverse subsemigroup

$$\mathcal{M}^* = \{(a^*,g,a^*) : a^* = gg^{-1}\},$$

isomorphic to G, satisfying (M1) and (M2).

We denote this semigroup by $\mathcal{M}(G;A,B;P)$.

Theorem 2.13. *Let S be semigroup with an associate inverse subsemigroup S^* satisfying (M1) and (M2). Then S is of the form prescribed in Construction 2.6.*

Proof. Let $A = \{ss^* : s \in S\}$ and $B = \{s^*s : s \in S\}$. Since S satisfies (M1), A and B are bands such that the set

$$E(S^*) = \{u^*u^{**} : u^* \in S^*\}$$

is a subset of both A and B. In fact, for all $x^* \in E(S^*)$,

$$\begin{aligned} x^* &= x^*x^{**}x^* & \\ &= x^*x^*x^{**} & [S^* \text{ Clifford}] \\ &= x^*x^{**} & [x^* \in E(S^*)] \end{aligned}$$

Moreover, by (M1), $E(S^*)$ is an inverse subsemigroup of both A and B. Also, given $ss^* \in A$, since $(ss^*)^* = s^{**}s^* \in E(S^*)$, we have that $s^{**}s^*$ is the least associate of ss^* in $E(S^*)$. Similarly, $s^{**}s^*$ is the least associate of $s^*s \in B$ in $E(S^*)$. Thus $E(S^*)$ is a common associate inverse subsemigroup (then a subsemilattice) of A and B. Let $G := S^*$.

Now, let $a = ss^* \in A$ and $x = t^*t \in B$. We show that $aa^* = a$, $x^*x = x$, $a^*a \in A \cap B$ and $xx^* \in A \cap B$. In fact,

$$aa^* = a,\ x^*x = x, \qquad \text{by Lemma 2.2,}$$

$$a \in A, a^* \in E(S^*) \subseteq A \Rightarrow a^*a \in A,$$

$$a^*a \in B, \text{by definition of } B,$$

$$x \in B, x^* \in E(S^*) \subseteq B \Rightarrow xx^* \in B,$$

$$xx^* \in A, \text{by definition of } A.$$

Define $P : B \times A \to G$ by

$$(s^*s, tt^*)P = p_{s^*s,tt^*} := (s^*stt^*)^{**}.$$

We show that (P1), (P2) and (P3) are satisfied by P:

$$\begin{aligned} \text{(P1) } p_{u^*us^*s,tt^*} &= (u^*us^*stt^*)^{**} & \\ &= ((stt^*)^*s^{**}(u^*u)^*)^* & [\text{(M2)}] \\ &= ((s^*stt^*)^*(u^*u)^*)^* & [\text{Lemma 2.5}] \\ &= (u^*u)^{**}(s^*stt^*)^{**} & [\text{Lemma 2.5}] \\ &= (u^*u)^*p_{s^*s,tt^*}; & [S^* \text{ inverse}] \end{aligned}$$

$$
\begin{aligned}
\text{(P2) } p_{s^*s,(tt^*)^*} &= p_{s^*s,t^{**}t^*} \\
&= (s^*st^{**}t^*)^{**} \\
&= (s^*s)^{**}t^{**}t^* && [\text{(M2)}] \\
&= (s^*s)^*(tt^*)^*; && [\text{Lemma 2.3 and } S^* \text{ inverse}]
\end{aligned}
$$

$$
\begin{aligned}
\text{(P3) } p_{s^*s,tt^*}p^{-1}_{s^*s,tt^*} &= (s^*stt^*)^{**}(s^*stt^*)^* \\
&= ((s^*stt^*)(s^*stt^*)^*)^* && [\text{Lemma 2.3}] \\
&= (s^*s(s^*s)^*tt^*(tt^*)^*)^* && [\text{(M1)}] \\
&= (tt^*(tt^*)^*)^*(s^*s)^{**}(s^*s)^* && [\text{(M2)}] \\
&= t^{**}t^*s^*s^{**} && [\text{Lemma 2.3}] \\
&= s^*s^{**}t^{**}t^* && [S^* \text{ inverse}] \\
&= (s^*s)^*(tt^*)^*; && [\text{Lemma 2.3}]
\end{aligned}
$$

Following Construction 2.6, we obtain the semigroup

$$\mathcal{M}(G;A,B;P) = \{(a,g,x) \in A \times G \times B : a^* = x^* = gg^{-1},\ a^*a = xx^*\}.$$

We show that $S \cong \mathcal{M}(G;A,B;P)$. Consider the mapping

$$
\begin{aligned}
\varphi :\ & S \to \mathcal{M}(G;A,B;P) \\
& s \mapsto (ss^*, s^{**}, s^*s)
\end{aligned}.
$$

φ maps S into $\mathcal{M}(G;A,B;P)$. In fact, for each $s \in S$,

$$(ss^*)^* = s^{**}s^* = s^*s^{**} = (s^*s)^*$$

since S^* is a Clifford semigroup and

$$
\begin{aligned}
(ss^*)^*ss^* &= s^{**}s^*ss^* \\
&= s^*s^{**}ss^* && [S^* \text{ is a Clifford semigroup}] \\
&= (s^*s)(s^*s)^* && [\text{(M1)}]
\end{aligned}
$$

Also, for each $s, t \in S$, we have

$$
\begin{aligned}
(st)\varphi &= (st(st)^*, (st)^{**}, (st)^*st) \\
&= (ss^*tt^*, ((st)^*)^*, s^*st^*t) && [\text{(M1)}] \\
&= (ss^*tt^*, (t^*(s^*stt^*)^*s^*)^*, s^*st^*t) && [\text{Lemma 2.4}] \\
&= (ss^*tt^*, s^{**}(s^*stt^*)^{**}t^{**}, s^*st^*t) \\
&= (ss^*, s^{**}, s^*s)(tt^*, t^{**}, t^*t) \\
&= s\varphi\, t\varphi.
\end{aligned}
$$

The homomorphism φ is injective since, for $s, t \in S$ such that

$$(ss^*, s^{**}, s^*s) = (tt^*, t^{**}, t^*t)$$

we have

$$s = ss^*s^{**}s^*s = tt^*t^{**}t^*t = t.$$

Finally, we show that φ is surjective. Let $(ss^*, u^{**}, t^*t) \in \mathcal{M}(G; A, B; P)$. Then,

$$(ss^*)^* = (t^*t)^* = u^{**}u^* = u^*u^{**}$$

and

$$s^{**}s^*ss^* = (ss^*)^*ss^* = t^*t(t^*t)^* = t^*tt^*t^{**}.$$

Consider $v := ss^*u^{**}t^*t \in S$. Then $v\varphi = (ss^*, u^{**}, t^*t)$, i.e.,

$$(vv^*, v^{**}, v^*v) = (ss^*, u^{**}, t^*t).$$

In fact, $vv^* = ss^*$ since

$$\begin{aligned}
vv^* &= ss^*u^{**}t^*t(ss^*u^{**}t^*t)^* && \\
&= ss^*u^{**}(ss^*u^{**})^*t^*t(t^*t)^* && [(M1)] \\
&= ss^*u^{**}u^*s^{**}s^*t^*tt^*t^{**} && [(M2)] \\
&= ss^*(ss^*)^*s^{**}s^*s^{**}s^*ss^* && [\text{hypothesis}] \\
&= ss^*s^{**}s^*ss^* && [\text{Lemma 2.2}] \\
&= ss^*.
\end{aligned}$$

Similarly, $v^*v = t^*t$. Moreover, $v^{**} = u^{**}$, since

$$\begin{aligned}
v^{**} &= (ss^*u^{**}t^*t)^{**} && \\
&= (ss^*)^{**}u^{**}(t^*t)^{**} && [(M2)] \\
&= (u^{**}u^*)^*u^{**}(u^*u^{**})^* && [\text{hypothesis}] \\
&= u^{**}.
\end{aligned}$$

□

Corollary 2.14. *Let S be a semigroup with an associate inverse subsemigroup S^* satisfying (M1) and (M2). Then*

$$s \in E(S) \Leftrightarrow s^* = (ss)^*.$$

Proof. Let $s \in S$ be such that $s^* = (ss)^*$. Considering the isomorphism φ defined in the proof of the previous theorem, we have

$$\begin{aligned}
ss &= (ss(ss)^*, (ss)^{**}, (ss)^*(ss))\varphi^{-1} && \\
&= (ss^*, s^{**}, s^*s)\varphi^{-1} && [(M1)] \\
&= s.
\end{aligned}$$

The converse is trivially true. □

Example 2.3. Let $S = \{1, a, b, c, d\}$ be the band with operation defined by

	1	a	b	c	d
1	1	a	b	c	d
a	a	a	b	a	a
b	b	a	b	b	b
c	c	a	b	c	d
d	d	a	b	c	d

Define a unary operation $*$ in S by

$$1^* = 1, \qquad a^* = b^* = b, \qquad c^* = d^* = c.$$

Then $S^* = \{1, b, c\}$ is an associate inverse subsemilattice of S. Moreover, conditions (M1) and (M2) are satisfied.

Example 2.4. Let $S = \{a, b, c, d\}$ be the band with operation defined by

	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

Let $x^* = a$, for all $x \in S$. Then S^* is an associate subgroup of S and (M1) and (M2) are satisfied.

Example 2.5. Consider the Rees matrix semigroup $S = \mathcal{M}[G; A, B; P]$, where $G = \{1, a\}$, $A = \{1, \mu\}$, $B = \{1, j\}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}$. Let $(i, g, \lambda)^* = (1, g, 1)$. Then $S^* = \{(1, 1, 1), (1, a, 1)\}$ is an associate subgroup of S and conditions (M1) and (M2) are satisfied. Note that the elements $(1, a, j), (\mu, a, 1) \in S$ are such that

$$((1, a, j)(\mu, a, 1))^* = (1, ap_{j,\mu}a, 1)^* = (1, a, 1)$$

and

$$(\mu, a, 1)^*(1, a, j)^* = (1, a, 1)(1, a, 1) = (1, 1, 1).$$

Thus, condition $(st)^* = t^*s^*$ is not satisfied.

3. Particular cases

In this section we consider two special cases in the class of semigroups described in Section 2. In the first one, by taking, in Construction 2.6, G to be a group and A and B to be left and right zero semigroups, respectively, we obtain a completely simple semigroup. In the second case, by considering S to be orthodox, the representation yields a subdirect product of a band and an inverse semigroup.

Proposition 3.1. *Let S be a semigroup with associate subgroup S^* that satisfies (M1). Then (M2) holds.*

Proof. Let 1 be the identity of S^* and s, t, u be arbitrary elements in S. Then,

$$s^*s \cdot 1 = s^*s(s^*s)^* = s^*s^{**}ss^* = 1 \cdot ss^*$$

and so $(1 \cdot ss^*)^* = (s^*s \cdot 1)^* = 1$.

Hence,

$$\begin{aligned}
(su^*t)^* &= t^*((su^*)^*su^*tt^*)^*(su^*)^* && \text{[Lemma 2.4]} \\
&= t^*(s^*su^{**}u^*tt^*)^*(su^*)^* && \text{[(M1)]} \\
&= t^*(s^*s1tt^*)^*[u^{**}(s^*su^*u^{**})^*s^*] && \text{[Lemma 2.4]} \\
&= t^*(1ss^*tt^*)^*[u^{**}(s^*s1)^*s^*] \\
&= t^*(1(st)(st)^*)^*(u^{**}1s^*) && \text{[(M1)]} \\
&= t^*1u^{**}s^* \\
&= t^*u^{**}s^*.
\end{aligned}$$

So (M2) holds. □

Theorem 3.2. *Let G be a group with identity element 1. Let A and B be bands such that $1 \in A \cap B$ and 1 is a right (left) identity of A (B). Let further $1a \in B$, $x1 \in A$, for each $a \in A$, $x \in B$. Let $P : B \times A \to G$, $(x, a) \mapsto p_{x,a}$ be a mapping satisfying*

(P1) $p_{yx,a} = p_{x,a} = p_{x,ab}$;

(P2) $p_{x,1} = 1 = p_{1,a}$.

Then $\mathcal{M}(G; A, B; P) = \{(a, g, x) \in A \times G \times B \mid 1a = x1\}$ is a semigroup with associate subgroup (isomorphic to G) that satisfies (M1) and (M2). Conversely each such semigroup can be constructed this way.

Proof. Clearly, G, A, B and P satisfy the conditions of Construction 2.6. By Theorem 2.12 and since $gg^{-1} = 1$,

$$\mathcal{M}(G; A, B; P) = \{(a, g, x) \in A \times G \times B \mid a^* = x^* = 1,\ 1a = x1\}$$

is a regular semigroup with associate subgroup $\mathcal{M}^* = \{(1, g, 1) \mid g \in G\}$, isomorphic to G, satisfying (M1) and (M2).

Conversely, if S contains an associate subgroup S^* satisfying (M1) and (M2), by Theorem 2.13, $S \cong \mathcal{M}(S^*; A, B; P)$ where the bands A and B and the mapping P are defined as in the proof of this theorem. The result follows. □

Corollary 3.3. *If A and B are left and right zero semigroups, respectively, in the above construction then $\mathcal{M}(G; A, B; P)$ is a normalized Rees-matrix semigroup.*

Corollary 3.4. *A semigroup S is completely simple if and only if it has an associate subgroup S^* satisfying $ss^*tt^* = ss^*$ and $s^*st^*t = t^*t$, for all $s, t \in S$.*

Proof. Let S be a completely simple semigroup. Then S is isomorphic to a Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ in which the matrix $P = [p_{\lambda i}]$ is normalized[[5], Theorem 3.4.2].

We can consider the sets I and Λ to be a left zero semigroup and a right zero semigroup, respectively. Let $P : \Lambda \times I \to G$ be defined by $p_{\lambda,i} = p_{\lambda i}$. Simple calculations show that all the conditions in the above construction are satisfied. Then, by Theorem 3.2, S has an associate subgroup S^* isomorphic to G. Moreover, $I \cong \{ss^* \mid s \in S\}$ and $\Lambda \cong \{s^*s \mid s \in S\}$.

The converse follows immediately from Theorem 3.2 and Corollary 3.3. □

Remark 3.5. *Note that* $ss^*tt^* = ss^*$ *and* $s^*st^*t = t^*t$ *imply* $(M1)$*. In fact, if* $ss^*tt^* = ss^*$ *then*

$$st(st)^* = ss^* \cdot st(st)^* = ss^* = ss^*tt^*.$$

Similarly, if $s^*st^*t = t^*t$ *then* $(st)^*st = s^*st^*t$*.*

Remark 3.6. *If, in Construction 2.6, A and B are assumed to be left and right zero, respectively, then $E(G)$, as common associate subsemilattice of A and B, must be trivial, whence G is a group.*

We proceed with the case where S is orthodox.

Lemma 3.7. *Let S be a semigroup with associate inverse subsemigroup S^* which satisfies (M1) and (M2). Then, S^* is orthodox if and only if $(st)^* = t^*s^*$.*

Proof. If S is orthodox, t^*s^* is an associate of st and so $(st)^* \leq t^*s^*$. To prove that $t^*s^* \leq (st)^*$ we show first that $t^*s^*(st)^{**} \in E(S^*)$. In fact, since

$$\begin{aligned}
(s^*stt^*)^* &= (s^*stt^*s^*stt^*)^* && \\
&= (s^*stt^*)^*t^{**}(s^*st)^* && [(\text{M2})] \\
&= (s^*stt^*)^*(s^*stt^*)^*, && [\text{Lemma } 2.5]
\end{aligned}$$

$(s^*stt^*)^{**} = (s^*stt^*)^*$ is also an idempotent of S^*. Then, by [[6], 1.4 Proposition 1], it follows from

$$\begin{aligned}
t^*s^*(st)^{**} &= t^*s^*(t^*(s^*stt^*)^*s^*)^* && [\text{Lemma } 2.4] \\
&= t^*s^*s^{**}(s^*stt^*)^{**}t^{**} && [S^* \text{ inverse}]
\end{aligned}$$

that $t^*s^*(st)^{**} \in E(S^*)$. Now we have:

$$
\begin{aligned}
t^*s^* &= t^*t^{**}t^*s^*s^{**}s^* \\
&= t^*s^*s^*s^{**}t^{**}t^* && [S^* \text{ Clifford}] \\
&= t^*s^*(s^*st^*t)^{**} && [\text{(M2)}] \\
&= t^*s^*((st)^*st)^{**} && [\text{(M1)}] \\
&= t^*s^*(st)^*(st)^{**} && [\text{Lemma 2.3}] \\
&= t^*s^*(st)^{**}(st)^* && [S^* \text{ Clifford}] \\
&\leq (st)^* && [t^*s^*(st)^{**} \in E(S^*)]
\end{aligned}
$$

Conversely, suppose that $(st)^* = t^*s^*$ and let $e, f \in E(S)$. Then,

$$e^*e^* = (ee)^* = e^*$$

and similarly $f^*f^* = f^*$. Hence,

$$((ef)^2)^* = (ef)^*(ef)^* = f^*e^*f^*e^* = f^*f^*e^*e^* = f^*e^* = (ef)^*$$

and so, by Corollary 2.14, $ef \in E(S)$ and therefore S is orthodox. □

Theorem 3.8. *Let S be a semigroup with associate inverse subsemigroup S^* which satisfies (M1) and (M2). Then the following are equivalent:*

(i) S is orthodox;

(ii) The representation $\mathcal{M}(G; A, B; P)$ yields a subdirect product of $A \times G \times B$, i.e., $gp_{x,b}h = gh$ for all $g, h \in G$, $b \in A$, $x \in B$ with $x^* = gg^{-1}$, $b^* = hh^{-1}$.

Proof. $(i) \implies (ii)$. If S is orthodox, $(st)^* = t^*s^*$. Then,

$$(st)^{**} = s^{**}(s^*stt^*)^{**}t^{**} = s^{**}s^*s^{**}t^{**}t^*t^{**} = s^{**}t^{**}.$$

The result follows by Theorem 2.13.

$(ii) \implies (i)$. Clear. □

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