

# THE VARIETY OF UNARY SEMIGROUPS WITH ASSOCIATE INVERSE SUBSEMIGROUP

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ABSTRACT. In [1] the authors introduced the notion of an associate inverse subsemigroup of a regular semigroup. This is a subsemigroup  $T$  of a regular semigroup  $S$  containing a least associate of each  $s \in S$ , in relation to the natural partial order  $\leq$ . In this paper we show that the class of regular semigroups containing an associate inverse subsemigroup can be considered as a variety of unary semigroups.

## 1. INTRODUCTION

In a semigroup  $S$ , an element  $t \in S$  is an associate of  $s \in S$  if  $s = sts$ . The concept of an associate inverse subsemigroup of a regular semigroup  $S$  was introduced in [1] and extends the concept of an associate subgroup of a semigroup first presented in [2]. An associate inverse subsemigroup of a regular semigroup  $S$  is a subsemigroup  $S^*$  of  $S$  containing a least associate  $x^*$  of each  $x \in S$ , in relation to the natural partial order  $\leq$ . Due to a simple characterisation of inverse semigroups in terms of the natural partial order on an arbitrary semigroup, such a semigroup  $S^*$  is necessarily inverse:

**Theorem 1.1.** [[1], Theorem 2.1] *Let  $S$  be a regular semigroup. The following are equivalent:*

- (i)  $S$  is inverse;
- (ii) for all  $a \in S$ , the set  $\{x \in S : a = axa\}$  contains a least element with respect to the natural partial order.

An axiomatic characterisation of a regular semigroup containing an associate inverse semigroup as a unary semigroup was established in [1]:

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**Theorem 1.2.** [[1], Theorem 2.5] *A regular semigroup  $S$  contains an associate inverse subsemigroup if and only if it has a unary operation  $x \mapsto x^*$  satisfying, for all  $s, t \in S$ ,*

- (i)  $s = ss^*s$ ;
- (ii)  $s^*t^* = (s^*t^*)^{**}$ ;
- (iii)  $s = st^*s \Rightarrow s^* \leq t^*$ .

This characterisation extends the axiomatic characterisation of semigroups with an associate subgroup, established in [[5], Theorem 3.1].

It is easily seen that some subclasses of the class of all semigroups with associate inverse subsemigroup can be defined in terms of identities and, therefore, form varieties of unary semigroups. We give some examples.

**Example 1.1.** Let  $S$  be a semigroup with a unary operation  $x \mapsto x^*$  satisfying, for all  $s, t \in S$ ,

- (1)  $s = ss^*s$ ;
- (2)  $s^*t^* = (s^*t^*)^{**}$ ;
- (3)  $s^*s^{**}t^*t^{**} = t^*t^{**}s^*s^{**}$ ;
- (4)  $(st)^* = t^*s^*$ .

By (1) and (4),  $S^* = \{s^* \mid s \in S\}$  is a regular subsemigroup of  $S$ . Also, for each  $s \in S$ , we have

$$\begin{aligned}
 s^* &= s^*(s^{**}s^*) && [(1)] \\
 &= s^*(ss^*)^* && [(4)] \\
 &= (s^*(ss^*)^*)^{**} && [(2)] \\
 &= (s^*s^{**}s^*)^* && [(4)] \\
 &= s^{***}. && [(1)]
 \end{aligned}$$

Moreover, for each idempotent  $s^*$  of  $S^*$ , we have

$$\begin{aligned}
 s^{**} &= s^{**}s^{***}s^{**} && [(1)] \\
 &= s^{**}s^*s^{**} && [s^{***} = s^*] \\
 &= s^{**}s^*s^*s^{**} && [s^* \text{ is idempotent}] \\
 &= s^*s^{**}s^{**}s^* && [(3) \text{ and } s^{***} = s^*] \\
 &= s^*(s^*s^*)^*s^* && [(4)] \\
 &= s^*s^{**}s^* && [s^* \text{ idempotent}] \\
 &= s^* && [(1)]
 \end{aligned}$$

and so  $s^* = s^*s^* = s^*s^{**}$ . It now follows from (3) that the subsemigroup  $S^*$  is inverse. We show next that  $s^*$  is the least associate of  $s \in S$  in  $S^*$ , since for any associate  $t^*$  of  $s$ ,

we have

$$\begin{aligned}
s = st^*s &\Rightarrow s^* = s^*t^{**}s^* && [(4)] \\
&\Rightarrow s^{**} \leq t^{**} && [S^* \text{ inverse; Theorem 1.1}] \\
&\Rightarrow s^* \leq t^* && [S^* \text{ inverse and } s^* = (s^{**})^{-1}]
\end{aligned}$$

Hence  $S$  is a regular semigroup with associate inverse subsemigroup  $S^* = \{s^* : s \in S\}$ .

Observe that this class of semigroups contains the class of inverse semigroups. In fact,  $S$  is an inverse semigroup if and only if  $S$  satisfies (1)-(4) and  $s^{**} = s$  for all  $s, t \in S$ .

**Example 1.2.** Let  $S$  be a semigroup with a unary operation  $x \mapsto x^*$  satisfying, for all  $s, t, u \in S$ ,

- (1)  $s = ss^*s$ ;
- (2)  $s^*t^* = (s^*t^*)^{**}$ ;
- (3)  $s^*s^{**}t^*t^{**} = t^*t^{**}s^*s^{**}$ ;
- (4)  $(su^*t)^* = t^*u^{**}s^*$ .

As in Example 1.1,  $S^*$  is an inverse subsemigroup of  $S$ . Also, for any associate  $t^*$  of  $s$ , we have, by (4),

$$s^* = (st^*s)^* = s^*t^{**}s^*$$

and, as in Example 1.1,  $s^* \leq t^*$ . Thus  $S$  is a regular semigroup with associate inverse subsemigroup  $S^* = \{s^* : s \in S\}$ .

This class of semigroups contains the class of completely simple semigroups. In fact, if  $S$  is the Rees matrix semigroup  $\mathcal{M}(G; I, \Lambda; P)$ , with  $P$  normalized, and  $(i, g, \lambda)^* = (1, g^{-1}, 1)$ , routine calculations show that  $S$  satisfies (1)-(4).

The forgoing examples naturally arise the question of whether the whole class of unary semigroups with an associate inverse subsemigroup is a variety of unary semigroups. The answer to this question is the main result of this paper and is presented in section 2.

## 2. A VARIETY OF UNARY SEMIGROUPS

We begin with a characterisation of the natural partial order on an inverse semigroup in terms of an identity.

**Lemma 2.1.** *Let  $V$  be an inverse semigroup and  $a, b \in V$ . Then*

$$a \leq b \Leftrightarrow bb^{-1} \cdot a \cdot b^{-1} \cdot a = a \Leftrightarrow a \cdot b^{-1} \cdot a \cdot b^{-1}b = a,$$

where  $\leq$  is the natural partial order.

*Proof.* We show that  $a \leq b \Leftrightarrow bb^{-1} \cdot a \cdot b^{-1} \cdot a = a$ . Let  $a, b \in V$ . Suppose first that  $a \leq b$ . Then  $a = be$ , for some  $e \in E(V)$ , and so  $bb^{-1}a = a$ . Since  $b^{-1}a = (b^{-1}b)e$  and  $V$  is inverse,  $b^{-1}a \in E(V)$ . So

$$bb^{-1}ab^{-1}a = bb^{-1}a = a.$$

Conversely, suppose that  $bb^{-1}ab^{-1}a = a$ . Then  $b^{-1}ab^{-1}a = b^{-1}a$ , giving  $b^{-1}a \in E(V)$ . Thus  $b(b^{-1}a) = a$  gives  $a \leq b$ .

Similarly,  $a \leq b \Leftrightarrow a \cdot b^{-1} \cdot a \cdot b^{-1}b = a$ . □

We are now in conditions to prove the following axiomatic characterisation of regular semigroups with an associate inverse subsemigroup using only identities.

**Theorem 2.2.** *Let  $S$  be a semigroup with a unary operation  $x \mapsto x^*$  satisfying, for all  $s, t \in S$ , the following axioms:*

- (1)  $s = ss^*s$ ;
- (2)  $s^*t^* = (s^*t^*)^{**}$ ;
- (3)  $s^*s^{**}t^*t^{**} = t^*t^{**}s^*s^{**}$ ;
- (4)  $t^*t^{**} \cdot (s(st^*s)^*st^*s(st^*s)^*s)^* \cdot t^{**} \cdot (s(st^*s)^*st^*s(st^*s)^*s)^* = (s(st^*s)^*st^*s(st^*s)^*s)^*$ .

*Then  $S$  is a semigroup with associate inverse subsemigroup  $S^* = \{s^* \mid s \in S\}$ .*

*Conversely every semigroup  $S$  with associate inverse subsemigroup  $S^* = \{s^* : s \in S\}$  where, for each  $s \in S$ ,  $s^*$  denotes the least associate of  $s$  in  $S^*$ , has a unary operation  $x \mapsto x^*$  satisfying axioms (1) - (4).*

*Proof.* Assume that there is a unary operation satisfying axioms (1)-(4). From (1), (2) and (3) it follows that  $S^*$  is an inverse subsemigroup of  $S$ . Let now  $s, t \in S$  be such that  $st^*s = s$ . We then have, by (4), that

$$t^*t^{**}(ss^*ss^*s)^*t^{**}(ss^*ss^*s)^* = (ss^*ss^*s)^*$$

and so, by (1),

$$t^*t^{**}s^*t^{**}s^* = s^*,$$

which, by Lemma 2.1, gives  $s^* \leq t^*$  and so, by Theorem 1.2,  $S^*$  is an associate inverse subsemigroup of  $S$ .

Conversely, if  $S^*$  is an associate inverse subsemigroup of  $S$ , conditions (1) - (3) hold. In order to prove (4), let  $s, t \in S$ . Then,  $(st^*s)^*st^*s$  is an idempotent and so

$$[s(st^*s)^*st^*s(st^*s)^*s] \cdot t^* \cdot [s(st^*s)^*st^*s(st^*s)^*s] = s(st^*s)^*st^*s(st^*s)^*s,$$

that is,  $t^*$  is an associate of  $s(st^*s)^*st^*s(st^*s)^*s$  in  $S^*$ . Thus  $(s(st^*s)^*st^*s(st^*s)^*s)^* \leq t^*$ . Axiom (4) now follows by Lemma 2.1. □

**Corollary 2.3.** *The class of regular semigroups containing an associate inverse subsemigroup is a variety of unary semigroups.*

**Remark 2.4.** *The natural partial order in a group is trivial and so, in case of  $S^*$  being an associate subgroup of  $S$ , because of Theorem 2.2 and its proof, (4) simplifies to*

$$(s(st^*s)^*st^*s(st^*s)^*s)^* = t^*.$$

*Also, a semigroup with a unary operation  $x \mapsto x^*$  satisfying (1), (2) and (4), for all  $s, t \in S$ , is a regular semigroup such that each element has a unique associate. Such semigroup contains a single idempotent and is, therefore, a group. Hence the following axiomatic characterisation of a semigroup containing an associate subgroup involving just identities holds.*

**Theorem 2.5.** *A regular semigroup  $S$  contains an associate subgroup if and only if it has a unary operation  $x \mapsto x^*$  satisfying, for all  $s, t \in S$ ,*

- (1)  $s = ss^*s$ ;
- (2)  $s^*t^* = (s^*t^*)^{**}$ ;
- (3)  $(s(st^*s)^*st^*s(st^*s)^*s)^* = t^*$ .

**Corollary 2.6.** *The class of regular semigroups containing an associate subgroup is a subvariety of the variety of regular semigroups containing an associate inverse subsemigroup.*

#### REFERENCES

- [1] B. Billhardt, E. Giraldez, P. Marques-Smith and P. Mendes Martins, *Associate inverse subsemigroups of regular semigroups*. Semigroup Forum **79** (2009) 101-118
- [2] T. S. Blyth, Emília Giraldez and M. Paula O. Marques-Smith. Associate subgroups of orthodox semigroups. Glasgow Math J. 36 (1994) 163-171
- [3] T. S. Blyth, P. Mendes Martins, *On associate subgroups of regular semigroups*. Communications in Algebra **25** (7) (1997), 2147-2156
- [4] J. M. Howie, *Fundamentals of semigroup theory*. Clarendon Press, Oxford (1995)
- [5] P. Mendes Martins, M. Petrich, *Unary Semigroups with an Associate Subgroup*, Communications in Algebra, **36** (6) (2008), 1999–2013

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