Regularity of Elastic Fields in Composites*

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Summary. It is well known that high stress concentrations can occur in elastic composites in particular due to the interaction of geometrical singularities like corners, edges and cracks and structural singularities like jumping material parameters. In the project C5 *Stress concentrations in heterogeneous materials* of the SFB 404 it was mathematically analyzed where and which kind of stress singularities in coupled linear and nonlinear elastic structures occur. In the linear case asymptotic expansions near the geometrical and structural peculiarities are derived, formulae for generalized stress intensity factors included. In the nonlinear case such expansions are unknown in general and regularity results are proved for elastic materials with power-law constitutive equations with the help of the difference quotient technique combined with a quasi-monotone covering condition for the subdomains and the energy densities. Furthermore, some applications of the regularity results to shape and structure optimization and the Griffith fracture criterion in linear and nonlinear elastic structures are discussed. Numerical examples illustrate the results.

Key words: Regularity, elasticity, composites, stress singularities, Ramberg-Osgood model

1 Introduction

Composites play an important role in the everyday life, examples are fiber-reinforced composites in car industry, piezo-electric stack actuators or semiconductor devices. From experience and experiments it is well known that very high mechanical stresses can occur in the composite in the vicinity of reentrant corners, edges, cracks and near interior surfaces, where the different materials of the composite come together. These stress concentrations have a strong influence on the strength and physical life of the structure. Their knowledge is fundamental for fracture and failure criteria.

* Research Project C5 "Stress Singularities in Heterogeneous Materials"

The project C5 *Stress concentrations in heterogeneous materials* was devoted to the mathematical analysis of stress singularities in linear and nonlinear elastic coupled nonsmooth structures. In the first years of this project we have focused on linear problems whereas in the last period we have concentrated on some nonlinear boundary transmission problems.

In this article we give an overview on the regularity results, shortly for linear and more detailed for nonlinear elastic composites. We consider bodies which are composed of several elastic substructures with different material properties. The whole body as well as the interfaces, which separate the substructures, may have corners or edges. Throughout the whole paper we assume small strains and consider constitutive laws which lead to linear elliptic systems of partial differential equations or to quasilinear elliptic systems of p-structure for the displacement fields. These PDEs have piecewise constant coefficients due to the heterostructure of the composite.

The Linear Case

In this case the substructures consist of linear elastic materials and the singular behavior of the displacement and stress fields can be completely characterized by means of an asymptotic expansion of the solution near the mentioned geometrical and structural peculiarities. In two dimensions this expansion reads in the neighborhood of a corner point or an interior cross point S for a displacement field u (polar coordinates with respect to S are used):

$$\eta^S u(r,\varphi) = u_{\text{reg}}(r,\varphi) + \eta^S \sum_{0<\Re\alpha<1} c_\alpha r^\alpha v_\alpha(\ln r,\varphi) \,. \tag{1}$$

The singular exponents $\alpha \in \mathbb{C}$ are eigenvalues of a corresponding nonlinear eigenvalue problem and the functions v_α consist of (generalized) eigenfunctions and powers of $\ln r$. The constants c_α are generalized stress intensity factors and depend on the given external loading. η^S is a cut-off function with respect to S and u_{reg} is a regular function. The regularity of the solution u is determined by the singular exponent α with the smallest positive real part. This exponent and the corresponding function v_α can be explicitly calculated for fixed geometries and material parameters and do not depend on the given external forces.

Expansions like (1) are well established for homogeneous materials and, more general, for linear elliptic systems with smooth coefficients, see for example [12, 20, 36]. In Sect. 2 we will demonstrate, that the Mellin-technique as an appropriate mathematical tool guarantees such expansions for solutions of general elliptic boundary-transmission problems in composites, too [43, 44]. Furthermore, we will formulate explicit formulae for the constants c_α in this case and give a numerical example. The computation of the singular terms in (1) is technically complicated in general and very small real parts of the exponents α can appear. In [27, 29, 45], we derived a criterion for linear elliptic systems with piecewise constant coefficients, which guarantees Re $\alpha \geq \frac{1}{2}$ in the two dimensional case and similar results for higher dimensions. This condition, the quasi-monotonicity condition, was originally defined for the Laplace operator with piecewise constant coefficients in a completely different context [16]. Its relevance for the regularity of weak solutions was discovered in [48] for the two dimensional Laplacian and in [45] for isotropic bi-materials. We discuss this condition in Sect. 2.4.

The Nonlinear Case

Much less is known about the regularity of displacement and stress fields of nonlinear elastic materials. For some classes of semilinear and quasilinear systems of partial differential equations (e.g. stationary Navier-Stokes equations, semiconductor equations) it can be shown [1, 6, 49] that the regularity of the solutions is dominated by assigned linearized problems. Furthermore, a comparison principle and barrier functions are used for a special class of scalar nonlinear equations on two-dimensional domains (see e.g. [8, 15, 55]) to get similar results.

Nonlinear elastic field equations of power-law type do not fit in this framework in general and it is an open question whether the behavior of the elastic fields can be completely characterized by leading terms in an asymptotic expansion similar to the linear case (1). First investigations into this direction were done at the end of the sixties in [22, 50] for homogeneous materials of Ramberg-Osgood type (power-law models). In order to describe the elastic fields near a crack tip an ansatz (HRR-field) of the form

$$u = r^{\alpha} v_{\alpha}(\varphi) \tag{2}$$

was inserted into the corresponding field equations and led to a fully nonlinear eigenvalue problem for the determination of the eigenpairs (α, v_α).

The field equations of the Ramberg-Osgood model are closely related to general systems of quasilinear elliptic partial differential equations of p-structure, see e.g. [18] for a definition. In [18, 19, 52], Ebmeyer, Frehse and Savaré obtained independently global regularity results for weak solutions of such systems on nonsmooth domains with a difference quotient technique. The difference quotient method is also applicable to the Ramberg-Osgood model [31, 32] and we cite and discuss the corresponding results in Sect. 4.2.

At the beginning of the last research period of the SFB 404 only very few regularity results for transmission problems of p-growth were reported in the literature; e.g. two subdomains were considered with either smooth interface and different p_i [39] or with nonsmooth interface and $p_1 = p_2 = 2$ [52]. In Sect. 3 we will present recently derived regularity results for transmission problems in composites where on the substructures Ω_i we have different quasilinear elliptic systems of p_i-structure. We do not restrict the number of subdomains and the growth properties of the differential operators may vary from subdomain to subdomain.

For obtaining the regularity results, the main idea is to combine the difference quotient technique with the concept of a quasi-monotone distribution of material parameters known from the linear problems. This leads to the new concept of a quasi-monotone covering condition for the subdomains and the energy densities which determine the differential operators on the subdomains. The very special case, the linear Laplace equation with piecewise constant coefficients, is included and our general quasi-monotone covering condition coincides in this case up to an additional geometric condition with the original definition of quasi-monotonicity from [16].

In the last section of this paper we discuss applications of the regularity results for linear and nonlinear elastic problems. These are shape and structure optimization problems in nonsmooth domains, sensitivity analysis for compound elastic structures and the Griffith fracture criterion for a nonlinear elastic model of power-law type. Relying on the proved regularity results formulae for shape derivatives and the energy release rate are derived which are suitable for computations.

2 Linear Elastic Composites

2.1 Weak Formulation

We start with the weak formulation of the elastic field equations in a composite. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain consisting of pairwise disjoint subdomains $\Omega_i \subset \Omega$, $1 \le i \le M$, with $\overline{\Omega} = \cup_{i=1}^M \overline{\Omega_i}$. We assume for simplicity that Ω_i are Lipschitz domains of polygonal or polyhedral type. We distinguish between exterior boundary pieces

$$\Gamma_i = \mathrm{int}\,(\partial\Omega_i \cap \partial\Omega) \tag{3}$$

and interior boundary pieces, the common interfaces of Ω_i and Ω_j,

$$\Gamma_{ij} = \mathrm{int}\,(\partial\Omega_i \cap \partial\Omega_j)\ . \tag{4}$$

In each domain Ω_i we consider the equilibrium equations for two classes of fields: potential fields (antiplanar case, (5)) and linear, anisotropic elastic fields, that means (6)

$$-\mathrm{div}\,(\mu_i \nabla u_i) = f_i \quad i = 1, \ldots, M\ , \tag{5}$$

$$-\mathrm{div}\,\sigma^i(u^i) = F_i \quad i = 1, \ldots, M\ . \tag{6}$$

Here, μ_i are given positive constants (shear moduli), u_i in (5) are the scalar potentials, $u^i = (u^i_1, \ldots, u^i_d)^\top$ in (6) the displacement fields and $\sigma^i = (\sigma^i_{kj})_{kj} \in \mathbb{R}^{d\times d}$ are the stress tensors. For small strains, Hooke's law yields

$$\sigma^i_{kj}(u^i) = \sum_{m,n=1}^{d} C^i_{kjmn}\varepsilon_{mn}(u^i), \quad i = 1, \ldots, M,\ k, j = 1, \ldots, d\ ,$$

where $\varepsilon_{mn}(u^i) = \frac{1}{2}(\frac{\partial u^i_m}{\partial x_n} + \frac{\partial u^i_n}{\partial x_m})$ are the components of the linearized strain tensor $\varepsilon(u^i) \in \mathbb{R}^{d\times d}$. It is assumed that the material tensors C^i are symmetric and positive definite

$$(C^i\xi) : \xi = \sum_{k,j,m,n=1}^{d} C^i_{kjmn}\xi_{kj}\xi_{mn} \geq M_i \sum_{k,j=1}^{d} |\xi_{kj}|^2 \tag{7}$$

for every $\xi = (\xi_{kj})_{kj} \in \mathbb{R}^{d\times d}$. We consider Dirichlet and Neumann boundary conditions on $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$. These conditions read for (6)

$$\sigma\mathbf{n} = g_i \quad \text{on } \Gamma_N \ , \tag{8}$$

$$u = 0 \quad \text{on } \Gamma_D \ , \tag{9}$$

where $\sigma = \sigma^i$ on $\Gamma_N \cap \Gamma_i$, $u = u^i$ on $\Gamma_D \cap \Gamma_i$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Furthermore, we assume that the subdomains Ω_i are bonded, which is expressed by the following transmission conditions on the skeleton $\Gamma = \cup_{i,j=1}^{M}\Gamma_{ij}$ ($\mathbf{n}_i$ denotes the exterior unit normal vector on Γ_{ij} with respect to Ω_i):

$$u^i = u^j \qquad \text{on } \Gamma_{ij} \ , \tag{10}$$

$$\sigma^i\mathbf{n}_i = \sigma^j\mathbf{n}_i \quad \text{on } \Gamma_{ij} \ . \tag{11}$$

We are now in a position to formulate the boundary-transmission problems in a weak sense. Let

$$V = \{u \in W^{1,2}(\Omega) : \, u\big|_{\Gamma_D} = 0\} \ , \tag{12}$$

where the restriction to Γ_D is to be understood in the trace-sense. We assume that $f \in V'$, $f_i = f$ on Ω_i, $g \in W^{-\frac{1}{2},2}(\Gamma_N)$, $g = g_i$ on $\Gamma_N \cap \Gamma_i$. The weak formulation reads

$$\text{Find } u \in V \text{ such that for every } v \in V$$
$$a(u,v) = \langle f, v\rangle_\Omega + \langle g, v\rangle_{\Gamma_N} \ , \tag{13}$$

where for the Poisson equations (5)

$$a(u,v) = \int_\Omega \mu(x)\nabla u \nabla v \, \mathrm{d}x = \sum_{i=1}^{M} \int_{\Omega_i} \mu_i \nabla u_i \nabla v_i \, \mathrm{d}x \tag{14}$$

and for the equations of linear elasticity (6)

$$a(u,v) = \int_\Omega \sigma(u) : \varepsilon(u) \, \mathrm{d}x = \sum_{i=1}^{M} \int_{\Omega_i} \sigma^i(u^i) : \varepsilon(u^i) \, \mathrm{d}x \ . \tag{15}$$

Both problems, (13)+(14) and (13)+(15), respectively, are elliptic boundary-transmission problems (see [43, 51]) due to the assumptions on the material parameters. Therefore it holds

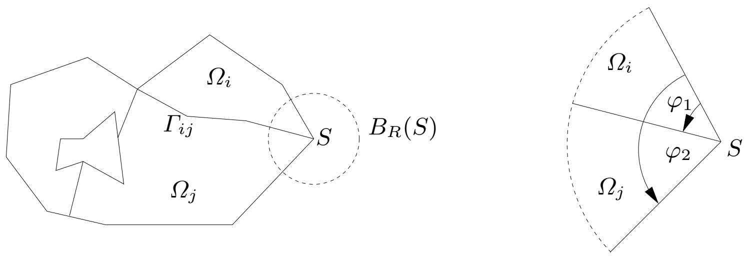

Fig. 1. Polygonal domain

1. If meas $\Gamma_D > 0$, then there exists a unique solution $u \in V$ of (13).
2. If meas $\Gamma_D = 0$ and $\langle f, r\rangle_\Omega + \langle g, r\rangle_{\Gamma_N} = 0$ for every $r \in \mathcal{R} \subset W^{1,2}(\Omega)$, where $\mathcal{R}$ is the set of rigid body displacements, then there exists a solution $u \in W^{1,2}(\Omega)$ which is unique up to elements from $\mathcal{R}$.

If Ω is a homogeneous body with a smooth boundary, then the regularity of a weak solution of problem (13) is determined by the regularity of the data f, g. But, if Ω is a composite and if the boundary of Ω has corners and edges, then such a regularity is not longer valid even if f and g are smooth. The geometric (corners, edges) and structural singularities (discontinuous material parameters) have an essential influence on the regularity of the solutions. Weak solutions of boundary-transmission problems of the form (13) can be decomposed into singular terms describing the behavior near corners, edges and jumps of the material parameters and into a more regular term. This decomposition can be described and calculated by Mellin-techniques, similar to pure boundary value problems. Special ansatzes frequently used in mechanics [23, 57, 58, 59] lead also to such results. In the following section we outline the use of the ansatzes and the Mellin technique for two dimensional transmission problems and we mention results for the 3D-case.

2.2 2D-Corner Singularities

Let $\mathcal{S}$ be the set of the corner points of $\partial\Omega_i$ and of the points of $\overline{\Gamma_D} \cap \overline{\Gamma_N}$, see Fig. 1. For a fixed point $S \in \mathcal{S}$ we introduce polar coordinates (r, φ) with respect to S and search a corner singularity of the form

$$u_{\text{sing}}(r, \varphi) = r^\alpha \Phi_\alpha(\varphi) , \tag{16}$$

The corner singularity (16) has to be a solution of

$$a(u_{\text{sing}}, v) = 0 \tag{17}$$

for $v \in V$ with $\operatorname{supp} v \subset \Omega_S = \overline{\Omega} \cap B_R(S)$, where $B_R(S)$ is an open ball centered at S with appropriate small radius and $a(\cdot, \cdot)$ is the bilinear form from (14) or (15). Choosing the special test function $v = \eta(r)\Psi(\varphi)$, where

$\Psi \in V_\varphi = \{\Psi \in W^{1,2}(0,\varphi_2) : \Psi|_{\Gamma_D} = 0\}$ and $\eta \in \mathcal{C}_0^\infty((-R,R))$, we obtain from (17) a bilinear form which depends on $S \in \mathcal{S}$ and the parameter α:

$$a_S(\alpha; \Phi_\alpha, \Psi) = 0 \; , \tag{18}$$

see for example [37, 41, 45] for explicit formulae. Relation (18) defines a quadratic eigenvalue problem and we refer to Sect. 2.4 for some examples.

By this procedure one can generate weak solutions having a singular behavior of the form (16) near a point $S \in \mathcal{S}$. Now the question is whether terms of the form (16) characterize the singular behavior of weak solutions completely. This problem can be answered using an appropriate integral transform, the Mellin transform [37]:

$$\mathcal{M}(u(r,\varphi))[\alpha] = \hat{u}(\alpha,\varphi) = \int_0^\infty r^{-\alpha-1} u(r,\varphi)\,\mathrm{d}r \; . \tag{19}$$

This transform maps $r\partial_r$ into the complex parameter α. The Mellin transform applied to the differential operators in (5), (6), (8)–(11), generates a Fredholm operator pencil $\mathcal{A}_S(\alpha)$ with

$$\mathcal{A}_S(\alpha) : V_\varphi \to V'_\varphi \; . \tag{20}$$

For explicit formulas for $\mathcal{A}_S(\alpha)$ see the examples in [37, 45]. The operator pencil $\mathcal{A}_S(\alpha)$ and the bilinear form $a_S(\alpha;\cdot,\cdot)$ from (18) are related as follows

$$\langle \mathcal{A}_S(\alpha)\Phi, \Psi\rangle = a_S(\alpha; \Phi, \Psi) \quad \text{for every } \Phi, \Psi \in V_\varphi \; . \tag{21}$$

The corresponding quadratic eigenvalue problem $\mathcal{A}_S(\alpha)\Phi_\alpha = 0$ has a finite number of eigenvalues in any strip [37, 45]

$$c_1 \le \mathrm{Re}\ \alpha \le c_2 \; . \tag{22}$$

The following regularity theorem is proved with the Mellin technique and connects the eigenvalue problems (18) to the global regularity of weak solutions.

Theorem 1 (Regularity theorem). *[20, 36, 37, 45] Let the volume force densities f of* (5) *and F of* (6) *be elements of $W^{l,2}(\Omega)$ and let the Neumann datum g be in $W^{l+\frac{1}{2},2}(\Gamma_N)$ with $l \in \mathbb{N}_0$. Assume that $\mathcal{A}_S(\alpha)$ is invertible on the line* Re $\alpha = l+1$. *Then the weak solution $u \in V$ of the boundary transmission problem admits the following decomposition:*

$$u = u_{reg} + \sum_{\substack{S\in\mathcal{S},\\ \gamma\in\Lambda_S}} \eta_S c_\gamma^S v_\gamma^S(r,\varphi) \; . \tag{23}$$

Here, $u_{reg}\big|_{\Omega_i} \in W^{l+2,2}(\Omega_i)$ and

$$\Lambda_S = \{\gamma = (\alpha,\mu,\kappa) : \alpha \textit{ is an eigenvalue of } \mathcal{A}_S(\alpha) \textit{ in the strip } \\ 0 < \mathrm{Re}\ \alpha < l+1;\ \mu = 1,\dots,I_\alpha^S;\ \kappa = 0,\dots,M_{\alpha,\mu}^S\} \; . \tag{24}$$

I_α^S denotes the geometrical multiplicity of α, $\{\Phi^S_{\alpha,\mu,\kappa},\ \mu = 1,\dots,I_\alpha^S; \kappa = 0,\dots,M^S_{\alpha,\mu}\}$ is a canonical system of Jordan chains of $\mathcal{A}_S(\alpha)$ with respect to eigenvalue α, $M^S_{\alpha,\mu}+1$ are the lengths of the Jordan chains, η_S are cut-off functions which equal to 1 near S and the singular functions v^S_γ are of the form

$$v^S_\gamma(r,\varphi) = r^\alpha \sum_{q=0}^{\kappa} \frac{(\ln r)^q}{q!} \Phi^S_{\alpha,\mu,\kappa-q}(\varphi)\,. \tag{25}$$

The constants c^S_γ are also called generalized stress intensity factors and depend on the data.

Coefficient Formulae

The coefficients c^S_γ in (23) express the intensity of the singular functions $v^S_\gamma(r,\varphi)$. In particular, they can vanish and then u is regular. Damage and crack criteria rely on these coefficients. The coefficients depend on the exterior forces, the elastic material parameters and on the geometry of the domain and can be calculated via so-called coefficient formulae. In the case of homogeneous materials these formulae are well known [37, 41]. In this section we derive analogous formulae of Mazya/Plamenevski type for potential fields in composites. Corresponding coefficient formulae for linear elastic fields in bonded structures are described in [5].

Lemma 1. *Let $\Omega, \Omega_1, \Omega_2 \subset \mathbb{R}^2$ be bounded polygons, $\overline{\Omega} = \cup_{i=1}^2 \overline{\Omega_i}$ and $S \in \partial\Omega \cap \partial\Omega_1 \cap \partial\Omega_2$ a corner point (see Fig. 1). For a weak solution u of the boundary-transmission problem* (13) *with bilinear form* (14) *and $M = 2$ the expansion* (23) *reads near the point S ($u_i = u\big|_{\Omega_i}$):*

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \eta_S c_0 r^{\alpha_0} M(\alpha_0) \begin{pmatrix} \sin(\alpha_0\varphi) \\ \cos(\alpha_0\varphi) \end{pmatrix} + \textit{remainder}\,. \tag{26}$$

Here, $M(\alpha_0)$ is a matrix depending on the kind of the boundary conditions near the corner S:

$$M(\alpha_0) = M_{D-D} = \begin{pmatrix} -\sin(\alpha_0(\varphi_2-\varphi_1)) & 0 \\ \cos(\alpha_0\varphi_2)\sin(\alpha_0\varphi_1) & -\sin(\alpha_0\varphi_2)\sin(\alpha_0\varphi_1) \end{pmatrix},$$

$$M(\alpha_0) = M_{N-N} = \begin{pmatrix} 0 & \cos(\alpha_0(\varphi_2-\varphi_1)) \\ \sin(\alpha_0\varphi_2)\cos(\alpha_0\varphi_1) & \cos(\alpha_0\varphi_1)\cos(\alpha_0\varphi_2) \end{pmatrix},$$

$$M(\alpha_0) = M_{N-D} = \begin{pmatrix} \cos(\alpha_0(\varphi_2-\varphi_1)) & 0 \\ \sin(\alpha_0\varphi_1)\sin(\alpha_0\varphi_2) & \sin(\alpha_0\varphi_1)\cos(\alpha_0\varphi_2) \end{pmatrix}.$$

The exponent α_0 is a zero of the equation

$$\mu_2\cos(\alpha(\varphi_2-\varphi_1))\sin(\alpha\varphi_1) + \mu_1\sin(\alpha(\varphi_2-\varphi_1))\cos(\alpha\varphi_1) = 0\ (D-D)\,,$$
$$\mu_2\cos(\alpha\varphi_1)\sin(\alpha(\varphi_2-\varphi_1)) + \mu_1\sin(\alpha\varphi_1)\cos(\alpha(\varphi_2-\varphi_1)) = 0\ (N-N)\,,$$
$$\mu_1\cos(\alpha\varphi_1)\cos(\alpha(\varphi_2-\varphi_1)) - \mu_2\sin(\alpha\varphi_1)\sin(\alpha(\varphi_2-\varphi_1)) = 0\ (N-D)\,.$$

The corresponding coefficient c_0 is given by

$$c_0 = \frac{1}{\alpha_0 K(\alpha_0)} \left(\int_\Omega \mu(f s_- + u \triangle s_-)\,\mathrm{d}x + \int_{\Gamma_N} \mu(g s_- - u\frac{\partial s_-}{\partial n})\mathrm{d}\sigma \right) , \quad (27)$$

where $s_- = \eta_S r^{-\alpha_0} M(\alpha_0)\,(\sin(\alpha_0\varphi), \cos(\alpha_0\varphi))^\top$ *and*

$$K(\alpha_0) = K_{D-D} = \mu_1\varphi_1 \sin^2\alpha_0(\varphi_2-\varphi_1) + \mu_2(\varphi_2-\varphi_1)\sin^2\alpha_0\varphi_1 \ ,$$
$$K(\alpha_0) = K_{N-N} = \mu_1\varphi_1 \cos^2\alpha_0(\varphi_2-\varphi_1) + \mu_2(\varphi_2-\varphi_1)\cos^2\alpha_0\varphi_1 \ ,$$
$$K(\alpha_0) = K_{N-D} = \mu_1\varphi_1 \cos^2\alpha_0(\varphi_2-\varphi_1) + \mu_2(\varphi_2-\varphi_1)\sin^2\alpha_0\varphi_1 \ .$$

For the meaning of the angles φ_1, φ_2 see Fig. 1.

Sketch of the proof. The formula (26) was derived via the Mellin technique in [45]. It remains to show (27): For a fixed cut-off function $\eta_S = \eta_S(r)$ we consider a family of balls $B_\delta(S) = \{x \in \mathbb{R}^2 : |x - S| = r < \delta\}$ such that $\eta \equiv 1$ on B_δ. We apply Green's formula on $\Omega_\delta = \Omega \backslash \overline{B_\delta(S)}$ and obtain

$$\begin{aligned}
&\int_{\Omega_\delta} (\mu(\Delta u s_- - u\Delta s_-)\,\mathrm{d}x \\
&= \sum_{i=1}^{2} \int_{\Omega_\delta \cap \Omega_i \cap \operatorname{supp}\eta_S} \mu_i(\Delta u_i s_{-,i} - u_i \Delta s_{-,i})\,\mathrm{d}x \\
&= \mu_1\delta \int_0^{\varphi_1} \left(u_1 \frac{\partial s_{-,1}}{\partial r} - \frac{\partial u_1}{\partial r} s_{-,1} \right) \mathrm{d}\varphi + \mu_2\delta \int_{\varphi_1}^{\varphi_2} \left(u_2 \frac{\partial s_{-,2}}{\partial r} - \frac{\partial u_2}{\partial r} s_{-,2} \right) \mathrm{d}\varphi \\
&\quad + \int_{\Gamma_N \cap \partial\Omega_\delta} \mu \left(g s_- - u \frac{\partial s_-}{\partial \mathbf{n}} \right) \mathrm{d}\sigma \ .
\end{aligned}$$

Inserting the expansion (26) of u and considering the limit $\delta \to 0$ we get the coefficient formula (27).

Computation of Stress Intensity Factors for Interface Cracks

As an example we consider the linear isotropic elasticity problem in a one sided clamped laminated structure with an interface crack, see Fig. 2. There are no volume forces and tensions of ± 2000 MPa. The Young moduli are $E_1 = 200000$ MPa and $E_2 = 400000$ MPa, the Poisson ratios are chosen as $\nu_1 = \nu_2 = 0.3$. The asymptotic expansion (23) near the crack tip S reads [5, 60]

$$u = \eta_S \left(c_1 r^{\frac{1}{2}+i\varepsilon} v_1(\varphi) + c_2 r^{\frac{1}{2}-i\varepsilon} v_2(\varphi) \right) + u_{\mathrm{reg}} \ ,$$

where

$$\varepsilon = \frac{1}{2\pi} \ln \frac{1+\beta}{1-\beta},\ \beta = \frac{\mu_2(1-2\nu_1) - \mu_1(1-2\nu_2)}{2(\mu_2(1-\nu_1) + \mu_1(1-\nu_2))} \ .$$

The coefficients c_i as well as the functions v_i are complex valued. The stress intensity factor $c_1 = K_1 + iK_2$ is computed via coefficient formulas similar to formula (27) and the real part and imaginary part are plotted in Fig. 2, [4]. Coefficient formulas for linear isotropic elasticity are derived in [5].

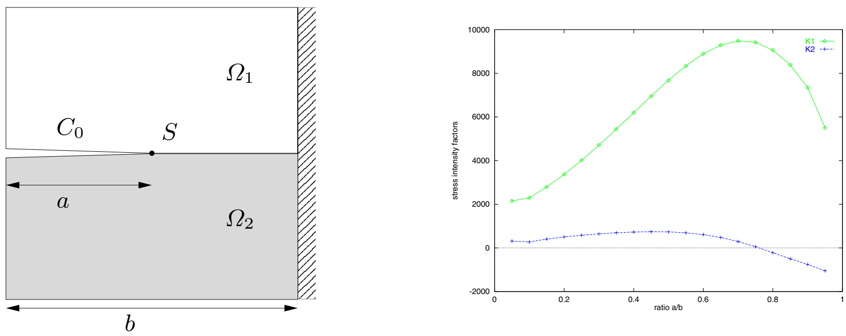

Fig. 2. **a)** interface crack, **b)** real part (K_1) and imaginary part (K_2) of c_1

2.3 3D Vertex and Edge Singularities

Let us assume that $\Omega_i \subset \mathbb{R}^3$ are polyhedrons and that the set $\Gamma_D \cap \Gamma_N$ consists of straight edges. We analyze now the behavior of the potential fields and elastic fields near the set $\mathcal{S}$:

$$\mathcal{S} = \{\text{edges of } \partial\Omega_i\} \cup \{\text{vertices of } \partial\Omega_i\} .$$

The *vertex singularities* can be described analogously to the 2D corner singularities taking an ansatz in spherical coordinates:

$$u_{\text{sing}}(r,\varphi,\theta) = r^{\alpha}\Phi_{\alpha}(\varphi,\theta) ,$$

where $r = |x - v|$ and v is a vertex as in Fig. 3. We proceed as in the 2D case and get (17), (18), (8) correspondingly and finally an operator pencil

$$\mathcal{A}_v(\alpha) : V(\Omega \cap S_R(v)) \to V(\Omega \cap S_R(v))' ,$$

where $S_R(v) = \{x \in \mathbb{R}^3 : |x - v| = R\}$ for sufficiently small fixed R and $V(\Omega \cap S_R(v)) = \{u \in W^{1,2}(\Omega \cap S_R(v)) : u|_{\Gamma_D} = 0\}$. There are finitely many eigenvalues of the Fredholm operator bundle $\mathcal{A}_v(\alpha)$ in any strip

$$c_1 \leq \text{Re } \alpha_v \leq c_2 .$$

We call an eigenvalue non-defective, if its algebraic and geometric multiplicities coincide. Assuming this for simplicity we get together with the eigensolutions $\Phi_{\alpha_v}(\varphi,\theta)$ singular vertex functions for weak solutions of the form

$$u_{\text{vertex}} = \sum_{-\frac{1}{2} < \text{Re } \alpha_v < \frac{1}{2}} \eta_v(r) c_{\alpha_v} r^{\alpha_v} \Phi_{\alpha_v}(\varphi,\theta) , \qquad (28)$$

where η_v is a cut-off function and c_{α_v} are constants which depend on the right hand sides.

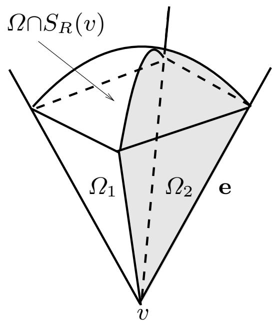

Fig. 3. The vertex neighborhood

The *edge singularities* are generated by 2D operator pencils. For a fixed edge $\mathbf{e}$, we introduce an orthogonal system of coordinates (y_1, y_2, y_3), where the y_3-axis is directed along the edge $\mathbf{e}$. We denote by $K_{\mathbf{e}}$ the straight plane angular cone of opening $\varphi_{\mathbf{e}}$ in the $\{y_1, y_2\}$-plane such that Ω coincides with $K_{\mathbf{e}} \times \mathbb{R}$ in a neighborhood of $\mathbf{e}$. Removing the derivative ∂_{y_3} in the operators L defined by (5) and (6) we get 2D-operators $L_{\mathbf{e}}$ in $K_{\mathbf{e}}$. Introducing polar coordinates $(\rho, \varphi_{\mathbf{e}})$ and applying the Mellin-technique we get edge-singularities of the weak solution of the following form provided the eigenvalues are nondefective:

$$u_{\text{edge}} = \sum_{0<\text{Re }\alpha_{\mathbf{e}}<1} c_{\alpha_{\mathbf{e}}}(y_3)\rho^{\alpha_{\mathbf{e}}}\Phi_{\alpha_{\mathbf{e}}}(\varphi_{\mathbf{e}}) \ . \tag{29}$$

The exponents $\alpha_{\mathbf{e}}$ are the eigenvalues and $\Phi_{\alpha_{\mathbf{e}}}(\varphi_{\mathbf{e}})$ the corresponding eigensolutions of the eigenvalue problem as formulated in (18).

The asymptotic expansion of weak solutions of (13), (14) and (15) reads in a vicinity of a vertex v, provided the eigenvalues are nondefective [2, 13, 37, 45]:

$$\begin{aligned} \eta_v u &= \eta_v\big(u_{\text{vertex}} + \sum_{\text{edges}} u_{\text{edge}}\big) + u_{reg} \\ &= \eta_v \sum_{-\frac{1}{2}<\text{Re }\alpha_v<\frac{1}{2}} c_{\alpha_v} r^{\alpha_v}\Phi_{\alpha_v}(\varphi,\theta) + \eta_v \sum_{\text{edges}} \sum_{0<\text{Re }\alpha_{\mathbf{e}}<1} \tilde{c}_{\alpha_{\mathbf{e}}}(y_3,r)\rho^{\alpha_{\mathbf{e}}}\Phi_{\alpha_{\mathbf{e}}}(\varphi_{\mathbf{e}}) \\ &\quad + u_{\text{reg}} \ , \end{aligned} \tag{30}$$

where $u_{\text{reg}}|_{\Omega_i} \in H^{2-\varepsilon}(\Omega_i)$. Here, we accumulated the interaction of vertex and edge singularities in the coefficient $\tilde{c}_{\alpha_{\mathbf{e}}}(y_3, r)$.

2.4 Examples

Vertex Exponents

We start with the Dirichlet problem for linear elastic fields in a composed Fichera domain with the Young moduli $E_1 = 1, E_2 = 10$ and Poisson ratios

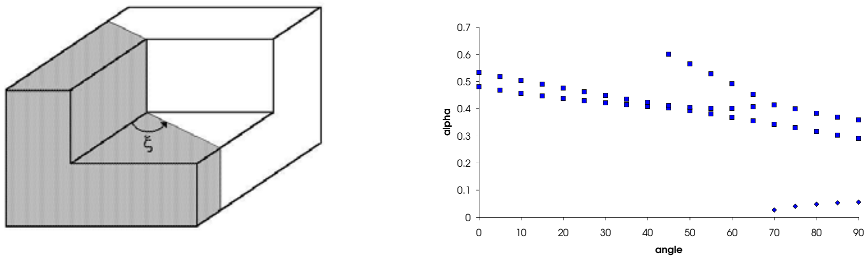

Fig. 4. a) Fichera corner, **b)** vertex exponent

$\nu_1 = \nu_2 = 0.3$, see Fig. 4. The plotted squares in Fig. 4 represent the real parts whereas the diamonds show the imaginary parts of the vertex exponents. Figure 4 is from [45], the computations where done by D. Leguillon.

Kellogg's Example

The following two dimensional example by R. B. Kellogg shows that the singular exponents in expansion (23) for solutions of the Laplace equation can have arbitrary small positive real parts.

Let be $\Omega = (-1,1)^2 \subset \mathbb{R}^2$, $\Omega_i = \{x = r\binom{\cos\varphi}{\sin\varphi} \in \Omega : (i-1)\pi/2 < \varphi < i\pi/2\}$ for $1 \leq i \leq 4$ and we consider the Laplace equation (5) on this domain. In the neighborhood of S, the solution u has the structure (23), the singular exponents α in (25) are real and there are no logarithmic terms. If we choose $\mu_1 = \mu_3 = 1$, $\mu_2 = \mu_4 = h > 0$, then $\alpha > 0$ is a singular exponent of (25) if and only if $\cos(\alpha\pi) = 1 - 8h/(1+h)^2$ [11, 24]. It follows from this relation that $\alpha_{\min} = \min\{\alpha > 0,\ \alpha \text{ is singular exponent}\}$ tends to 0 for $h \to 0$ or $h \to \infty$, see also Fig. 5. Therefore, the regularity of weak solutions can get arbitrarily low, i.e. for general situations it can only be guaranteed that $u\big|_{\Omega_i} \in W^{1+\epsilon,2}(\Omega_i)$, $\epsilon > 0$ small.

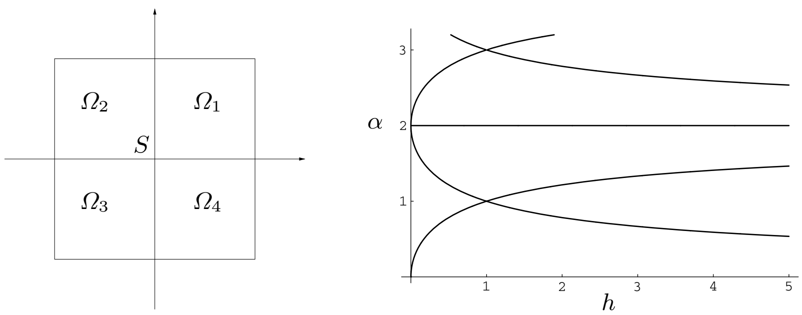

Fig. 5. Domain and eigenvalues α for Kellogg's example

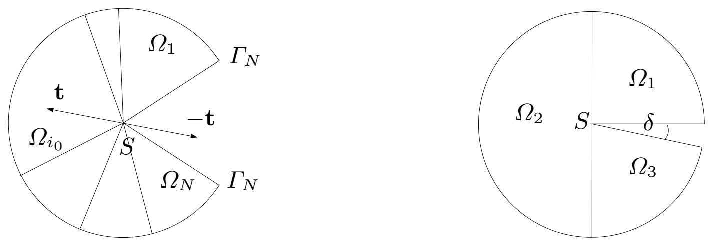

Fig. 6. Example for the quasi-monotonicity condition

Quasi-Monotonicity

Different authors investigated in detail the dependence of the singular exponents in (25) and (30) on the geometry, material parameters and the number of subdomains and we refer to [27, 42, 45, 48] for the Laplace equation and to [27, 45] for linear elasticity. M. Petzoldt observed and proved for two dimensional polygons that a *quasi-monotone* distribution of the parameters μ_i in the Laplace equation (5) leads to general positive lower bounds for Re $\alpha_{\min}$ and thus guarantees a higher minimum regularity of weak solutions [48]. In [26, 27, 29] this condition was slightly modified and extended to composites of linear elastic materials in 2D and 3D polyhedral domains and the real parts of the singular exponents were estimated. We give here the definition of quasi-monotonicity from [29] for a cross point S which is situated on the Neumann boundary of a two dimensional polygon and formulate the corresponding regularity result.

Let $R > 0$, $N \in \mathbb{N}$. For $\Phi_0 < \Phi_1 < \ldots < \Phi_N \leq \Phi_0 + 2\pi$ we introduce

$$\Omega_i = \{x \in \mathbb{R}^2 : 0 < |x| < R, \Phi_{i-1} < \varphi < \Phi_i\}, \ 1 \leq i \leq N$$

and $\Omega = \{x \in \mathbb{R}^2 : 0 < |x| < R, \Phi_0 < \varphi < \Phi_N\}$. Let furthermore C^i be the elasticity tensor corresponding to the subdomain Ω_i. It is assumed that the boundaries $\Gamma_i = \{x \in \mathbb{R}^2 : |x| \leq R, \varphi = \Phi_i\}$, $i \in \{0, N\}$, are parts of the Neumann boundary. Then the quasi-monotonicity condition is satisfied if there exists an index $i_0 \in \{0, \ldots, N\}$ such that (Fig. 6)

QM1 $C^1 \leq \ldots \leq C^{i_0-1} \leq C^{i_0} \geq C^{i_0+1} \geq \ldots \geq C^N$,
QM2 There exists $\mathbf{t} \in \mathbb{R}^2 \backslash \{0\}$ such that $\mathbf{t} \in \Omega_{i_0}$ and $-\mathbf{t} \notin \overline{\Omega}$.

By $C^i \geq C^j$ we mean that $(C^i A) : A \geq (C^j A) : A$ for every $A \in \mathbb{R}^{2\times 2}$. If the materials are isotropic, then $C^i\varepsilon = \lambda_i \mathrm{tr}\,(\varepsilon) I + 2\mu_i\varepsilon$ for $\varepsilon \in \mathbb{R}^{2\times 2}$ with Lamé constants λ_i, μ_i and **QM1** is equivalent to

$$\mu_1 \leq \ldots \leq \mu_{i_0} \geq \ldots \geq \mu_N \ , \tag{31}$$

$$\lambda_1 + \mu_1 \leq \ldots \leq \lambda_{i_0} + \mu_{i_0} \geq \ldots \geq \lambda_N + \mu_N \ . \tag{32}$$

Theorem 2. *[26, 27, 29, 45] Let* $\Omega \subset \mathbb{R}^2$ *be as described above and let* $u \in W^{1,2}(\Omega)$ *satisfy* (13) *with bilinear form* (15) *for every* $v \in W^{1,2}(\Omega)$ *with* $v\big|_{\partial\Omega\cap\partial B_R(0)} = 0$*. Assume furthermore that* $f \in L^2(\Omega)$ *and* $g_i = 0$ *(for simplicity). Let finally the quasi-monotonicity condition* **QM1**, **QM2** *be satisfied. Then it holds for the exponents* α *in the asymptotic expansion* (23) *with respect to* S*:* $\operatorname{Re}\alpha \geq \frac{1}{2}$*. Thus* $\eta_S u\big|_{\Omega_i} \in W^{\frac{3}{2}-\epsilon,2}(\Omega_i)$ *for every* $\epsilon > 0$ *and a cut-off function* η_S*. Moreover, if the materials are isotropic for every* i*, then* $\operatorname{Re}\alpha > \frac{1}{2}$ *and* $\eta_S u\big|_{\Omega_i} \in W^{\frac{3}{2}+\epsilon,2}(\Omega_i)$ *for an appropriate* $\epsilon > 0$*.*

The isotropic case is proved in [26, 27, 45] with a homotopy method. The proof of the general case relies on a difference quotient technique and we go into details in Sect. 3, see also [29, 31]. Analogous results are derived for more general linear elliptic systems in two and three dimensions [29, 31, 45].

As a special application of Theorem 2 we consider an elastic, isotropic bimaterial with a crack perpendicular to the interface. This example is studied in [35]. Let $\Phi_0 = 0$, $\Phi_1 = \frac{\pi}{2}$, $\Phi_2 = \frac{3\pi}{2}$ and $\Phi_4 = 2\pi - \delta$ for small $\delta > 0$. We assume that vanishing Neumann conditions are imposed near the cross point $S = (0,0)$, see Fig. 6. Let Ω_1 and Ω_3 be occupied by zirconia oxide ZrO_2 and Ω_2 by aluminum oxide Al_2O_3. The corresponding material parameters are $\mu_1 = \mu_3 = 0.73\ [10^5\,\mathrm{N/mm}^2]$, $\lambda_1 = \lambda_3 = 1.096\ [10^5\,\mathrm{N/mm}^2]$ and $\mu_2 = 1.46\ [10^5\,\mathrm{N/mm}^2]$, $\lambda_2 = 2.19\ [10^5\,\mathrm{N/mm}^2]$. The quasi-monotonicity condition **QM1**, **QM2** is satisfied for $\delta > 0$ and thus $\operatorname{Re}\alpha_{\min} \geq \frac{1}{2}$ due to Theorem 2. The numerical calculations in [35] confirm this with $\alpha_{\min} = 0.57$ for $\delta = 0$. If on the other hand the materials are interchanged, i.e. Ω_1 and Ω_3 are occupied by Al_2O_3 and Ω_2 by ZrO_2, then the quasi-monotonicity condition is violated and the calculations from [35] give $\alpha_{\min} = 0.42$ for $\delta = 0$. Following the discussion in [35] it seems that the quasi-monotonicity condition does not only lead to higher regularity results but also describes a class of composites which can sustain higher loads before breaking.

3 Nonlinear Elliptic Systems of *p*-Structure

In this section we consider boundary transmission problems for quasilinear elliptic equations and systems of p-structure, where the p-Laplace equation is a typical example. We admit that the growth properties of the differential operators vary from subdomain to subdomain. Unlike the linear case it is to the authors' knowledge an unsolved problem whether the behavior of weak solutions of such nonlinear transmission problems can completely be characterized by asymptotic expansions similar to (23). By a difference quotient method, Savaré [52] and Ebmeyer and Frehse [18, 19] obtained global regularity results for quasilinear elliptic boundary value problems with smooth coefficients on Lipschitz domains. They assumed that the domains satisfy an additional geometrical condition near those points, where the boundary conditions change.

Their results describe a minimum regularity in Sobolev-Slobodeckij spaces for weak solutions on this class of domains.

The main idea for obtaining global regularity results also for transmission problems is to combine the difference quotient technique with the quasi-monotonicity condition, which originally was introduced for linear elliptic transmission problems (Sect. 2.4). We explain here this concept in detail.

3.1 Systems of p-Structure

It is assumed that the differential operators under consideration can be derived from convex minimization problems. Let $\overline{\Omega} = \cup_{i=1}^{M} \overline{\Omega_i} \subset \mathbb{R}^d$ and assume that M functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ are given for some $m \geq 1$. The conditions on the functions W_i are specified later. The boundary transmission problem reads for $u : \Omega \to \mathbb{R}^m$, $u\big|_{\Omega_i} = u_i$ and given functions f, g:

$$\operatorname{div}\left(DW_i(\nabla u_i)\right) + f_i = 0 \quad \text{in } \Omega_i \, , \tag{33}$$

$$u_i - u_j = 0 \quad \text{on } \Gamma_{ij} \, , \tag{34}$$

$$DW_i(\nabla u_i)\mathbf{n}_i + DW_j(\nabla u_j)\mathbf{n}_j = 0 \quad \text{on } \Gamma_{ij} \, , \tag{35}$$

$$u = 0 \quad \text{on } \Gamma_D \, , \tag{36}$$

$$DW_i(\nabla u_i)\mathbf{n}_i = g \quad \text{on } \Gamma_N \cap \Gamma_i \, . \tag{37}$$

Here, we use the notation $DW_i(A) = \left(\frac{\partial W_i(A)}{\partial A_{kl}}\right)_{kl} \in \mathbb{R}^{m\times d}$ and $A : B = \sum_{k=1}^{m}\sum_{l=1}^{d} A_{kl}B_{kl}$, $|A| = \sqrt{A:A}$ for $A, B \in \mathbb{R}^{m\times d}$. It is assumed that the differential operators (33) are of p-structure. This means that the energy densities W_i satisfy **H1**–**H4** here below for some $p_i \in (1,\infty)$:

H1 $W_i \in \mathcal{C}^1(\mathbb{R}^{m\times d},\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}^{m\times d}\backslash\{0\},\mathbb{R})$.

H2 There exist $c_0^i \in \mathbb{R}$, $c_1^i, c_2^i > 0$ such that for every $A \in \mathbb{R}^{m\times d}$

$$c_0^i + c_1^i \, |A|^{p_i} \leq W_i(A) \leq c_2^i(1 + |A|^{p_i}) \, .$$

H3 There exist $c^i > 0$ such that for every $A \in \mathbb{R}^{m\times d}$:

$$|DW_i(A)| \leq c^i(1 + |A|^{p_i - 1}) \, , \quad \left|D^2W_i(A)\right| \leq c^i \, |A|^{p_i-2} \, .$$

H4 There exist $c_i > 0$ such that for every $A, B \in \mathbb{R}^{m\times d}$, $A \neq 0$:

$$D^2W_i(A)[B,B] = \sum_{k,j=1}^{m}\sum_{r,s=1}^{d} \frac{\partial^2 W_i(A)}{\partial A_{ks}\partial A_{jr}} B_{ks}B_{jr} \geq c_i \, |A|^{p_i-2} \, |B|^2 \, .$$

Condition **H4** implies that the functions W_i are strictly convex and that the corresponding differential operators (33) are elliptic. The p-Laplace equation is included here with $W_i(\nabla u) = \frac{1}{p_i} \, |\nabla u|^{p_i}$ for $u : \Omega \to \mathbb{R}$.

Appropriate function spaces for a weak formulation of (33)–(37) were first introduced and studied by W.B. Liu [39]. Let $\mathbf{p} = (p_1, \ldots, p_M)$ with $p_i \in (1, \infty)$ corresponding to W_i and $p_{\min} = \min\{p_i,\, 1 \le i \le M\}$. Then

$$W^{1,\mathbf{p}}(\Omega) = \{u \in W^{1,p_{\min}}(\Omega) :\, u\big|_{\Omega_i} \in W^{1,p_i}(\Omega_i)\} \,.$$

Since $W^{1,\mathbf{p}}(\Omega) \subset W^{1,p_{\min}}(\Omega)$, traces are well defined for elements of $W^{1,\mathbf{p}}(\Omega)$ and thus, the following definition is meaningful:

$$V^{\mathbf{p}}(\Omega) = \{u \in W^{1,\mathbf{p}}(\Omega) :\, u\big|_{\Gamma_D} = 0\} \,.$$

Assume now for simplicity that the Neumann datum g in (37) vanishes and that $\Gamma_D \neq \emptyset$. The weak formulation to (33)–(37) reads for given $f \in L^{\mathbf{q}}(\Omega)$ with $\mathbf{q} = (q_1, \ldots, q_M)$ and $q_i^{-1} + p_i^{-1} = 1$: Find $u \in V^{\mathbf{p}}(\Omega)$ such that for every $v \in V^{\mathbf{p}}(\Omega)$

$$\sum_{i=1}^{M} \int_{\Omega_i} DW_i(\nabla u) : \nabla v \,\mathrm{d}x = \int_{\Omega} f v \,\mathrm{d}x \,. \tag{38}$$

It follows from the main theorem on monotone operators that (38) has a unique weak solution $u \in V^{\mathbf{p}}(\Omega)$.

3.2 The Quasi-Monotone Covering Condition and Regularity

Kellogg's example in Sect. 2.4 shows that even in the linear case one cannot expect to obtain general minimum regularity results without any further assumptions on the geometry of the subdomains or the distribution of the coefficients. Furthermore, when proving the regularity results with a difference quotient technique, one has to ensure that functions of the form $\eta^2(x)(u(x + he_j) - u(x))$, where η is a cut-off function, $\{e_1, \ldots, e_d\}$ a basis of $\mathbb{R}^d$, $h > 0$ and $u \in W^{1,\mathbf{p}}(\Omega)$, are admissible test functions. In particular, the translated function $\eta^2 u(\cdot + he_j)$ should be an element of $W^{1,\mathbf{p}}(\Omega)$ as well. This cannot be guaranteed for an arbitrary geometry and an arbitrary distribution of the parameters p_i.

Our main assumption on the boundary transmission problem is that the subdomains Ω_i together with the energy densities W_i satisfy the quasi-monotone covering condition. We formulate here this condition for an interior cross point S and refer to [29, 31] for the general case.

Definition 1. *Let $S \in \Omega \subset \mathbb{R}^d$ be an interior cross point of the subdomains Ω_i, $1 \le i \le N$ and $R > 0$ such that $B_R(S) \Subset \Omega$. Let W_i, $1 \le i \le N$, be the energy densities corresponding to the subdomains Ω_i.*

The pairs $\{(\Omega_i, W_i),\, 1 \le i \le N\}$ satisfy the quasi-monotonicity condition on $B_R(S)$ if there exist numbers $k_i \in \mathbb{R}$ and an open cone $\mathcal{K} \subset \mathbb{R}^d$ with vertex in 0 such that for every $h \in \mathcal{K}$, $1 \le i, j \le N$ and $A \in \mathbb{R}^{m\times d}$ it holds

$$\text{if } \Omega_i + h \cap \Omega_j \neq \emptyset, \text{ then } W_j(A) + k_j \ge W_i(A) + k_i \,. \tag{39}$$

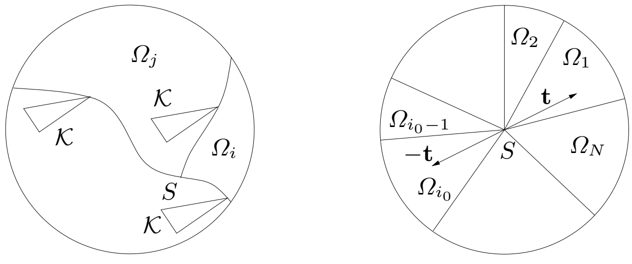

Fig. 7. Examples for the quasi-monotonicity condition

See Fig. 7 (left) for an example. For interior cross points on two dimensional domains which are composed of polygonal subdomains, this definition can be reformulated in a more illustrative way. Let $\Omega = B_R(0) \subset \mathbb{R}^2$ and $\Omega_i = \{x \in \mathbb{R}^2 : \Phi_{i-1} < \varphi < \Phi_i,\, |x| < R\}$ for $\Phi_0 < \ldots < \Phi_N = \Phi_0 + 2\pi$. The quasi-monotonicity condition is satisfied if there exist an index $i_0 \in \{2, \ldots, N\}$ and numbers $k_i \in \mathbb{R}$ such that it holds (see Fig. 7, right):

QM3 For every $A \in \mathbb{R}^{m\times 2}$ we have

$$W_1(A) + k_1 \geq W_2(A) + k_2 \geq \ldots \geq W_{i_0}(A) + k_{i_0} \leq \\ \ldots \leq W_N(A) + k_N \leq W_1(A) + k_1 \,.$$

QM4 There exists $\mathbf{t} \in \mathbb{R}^2 \backslash \{0\}$ such that $\mathbf{t} \in \Omega_1$ and $-\mathbf{t} \in \Omega_{i_0}$.

Example 1. Let $W_i(A) = \frac{1}{p_i} |A|^{p_i}$ for $A \in \mathbb{R}^d$. Then **QM3** is equivalent to

$$p_1 \geq p_2 \geq \ldots \geq p_{i_0} \leq \ldots \leq p_N \leq p_1 \,.$$

The energy densities $W_i(A) = \frac{\mu_i}{2} |A|^2$, $A \in \mathbb{R}^2$, in Kellogg's example (Sect. 2.4) do not satisfy the quasi-monotonicity condition with respect to $S = (0,0)$.

Example 2. Let $\Omega, \Omega_1, \Omega_2 \subset \mathbb{R}^d$ be bounded Lipschitz domains with $\Omega_1 \cap \Omega_2 = \emptyset$, $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$ and $\Omega_2 \Subset \Omega$. Let furthermore $\partial\Omega = \Gamma_D$ or $\partial\Omega = \Gamma_N$ and assume that W_1, W_2 satisfy **H1**–**H4** with $p_1 \neq p_2$. Then the quasi-monotone covering condition is satisfied for the pairs $\{(\Omega_i, W_i),\, i = 1, 2\}$, see Fig. 8. We refer to [29, 31] for further examples.

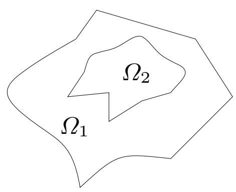

Fig. 8. Nested Lipschitz domains

Theorem 3. *[29, 30, 31] Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with $\overline{\Omega} = \cup_{i=1}^M \overline{\Omega_i}$. We assume that the functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$, $1 \le i \le M$, satisfy* **H1**–**H4** *for some $p_i \in (1,\infty)$ and the quasi-monotone covering condition. Let finally $f \in L^{\mathbf{q}}(\Omega)$ with $p_i^{-1} + q_i^{-1} = 1$.*

Then the weak solution $u \in V^{\mathbf{p}}(\Omega)$ of (38) and the stress field $\sigma = DW(\nabla u) \in L^{\mathbf{q}}(\Omega)$ have the following regularity: For every $\delta > 0$

$$u\big|_{\Omega_i} \in W^{1+\frac{1}{p_i}-\delta,p_i}(\Omega_i),\ \ \sigma\big|_{\Omega_i} \in W^{\frac{1}{2}-\delta,r(q_i)}(\Omega_i)\ \ \text{if } p_i \in [2,\infty)\,, \tag{40}$$

$$u\big|_{\Omega_i} \in W^{\frac{3}{2}-\delta,r(p_i)}(\Omega_i),\ \ \sigma\big|_{\Omega_i} \in W^{\frac{1}{q_i}-\delta,q_i}(\Omega_i)\quad \text{if } p_i \in (1,2]\,. \tag{41}$$

Here, $r(s) = \frac{2ds}{2d-2+s}$ and for $s \in (1,2]$ it is $s \le r(s) \le 2$.

Non vanishing Dirichlet and Neumann conditions are treated in [29, 31]. The regularity theorem corresponds well with the results of references [18, 19, 52] for pure boundary value problems. Moreover, the results of Theorem 2 (linear case) are recovered by Theorem 3. Let us remark that Theorem 3 is applicable in the situation described in Example 2.

Remarks on the proof. The proof of Theorem 3 is carried out with a difference quotient technique, where the domain Ω is covered by a finite number of balls and Theorem 3 is proved for each ball separately. We give here a sketch for an interior cross point S for which the quasi-monotonicity condition of Definition 1 is satisfied. For simplicity we assume that $p_i \ge 2$ for every i. For the full proof we refer to [29, 31].

The goal is to show the following estimate for $h \in \mathcal{K}$, where $\mathcal{K}$ is the cone of Definition 1:

$$\sum_{i=1}^N \int_{\Omega_i \cap B_R(S)} |\nabla u(x+h) - \nabla u(x)|^{p_i}\ \mathrm{d}x \le c\,|h|\ , \tag{42}$$

and the constant c is independent of $h \in \mathcal{K}$. Inequality (42) implies that $u\big|_{\Omega_i\cap B_R(S)}$ is an element of the Nikolskii space $\mathcal{N}^{1+\frac{1}{p_i},p_i}(\Omega_i \cap B_R(S))$ [56]. The embedding theorems for Nikolskii and Sobolev-Slobodeckij spaces lead to $u\big|_{\Omega_i\cap B_R(S)} \in W^{1+\frac{1}{p_i}-\delta,p_i}(\Omega_i \cap B_R(S))$ for every $\delta > 0$ [56]. We prove now estimate (42). **H1**–**H4** imply the following convexity inequality (see e.g. [29])

$$\begin{aligned}\sum_{i=1}^N \int_{\Omega_i} c_i\eta^2\,|\nabla u(x+h) - \nabla u(x)|^{p_i}\ \mathrm{d}x \qquad&\\ \le \sum_{i=1}^N \int_{\Omega_i} \eta^2 DW(\nabla u) : \nabla(u(x+h) - u(x))\,\mathrm{d}x&\\ + \sum_{i=1}^N \int_{\Omega_i} \eta^2\left(W_i(\nabla u(x+h) - W_i(\nabla u(x))\right)\mathrm{d}x\ ,&\end{aligned} \tag{43}$$

where η is a smooth cut-off function with $\eta = 1$ on $B_R(S)$. The quasi-monotonicity condition guarantees that $v(x) = \eta^2(u(x+h) - u(x))$, $h \in \mathcal{K}$, is an an element of $V^{\mathbf{p}}(\Omega)$ and thus is an admissible test function for the weak formulation. Using (38), the first term on the right hand side in (43) can be controlled after some technical calculations by $c\,|h|$. The second term in (43) can be rewritten as follows with $\triangle_h w(x) = w(x+h) - w(x)$ and the numbers k_i from Definition 1:

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega_i} \eta^2 \left(W_i(\nabla u(x+h)) + k_i - W_i(\nabla u(x)) - k_i\right) \mathrm{d}x \\ = \sum_{i=1}^{N} \int_{\Omega_i} \triangle_h(\eta^2(W_i(\nabla u) + k_i))\,\mathrm{d}x \\ - \sum_{i=1}^{N} \int_{\Omega_i} (\triangle_h \eta^2)(W_i(\nabla u(x+h)) + k_i)\,\mathrm{d}x \,. \end{aligned} \tag{44}$$

The quasi-monotonicity condition implies that the first term on the right hand side in (44) is ≤ 0. Since η is smooth, the second term can be estimated by $c\,|h|$. Thus (42) holds and the proof is finished.

Inequalities like convexity inequality (43) are essential for obtaining regularity results via a difference quotient technique. On the basis of inequalities like (43) we proved a global regularity result for a shear thinning fluid of power-law type [28] and extended a local regularity result by Carstensen and Müller [9] for stress fields of not strictly convex energies to a global one [30].

4 Application of the Regularity Results

The derived regularity results and coefficient formulae can be applied in different fields. We discuss here two of them in detail, namely sensitivity analysis for linear elastic fields and the derivation of formulas in fracture mechanics for a nonlinear elastic model of power-law type.

4.1 Sensitivity Analysis

The goal of shape and structure optimization in elasticity is to determine an elastic body or composite which is optimal with respect to objective and constraint functionals. For example, if one wants to avoid plastification the values of the von Mises yield functional should be small enough, or if one wants to avoid crack growth, the energy release rate (or the stress intensity factors) should not exceed their critical values. The influence of the shape or the structure of the domain on the stress behavior has been studied by many authors [21, 53] and the corresponding sensitivity analysis is well developed for problems in smooth domains. Here, we focus on the sensitivity analysis for linear elastic fields in two-dimensional non smooth domains and study a class of functionals with respect to shape perturbation.

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. We introduce a family of mappings $\{\Phi_\varepsilon \in [\mathcal{C}^3(\overline{\Omega})]^2,\ \varepsilon \in [0,\varepsilon_0]\}$ which admit Taylor expansions

$$\Phi_\varepsilon(x) = x + \varepsilon\Phi(x) + \varepsilon^2\Phi_R(\varepsilon, x)$$

with $\Phi, \Phi_R \in [\mathcal{C}^3(\overline{\Omega})]^2$. The function $\Phi_R(\varepsilon, x)$ is bounded with respect to ε for every $x \in \Omega$. The perturbations $(\Omega_\varepsilon, \Gamma_\varepsilon, \Gamma_\varepsilon^D, \Gamma_\varepsilon^N)$ of the reference configuration $(\Omega, \Gamma, \Gamma^D, \Gamma^N)$ are defined by

$$\Omega_\varepsilon = \Phi_\varepsilon(\Omega)\ , \quad \Gamma_\varepsilon = \Phi_\varepsilon(\Gamma)\ , \quad \Gamma_\varepsilon^D = \Phi_\varepsilon(\Gamma^D)\ , \quad \Gamma_\varepsilon^N = \Phi_\varepsilon(\Gamma^N)\ .$$

Since $\Phi_\varepsilon \in [\mathcal{C}^3(\overline{\Omega})]^2$ the number of singular points in Ω_ε is independent of ε.

Let be u_ε a solution of (6) in Ω_ε with mixed boundary conditions on $\Gamma_\varepsilon^D, \Gamma_\varepsilon^N$ and corresponding interface conditions on Γ_ε. We consider functionals associated with the elastic fields u_ε and $\sigma(u_\varepsilon)$

$$J(\Omega_\varepsilon) = \int_{\Omega_\varepsilon} F(u_\varepsilon, \sigma(u_\varepsilon))dx_\varepsilon\ , \tag{45}$$

where the function F satisfies for a positive constant c the growth conditions

$$F(p,q) \le a(p)(c + |q|^2),\ \partial_q F(p,q) \le a(p)(c + |q|) \tag{46}$$

for some $a \in \mathcal{C}(\mathbb{R}^2)$ and all $p \in \mathbb{R}^2,\ q \in \mathbb{R}^4$.

Our goal is to derive formulae for the sensitivity of the functional J with respect to the perturbation mapping Φ_ε, i.e. we want to calculate the shape derivative

$$dJ(\Omega, \Phi) = \lim_{\varepsilon \to 0} \frac{J(\Phi_\varepsilon(\Omega)) - J(\Omega)}{\varepsilon} \tag{47}$$

and to express $dJ(\Omega, \Phi)$ as an integral over $\partial\Omega$.

Sensitivity Formulae with Material and Shape Derivatives

We give different formulae for $dJ(\Omega, \Phi)$ with material and shape derivatives. The *material derivatives* of u_ε and σ_ε are defined as

$$\dot{u} := \left.\frac{d(u_\varepsilon \circ \Phi_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}\ , \tag{48}$$

whereas the *shape derivative* is given as

$$u' := \left.\frac{du_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \dot{u} - Du_0\Phi\ . \tag{49}$$

It is proved in [7] that $\dot{u}$ and u' are well defined. We assume that the transformed force densities $f_\varepsilon \circ \Phi_\varepsilon$ and $g_\varepsilon \circ \Phi_\varepsilon$ depend smoothly on ε

$$\begin{aligned} f_\varepsilon \circ \Phi_\varepsilon &= f_0 + \varepsilon f_1 + \varepsilon^2 f_R(\varepsilon) \; , \\ g_\varepsilon \circ \Phi_\varepsilon &= g_0 + \varepsilon g_1 + \varepsilon^2 g_R(\varepsilon) \; . \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathbf{n}_\varepsilon \circ \Phi_\varepsilon &= \quad \mathbf{n}_0 + \varepsilon \dot{\mathbf{n}} + \varepsilon^2 \mathbf{n}_R(\varepsilon) \; , \\ (u_\varepsilon \circ \Phi_\varepsilon)(x) &= u_0(x) + \varepsilon \dot{u}(x) + O(\varepsilon^2) \; . \end{aligned}$$

Theorem 4. *[7] Let F in* (45) *be continuously differentiable with respect to all its arguments and satisfy the growth conditions* (46). *Furthermore, let $\alpha* = \min\{\mathrm{Re}\ \alpha_j \in (0,1)\}$, where α_j is defined by (24). Then*

$$dJ(\Omega, \Phi) = \int_\Omega (\partial_u F(u_0, \sigma_0) \cdot \dot{u} + \partial_\sigma F(u_0, \sigma_0) : \dot{\sigma} + F(u_0, \sigma_0) \mathrm{div}\, \Phi) \; \mathrm{d}x \; .$$

If additionally $\alpha* \geq \frac{1}{2}$, then

$$dJ(\Omega, \Phi) = \int_\Omega (\partial_u F(u_0, \sigma_0) \cdot u' + \partial_\sigma F(u_0, \sigma_0) : \sigma') \, \mathrm{d}x + \int_{\partial\Omega} F(u_0, \sigma_0) \Phi \cdot \mathbf{n}_0 \mathrm{d}s_x \; .$$

If $\alpha* > \frac{1}{2}$, then

$$\begin{aligned} dJ(\Omega, \Phi) = & \int_{\partial\Omega} F(u_0, \sigma_0)(\Phi \cdot \mathbf{n}_0) \mathrm{d}s_x \\ & + \int_{\Gamma^N} w \cdot ((\Phi \cdot \mathbf{n}_0)(f + \kappa g) - \mathrm{div}\,_\Gamma ((\Phi \cdot \mathbf{n}_0) \sigma_T(u_0))) \, \mathrm{d}s_x \\ & - \int_{\Gamma^D} (\Phi \cdot \mathbf{n}_0)(C \partial_\sigma F(u_0, \sigma_0) - \sigma(w)) \mathbf{n}_0 \cdot \partial_n u_0 \mathrm{d}s_x \; . \end{aligned} \tag{50}$$

$\sigma_T(u_0)$ is the tangential component of the stress tensor on $\partial\Omega$, κ is the curvature of $\partial\Omega$, the tangential divergence operator $\mathrm{div}\,_\Gamma$ *is defined by*

$$\mathrm{div}\,_\Gamma v = \mathrm{div}\, v - D v \mathbf{n}_0 \cdot \mathbf{n}_0 \; ,$$

and w is the so-called adjoint displacement field, see [7].

Remark 1. If $\alpha* = \frac{1}{2}$ and a homogeneous material is given, then we have to add to (50) stress intensity factors [7]. This yields for straight propagation of cracks in linear, isotropic, elastic materials with the energy functional

$$J(\Omega_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} \sigma(u_\varepsilon) : \varepsilon(u_\varepsilon) dx_\varepsilon \tag{51}$$

to the well-known Irwin formula

$$dJ(\Omega, \Phi) = \gamma \sum_{i=1}^{2} K_i(u_0)^2 \; .$$

Here, K_i are the classical stress intensity factors and γ is a material constant.

4.2 Griffith's Fracture Criterion for a Power-Law Model

A special case in sensitivity analysis is the derivation of the formulas for the energy release rate of bodies with pre-existing cracks. We describe here recently derived results for a power-law model.

The Ramberg-Osgood Model and Regularity

The Ramberg-Osgood model is applied to describe materials with low proportionality limit and with strain hardening behavior [10, 47]. The field equations for the displacement field $u : \Omega \to \mathbb{R}^d$ and stress field $\sigma : \Omega \to \mathbb{R}^{d\times d}$ read as follows:

$$\operatorname{div} \sigma + f = 0 \quad \text{in } \Omega\,, \tag{52}$$

$$\varepsilon(u) - A\sigma - \alpha \left|\sigma^D\right|^{q-2} \sigma^D = 0 \quad \text{in } \Omega \tag{53}$$

together with boundary conditions on $\partial\Omega$. Here, $\sigma^D = \sigma - \frac{1}{d}\operatorname{tr}\sigma I$ denotes the deviator of σ, $q \geq 2$ is the strain hardening parameter, A the tensor of elastic compliances ($A^{-1} = C$ with C from (7)) and $\alpha > 0$ a further material constant. The field equations (52)–(53) are closely related to quasilinear elliptic systems of p-structure (Sect. 3). The following global regularity results are derived in [31, 32] with a difference quotient technique for weak solutions on admissible domains ($q \geq 2$, $p^{-1} + q^{-1} = 1$):

$$u \in W^{\frac{3}{2}-\delta, \frac{2dp}{2d-2+p}}(\Omega)\,, \quad \sigma, \operatorname{div} u \in W^{\frac{1}{2}-\delta,2}(\Omega) \cap W^{\frac{1}{q}-\delta,2}(\Omega) \tag{54}$$

for every $\delta > 0$. We call a domain admissible if either **A1** or **A2** here below is satisfied:

A1 $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$.

A2 $\Omega \subset \mathbb{R}^d$ is a Lipschitz polyhedron where at most d faces intersect near points $S \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$. Furthermore, the interior opening angle between Γ_D and Γ_N is less than π, see Fig. 9 and [17].

A slightly more general definition of admissible domains is given in [32]. Local regularity results are proved by Bensoussan and Frehse in [3].

As in the case of quasilinear elliptic systems of p-structure, it is also for the Ramberg-Osgood model an unsolved problem, whether the behavior of weak solutions near re-entrant corners, edges or crack tips can be completely characterized by asymptotic expansions. A comparison between singularities obtained with ansatzes of the form $u(r, \varphi) = r^\alpha v(\varphi)$ and between the regularity results (54) shows good agreement:

Let $\Omega_{2\pi} = \{x \in \mathbb{R}^2 : -\pi < \varphi < \pi, |x| < R\}$ for $R > 0$, and assume that $\Gamma_N \supset \{x : \varphi = \pm\pi\}$, i.e. $\Omega_{2\pi} \subset \mathbb{R}^2$ is a domain with a crack on the negative x_1-axis. First investigations on crack tip singularities for Ramberg-Osgood materials were done by Hutchinson [22] and Rice and Rosengren [50]. Based

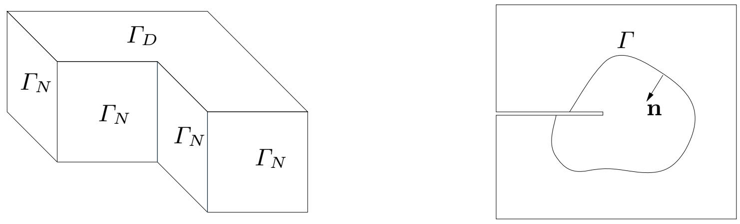

Fig. 9. Examples for an admissible domain and a cracked domain

on the assumption that the displacement and stress fields have an asymptotic structure like in the linear case (23), they derived a strongly nonlinear eigenvalue problem from which they calculated the leading terms in the asymptotic expansion. In particular, they obtained

$$u(r,\varphi) = r^{\frac{1}{q}} v(\varphi) + u_{\mathrm{reg}}(r,\varphi)\,, \quad \sigma(r,\varphi) = r^{-\frac{1}{q}} \tau(\varphi) + \sigma_{\mathrm{reg}}(r,\varphi) \tag{55}$$

near the crack tip. The singular terms are called HRR-fields. Relation (55) fits well with the regularity results (54) since the function $v(x) = |x|^{\gamma}$ is an element of $W^{\frac{3}{2}-\delta,\frac{4p}{2+p}}(\varOmega)$, which is the space from (54) for $d = 2$, if and only if $\gamma \geq \frac{1}{q}$. Furthermore, numerical investigations (see e.g. [61]) show the dependence of the exponents in (55) on the opening angle of the domain and indicate that the singular behavior of weak solutions is completely characterized by asymptotic expansions as in the linear case. But to our knowledge there is no rigorous proof of this conjecture and therefore, we do not use relation (55) for the derivation of formulas for the energy release rate.

Griffith's Fracture Criterion and Energy Release Rate

Griffith's fracture criterion is an energetic criterion and reads as follows for a domain $\varOmega_0$ with pre-existing crack C_0 and loading F [38]:

The crack C_0 is stationary with respect to the applied loading F if the total potential energy $\varPi$ of the body in the actual configuration is minimal compared to every admissible neighboring configuration.

We consider here the simplest case and assume that plane strain conditions hold, that the crack is part of a straight line and that the crack can grow straight on, only. Admissible neighboring configurations are characterized as follows: For $\delta \in \mathbb{R}$ let $S_\delta = \{x \in \mathbb{R}^2 : x_1 \leq \delta, x_2 = 0\}$ and let $\tilde{\varOmega} \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary such that $\delta \binom{1}{0} \in \tilde{\varOmega}$ for $|\delta| < \delta_0$. For $|\delta| < \delta_0$ we define $\varOmega_\delta = \tilde{\varOmega} \backslash S_\delta$, $C_\delta = \tilde{\varOmega} \cap S_\delta$ and call $\varOmega_0$ actual configuration with crack C_0. The domains $\varOmega_\delta$, $\delta > 0$, are admissible neighboring configurations with cracks C_δ.

The total potential energy $\varPi$ has the following form for a displacement field u and external forces F:

$$\Pi(\Omega_\delta) = I_{\text{el}}(\Omega_\delta, u) - W(\Omega_\delta, u, F) + D(\Omega_\delta)\ . \tag{56}$$

Here, I_{el} denotes the elastic strain energy, W the work of the external forces and D is a dissipative energy which we assume to be proportional to the crack length: $D(\Omega_\delta) = D(\Omega_0) + 2\gamma\delta$. The constant $\gamma > 0$ is the specific surface energy or fracture toughness and depends on the material. The elastic strain energy of the Ramberg-Osgood model reads for a displacement field u_δ, a corresponding stress field σ_δ with $\varepsilon(u_\delta) = A\sigma_\delta + \alpha \left|\sigma_\delta^D\right|^{q-2} \sigma_\delta^D = D_A W_c(\sigma_\delta)$ and complementary energy density $W_c(\sigma_\delta) = \frac{1}{2}A\sigma_\delta : \sigma_\delta + \frac{\alpha}{q}\left|\sigma_\delta^D\right|^q$:

$$I_{\text{el}}(\Omega_\delta, u_\delta) = \int_{\Omega_\delta} \sigma_\delta : \varepsilon(u_\delta) - W_c(\sigma_\delta)\,\mathrm{d}x\ . \tag{57}$$

Let $f \in L^q(\tilde{\Omega})$, $g \in \left(W^{\frac{1}{q},q}(\Gamma_N)\right)'$ and $F = (f, g)$. Then $W(\Omega_\delta, u_\delta, F) = \int_{\Omega_\delta} f u_\delta\,\mathrm{d}x + \langle u_\delta, g\rangle_{\Gamma_N}$. The quantity $E(\Omega_\delta, u_\delta, F) = I_{\text{el}}(\Omega_\delta, u_\delta) - W(\Omega_\delta, u_\delta, F)$ describes the potential deformation energy. Let $(u_\delta, \sigma_\delta)$ be a weak solution of (52)–(53). Griffith's fracture criterion takes now the form: If

$$E(\Omega_0, u_0, F) - E(\Omega_\delta, u_\delta, F) < D(\Omega_\delta) - D(\Omega_0) = 2\gamma\delta \tag{58}$$

for every small $\delta > 0$, then the crack C_0 is stationary. This motivates the following definition, which is a special case of (47) with $J = -E$:

Definition 2 (Energy release rate). *For $\delta \geq 0$ let u_δ, σ_δ be a weak solution of* (52)–(53)*. The energy release rate, shortly ERR, for the domain Ω_0 with crack C_0 and exterior forces F is defined as*

$$ERR(\Omega_0, F) = \lim_{\delta \searrow 0} \frac{1}{\delta}(E(\Omega_0, u_0, F) - E(\Omega_\delta, u_\delta, F))\ .$$

With this definition the fracture criterion reads:

If $ERR(\Omega_0, F) < 2\gamma$, then the crack is stationary, otherwise it will grow.

The question is whether the energy release rate is well defined and whether there exist formulas for calculating this quantity. In the case of linear elastic materials such formulas (Griffith's formula, J-integral, formulas based on the stress intensity factors) are rigorously proved in [14, 25, 40]. For nonlinear elastic models these formulas were derived in the literature under the assumption that the elastic fields u_0, σ_0 are smooth enough or that they can be characterized by certain asymptotic expansions near the crack tip. However, such regularity results are not known in general. Using the regularity results for the Ramberg-Osgood model from Sect. 4.2 we proved the following theorem:

Theorem 5. *[33, 31] Let $\theta \in \mathcal{C}_0^\infty(\tilde{\Omega})$ with $\theta = 1$ in a neighborhood of the crack tip. Let furthermore Γ be a non-intersecting, Lipschitz-continuous path around the crack tip with normal vector $\mathbf{n} = (n_1, n_2)^\top$ pointing towards the* *crack tip. Let finally* $f \in \mathcal{C}^1(\overline{\tilde{\Omega}})$ *with* $\frac{\partial}{\partial x_1} f = 0$ *in a neighborhood of the crack tip and* Γ *(see Fig. 9). Then the energy release rate is well defined for the Ramberg-Osgood model and the* Griffith-formula *holds:*

$$ERR(\Omega_0, F) = \int_{\Omega_0} \sigma_0 : (\partial_1 u_0 \otimes \nabla\theta)_{sym}\,\mathrm{d}x + \int_{\Omega_0} u_0 \cdot \partial_1(\theta f)\,\mathrm{d}x \\ - \int_{\Omega_0} \big(\sigma_0 : \varepsilon(u_0) - W_c(\sigma_0)\big)\partial_1\theta\,\mathrm{d}x\,. \quad (59)$$

Furthermore, after integration by parts,

$$ERR(\Omega_0, F) = \int_\Gamma (\sigma_0 \mathbf{n}) \cdot \partial_1 u_0 \mathrm{d}s - \int_\Gamma \big(\sigma_0 : \varepsilon(u_0) - W_c(\sigma_0)\big) n_1 \mathrm{d}s + \int_\Gamma u \cdot f n_1 \mathrm{d}s\,. \quad (60)$$

This path integral is called J-integral. *The integrands of* (59) *and* (60) *are* L^1*-functions and* (59)–(60) *are independent of* θ *and of the path* Γ*.*

The formulas for the energy release rate have the same structure as in the linear case. Moreover, the proof of Theorem 5 runs parallel to the linear case and is based on the mapping $T_\delta(x) = x - \delta \begin{pmatrix} \theta(x) \\ 0 \end{pmatrix}$, which is a diffeomorphism from the domain Ω_δ to Ω_0. The J-integral is meaningful due to the regularity results in (54). In a recent paper we extended Theorem 5 to geometrically nonlinear elastic models with polyconvex energy densities [34], results for dynamical crack propagation (linear case) are proved in [46].

Numerical Examples for the Energy Release Rate

The following example is studied in [54]. Let $\Omega_0 = (-5,5)^2 \backslash S_0$ be a compound of two materials with an interface crack and energy densities corresponding to modified p_i-Laplace operators, i.e. $W_i(A) = p_i^{-1}(\kappa_i + |A|^2)^{\frac{p_i}{2}}$ for $A \in \mathbb{R}^2$, $i = 1, 2$. This example can be interpreted as an anti-plane case of the Ramberg-Osgood model. The same notation as in the previous section is used here, see also Fig. 2. The field equations for $u : \Omega_0 \to \mathbb{R}$ read

$$\operatorname{div} DW_i(\nabla u_i) = 0 \quad \text{in } \Omega_i\,, \qquad DW_i(\nabla u_i)\mathbf{n}_i = 0 \quad \text{on } \Gamma_N \cup C_0$$

together with the Dirichlet conditions

$$u(x) = \begin{cases} -2x_1 + x_2 + 15 & \text{if } x_2 = -5,\ x_1 \in (-5,5)\,, \\ 0 & \text{if } x_1 = 5\,, \\ 2x_1 + x_2 - 15 & \text{if } x_2 = 5,\ x_1 \in (-5,5)\,. \end{cases}$$

The energy release rate can be expressed by the Griffith formula

$$ERR(\Omega_0) = \sum_{i=1}^{2} \int_{\Omega_i} \partial_{x_1} u_i DW_i(\nabla u_i) \cdot \nabla\theta - W_i(\nabla u_i)\partial_{x_1}\theta\,\mathrm{d}x\,,$$

where θ is a cut-off function centered at the crack tip. Figure 10 shows the energy release rate for $\kappa_1 = \kappa_2 = 10^{-7}$, different lengths of the crack C_0 and varying parameters μ_i and p_i.

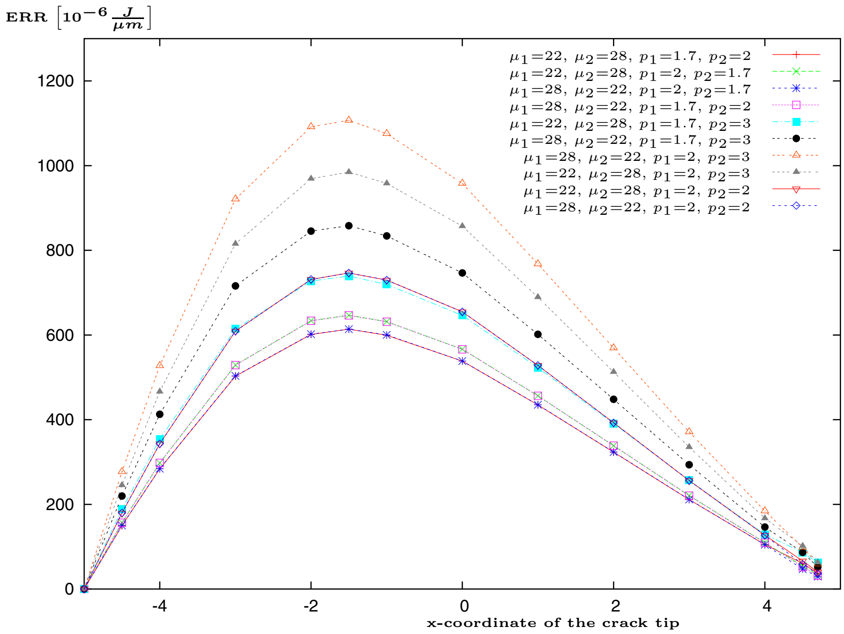

Fig. 10. Energy release rate versus crack length, μ_1, μ_2, p_1, p_2

5 Conclusions

High mechanical stresses can occur in linear and nonlinear elastic composites in the vicinity of re-entrant corners, edges, cracks and near interior surfaces, where the different materials of the composite come together. These stress concentrations have a strong influence on the strength and physical life of the structure. Their knowledge is fundamental for fracture and failure criteria.

In the linear case the substructures consist of linear elastic materials and the singular behavior of the displacement and stress fields can be completely characterized by means of an asymptotic expansion of the solution near the mentioned geometrical and structural peculiarities. Detailed formulas are derived. For some classes of semilinear and quasilinear systems of partial differential equations (e.g. stationary Navier-Stokes equations, semiconductor equations) it can be shown that the regularity of the solutions is dominated by assigned linearized problems.

Nonlinear elastic field equations of power-law type do not fit in this framework in general and it is an open question whether the behavior of the elastic fields can be completely characterized by leading terms in an asymptotic expansion similar to the linear case. In order to obtain global regularity results for nonlinear elastic field equations of power-law type a combination of the difference quotient technique with the concept of a quasi-monotone distribution of material parameters was used. This leads to the new concept of a quasi-monotone covering condition for the subdomains and the energy densities which determine the differential operators on the subdomains.

The regularity results for linear and nonlinear elastic problems can be applied in shape and structure optimization problems in nonsmooth domains, sensitivity analysis for compound elastic structures and the Griffith fracture criterion for a nonlinear elastic model of power-law type. Relying on the proved regularity results formulae for shape derivatives and the energy release rate are derived. Numerical experiments show their relevance for computations.

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