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Computational aspects of quasi-static crack propagation

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Abstract

The focus of this note lies on the numerical analysis of models describing the propagation of a single crack in a linearly elastic material. The evolution of the crack is modeled as a rate-independent process based on the Griffith criterion. We follow two different approaches for setting up mathematically well defined models: the global energetic approach and an approach based on a viscous regularization.

We prove the convergence of solutions of fully discretized models (i.e. with respect to time and space) and derive relations between the discretization parameters (mesh size, time step size, viscosity parameter, crack increment) which guarantee the convergence of the schemes. Further, convergence rates are provided for the approximation of energy release rates by certain discrete energy release rates. Thereby we discuss both, models with self-contact conditions on the crack faces as well as models with pure Neumann conditions on the crack faces. The convergence proofs rely on regularity estimates for the elastic fields close to the crack tip and local and global finite element error estimates. Finally the theoretical results are illustrated with some numerical calculations.

1 Introduction

The prediction of the growth of cracks in brittle materials is of importance in many practical applications. However, mathematical models involving the full elastic interaction as well as the evolution of a freely growing crack are rare. Even in the simpler case of a single crack which propagates along a prescribed path there are only few mathematical contributions investigating evolution models for crack propagation analytically, see [NS07, NO08, KKT08, KMZ08, KZM10, LT10] and the references therein. Moreover, a rigorous convergence analysis for numerical schemes for crack propagation is only available in [NO08] for the out-of-plane case with a given crack path and in [GP06] for a crack evolution model that is based on the global minimization of the total energy (stored energy and dissipation).

In this paper we study the evolution of a single crack in a two dimensional elastic body, where the crack can propagate along a given straight line. On the crack faces, non-penetration conditions (self-contact conditions) are imposed. The basis for the crack evolution models studied here is the Griffith fracture criterion. We assume that inertia terms can be neglected in the force balance and investigate rate-independent models, which are relevant for cases, where the external loading via time-dependent forces is much smaller than internal relaxation times.

It is intrinsic to rate independent evolution models that in spite of time-continuous data discontinuous solutions may occur, i.e. the function $s : [0, T] \to [0, L]$ describing the position of the crack tip at time t might develop jumps. Therefore, the Griffith criterion describing the evolution of the crack has to be completed with suitable jump criteria. In literature, essentially two approaches are followed leading to different predictions of the discontinuities: In the global energetic approach, cf. [Mie05], the jump criteria are determined by global minimization principles for the total energy (elastic energy plus dissipation), whereas in the so-called BV-setting, cf. [MRS09, KMZ08, KZM10], the jump criteria are derived by a vanishing viscosity limit of models including a viscous regularization.

In this paper we focus on the numerical realization of both, the global energetic model and the BV-approach and highlight the different predictions of the models. Moreover, we prove the convergence of solutions of the fully discretized models (FE in space, implicit Euler scheme in time) to solutions of the original models provided that the discretization parameters (time step size, mesh size, crack increment, viscosity) are chosen appropriately. We emphasize that non-penetration conditions on the crack faces are taken into account.

While in the global energetic setting the convergence of the fully discretized solutions follows from a convergence theorem in [MRS08] under quite general assumptions, Section 3, the analysis in the vanishing viscosity setting is more delicate. Here, the main step is to prove the convergence of sequences of discrete energy release rates to the continuous one. To be more precise, if $\mathcal{E} : H^1(\Omega_s, \mathbb{R}^2) \times [0, L] \to \mathbb{R}_\infty$ denotes the elastic energy depending on the displacement field $u \in H^1(\Omega_s, \mathbb{R}^2)$ and the crack of length $s \in [0, L]$, we have to show that

$$\sigma_N^{-1}\big(\mathcal{E}_N(u_N, s_N + \sigma_N) - \mathcal{E}_N(u_N, s_N)\big) \to \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(u(s), s) \text{ for } \sigma_N \searrow 0 \text{ and } s_N \to s, \tag{1.1}$$

see Theorem 4.5. Here, $u(s)$ is the minimizer of the elastic energy $\mathcal{E}$ for the crack length s and $\mathcal{E}_N : V^N \times Z^N \to \mathbb{R}_\infty$ is a spatially discretized version of $\mathcal{E}$, $V^N \subset H^1_{\Gamma_D}(\Omega_s, \mathbb{R}^2)$ is a finite dimensional subspace and $Z^N \subset [0, L]$ a finite set. The function $u_N \in V^N$ is the minimizer of $\mathcal{E}_N(\cdot, s_N)$.

We verify (1.1) for two cases, namely for models with contact conditions on the crack surface and for models without such conditions (pure Neumann conditions). The proofs rely on the regularity properties of the displacement field $u(s)$ in a neighborhood of the crack tip. Higher differentiability results are well known in the case without contact conditions on the crack faces and very detailed descriptions of the crack tip singularities are available, see e.g. [Gri89]. For cracks with contact conditions we use the result derived in [KS11], which states that the displacement field u belongs to $H^{\frac{3}{2}-\delta}(\Omega_s)$ for all $\delta > 0$. This is in accordance with the results for pure Neumann conditions. Using this property the proof of (1.1) is carried out for models with contact conditions in Section 4.2.1. Applying local FE-error estimates improved convergence properties are then derived for models without contact conditions in Section 4.2.2.

In Section 5, we present some numerical experiments which shed light on the different predictions of the global energetic model and the model based on vanishing viscosity. We note that the BV-model is possibly more realistic from a physical point of view. Our aim is to illustrate the interplay of discretization parameters. Since this case seems to be closer to reality, we only consider the contact case in our numerical experiments. We use finite elements with continuous, piecewise bilinear ansatz functions on uniformly refined quadrilateral meshes to discretize the variational inequalities arising from the non-penetration condition of self-contact at the crack surface, cf. [KO88]. For simplicity, we assume that the crack is partitioned by the edges of the mesh so that the crack increment is determined by the edge lengths. We simply double the edges along the crack to construct such partitionings. To provide for arbitrary crack increments one may use remeshing or incorporate the crack into the discretization via, e.g., an extended finite element approach (XFEM).

We study two algorithms computing the incremental solutions as a sequence of time-steps and crack lengths. They rely on solving a sequence of time-incremental minimization problems defined via the discrete energy and dissipation. The convergence of the first algorithm is analyzed in this paper. The second algorithm is an extension of the first algorithm where some derivative information of the interpolant of the discrete energy is used. Both algorithms are applicable to compute approximations of the BV-solutions as well as of global energetic solutions. As a result of the convergence analysis, the first algorithm exhibits a certain sensitivity with respect to the discretization parameters. This can also be observed in the numerical experiments. In the second algorithm this sensitivity is significantly reduced. However, the analysis of this improved algorithm is still in progress.

For general applications it is often too restrictive to assume that the crack path is known in advance. An exception is the study of an interface crack. Arbitrary crack geometries (including branching and kinking) are included in crack evolution models developed and analyzed in the global energetic framework [FM98, DFT05]. There, the displacement field belongs to the space $SBV(\Omega)$ (special functions of bounded variation) and the crack is related with the discontinuity set of the displacement field. We refer to [GP06], where the convergence of fully discretized approximation schemes for these models are shown. It is an open question how to transfer this general approach to the vanishing viscosity setting. One of the main challenges is to find a suitable notion for the energy release rate for such general crack geometries.

2 The global energetic model and BV-solution models

2.1 Geometric assumptions and basic properties of the energy release rate

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary describing the undamaged physical body. The boundary $\partial\Omega$ is divided into a part Γ_D, where the displacements are prescribed, and into a part Γ_N, where the surface forces are imposed. We define $\Omega_+ = \{\, x = (x_1, x_2) \in \Omega\,;\, x_2 > 0 \,\}$ and Ω_- is defined analogously. It is assumed that $\left|\Gamma_D \cap \overline{\Omega_+}\right| > 0$ and $\left|\Gamma_D \cap \overline{\Omega_-}\right| > 0$. Further, $(0,0)^\top \in \partial\Omega$ and there exists $L > 0$ such that for all $s \in (0, L)$ we have

$$C_s := \{\, x \in \mathbb{R}^2\,;\, x = (\sigma, 0)^\top,\, \sigma \in (0, s] \,\} \subset \Omega$$

and $(L,0)^\top \in \partial\Omega$. The line C_s describes a crack of length s with crack tip $x_s = (s,0)^\top$. Moreover, $\Omega_s = \Omega \backslash C_s$ is the domain with crack C_s, see Fig. 1(a).

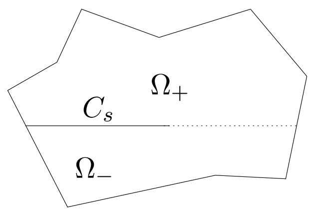


Figure 1: Example for an admissible domain with crack C_s.

For a given crack of length s the set of admissible displacements with vanishing Dirichlet-boundary conditions on Γ_D is

$$V_s = \{\, u \in H^1(\Omega_s, \mathbb{R}^2)\,;\, u|_{\Gamma_D} = 0 \,\}, \quad V := V_L.$$

Since it is assumed that $\left|\Gamma_D \cap \overline{\Omega_\pm}\right| > 0$, Korn's inequality is valid in V_s for all $s \in [0, L]$. The convex cone of admissible displacements satisfying in addition non-penetration conditions on C_s is defined as

$$K_s = \{\, u \in V_s\,;\, [u] \cdot \mathbf{n} \geq 0 \text{ on } C_s \,\}, \tag{2.1}$$

where $\mathbf{n}$ is the unit normal vector on C_s and $[u]$ denotes the jump of u across the crack. For given time t, crack tip position s and displacement field u the elastic energy $\mathcal{E} : [0, T] \times V \times [0, L] \to \mathbb{R} \cup \{\infty\}$ is given by

$$\mathcal{E}(t, u, s) = \begin{cases} \int_{\Omega_s} \frac{1}{2}(\mathbf{C}\varepsilon(u)) : \varepsilon(u)\,\mathrm{d}x - \int_{\Gamma_N} \ell(t) \cdot u \mathrm{d}\Gamma & \text{if } u \in K_s \\ \infty & \text{otherwise} \end{cases}.$$

Here, $\ell \in C^1([0, T]; L^2(\Gamma_N, \mathbb{R}^2))$ describes the applied surface forces and $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ is the linearized strain tensor. The fourth order tensor $\mathbf{C}$ denotes the elasticity tensor, which is assumed to be constant, symmetric and positive definite on $\mathbb{R}^{2\times 2}_{\text{sym}}$, i.e. for all $\eta, \xi \in \mathbb{R}^{2\times 2}_{\text{sym}}$ it holds $\mathbf{C}\eta : \xi = (\mathbf{C}_{ijkl}\eta_{kl})\xi_{ij} = (\mathbf{C}_{ijkl}\xi_{kl})\eta_{ij}$ and $\mathbf{C}\eta : \eta \geq c_{\mathbf{C}} |\eta|^2$. In the sequel we will frequently use the notation $A : B = \sum_{ij} A_{ij} B_{ij}$ for the inner product for tensors $A, B \in \mathbb{R}^{2\times 2}$ and $|A| = \sqrt{A : A}$ for the corresponding Frobenius norm.

For every fixed $t \in [0, T]$ and $s \in [0, L]$ the energy $\mathcal{E}(t, \cdot, s)$ has a unique minimizer $u(t, s) \in K_s$. Observe, that the minimizer is the unique solution of the following variational inequality: for all $v \in K_s$ it holds

$$a_s(u(t, s), u(t, s) - v) \leq \int_{\Gamma_N} \ell(t) \cdot (u(t, s) - v)\,\mathrm{d}\Gamma \tag{2.2}$$

with $a_s(u, v) = \int_{\Omega_s} \mathbf{C}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x$ for $u, v \in H^1(\Omega_s)$. For later use we also introduce the associated linear elliptic differential operator $\mathcal{A}_s : V_s \to V_s^*$, which is defined via $\langle \mathcal{A}_s(u), v\rangle = a_s(u, v)$ for all $u, v \in V_s$.

The following uniform estimate for minimizers relies on Korn's inequality and the continuity of a_s: There exists a constant $c > 0$ such that

$$\sup_{(t,s)\in[0,T]\times[0,L]} \|u(t, s)\|_{H^1(\Omega_L)} \leq c\,\|\ell\|_{C^0([0,T];L^2(\Gamma_N))}\,. \tag{2.3}$$

We denote by

$$\mathcal{I} : [0,T] \times [0,L] \to \mathbb{R},\ \mathcal{I}(t,s) = \min_{v \in K_s} \mathcal{E}(t,v,s) = \mathcal{E}(t,u(t,s),s)$$

the corresponding reduced energy functional.

A central quantity in the crack evolution models, which we investigate in this paper, is the energy release rate $\mathcal{G}(t,s)$. This quantity is defined as $\mathcal{G}(t,s) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(t,s)$ and has the following properties:

Theorem 2.1. *If $\ell \in C^1([0,T]; L^2(\Gamma_N))$, then $\mathcal{I} \in C^1([0,T] \times (0,L))$ and $\mathcal{G} \in C^0([0,T] \times (0,L))$. Moreover, if $(t_n, s_n) \to (t,s) \in [0,T] \times (0,L)$, then $u(t_n, s_n) \to u(t,s)$ strongly in V.*

For $s \in (0,L)$ the energy release rate $\mathcal{G}$ can be expressed by the Griffith formula via $\mathcal{G}(t,s) = G(s, u(t,s))$ where $u(t,s) = \operatorname{argmin}\{\, \mathcal{E}(t,s,v)\,;\, v \in K_s \,\}$ and

$$G(s,v) = \int_{\Omega_s} \mathbb{E}(\nabla v) : \nabla \rho_s \,\mathrm{d}x \tag{2.4}$$

for $v \in V_s$. The Eshelby tensor $\mathbb{E}$ is defined as $\mathbb{E}(F) = F^\top D_F W(F) - W(F)\mathbb{I}$ for $F \in \mathbb{R}^{2\times 2}$ with $W(F) = \frac{1}{2}\mathbf{C}F_{sym} : F_{sym}$ and $\rho_s = \theta_s(x)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, where $\theta_s \in C_0^\infty(\Omega)$ is an arbitrary cut-off function with $\theta_s = 1$ in a neighborhood of the crack tip $(s,0)^\top$.

The existence of the energy release rate was proved in [DD81, KS00] for quadratic energies and extended in [KMZ08] to more general strictly convex energy densities with p-growth. Furthermore, in [KMZ08] the continuity properties were investigated. We refer to [KM08, KZM10] for the finite strain case. It is also shown in these references that formula (2.4) does not depend on the particular choice of the cut-off function θ_s.

Since θ_s is constant outside a certain annulus centered at the crack tip, the support of $\nabla\theta_s$ is a subset of this annulus and does not contain the crack tip. Hence, the integration domain Ω_s in (2.4) can be reduced to this annulus. This observation is the basis for the refined estimates which we carry out in Section 4.2.2.

We also need the following refined continuity property of $\partial_s\mathcal{I}$:

Theorem 2.2. *Let $\ell \in C^1([0,T]; L^2(\Gamma_N))$. Then the energy release rate $-\partial_s\mathcal{I}$ is locally Lipschitz continuous on $[0,T] \times (0,L)$, i.e. for every $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that for all $t_1, t_2 \in [0,T]$ and $s_1, s_2 \in [\epsilon, L-\epsilon]$ it holds*

$$|\partial_s\mathcal{I}(t_1,s_1) - \partial_s\mathcal{I}(t_2,s_2)| \le c_\epsilon\big(\,|t_1 - t_2| + |s_1 - s_2|\,\big). \tag{2.5}$$

In Theorem 2.2 quantities are compared, which are defined with respect to different crack lengths. In order to use minimizers for different crack-lengths as mutual test functions for the corresponding variational inequalities, a transformation of these inequalities to a domain with a fixed crack length has to be carried out. The advantage then is that the crack parameter occurs in the coefficients of the corresponding bilinear forms and not any more in the domain of integration. For the spatial transformations we use special inner variations, see [GH96], which map cracks of different lengths onto each other, see also [DD81, KS00, KMZ08].

Proof. We first construct spatial transformations in the spirit of [DD81, KMZ08].

Let $\epsilon > 0$ and choose $R_\epsilon > 0$ such that for every $s \in [\epsilon, L-\epsilon]$ the ball $B_{4R_\epsilon^2}(x_s)$ is compactly contained in Ω. Let $\theta \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$ be a cut-off function with $\operatorname{supp}\theta \subset B_{4R_\epsilon^2}(0)$ and $\theta\big|_{B_{R_\epsilon^2}(0)} = 1$. For $s \in [\epsilon, L-\epsilon]$ and $x \in \mathbb{R}^2$ let $\theta_s(x) = \theta(|x-x_s|^2)$. Obviously, $\theta_s \in C_0^\infty(B_{2R_\epsilon}(x_s))$ with $\theta\big|_{B_{R_\epsilon}(x_s)} = 1$. For $\rho \in \mathbb{R}$ we define the following family of mappings

$$T_{s,\rho} : \mathbb{R}^2 \to \mathbb{R}^2, \quad x \mapsto x + \rho \begin{pmatrix} \theta_s(x) \\ 0 \end{pmatrix}. \tag{2.6}$$

Roughly speaking, $T_{s,\rho}$ describes a translation which is tangential to the crack and of local length $\rho\theta_s$. There exists a constant $\rho_\epsilon > 0$ such that for all $|\rho| \le \rho_\epsilon$ and $s \in [\epsilon, L-\epsilon]$ the mappings $T_{s,\rho}$ are diffeomorphisms with $T_{s,\rho}(\Omega_s) = \Omega_{s+\rho}$. Moreover, for these ρ the mappings $T_{s,\rho}$ induce isomorphisms between the spaces V_s and $V_{s+\rho}$, and $v \in K_{s+\rho}$ if and only if $v \circ T_{s,\rho} \in K_s$. Observe that the expression $\|T_{s,\rho}\|_{C^1(\mathbb{R}^2)} + \left\|T_{s,\rho}^{-1}\right\|_{C^1(\mathbb{R}^2)}$ is uniformly bounded on parameter sets $(s,\rho) \in [\epsilon, L-\epsilon] \times [-\rho_\epsilon, \rho_\epsilon]$.

We will next transform the energies and variational inequalities to a domain with a fixed crack length. Thereto let the coefficient tensor $\mathbf{B}_s(\rho) \in C^\infty(\mathbb{R}^2;\mathbb{R}^{2\times2\times2\times2})$ be defined as follows: for all $\eta_1, \eta_2 \in \mathbb{R}^{2\times2}$ and $y \in \mathbb{R}^2$

$$\mathbf{B}_s(\rho,y)\eta_1 : \eta_2 = |\det \nabla T_{s,\rho}(y)| \left(\mathbf{C}(\eta_1(\nabla T_{s,\rho}(y))^{-1})_{\mathrm{sym}}\right) : (\eta_2(\nabla T_{s,\rho}(y))^{-1})_{\mathrm{sym}}. \tag{2.7}$$

Clearly, $\mathbf{B}_s(0,y)\eta_1 : \eta_2 = \mathbf{C}\eta_{1,\mathrm{sym}} : \eta_{2,\mathrm{sym}}$.

With this notation, $u(t, s+\rho) \in K_{s+\rho}$ is a minimizer of $\mathcal{E}(t,\cdot,s+\rho)$ if and only if the following transformed variational inequality is satisfied by $\widetilde{u}_{s,\rho}(t) := u(t,s+\rho) \circ T_{s,\rho}$: for all $v \in K_s$

$$\int_{\Omega_s} \mathbf{B}_s(\rho,y)\nabla\widetilde{u}_{s,\rho}(t) : \nabla(v - \widetilde{u}_{s,\rho}(t))\,\mathrm{d}y \ge \int_{\Gamma_N} \ell(t)\cdot(v - \widetilde{u}_{s,\rho}(t))\,\mathrm{d}\Gamma.$$

Observe that the integration is carried out on the fixed domain Ω_s and the different crack lengths are represented by the parameter ρ.

Let $t_1, t_2 \in [0,T]$ and $s_1, s_2 \in [\epsilon, L-\epsilon]$ with $|s_1 - s_2| \le \rho_\epsilon/2$. From the previous variational inequality with $s = s_1$, $t = t_2$, $\rho = s_2 - s_1$ and $v = u(t_1,s_1)$ together with the variational inequality for the minimizer $u(t_1,s_1)$ tested with $\widetilde{u}_{s_1,s_2-s_1}(t_2) = u(t_2,s_2)\circ T_{s_1,s_2-s_1} \in K_{s_1}$ we obtain

$$\begin{aligned}
&\int_{\Omega_{s_1}} \mathbf{B}_{s_1}(0,y)\nabla(\widetilde{u}_{s_1,s_2-s_1}(t_2) - u(t_1,s_1)) : \nabla(\widetilde{u}_{s_1,s_2-s_1}(t_2) - u(t_1,s_1))\,\mathrm{d}y \\
&\quad\le \int_{\Omega_{s_1}} \big(\mathbf{B}_{s_1}(s_2-s_1,y) - \mathbf{B}_{s_1}(0,y)\big)\nabla\widetilde{u}_{s_1,s_2-s_1}(t_2) : \nabla(u(t_1,s_1) - \widetilde{u}_{s_1,s_2-s_1}(t_2))\,\mathrm{d}y \\
&\quad\quad - \int_{\Gamma_N} (\ell(t_2) - \ell(t_1))\cdot(u(t_1,s_1) - \widetilde{u}_{s_1,s_2-s_1}(t_2))\,\mathrm{d}\Gamma.
\end{aligned}$$

Technical calculations show that there exists a constant $\widetilde{c}_\epsilon > 0$ such that for all $y \in \mathbb{R}^2$, $s_1, s_2 \in [\epsilon, L-\epsilon]$ such that $|s_1 - s_2| \le \rho_\epsilon$ it holds (see e.g. [GH96] or [KMZ08, Lemma 3.1])

$$\sup\big(|\mathbf{B}_{s_1}(s_2-s_1,y) - \mathbf{B}_{s_1}(0,y)| + |\partial_\rho\mathbf{B}_{s_1}(s_2-s_1,y) - \partial_\rho\mathbf{B}_{s_1}(0,y)|\big) \le \widetilde{c}_\epsilon\,|s_1 - s_2|\,. \tag{2.8}$$

Hence, by Korn's inequality (applied on Ω_L) we arrive at the estimate

$$\|u(t_1, s_1) - \widetilde{u}_{s_1, s_2 - s_1}(t_2)\|_{H^1(\Omega_{s_1})} \leq c_\epsilon\big(|s_1 - s_2| + |t_1 - t_2| \big) \, \|\ell\|_{C^0([0,T];L^2(\Gamma_N))} \,. \tag{2.9}$$

Let now $s \in [\epsilon, L - \epsilon]$ be fixed and $|\rho_0| \leq \rho_\epsilon/2$. Observe that the following identity holds true:

$$\mathcal{I}(t, s + \rho_0) = \int_{\Omega_s} \frac{1}{2} \mathbf{B}(y, \rho_0) \nabla \widetilde{u}_{s,\rho_0}(t_2) : \nabla \widetilde{u}_{s,\rho_0}(t_2) \, \mathrm{d}y - \int_{\Gamma_N} \ell(t) \cdot \widetilde{u}_{s,\rho_0}(t_2) \mathrm{d}\Gamma,$$

where, as before, $\widetilde{u}_{s,\rho_0}(t_2)(y) = u(t_2, s + \rho_0) \circ T_{s,\rho_0}$. It follows from Theorem 3.1 and Theorem 3.2 in [KMZ08] that for $t_1, t_2 \in [0, T]$ we have

$$\partial_s \mathcal{I}(t_1, s) = \int_{\Omega_s} \frac{1}{2} \partial_\rho \mathbf{B}_s(0, y) \nabla u(t_1, s) : \nabla u(t_1, s) \, \mathrm{d}y, \tag{2.10}$$

$$\partial_s \mathcal{I}(t_2, s + \rho_0) = \int_{\Omega_s} \frac{1}{2} \partial_\rho \mathbf{B}_s(\rho_0, y) \nabla (\widetilde{u}_{s,\rho_0}(t_2)) : \nabla (\widetilde{u}_{s,\rho_0}(t_2)) \, \mathrm{d}y. \tag{2.11}$$

Hence, combining these formulas with (2.8) and (2.9), estimate (2.5) follows for $s_1, s_2 \in [\epsilon, L - \epsilon]$ such that $|s_1 - s_2| \leq \rho_\epsilon$. Covering $[\epsilon, L - \epsilon]$ with finitely many intervals of length $2\rho_\epsilon$ we finally arrive at estimate (2.5) for s_i, t_i as stated in Theorem 2.2. □

2.2 Evolution models based on the Griffith criterion

In the Griffith fracture criterion the energy, which is dissipated due to the crack growth, is assumed to be proportional to the crack increment. This is characterized with the material dependent fracture toughness $\kappa \in C^0([0, L])$, $\kappa > 0$. The Griffith criterion implies to the following quasistatic, rate independent model for crack propagation in Karush–Kuhn–Tucker form:

Definition 2.3. The function $s : [0, T] \to (0, L)$ is a solution of the (continuous) crack propagation model if s is non-decreasing, $\dot{s}(t) = \frac{\mathrm{d}}{\mathrm{d}t} s(t)$ exists for all $t \in [0, T]$ and if for all $t \in [0, T]$ we have

(a) local stability: $\kappa(s(t)) - \mathcal{G}(t, s(t)) \geq 0$,

(b) complementarity condition: $\dot{s}(t)\big(\kappa(s(t)) - \mathcal{G}(t, s(t))\big) = 0$.

Conditions (a) and (b) imply: if $\kappa(s(t)) > \mathcal{G}(t, s(t))$, then $\dot{s}(t) = 0$ and the crack does not move. The crack can only propagate if the local force balance $\kappa(s(t)) = \mathcal{G}(t, s(t))$ is satisfied.

However, Definition 2.3 is too strong in the sense that solutions might exist only for short time intervals. This can be seen as follows: Assume that during the evolution a point (t_*, s_*) is reached with $\kappa(s_*) - \mathcal{G}(t_*, s_*) = 0$ and $\kappa(s_* + \epsilon) - \mathcal{G}(t_* + \delta, s_* + \epsilon) < 0$ for every $\delta \in (0, \delta_0)$, $\epsilon \in (0, \epsilon_0)$. Then every continuous non-decreasing function $s : [t_*, t_* + \delta_0] \to (0, L)$ with $s(t_*) = s_*$ violates the local stability condition (a) and hence there is no continuous solution beyond the time t_*. In the example in Section 5.1 such a situation is described. Thus, the model in Definition 2.3

is not satisfactory and has to be refined for example by allowing for discontinuous solutions and by adding further conditions for the discontinuities.

Let $BV([0,T])$ denote the space of functions from $[0,T]$ into $\mathbb{R}$ with bounded variation. For $s \in BV([0,T])$ the set $J(s) \subset [0,T]$ is the jump set and consists of the discontinuity points of s.

Definition 2.4. A function $s \in BV([0,T])$ is a *local solution* (LS) to the crack problem if it is non-decreasing and satisfies

(a) Local stability: $t \in [0,T] \backslash J(s) \quad \Rightarrow \quad \kappa(s(t)) - \mathcal{G}(t, s(t)) \geq 0$,

(b) Energy inequality: For all $0 \leq t_1 < t_2 \leq T$ we have
$\mathcal{I}(t_2, s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma) \mathrm{d}\sigma \leq \mathcal{I}(t_1, s(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{I}(t, s(t)) \mathrm{d}t$.

Note that (a) and (b) of Definition 2.4 imply that the complementarity condition in Definition (2.3)(b) is satisfied for almost every $t \in [0,T]$. The model in Definition 2.4 is thermodynamically admissible, but it allows for a great variety of solutions. By adding either a global minimization criterion or a criterion based on a vanishing viscosity approach, one can select particular local solutions. These two approaches are discussed in the next sections.

2.2.1 The global energetic model

The global energetic model is based on the elastic energy $\mathcal{E}(t, u, s)$ and the dissipation distance

$$\mathcal{D}(s_1, s_2) = \begin{cases} \int_{s_1}^{s_2} \kappa(\sigma) \mathrm{d}\sigma & \text{if } s_2 \geq s_1 \\ \infty & \text{otherwise} \end{cases},$$

which quantifies the dissipated energy when passing from a crack with length s_1 to a crack with length s_2. The dissipation distance takes into account the irreversibility of the crack propagation process, i.e. the healing of the crack is excluded.

Definition 2.5. A pair $(s, u) \in BV([0,T], [0,L]) \times BV([0,T], V)$ is a *global energetic solution* (GES) to the initial values $(s_0, u_0) \in [0,L] \times K_{s_0}$ if $(s(0), u(0)) = (s_0, u_0)$ and if for all $t \in [0,T]$ it holds

(a) Global stability: For all $\tilde{s} \in [0,L]$, $v \in V$ we have
$$\mathcal{E}(t, u(t), s(t)) \leq \mathcal{E}(t, v, \tilde{s}) + \mathcal{D}(s(t), \tilde{s}),$$

(b) Energy equality:
$$\mathcal{E}(t, u(t), s(t)) + \mathrm{Diss}_{\mathcal{D}}(s; [0,t]) = \mathcal{E}(0, u(0), s(0)) + \int_0^t \partial_t \mathcal{E}(\tau, u(\tau), s(\tau)) \mathrm{d}\tau.$$

The total dissipation along a given path $s : [t_1, t_2] \to \mathbb{R}$ is defined as

$$\mathrm{Diss}_{\mathcal{D}}(s; [t_1, t_2]) = \sup_{\text{partitions } \{\tau_i\} \text{ of } [t_1, t_2]} \Big\{ \sum_{i=1}^{N} \mathcal{D}(s(\tau_{i-1}), s(\tau_i)) \, ; \, t_1 = \tau_0 < \cdots < \tau_N = t_2 \Big\}$$

For the particular dissipation distance $\mathcal{D}$ of the crack model we obtain for nondecreasing curves $s : [t_1, t_2] \to \mathbb{R}$ the expression $\mathrm{Diss}_{\mathcal{D}}(s; [t_1, t_2]) = \mathcal{D}(s(t_1), s(t_2))$.

The global energetic formulation is a general concept for modeling rate independent problems and we refer to [Mie05, MRS08] for a survey. The existence of a GES for the crack evolution problem follows from this general framework ([MRS08, Theorem 3.3]):

Theorem 2.6. *Let $\kappa \in L^\infty(0, L)$ with $0 < \kappa_0 \le \kappa(\sigma)$ for a.e. $\sigma \in [0, L]$. Let further $\ell \in C^1([0, T]; L^2(\Gamma_N))$. Then for every stable initial datum $(s_0, u_0) \in [0, L] \times K_{s_0}$ there exists a global energetic solution $(s, u) \in BV([0, T], [0, L]) \times BV([0, T], V)$.*

The initial datum (s_0, u_0) is called stable if it satisfies the global stability condition for $t = 0$. Note that every GES is a special LS and that in particular the local stability condition (a) in Definition 2.4 is satisfied for almost every t. The behavior of GES is illustrated in Section 5.1.

2.2.2 The vanishing viscosity approach and BV-solutions

To generate solutions staying in local minimizers a vanishing viscosity approach is applied, which is close to the physical modeling. In fact, true physical systems are not strictly rate-independent but have some internal time scales (relaxation times) that are usually neglected when very slow loading is considered. However, if rate-independent solutions are not continuous, then the corresponding solution with small viscosity develops very large velocities. The aim is to derive jump criteria for the rate-independent model by studying the limits of viscous solutions when the viscosity tends to zero. We refer to [MRS09, EM06] for the general philosophy and to [KMZ08, KZM10] for the application to the crack model for strictly convex elastic energies and for the finite strain case.

The starting point for deriving the BV-model as a vanishing viscosity limit is a viscous regularization of the model presented in Definition 2.3: Given a viscosity parameter $\nu > 0$, the function $s^\nu \in H^1([0, T]; \mathbb{R})$ is a viscous solution of the crack model if for a.e. $t \in [0, T]$ we have $\dot{s}(t) \ge 0$ together with

(a) local stability: $\kappa(s^\nu(t)) + \nu \dot{s}^\nu(t) - \mathcal{G}(t, s^\nu(t)) \ge 0$,

(b) complementarity condition: $\dot{s}^\nu(t)\big(\kappa(s^\nu(t)) + \nu \dot{s}^\nu(t) - \mathcal{G}(t, s^\nu(t))\big) = 0$.

It is shown in [KMZ08] that viscous solutions (i.e. s^ν) exist and that (sub)sequences of viscous solutions converge weakly* in $BV([0, T]; \mathbb{R})$ to so-called BV-solutions or vanishing viscosity solutions described in the definition here below:

Definition 2.7. A nondecreasing function $s \in BV([0, T]; \mathbb{R})$ is called a BV-solution of the crack evolution model with initial value s_0 if $s(0) = s_0$ and if for all $t \in [0, T]$ conditions (a)-(d) here below are satisfied

(a) $\kappa(s(t)) - \mathcal{G}(t, s(t)) \ge 0$ if $t \notin J(s)$ and $s(t) < L$.

(b) if $\kappa(s(t)) - \mathcal{G}(t, s(t)) > 0$, then $t \in D(s)$ and $\dot{s}(t) = 0$.

(c) For all $t \in J(s)$ and all $s_* \in [s(t-), s(t+)]$ we have $\kappa(s_*) - \mathcal{G}(t, s_*) \leq 0$. If $s(t+) = L$, then the inequality holds for $s_* \in [s(t-), L)$.

Here, the set $J(s)$ is the set of discontinuity points and $D(s)$ is the set of differentiable points of $s \in BV([0,T];\mathbb{R})$. The case $s(t) = L$ plays a special role since it is not clear, whether here the energy release rate is well defined. BV-solutions satisfy an energy equality and are special local solutions, see [KMZ08].

3 Numerical approximation and convergence analysis for the global energetic model

In order to calculate global energetic solutions and BV-solutions numerically, the models are discretized using finite elements in the space and an implicit Euler scheme in time. In this section we prove the convergence of the fully discretized problems to global energetic solutions. In Section 4, BV-solutions are treated.

Let $Z = [0, L]$. For $N \in \mathbb{N}$ let $Z^N \subset Z$ be a finite partition with $L \in Z^N$ and $\overline{\cup_{N\in\mathbb{N}} Z_N} = Z$. Let further $V^N \subset V$ be a closed subspace. We define $V_s^N := V^N \cap V_s$ and $K_s^N := V_s^N \cap K_s$. Observe that the following closedness property holds true:

Lemma 3.1. *For all sequences $\{\, s_N \,;\, N \in \mathbb{N} \,\} \subset Z$ with $s_N \in Z^N$ and all $\{\, u_N \,;\, N \in \mathbb{N} \,\} \subset V_L$ with $u_N \in K_{s_N}^N$ such that $s_N \to s$ and $u_N \rightharpoonup u$ weakly in V_L it holds $u \in K_s$.*

Proof. The weak convergence in V_L implies the strong convergence of the traces $u_N^{\pm}\big|_{C_L}$ in $L^2(C_L)$ to $u^{\pm}\big|_{C_L}$. Hence, at least for a subsequence we have $[u_N(x)] \cdot \mathbf{n} \to [u(x)] \cdot \mathbf{n}$ for almost every $x \in C_L$, which shows that $u \in K_s$. □

The following approximation property is assumed to hold:

For all $s \in Z$, $v \in K_s$, $N \in \mathbb{N}$ and all sequences $\{\, s_N \,;\, N \in \mathbb{N} \,\} \subset Z$ with $s_N \in Z^N$ and $s_N \to s$ there exists $v_N \in K_{s_N}^N$ such that $v_N \to v$ strongly in V for $N \to \infty$. (3.1)

The discrete energy $\mathcal{E}_N : [0,T] \times V \times Z \to \mathbb{R}_\infty$ is defined as

$$\mathcal{E}_N(t, v, s) = \begin{cases} \mathcal{E}(t, v, s) & \text{if } s \in Z^N \text{ and } v \in K_s^N \\ \infty & \text{otherwise} \end{cases} . \tag{3.2}$$

Observe that if $(s, u) \in Z \times K_s$ and if $(s_N, u_N)_{N\in\mathbb{N}}$ is a sequence according to (3.1), then $\mathcal{E}_N(t, u_N, s_N) \to \mathcal{E}(t, u, s)$ for $N \to \infty$.

Finally, for $N \in \mathbb{N}$ let Π_N be a partition of the time interval into $0 = t_N^0 < t_N^1 < \ldots < t_N^N = T$ with fineness $f(\Pi_N) = \max_{1 \leq i \leq N}(t_N^i - t_N^{i-1})$.

The approximation of the global energetic crack propagation model from Definition 2.5 relies on a sequence of time-incremental minimization problems defined via the discrete energy $\mathcal{E}_N$ and the dissipation $\mathcal{D}_N$. Thereby the dissipation $\mathcal{D}_N$ is defined as

$$\mathcal{D}_N(s_1, s_2) = \begin{cases} \int_{s_1}^{s_2} \kappa_N(\sigma)\mathrm{d}\sigma & \text{if } s_1 \le s_2 \\ \infty & \text{otherwise} \end{cases}$$

with functions $\kappa_N \in L^\infty(0, L)$, $0 < \kappa_0 \le \kappa_N \le \|\kappa\|_{L^\infty(0,L)}$. For example the κ_N can be chosen as piecewise constant approximations of the fracture toughness κ.

Let $N \in \mathbb{N}$ be given and let $(u_N^0, s_N^0) \in V^N \times Z^N$ be stable initial values. This means that

$$\mathcal{E}_N(0, u_N^0, s_N^0) \le \mathcal{E}_N(0, v, s) + \mathcal{D}_N(s_N^0, s) \text{ for all } (v, s) \in V^N \times Z^N.$$

For $k \in \{1, \dots, N\}$ the incremental solutions (u_N^k, s_N^k) are determined from the minimization problem

$$(u_N^k, s_N^k) \in \operatorname{Argmin}\{\, \mathcal{E}_N(t_N^k, v, s) + \mathcal{D}_N(s_N^{k-1}, s)\,;\, s \in Z^N,\, v \in K_s^N \,\}. \tag{3.3}$$

From the minimizers we construct the piecewise constant functions $\overline{u}_N : [0, T] \to V^N$ and $\overline{s}_N : [0, T] \to Z^N$ via

$$\overline{u}_N(t) = u_N^k \text{ for } t \in (t_N^{k-1}, t_N^k], \qquad \overline{s}_N(t) = s_N^{k-1} \text{ for } t \in (t_N^{k-1}, t_N^k].$$

As an application of the abstract convergence Theorem 3.3 in [MRS08] we obtain:

Theorem 3.2. *Let the assumptions of Theorem 2.6 be satisfied and choose a sequence of partitions Π_N with $f(\Pi_N) \to 0$ for $N \to \infty$. Assume that the functions κ_N converge to κ strongly in $L^1(\Omega)$. Let furthermore $(u_N^0, s_N^0) \in V^N \times Z^N$ be stable initial values with $u_N^0 \to u^0$ strongly in V and $s_N^0 \to s^0$ for $N \to \infty$.*

Then for every N the corresponding incremental problems (3.3) *have minimizers and there exists a subsequence $(\overline{u}_{N_j}, \overline{s}_{N_j})_{j\in\mathbb{N}}$ and a pair of functions $(u, s) \in BV([0, T], V) \times BV([0, T], Z)$ such that (u, s) is a global energetic solution with initial values (u^0, s^0) and for all $t \in [0, T]$ we have $s_{N_j}(t) \to s(t)$ and $u_{N_j}(t) \to u(t)$ strongly in V. In addition, for all t the energies converge, i.e. $\mathcal{E}_{N_j}(t, \overline{u}_{N_j}(t), \overline{s}_{N_j}(t)) \to \mathcal{E}(t, u(t), s(t))$, and any function $(\tilde{u}, \tilde{s}) : [0, T] \to V \times Z$ obtained as such a limit is a global energetic solution of the crack problem.*

Proof. The assumptions of Theorem 3.3 in [MRS08] can easily be verified. In particular, the approximation property (3.1) implies the required *conditioned upper semicontinuity of stable sets.* Hence, Theorem 3.3 in [MRS08] implies Theorem 3.2, but with weak convergence of $u_{N_j}(t) \rightharpoonup u(t)$. The uniform convexity of $\mathcal{E}$ with respect to u and the convergence of the energies ensure the strong convergence $u_{N_j}(t) \to u(t)$ in V. □

Since the solutions of the global energetic model in general are not unique, one cannot expect the whole sequence to converge.

4 Numerical approximation and convergence analysis for the BV-model

We first introduce the notation and the fully discretized model. Based on an assumption concerning the convergence of certain discrete energy release rates, we prove the convergence of solutions of the fully discretized model (with viscosity) to BV-solutions. In Section 4.2 we present conditions, which are sufficient to guarantee the above mentioned convergence of energy release rates. We will consider both, models with and models without contact conditions on the crack faces.

4.1 Convergence analysis for the fully discretized vanishing viscosity model

For $N \in \mathbb{N}$ let $\Pi_N = \{0 = t_N^0 < t_N^1 < \ldots < t_N^N = T\}$ be a partition of the time interval with fineness $\tau_N = \max_k\{t_N^k - t_N^{k-1}\}$, $\underline{\tau}_N := \min_k\{t_N^k - t_N^{k-1}\}$ and local time step size $\tau_N^k = t_N^k - t_N^{k-1}$.

The assumptions on the sets $Z^N \subset Z$ of discrete crack lengths are slightly stronger in comparison to Section 3: Let $\mathfrak{s}_0 \in (0, L)$ and $Z = [\mathfrak{s}_0, L]$. Let $(M_N)_N \subset \mathbb{N}$ be a sequence with $M_N \to \infty$ for $N \to \infty$. We define $\sigma_N := (L - \mathfrak{s}_0)/M_N$ and $Z^N := \{\, \mathfrak{s}_0 + k\sigma_N\, ;\, k \in \mathbb{N},\, 0 \le k \le M_N\,\}$. The set Z^N describes admissible discrete crack lengths. The assumption that the elements of Z^N are equally spaced is for notational simplicity.

Finally, let again $V^N \subset V$ be a family of closed subspaces and $K_s^N = V^N \cap K_s$. Assume that for all N we have $K_{\mathfrak{s}_0}^N \neq \emptyset$. Further compatibility conditions between the spaces V^N, K_s^N and the sets Z^N are implicitly formulated here below in assumption (4.1) on the uniform convergence of discrete energy release rates: For $t \in [0, T]$, $s \in [0, L]$ the reduced energy is defined as

$$\mathcal{I}_N(t, s) = \min\{\, \mathcal{E}(t, v, s)\, ;\, v \in K_s^N\,\}.$$

Observe that in general $\mathcal{I}_N$ is not continuous with respect to s. We assume

For every $\epsilon, \mu > 0$ exists $N_{\epsilon,\mu} \in \mathbb{N}$ such that for all $N \ge N_{\epsilon,\mu}$, $s \in [\epsilon, L-\epsilon] \cap Z^N$ and $t \in [0, T]$ it holds $\left| \frac{1}{\sigma_N} \left(\mathcal{I}_N(t, s \pm \sigma_N) - \mathcal{I}_N(t, s)\right) \mp \partial_s \mathcal{I}(t, s) \right| \le \mu.$ (4.1)

Taking into account the uniform boundedness of $\partial_s \mathcal{I}(t, s)$ on sets $[0, T] \times [\epsilon, L-\epsilon]$, see Theorem 2.1, assumption (4.1) implies that for all $\epsilon > 0$ it holds

$$\sup_{N \in \mathbb{N}} \sup_{s \in Z^N \cap [\epsilon, L-\epsilon]} \sup_{t \in [0,T]} \left| \frac{1}{\sigma_N} \big(\mathcal{I}_N(t, s \pm \sigma_N) - \mathcal{I}_N(t, s)\big) \right| < \infty. \tag{4.2}$$

In Sections 4.2.1 and 4.2.2 we give concrete examples for settings where condition (4.1) is satisfied.

Let $\{\, \nu_N\, ;\, N \in \mathbb{N}\,\} \subset (0, \infty)$ be a sequence of viscosity parameters. For $k \in \{1, \ldots, N\}$ the functions $(u_N^k, s_N^k) \in K_{s_N^k}^N \times Z^N$ are determined from the following time-incremental, viscous minimization problem: Given initial values $s_N^0 \in Z^N$ and $u_N^0 = \operatorname{argmin}\{\, \mathcal{E}(0, v, s_N^0)\, ;\, v \in K_{s_N^0}^N\,\}$ find

$$(u_N^k, s_N^k) \in \operatorname{Argmin}\{\, \mathcal{E}(t_N^k, v, s) + \tau_N^k \mathcal{R}_{\nu_N}(s_N^{k-1}; \frac{s - s_N^{k-1}}{\tau_N^k})\, ;\, s \in Z^N,\, v \in K_s^N\,\}. \tag{4.3}$$

Here, $\mathcal{R}_\nu(s;v) := \kappa(s)v + \frac{\nu}{2}|v|^2$ if $v \geq 0$ and $\mathcal{R}_\nu(s;v) = \infty$ otherwise. Observe that in contrast to the minimization problem (3.3) now viscosity terms are present in (4.3).

As before, we construct the piecewise constant functions $\underline{s}_N$, $\overline{s}_N : [0,T] \to Z^N$ via

$$\overline{s}_N(t) = s_N^k \text{ for } t \in (t_N^{k-1}, t_N^k],$$
$$\underline{s}_N(t) = s_N^{k-1} \text{ for } t \in [t_N^{k-1}, t_N^k)$$

and similar for $\overline{u}_N, \underline{u}_N : [0,T] \to V^N$. Moreover, the continuous and piecewise affine interpolant is defined as

$$\hat{s}_N(t) = s_N^{k-1} + \frac{t - t_N^{k-1}}{\tau_N^k}(s_N^k - s_N^{k-1}) \text{ for } t \in (t_N^{k-1}, t_N^k],\ \hat{s}_N(0) = s_N^0.$$

The main result is the following convergence theorem:

Theorem 4.1. *Let $\kappa \in C^0([0,L])$, $\kappa > 0$, $\ell \in C^1([0,T]; L^2(\Gamma_N))$ and $s_0 \in [\mathfrak{s}_0, L)$. Let further condition (4.1) be satisfied and assume that $(u_N^0, s_N^0) \to (u_0, s_0) \in K_{s_0} \times [\mathfrak{s}_0, L)$ with $u_0 = \operatorname{argmin}\{\, \mathcal{E}(0, v, s_0)\,;\, v \in K_{s_0}\,\}$. Assume finally that for $N \to \infty$ it holds*

$$\sigma_N \to 0,\ \tau_N \to 0,\ \nu_N \to 0, \sigma_N \nu_N / \underline{\tau}_N \to 0,\ \tau_N/\nu_N \to 0. \tag{4.4}$$

Then there exists a subsequence of $(\hat{s}_N)_{N \in \mathbb{N}}$, a nondecreasing function $s \in BV([0,T];\mathbb{R})$ with $s(0) = s_0$ and a function $u : [0,T] \to H^1(\Omega_L)$ such that for $n \to \infty$ it holds

$$\begin{aligned}
&\hat{s}_{N_n} \overset{*}{\rightharpoonup} s \quad \text{in } BV[0,T],\\
&\hat{s}_{N_n}(t) \to s(t) \quad \text{for all } t \in [0,T],\\
&\overline{u}_{N_n}(t),\ \underline{u}_{N_n}(t) \to u(t) \quad \text{strongly in } H^1(\Omega_L) \text{ for all } t \in [0,T].
\end{aligned}$$

Moreover, the limit function s is a BV-solution in the sense of Definition 2.7 and $u(t) = \operatorname{argmin}\{\, \mathcal{E}(t, v, s(t))\,;\, v \in K_{s(t)}\,\}$ for all t.

The proof follows closely the lines in [KMZ08, KZM10] investigating carefully the dependence of the estimates on σ_N, τ_N and ν_N. Roughly speaking, the proof is a discrete version of the proof of Theorem 5.1 in [KMZ08]: The additional technical difficulty comes from the fact that the energy release rate $\partial_s \mathcal{I}(t,s)$ has to be approximated by difference quotients of the type $\sigma_N^{-1}(\mathcal{I}_N(t, s+\sigma_N) - \mathcal{I}_N(t,s))$ and the energies $\mathcal{I}_N(t,\cdot)$ in general are not continuous with respect to the second variable.

In the proof we also take care of what happens if s reaches the length L, for which the body is broken into two pieces. This extends the existence result in [KMZ08] by avoiding the artificial stopping criterion formulated there. Let us finally remark that time dependent Dirichlet conditions can be treated in a similar way.

Proof. The following estimates are valid due to the boundedness of the set Z^N and the coercivity of the energy $\mathcal{E}$: There exists a constant $c > 0$ such that

$$\sup_{N\in\mathbb{N}} \Big(\|\overline{s}_N\|_{L^\infty(0,T)} + \|\underline{s}_N\|_{L^\infty(0,T)} + \|\hat{s}_N\|_{L^\infty(0,T)} \Big) \le L, \tag{4.5}$$

$$\sup_{N\in\mathbb{N}} \Big(\|\overline{u}_N\|_{L^\infty(0,T;H^1(\Omega_L))} + \|\underline{u}_N\|_{L^\infty(0,T;H^1(\Omega_L))} \Big) \le c\,\|\ell\|_{C^0([0,T];L^2(\Gamma_N))} \,. \tag{4.6}$$

In terms of the reduced energy $\mathcal{I}_N(t,s)$, the minimization problem (4.3) can be rewritten as follows ($\tau_N^k = t_N^k - t_N^{k-1}$)

$$s_N^k \in \operatorname{Argmin}\{\, \mathcal{I}_N(t_N^k, \widetilde{s}) + \tau_N^k \mathcal{R}_{\nu_N}\big(s_N^{k-1}; \frac{\widetilde{s} - s_N^{k-1}}{\tau_N^k}\big) \,;\, \widetilde{s} \in Z^N \,\}. \tag{4.7}$$

Hence, from the minimality we obtain with $\widetilde{s} = s_N^{k-1}$

$$\begin{aligned} \mathcal{I}_N(t_N^k, s_N^k) + \tau_N^k \mathcal{R}_{\nu_N}\big(s_N^{k-1}; \frac{s_N^k - s_N^{k-1}}{\tau_N^k}\big) &\le \mathcal{I}_N(t_N^k, s_N^{k-1}) + \tau_N^k \mathcal{R}_{\nu_N}\big(s_N^{k-1}; \frac{s_N^{k-1} - s_N^{k-1}}{\tau_N^k}\big) \\ &= \mathcal{I}_N(t_N^{k-1}, s_N^{k-1}) + \int_{t_N^{k-1}}^{t_N^k} \partial_t \mathcal{I}_N(\rho, s_N^{k-1}) \mathrm{d}\rho. \end{aligned}$$

Summation over all time steps leads to

$$\mathcal{I}_N(t_N^k, s_N^k) + \int_0^{t_N^k} \mathcal{R}_{\nu_N}(\underline{s}_N(\rho); \hat{s}_N'(\rho)) \mathrm{d}\rho \le \mathcal{I}_N(0, s_N^0) + \int_0^{t_N^k} \partial_t \mathcal{I}_N(\rho, \underline{s}_N(\rho)) \mathrm{d}\rho. \tag{4.8}$$

Observe ([KMZ08]) that $\partial_t \mathcal{I}_N(t, \underline{s}_N(t)) = -\int_{\Gamma_N} \ell'(t) \cdot u_N(t)\, \mathrm{d}\Gamma$ with

$$u_N(t) = \operatorname{argmin}\{\, \mathcal{E}(t, v, \underline{s}_N(t))\,;\, v \in K^N_{\underline{s}_N(t)} \,\}.$$

Again, the uniform bound $\sup_N \|u_N\|_{L^\infty(0,T;H^1(\Omega_L))} \le c\,\|\ell\|_{C^0([0,T];L^2(\Gamma_N))}$ is valid. Thus,

$$\sup_N |\partial_t \mathcal{I}_N(\cdot, \underline{s}_N(\cdot))|_{L^\infty(0,T)} \le c\,\|\ell\|^2_{C^1([0,T];L^2(\Gamma_N))} \,,$$

which implies in connection with (4.8), (4.6) and the definition of $\mathcal{R}_\nu$ that there exists a constant $c > 0$ such that

$$\sup_{N\in\mathbb{N}} \sqrt{\nu_N}\, \big\|\hat{s}_N'\big\|_{L^2(0,T)} \le c. \tag{4.9}$$

Like in [KMZ08, Lemma 4.1] we conclude that

$$\|\overline{s}_N - \hat{s}_N\|_{L^\infty(0,T)} + \|\underline{s}_N - \hat{s}_N\|_{L^\infty(0,T)} \le c(\tau_N/\nu_N)^{\frac{1}{2}}. \tag{4.10}$$

Since the sequence $\{\overline{s}_N,\, N \in \mathbb{N}\}$ is bounded from above and since the $\overline{s}_N$ are monotone functions, Helly's selection principle, see e.g. [Rud76], yields the existence of a subsequence (not relabeled) and of a function $s \in BV([0,T],\mathbb{R})$ with the properties

$$\overline{s}_N, \hat{s}_N, \underline{s}_N \overset{*}{\rightharpoonup} s \text{ weakly in } BV([0,T]) \text{ and } \overline{s}_N(t), \hat{s}_N(t), \underline{s}_N(t) \to s(t) \text{ for all } t \in [0,T]. \tag{4.11}$$

It remains to show that s is a BV-solution in the sense of Definition 2.7. Let

$$T_* := \begin{cases} \inf\{\, t \in [0,T]\,;\, s(t) = L\,\} & \text{if } s(T) = L, \\ T & \text{if } s(T) < L. \end{cases}$$

Assume that $T_* > 0$ (the case $T_* = 0$ is treated at the end of the proof). Then for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for all $0 \le t \le T_* - \delta_\epsilon$ it holds $s(t) \le L - \epsilon$. Fix now $\epsilon, \delta_\epsilon > 0$ and let $T_\epsilon = T_* - \delta_\epsilon$. From (4.11) it follows that there exists $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$ we have

$$\sigma_N \le \frac{\epsilon}{4} \quad \text{and} \quad \overline{s}_N(T_\epsilon), \hat{s}_N(T_\epsilon), \underline{s}_N(T_\epsilon) \le L - \frac{\epsilon}{2}.$$

Proof of condition (a) of Definition 2.7. The choice $\widetilde{s} = s_N^k + \sigma_N \le L - \frac{\epsilon}{4}$ and $t_N^k \le T_\epsilon$ in the minimization problem (4.7), implies for $t \in [t_N^{k-1}, t_N^k)$ that

$$0 \le \frac{1}{\sigma_N}\big(\mathcal{I}_N(t_N^k, s_N^k + \sigma_N) - \mathcal{I}_N(t_N^k, s_N^k)\big) + \kappa(s_N^{k-1}) + \nu_N \hat{s}_N'(t_N^k) + \frac{\nu_N \sigma_N}{2\underline{\mathcal{I}}_N}.$$

The previous inequality is a discrete, viscous version of condition (a) of Definition 2.7. For all $\psi \in L^2(0, T_\epsilon)$ with $\psi \ge 0$ we obtain

$$\begin{aligned} 0 \le \int_0^{T_\epsilon} \psi(\rho)\Big(\frac{1}{\sigma_N}\big(\mathcal{I}_N(\overline{t}_N(\rho), \overline{s}_N(\rho) + \sigma_N) - \mathcal{I}_N(\overline{t}_N(\rho), \overline{s}_N(\rho))\big) \\ + \kappa(\underline{s}_N(\rho)) + \nu_N \hat{s}_N'(\rho) + \frac{\nu_N \sigma_N}{2\underline{\mathcal{I}}_N}\Big)\mathrm{d}\rho, \end{aligned} \tag{4.12}$$

where $\overline{t}_N(\rho) = t_N^k$ for $\rho \in [t_N^{k-1}, t_N^k)$. By (4.2) the discrete energy release rate satisfies

$$\sup_{\rho \in [0,T_\epsilon],\, N \in \mathbb{N}} \Big| \tfrac{1}{\sigma_N}(\mathcal{I}_N(\overline{t}_N(\rho), \overline{s}_N(\rho) + \sigma_N) - \mathcal{I}_N(\overline{t}_N(\rho), \overline{s}_N(\rho))) \Big| < \infty.$$

Moreover, the Lipschitz continuity of $\partial_s \mathcal{I}$ on the set $[0,T] \times [\mathfrak{s}_0, L - \epsilon]$, see Theorem 2.2, and assumption (4.1) imply that the discrete energy release rate converges pointwise to $\partial_s \mathcal{I}(\rho, s(\rho))$ for $N \to \infty$. Hence, with the Lebesgue Theorem, assumption (4.4), estimate (4.9) and the continuity of κ we conclude in the same way as in [KMZ08] that the right hand side in (4.12) converges to $\int_0^{T_\epsilon} \psi(\rho)(\partial_s \mathcal{I}(\rho, s(\rho)) + \kappa(s(\rho)))\mathrm{d}\rho$. Since $\psi \ge 0$ is arbitrary, we arrive at condition (a) of Definition 2.7.

Proof of condition (c) of Definition 2.7. Again, the proof is a discrete version of the proof of Theorem 5.1 in [KMZ08]. Assume that $s_N^k > s_N^{k-1}$. Then $\tilde{s} = s_N^k - \sigma_N \ge s_N^{k-1}$ is an admissible test for the minimality condition (4.3) leading to a finite value of $\mathcal{R}_{\nu_N}$. Hence, evaluating the minimality condition for this particular choice gives

$$0 \ge \kappa(s_N^{k-1}) + \nu_N \hat{s}_N'(t_N^k) - \frac{1}{\sigma_N}\Big(\mathcal{I}_N(t_N^k, s_N^k - \sigma_N) - \mathcal{I}_N(t_N^k, s_N^k)\Big) - \frac{\nu_N \sigma_N}{2\underline{\mathcal{I}}_N} \tag{4.13}$$

Since $\hat{s}'_N(t) \geq 0$ for all $t \in [0, T_\epsilon] \backslash \Pi_N$ and since $\hat{s}'_N(t) = 0$ if $s_N^k = s_N^{k-1}$, it follows that for all $t \in [0, T_\epsilon] \backslash \Pi_N$ it holds

$$0 \geq \hat{s}'_N(t) \Big(- \frac{\nu_N \sigma_N}{2 \underline{\mathcal{I}}_N} + \kappa(\underline{s}_N(t)) - \frac{1}{\sigma_N} \big(\mathcal{I}_N(\bar{t}_N(t), \bar{s}_N(t) - \sigma_N) - \mathcal{I}_N(\bar{t}_N(t), \bar{s}_N(t)) \big) \Big) \quad (4.14)$$

Let $t_* \in J(s) \cap [0, T_\epsilon]$ and choose s_a, s_b with $s(t_*-) \leq s_a < s_b \leq s(t_*+)$. Since the functions $\hat{s}_N$ are continuous and not decreasing, for N large enough there exist $t_N^a < t_N^b$ with $\hat{s}_N(t_N^a) = s_a$, $\hat{s}_N(t_N^b) = s_b$ and $t_N^a \to t_*$, $t_N^b \to t_*$ for $N \to \infty$. For all $\varphi \in L^2(s_a, s_b)$ with $\varphi \geq 0$ it follows with (4.14):

$$\begin{aligned} 0 \geq \int_{t_N^a}^{t_N^b} \varphi(\hat{s}_N(t)) \hat{s}'_N(t) \Big(&- \frac{\nu_N \sigma_N}{2 \underline{\mathcal{I}}_N} + \kappa(\underline{s}_N(t)) \\ &- \frac{1}{\sigma_N} \big(\mathcal{I}_N(\bar{t}_N(t), \bar{s}_N(t) - \sigma_N) - \mathcal{I}_N(\bar{t}_N(t), \bar{s}_N(t)) \big) \Big) \,\mathrm{d}t \end{aligned}$$

Now, as in the proof of [KMZ08, Thm. 5.2], we change variables:

$$\hat{t}_N(\sigma) := \min\{\, t \in [t_N^a, t_N^b] \,;\, \hat{s}_N(t) = \sigma \,\}.$$

Observe that $\hat{s}_N(\hat{t}_N(\sigma)) = \sigma$. With this, the previous inequality can be rewritten as

$$\begin{aligned} 0 \geq \int_{s_a}^{s_b} \varphi(\sigma) \Big(&- \frac{\nu_N \sigma_N}{2 \underline{\mathcal{I}}_N} + \kappa(\underline{s}_N(\hat{t}_N(\sigma))) \\ &- \frac{1}{\sigma_N} \big(\mathcal{I}_N(\bar{t}_N(\hat{t}_N(\sigma)), \bar{s}_N(\hat{t}_N(\sigma)) - \sigma_N) - \mathcal{I}_N(\bar{t}_N(\hat{t}_N(\sigma)), \bar{s}_N(\hat{t}_N(\sigma))) \big) \Big) \mathrm{d}\sigma \end{aligned} \quad (4.15)$$

By (4.10) it holds $\big|\underline{s}_N(\hat{t}_N(\sigma)) - \sigma\big| = \big|\underline{s}_N(\hat{t}_N(\sigma)) - \hat{s}_N(\hat{t}_N(\sigma))\big| \leq c(\tau_N/\nu_N)^{1/2}$, and the right hand side tends to 0 for $N \to \infty$ by assumption (4.4). Furthermore, by the definition of $\bar{t}_N$ and $\hat{t}_N$ we find $\big|\bar{t}_N(\hat{t}_N(\sigma)) - t_*\big| \leq \tau_N + \max\{|t_N^a - t_*|\,, \big|t_N^b - t_*\big|\} \to 0$ for $N \to \infty$, uniformly in σ. Hence with assumption (4.4), the Lebesgue Theorem and assumption (4.1) we obtain from (4.15) the following estimate in the limit $N \to \infty$: for all $\varphi \in L^2(s_a, s_b)$ with $\varphi \geq 0$ it holds

$$0 \geq \int_{s_a}^{s_b} \varphi(\sigma) \big(\kappa(\sigma) + \partial_s \mathcal{I}(t, \sigma) \big) \mathrm{d}\sigma.$$

Since $s_a, s_b \in [s(t_*-), s(t_*+)]$ are arbitrary we finally arrive at condition (c) of Definition 2.7 on the time interval $[0, T_*]$.

Proof of condition (b) of Definition 2.7. The basic properties used are the uniform continuity of $\partial_s \mathcal{I}(\cdot, \cdot)$ on sets of the type $[0, T] \times [\mathfrak{s}_0, \mathfrak{s}_1]$ with $\mathfrak{s}_1 < L$ and the uniform convergence of the discrete energy release rates to $\partial_s \mathcal{I}$ formulated in assumption (4.1). Again, the proof is a discrete version of the corresponding part of the proof of Theorem 5.1 in [KMZ08].

Let $t_* \in (0, T_\epsilon)$ with $\kappa(s(t_*)) + \partial_s \mathcal{I}(t_*, s(t_*)) =: \eta > 0$. The goal is to show that $t_* \in D(s)$ and that $\dot{s}(t_*) = 0$. Thereto we show that there exist constants $\widetilde{\delta} > 0$ and $\widetilde{N} \in \mathbb{N}$ such that for all

$N \geq \widetilde{N}$ the functions $\overline{s}_N$ are constant on $(t_* - \widetilde{\delta}, t_* + \widetilde{\delta})$. Due to the pointwise convergence of the sequence $\{ \overline{s}_N \, ; \, N \in \mathbb{N} \}$ to the limit function s it then follows that s is constant on $(t_* - \widetilde{\delta}, t_* + \widetilde{\delta})$ as well and hence $\dot{s}(t_*) = 0$.

Let $s_* = s(t_*)$. From the continuity of κ and $\partial_s \mathcal{I}$ it follows that there exist constants $\epsilon_0, \delta_0 > 0$ such that for all $\tilde{s}, t$ with $|\tilde{s} - s_*| \leq \epsilon_0$, $|t_* - t| \leq \delta_0$ it holds

$$\kappa(\tilde{s}) + \partial_s \mathcal{I}(t, \tilde{s}) \geq \frac{\eta}{2}.$$

Further, there exists $\epsilon_1 > 0$ such that for all t with $|t_* - t| \leq \delta_0$ and $s_1, s_2 \in (s_* - \frac{\epsilon_0}{2}, s_* + \frac{\epsilon_0}{2})$ with $|s_1 - s_2| < \epsilon_1$ it holds

$$\kappa(s_1) + \partial_s \mathcal{I}(t, s_2) \geq \frac{\eta}{4}. \tag{4.16}$$

Due to condition (c), $t_* \notin J(s)$ and hence, the limit function s is continuous in t_*. This implies that there exists a constant $\delta_1 \in (0, \delta_0)$ such that for all t with $|t - t_*| \leq \delta_1$ we have $|s(t) - s_*| \leq \epsilon_0/4$ and hence $\kappa(s(t)) - \partial_s \mathcal{I}(t, s(t)) \geq \frac{\eta}{2}$. Since the piecewise constant functions $\underline{s}_N$ and $\overline{s}_N$ converge pointwise to s and since $\underline{s}_N$, $\overline{s}_N$ and s are monotone, there exists a constant $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$ and all t with $|t - t_*| \leq \delta_1$ it holds

$$|\underline{s}_N(t) - s_*| \leq \epsilon_0/2, \quad |\overline{s}_N(t) - s_*| \leq \epsilon_0/2. \tag{4.17}$$

Let $R_N^{\pm}(t, \tilde{s}) = \partial_s \mathcal{I}(t, \tilde{s}) \mp \sigma_N^{-1} \left(\mathcal{I}_N(t, \tilde{s} \pm \sigma_N) - \mathcal{I}_N(t, \tilde{s}) \right)$ for $\tilde{s} \in Z^N$, cf. (4.1). Due to assumption (4.1) we may finally choose N_1 in such a way that in addition for all $N \geq N_1$, all t with $|t - t_*| \leq \delta_1$ and all $\tilde{s}$ with $|\tilde{s} - s_*| \leq \epsilon_0/2$ we have $\left| R_N^{\pm}(t, \tilde{s}) \right| \leq \frac{\eta}{16}$.

Consider now the following incremental minimization problem for given $t = t_N^k \in \Pi_N$ and $s_0 \in Z^N$:

$$s_1 \in \operatorname{Argmin}\{ \mathcal{I}_N(t, \tilde{s}) + \tau_N^k \mathcal{R}_{\nu_N}\big(s_0; \frac{\tilde{s} - s_0}{\tau_N^k}\big) \, ; \, \tilde{s} \in Z^N \}. \tag{4.18}$$

Claim: There exists $N_2 \geq N_1$ such that for all $N \geq N_2$ the following is valid: Let $(t, s_0) \in (\Pi_N \times Z^N) \cap \big((t_* - \delta_1, t_* + \delta_1) \times [s_* - \epsilon_0/2, s_* + \epsilon_0/2] \big)$. If $s_1 \in [s_* - \epsilon_0/2, s_* + \epsilon_0/2]$ satisfies (4.18) and if $|s_1 - s_0| \leq \epsilon_1$, then $s_1 = s_0$.

Proof of the claim: Choose $N_2 \geq N_1$ such that for all $N \geq N_2$ we have $\nu_N \sigma_N / (2 \underline{\tau}_N) \leq \eta/16$ and $\epsilon_1 \geq 2c(\tau_N/\nu_N)^{\frac{1}{2}}$ (from (4.10)). Let s_1 be a minimizer as described above and assume that $s_1 > s_0$. Then similar to (4.13) it follows from (4.18) that

$$\begin{aligned} 0 &\geq \kappa(s_0) + \nu_N \hat{s}_N'(t) - \sigma_N^{-1} \Big(\mathcal{I}_N(t, s_1 - \sigma_N) - \mathcal{I}_N(t, s_1) \Big) - \nu_N \sigma_N / (2 \underline{\tau}_N) \\ &\geq \kappa(s_0) + \partial_s \mathcal{I}(t, s_1) - R_N^-(t, s_1) - \eta/16 \\ &\geq \eta/8 > 0, \end{aligned}$$

which is a contradiction. Hence, $s_1 = s_0$.

We now turn back to the proof of (b). As already announced, the goal is to show that for $N \geq N_2$ the function $\bar{s}_N$ is constant on the fixed interval $(t_* - \delta_1/2, t_* + \delta_1/2)$. For this purpose let $N \geq N_2$ and define $t_{N,1} = \min\{ t_N^k \,;\, t_N^k \geq t_* - \delta_1 + \tau_N,\, t_N^k \in \Pi_N \}$. Then $\bar{s}_N(t_{N,1}), \underline{s}_N(t_{N,1}) \in [s_* - \epsilon_0/2, s_* + \epsilon_0/2]$ due to (4.17) and $\bar{s}_N(t_{N,1})$ satisfies (4.18) with $t = t_{N,1}$ and $s_0 = \underline{s}_N(t_{N,1})$. Hence, by the above proven claim in combination with estimate (4.10), it follows that $\bar{s}_N(t_{N,1}) = \underline{s}_N(t_{N,1})$. We now repeat the argument with $t = t_{N,2} = t_{N,1} + \tau_N^k$ and $s_0 = \underline{s}_N(t_{N,2}) = \bar{s}_N(t_{N,1})$ until the time $t_* + \delta_1$ is reached. This shows that the function $\bar{s}_N$ is constant on $(t_* - \delta_1/2, t_* + \delta_1/2)$. Since $\bar{s}_N$ converges pointwise to s, this implies that also s is constant on the interval $(t_* - \delta_1/2, t_* + \delta_1/2)$ and (b) is proved for the time interval $[0, T_*]$.

Let us finally discuss the case $T_* = 0$. Then $s(0) = s_0$ and for all $t > 0$ we have $s(t) = L$. Hence, $J(s) = \{0\}$ and we only have to verify condition (c). This means, we have to show that for all $s_* \in [s_0, L)$ it holds $\kappa(s_*) - \mathcal{G}(0, s_*) \leq 0$. But this follows similar to the previous proof of (c) with obvious modifications. $\square$

4.2 Convergence of discrete energy release rates

The goal of this section is to present two sufficient conditions, under which assumption (4.1) on the uniform convergence of discrete energy release rates is valid. Assumption (4.1) implicitly requires a compatibility condition between the discrete spaces V^N and the crack increments Z^N. Regularity and interpolation properties play a fundamental role in the construction of suitable spaces V^N and Z^N. We discuss here both cases, models with contact conditions on the crack surface and models without contact conditions. In the second case better relations between Z^N and V^N can be obtained under slightly stronger assumptions on the meshes.

In this section we need the following spaces defined on the domains Ω_s: For non-integers γ, the Sobolev-Slobodeckij spaces on Ω_s are defined as complex interpolation spaces, [LM72]: Let $\gamma \in (0,1)$, $k \in \mathbb{N}_0$. Then

$$H^{k+\gamma}(\Omega_s) := [H^{k+1}(\Omega_s), H^k(\Omega_s)]_{(1-\gamma)}, \qquad H^\gamma_{\Gamma_D}(\Omega_s) := [V_s, L^2(\Omega_s)]_{(1-\gamma)}.$$

Moreover, the intermediate Besov or Nikolskii space $B^{\frac{3}{2}}_{2,\infty}(\Omega_s)$ is defined as a real interpolation space (cf. [Tri10]) in the following way:

$$B^{\frac{3}{2}}_{2,\infty}(\Omega_s) = (H^1(\Omega_s), H^2(\Omega_s))_{\frac{1}{2},\infty}.$$

For every $\delta > 0$ the space $B^{3/2}_{2,\infty}$ is continuously embedded in $H^{3/2-\delta}$, [Tri10].

As a general assumption on the datum ℓ we require

$$\ell = H\big|_{\Gamma_N}\mathbf{n} \text{ for some } H \in C^0([0,T]; H^{1+\gamma}(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}})) \text{ and some (small) } \gamma > 0. \tag{4.19}$$

This is a sufficient condition for the subsequent analysis. In some of the following statements the assumptions on ℓ can be weakened.

4.2.1 The case with contact conditions

The same notation as in Section 4.1 is used. In particular, the minimizing displacement fields are denoted by

$$u(t,s) = \operatorname{argmin}_{v\in K_s} \mathcal{E}(t,v,s), \qquad u_N(t,s) = \operatorname{argmin}_{v\in K_s^N} \mathcal{E}(t,v,s).$$

Observe that for all $v_N \in K_s^N$ the minimizer $u_N(t,s)$ satisfies the variational inequality

$$a_s(u_N(t,s), u_N(t,s) - v_N) \leq \int_{\Gamma_N} \ell(t)\cdot(u_N(t,s) - v_N)\,\mathrm{d}\Gamma. \tag{4.20}$$

Hence, there exists a constant $c > 0$ such that

$$\sup_{\substack{N\in\mathbb{N},\\ s\in Z^N,\, t\in[0,T]}} \|u_N(t,s)\|_{H^1(\Omega_L)} \leq c\,\|\ell\|_{C^0([0,T];L^2(\Gamma_N))}\,. \tag{4.21}$$

The next interpolation assumption plays a crucial role in this section:

> There exists a sequence $(h_N)_{N\in\mathbb{N}} \subset (0,\infty)$ with $h_N \to 0$ for $N\to\infty$ and parameters $\alpha,\beta > 0$ such that for all $\epsilon > 0$ it holds: there exists a constant $c_\epsilon > 0$ such that for all $t\in[0,T]$, $N\in\mathbb{N}$, $s_N \in Z^N \cap [\epsilon, L-\epsilon]$ and all minimizers $u(t,s_N)\in K_{s_N}$ of $\mathcal{E}(t,\cdot,s_N)$ there exists an element $\widetilde{u}^N_{t,s_N} \in K^N_{s_N}$ satisfying $\big\|u(t,s_N) - \widetilde{u}^N_{t,s_N}\big\|_{H^1(\Omega_L)} \leq c_\epsilon h_N^\alpha$ and $\big\|u(t,s_N) - \widetilde{u}^N_{t,s_N}\big\|_{L^2(\Omega_L)} \leq c_\epsilon h_N^\beta$. (4.22)

The sequence $(h_N)_N$ for example can be interpreted as mesh parameters of finite element meshes defining the spaces V^N. The estimates in terms of powers of h_N then can be obtained from regularity results for minimizers in combination with suitable projection/interpolation operators. We give an example in Section 5.1.

The following uniform regularity estimate is valid in a neighborhood of the crack tip:

Lemma 4.2. *For every $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ and a radius $R_\epsilon < \epsilon$ such that for all $t\in[0,T]$ and $s\in[\epsilon, L-\epsilon]$ it holds $u(t,s)\in B^{3/2}_{2,\infty}(B_{R_\epsilon}(x_s)\cap\Omega_s)$ and*

$$\|u(t,s)\|_{B^{\frac{3}{2}}_{2,\infty}(B_{R_\epsilon}(x_s)\cap\Omega_s)} \leq c_\epsilon. \tag{4.23}$$

Proof. The regularity result is derived in [KS11]. A close inspection of the proof in [KS11] shows that a uniform estimate is valid on parameter sets $[0,T]\times[\epsilon,L-\epsilon]$ for every $\epsilon > 0$. ☐

Motivated by this regularity property, we impose the following uniform regularity assumption on the minimizing displacement fields:

> For every $(t,s)\in[0,T]\times[0,L]$ the minimizers of $\mathcal{E}(t,\cdot,s)$ with respect to K_s satisfy $u(t,s)\in B^{3/2}_{2,\infty}(\Omega_s)$ and for every $\epsilon > 0$ exists a constant $c_\epsilon > 0$ such that
>
> $$\sup_{t\in[0,T],\, s\in[\epsilon,L-\epsilon]} \|u(t,s)\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega_s)} \leq c_\epsilon. \tag{4.24}$$

In order to have $B^{3/2}_{2,\infty}$-regularity globally on Ω_s, a sufficient condition is to assume that Ω is a polygonal domain, which is convex in those points, where the Dirichlet and Neumann-boundaries intersect, and that ℓ satisfies (4.19), see for example[NS99, EF99] and the references therein. Moreover, under these assumptions the same regularity is valid in a neighborhood of those points, where the crack intersects $\partial\Omega$, [KS11].

It follows from assumption (4.24) that $u(t,s) \in H^{\frac{3}{2}-\delta}(\Omega_s)$ for every $\delta > 0$ and that there exists a constant $c_{\epsilon,\delta} > 0$ such that

$$\sup_{t\in[0,T],\, s\in[\epsilon,L-\epsilon]} \|u(t,s)\|_{H^{\frac{3}{2}-\delta}(\Omega_s)} \le c_{\epsilon,\delta}. \tag{4.25}$$

Let $\mathcal{A}_s : V_s \to V_s^*$ be the differential operator introduced in Section 2.1 via $\langle \mathcal{A}_s(u), v\rangle = a_s(u,v)$ for all $u, v \in V_s$. On the basis of the regularity estimates it follows that for all $\widetilde{\delta} > 0$ the functional $\mathcal{A}_s(u(t,s)) \in V_s^*$ can be extended to a linear and continuous functional on $H^{\frac{1}{2}+\widetilde{\delta}}_{\Gamma_D}(\Omega_s)$, i.e.

$$\mathcal{A}_s(u(t,s)) \in (H^{\frac{1}{2}+\widetilde{\delta}}_{\Gamma_D}(\Omega_s))^* =: W_{s,\tilde{\delta}}. \tag{4.26}$$

Moreover, for every fixed $\widetilde{\delta} > 0$ the operator norm of $\mathcal{A}_s(u(t,s))$ with respect to $W_{s,\tilde{\delta}}$ is uniformly bounded on parameter sets of the type $(t,s) \in [0,T]\times[\epsilon, L-\epsilon]$. This is an immediate consequence of the regularity estimate (4.25).

Using the Falk approximation theorem for variational inequalities [Fal74] we obtain

Proposition 4.3. *Let $\epsilon, \delta > 0$ and assume that conditions (4.19) and (4.24) are satisfied. Then there exists a constant $c_{\epsilon,\delta} > 0$ such that for all $t \in [0,T]$, $s \in Z^N \cap [\epsilon, L-\epsilon]$ and all $N \in \mathbb{N}$ it holds*

$$\begin{aligned} &\|u_N(t,s) - u(t,s)\|_{H^1(\Omega_s)} \\ &\quad \le c_{\epsilon,\delta} \inf_{v_N \in K_s^N} \Big(\|u(t,s) - v_N\|^2_{H^1(\Omega_s)} \\ &\qquad\qquad + \big(\|\mathcal{A}_s(u(t,s))\|_{W_{s,\delta}} + \|\ell\|_{C^0([0,T];L^2(\Gamma_N))} \big) \|u(t,s) - v_N\|_{H^{\frac{1}{2}+\delta}(\Omega_s)} \Big)^{\frac{1}{2}}. \end{aligned} \tag{4.27}$$

If in addition condition (4.22) is satisfied, then

$$\|u_N(t,s) - u(t,s)\|_{H^1(\Omega_s)} \le c_{\epsilon,\delta} \big(h_N^{2\alpha} + h_N^{\beta + (\frac{1}{2}+\delta)(\alpha-\beta)} \big)^{\frac{1}{2}}. \tag{4.28}$$

Proof. By assumption we have $K_s^N \subset K_s$. Hence, in the same way as in the proof of [Fal74, Theorem 1] we obtain that for all $v_N \in K_s^N$ it holds with $u := u(t,s)$ and $u_N := u_N(t,s)$

$$a_s(u-u_N, u-u_N) \le \int_{\Gamma_N} \ell(t)\cdot(u-v_N)\,\mathrm{d}\Gamma - \langle \mathcal{A}_s(u), u-v_N\rangle + a_s(u-u_N, u-v_N).$$

The mapping properties of $\mathcal{A}_s(u)$, see (4.26), together with Korn's and Young's inequality now imply (4.27).

Estimate (4.28) follows from (4.27) and assumption (4.22) by the interpolation inequality. □

An immediate consequence of the previous proposition is

Corollary 4.4. *Let $\epsilon > 0$ and assume that* (4.22) *and* (4.19) *with C^1 instead of C^0 are valid. Then for every sequence $(t_N, s_N)_N \subset [0,T] \times [\epsilon, L-\epsilon]$ with $s_N \in Z^N$ and $(t_N, s_N) \to (t,s)$ it holds: There exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have*

$$\|u_N(t_N,s_N) - u(t,s)\|_{H^1(\Omega_L)} \leq c_{\epsilon,\delta}\left(|t-t_N| + |s-s_N| + |s-s_N|^{\frac{1}{2}} + \left(h_N^{2\alpha} + h_N^{\beta+(\frac{1}{2}+\delta)(\alpha-\beta)}\right)^{\frac{1}{2}}\right),$$

$$\mathcal{I}_N(t_N,s_N) = \mathcal{E}(t_N, u_N(t_N,s_N), s_N) \to \mathcal{E}(t, u(t,s), s) = \mathcal{I}(t,s).$$

Proof. Let ρ_ϵ be the radius defined in the proof of Theorem 2.2 and let $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have $|s - s_N| \leq \rho_\epsilon$. Then, in the notation of the proof of Theorem 2.2, it holds

$$\begin{aligned}\|u_N(t_N,s_N) - u(t,s)\|_{H^1(\Omega_L)} &\leq \|u_N(t_N,s_N) - u(t_N,s_N)\|_{H^1(\Omega_L)} \\ &\quad + \|u(t,s) - \widetilde{u}_{s,s_N-s}(t_N)\|_{H^1(\Omega_L)} + \|\widetilde{u}_{s,s_N-s}(t_N) - u(t_N,s_N)\|_{H^1(\Omega_L)}.\end{aligned}$$

The first term on the right hand side can be estimated with (4.28) and the second term with (2.9). The estimate for the last term relies on the regularity assumption (4.24), on Lemma 4.1 in [KM08] and on the interpolation inequality applied to $B^{3/2}_{2,\infty} = (H^1, H^2)_{1/2,\infty}$, which all imply that $\|\widetilde{u}_{s,s_N-s}(t_N) - u(t_N,s_N)\|_{H^1(\Omega_L)} \leq c_\epsilon \sqrt{|s_N - s|}\, \|u\|_{B^{3/2}_{2,\infty}(\Omega_s)}$. □

Theorem 4.5. *Assume that conditions* (4.19), (4.22) *and* (4.24) *are valid. Then for every $\epsilon, \delta > 0$ there exist constants $c_{\epsilon,\delta} > 0$ and $N_\epsilon \in \mathbb{N}$ such that for all $t \in [0,T]$, $N \geq N_\epsilon$ and $s \in Z^N \cap [\epsilon, L-\epsilon]$ it holds*

$$\left|\frac{1}{\sigma_N}\big(\mathcal{I}_N(t, s\pm\sigma_N) - \mathcal{I}_N(t,s)\big) \mp \partial_s\mathcal{I}(t,s)\right| \leq c_{\epsilon,\delta}(\sigma_N + \sigma_N^{-1}(h_N^{2\alpha} + h_N^{\beta+(\frac{1}{2}+\delta)(\alpha-\beta)})^{\frac{1}{2}}). \quad (4.29)$$

Hence, condition (4.1) *is satisfied provided that the right hand side in* (4.29) *tends to zero for $N \to \infty$.*

Estimate (4.29) determines the relation between the mesh size h_N associated with the spaces V^N and the crack increment σ_N associated with Z^N. In Section 5.1 we study an example, where $\alpha = \frac{1}{2} - \delta$ and $\beta = \alpha + 1$. In this case one obtains $\sigma_N + \sigma_N^{-1}\big(h_N^{2\alpha} + h_N^{\beta+(\frac{1}{2}+\delta)(\alpha-\beta)}\big)^{\frac{1}{2}} \approx \sigma_N + \sigma_N^{-1} h_N^{\frac{1}{2}-\delta}$ in (4.29). If one neglects contact conditions on the crack surface, then this relation can be improved. This will be discussed in the next section.

Proof. Let $\epsilon > 0$ and choose $N_\epsilon \in \mathbb{N}$ such that for all $N \geq N_\epsilon$ we have $\sigma_N \leq \rho_\epsilon$ with ρ_ϵ from the proof of Theorem 2.2. For $N \geq N_\epsilon$ and $s \in Z^N$ it holds

$$\begin{aligned}&\left|\frac{1}{\sigma_N}\big(\mathcal{I}_N(t,s+\sigma_N) - \mathcal{I}_N(t,s)\big) - \partial_s\mathcal{I}(t,s)\right| \\ &\qquad \leq \frac{1}{\sigma_N}\big(\,|\mathcal{I}_N(t,s+\sigma_N) - \mathcal{I}(t,s+\sigma_N)| + |\mathcal{I}_N(t,s) - \mathcal{I}(t,s)|\,\big) \\ &\qquad\quad + \left|\frac{1}{\sigma_N}\big(\mathcal{I}(t,s+\sigma_N) - \mathcal{I}(t,s)\big) - \partial_s\mathcal{I}(t,s)\right|. \end{aligned} \quad (4.30)$$

Due to the quadratic structure of $\mathcal{E}$ the first two terms on the right hand side can be estimated as follows using (4.28) and (4.21):

$$\begin{aligned}
&\frac{1}{\sigma_N} \left| \mathcal{I}_N(t,s) - \mathcal{I}(t,s) \right| \\
&\quad \le \sigma_N^{-1} c\big(\|u_N(t,s)\|_{H^1(\Omega_s)} + \|u(t,s)\|_{H^1(\Omega_s)} + \|\ell\|_{C^0([0,T];L^2(\Gamma_N))} \big) \|u_N(t,s) - u(t,s)\|_{H^1(\Omega_s)} \\
&\quad \le c_{\epsilon,\delta} \|\ell\|_{C^0([0,T];L^2(\Gamma_N))} \, \sigma_N^{-1} (h_N^{2\alpha} + h_N^{\beta + (\frac{1}{2}+\delta)(\alpha-\beta)})^{\frac{1}{2}} .
\end{aligned}$$

For estimating the last term in (4.30) we apply Theorem 2.2:

$$\left| \frac{1}{\sigma_N} \big(\mathcal{I}(t, s+\sigma_N) - \mathcal{I}(t,s) \big) - \partial_s \mathcal{I}(t,s) \right| \le \int_0^1 \left| \partial_s \mathcal{I}(t, s + r\sigma_N) - \partial_s \mathcal{I}(t,s) \right| \mathrm{d}r \le c_\epsilon \sigma_N .$$

Combining the above considerations gives (4.29). □

4.2.2 The case without contact conditions

In the previous section global regularity results, in particular the $B^{3/2}_{2,\infty}$-smoothness close to the crack tip, were combined with Falk's Approximation Theorem for variational inequalities to deduce a relation between the discretization parameters h_N and σ_N, see (4.29). A closer look at the representation formula for the energy release rate in Theorem 2.1 shows that in fact the integration is taken with respect to an annulus, which does not contain the crack tip. Inside this annulus the displacement fields are H^2-regular. Using local finite element error estimates from [NS74] weaker relations between h_N and σ_N can be formulated, which still guarantee the convergence of discrete energy release rates. Such local error estimates to the author's knowledge are known for equations without contact conditions, only. Hence, in this section we restrict ourselves to the crack propagation model without contact conditions on the crack faces.

Given $t \in [0,T]$ and $s \in [0,L]$, the function $u(t,s) \in V_s$ is now defined as

$$u(t,s) = \operatorname{argmin}\{ \mathcal{E}(t,v,s) \, ; \, v \in V_s \},$$

or, equivalently, as the unique solution of the equation

$$\int_{\Omega_s} \mathbf{C}\varepsilon(u(t,s)) : \varepsilon(v) \, \mathrm{d}x = \int_{\Gamma_N} \ell(t) \cdot v \, \mathrm{d}\Gamma \qquad \text{for all } v \in V_s .$$

Clearly, there exists a constant $c > 0$ such that

$$\sup_{t \in [0,T],\, s \in [0,L]} \|u(t,s)\|_{H^1(\Omega_s)} \le c \, \|\ell\|_{C^0([0,T];L^2(\Gamma_N))} .$$

Furthermore, Lemma 4.2 is valid as well.

Based on this regularity estimate in the sequel we assume that minimizers $u(t,s)$ are elements of $B^{3/2}_{2,\infty}(\Omega_s)$. More precisely we assume the following regularity estimate to hold true for the linear elliptic operator $\mathcal{A}_s$ associated with $a_s(\cdot,\cdot)$:

For all $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that for all $s \in [\epsilon, L-\epsilon]$ it holds: If $f \in L^2(\Omega)$, if ℓ satisfies (4.19), and if $v_s \in V_s$ is the unique solution of $a_s(v_s, v) = \int_{\Omega_s} f \cdot v \,\mathrm{d}x + \int_{\Gamma_N} \ell \cdot v \,\mathrm{d}\Gamma$ for $v \in V$, then

$$\|v_s\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega_s)} \le c_\epsilon \left(\|f\|_{L^2(\Omega)} + \|H\|_{H^{1+\gamma}(\Omega)} \right). \tag{4.31}$$

Remark 4.6. As in the case with contact conditions on C_s, a sufficient geometrical condition to guarantee (4.31) is to assume that $\partial\Omega$ is a polygon which is convex in a neighborhood of those points, where the type of the boundary conditions changes, see for example [NS99, EF99]. Assumption (4.31) is formulated in order to reduce the technicalities in the derivation of our final result, Theorem 4.12. For example by choosing suitably adapted finite element meshes one could also treat situations, where stronger singularities occur at the boundary far from the crack tip.

On the finite dimensional subspaces V_s^N of V_s and the discrete crack sets Z^N we impose the following interpolation condition:

For every $N \in \mathbb{N}$, $s \in Z^N$ there exists a linear operator $Q_s^N : V_s \to V_s^N$ with the following properties: For all $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that for all $N \in \mathbb{N}$, $s \in Z^N \cap [\epsilon, L-\epsilon]$, $l \in \{0,1\}$, $m \in \{1,2\}$ and $v \in H^m(\Omega_s)$ it holds $\left\|v - Q_s^N(v)\right\|_{H^l(\Omega_s)} \le c_\epsilon h_N^{m-l} \|v\|_{H^m(\Omega_s)}$. (4.32)

As in the previous section, we define $u_N(t,s) = \operatorname{argmin}_{v \in V_s^N} \mathcal{E}(t,v,s)$ and obtain the estimate

$$\sup_{t\in[0,T], N\in\mathbb{N}, s\in Z^N} \|u_N(t,s)\|_{H^1(\Omega_s)} < c\,\|\ell\|_{C^0([0,T];L^2(\Gamma_N))}. \tag{4.33}$$

As a conclusion of the regularity estimate and the assumptions on the spaces V_s^N one obtains the following version of the Aubin-Nitsche estimate, which we need in the further analysis:

Corollary 4.7. *Assume that conditions* (4.19), (4.31) *and* (4.32) *are satisfied. Then there exists a constant $c_\epsilon > 0$ such that for all $s \in Z^N \cap [\epsilon, L-\epsilon]$ and $t \in [0,T]$ it holds*

$$\|u(t,s) - u_N(t,s)\|_{L^2(\Omega_s)} \le c_\epsilon h_N \|u(t,s)\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega_s)}. \tag{4.34}$$

Proof. Let $z_s \in V_s$ be the unique solution of the equation $a_s(z,v) = \int_{\Omega_s} (u(t,s) - u_N(t,s)) \cdot v \,\mathrm{d}x$ for $v \in V_s$. By assumption (4.31) the solution z_s belongs to $B^{3/2}_{2,\infty}(\Omega_s)$ and there exists a constant $c_\epsilon > 0$ such that $\|z_s\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega_s)} \le c_\epsilon \|u(t,s) - u_N(t,s)\|_{L^2(\Omega_s)}$. Hence, from the Galerkin orthogonality one finds that for all $v \in V_s^N$ it holds

$$\|u(t,s) - u_N(t,s)\|^2_{L^2(\Omega_s)} \le c\,\|u(t,s) - u_N(t,s)\|_{H^1(\Omega_s)} \|z_s - v\|_{H^1(\Omega_s)}.$$

Choosing $v = Q_s^N(z)$ with Q_s^N from assumption (4.32), applying the Cea estimate to the first factor and taking into account the interpolation identity $B^{3/2}_{2,\infty} = (H^1, H^2)_{\frac{1}{2},\infty}$ we finally arrive at (4.34). □

In addition to the Aubin-Nitsche estimates our further analysis also relies on the local error estimates due to Nitsche and Schatz, [NS74]. We will apply them to annuli which are centered at the crack tip of Ω_s.

For $x_0 \in \mathbb{R}^2$, $R > \rho > 0$ the annulus $A_{\rho,R}(x_0)$ centered at x_0 is defined by $A_{\rho,R}(x_0) := E_R(x_0)\backslash E_\rho(x_0)$. Here, $E_r(x_0) = x_0 + (-r, r)^2$ is the cube with center x_0 and side length $2r$.

Let x_s be the vertex of the crack of the domain Ω_s and choose $R > \rho > 0$ such that $\overline{A_{\rho,R}(x_s)}$ is contained in the interior of $\overline{\Omega_s}$. Since the crack is assumed to be a straight line and since the volume forces are equal to 0 it follows that $u(t,s)\big|_{\Omega_s \cap A_{\rho,R}(x_s)} \in H^2(\Omega_s \cap A_{\rho,R}(x_s))$. Moreover, for every $\epsilon > 0$ there exist constants $c_\epsilon > 0$ and $\rho_\epsilon > 0$ such that for all $t \in [0,T]$, $s \in [\epsilon, L-\epsilon]$ it holds

$$\cup_{s\in[\epsilon,L-\epsilon]} E_{8\rho_\epsilon}(x_s) \Subset \Omega, \tag{4.35}$$

$$\|u(t,s)\|_{H^2(A_{\rho_\epsilon,8\rho_\epsilon}(x_s)\cap\Omega_s)} \le c_\epsilon. \tag{4.36}$$

The version of the Nitsche-Schatz estimates adapted to these annuli reads as follows:

Corollary 4.8. *Assume that conditions* (4.19), (4.31) *and* (4.32) *are valid. For every* $\epsilon > 0$ *there exist constants* $\widetilde{c}_\epsilon, c_\epsilon > 0$ *such that for all* $t \in [0,T]$, $s \in [\epsilon, L-\epsilon]$ *it holds with* ρ_ϵ *from above:*

$$\begin{aligned}\|u(t,s) - u_N(t,s)\|_{H^1(A_{2\rho_\epsilon,7\rho_\epsilon}(x_s)\cap\Omega_s)} &\le \widetilde{c}_\epsilon h_N (\|u\|_{H^2(A_{\rho_\epsilon,8\rho_\epsilon}(x_s)\cap\Omega_s)} + \|u(t,s)\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega_s)}) \\ &\le c_\epsilon h_N.\end{aligned}$$

Proof. Corollary 4.8 is a combination of Theorem 5.1 from [NS74] with Corollary 4.7 and estimate (4.36). ☐

Remark 4.9. The original proof of Theorem 5.1 from [NS74] is derived for subdomains Ω_1, which are compactly contained in Ω_s. A careful inspection of the proof reveals that the arguments can be transferred also to the annuli we study, possibly with a slightly modified geometry at the points, where the annuli intersect the crack C_s. The essential ingredients are again regularity results for solutions to the equations of linear elasticity. In particular, it is needed in [NS74] that on cubes $E \Subset \Omega_s$ the equation $a_s(w,v) = \int_E f \cdot v \,\mathrm{d}x$ for $v \in H^1_0(E)$ has a unique solution $w \in H^2(E) \cap H^1_0(E)$ provided that $f \in L^2(E)$. In order to extend the estimates to the boundary, it is additionally needed that there exists an angle $\omega \in (0, \pi/2]$ such that on trapezoids $\mathcal{T}_\omega$ as drawn in Figure 2, the equation $a_s(w,v) = \int_{\mathcal{T}_\omega} f \cdot v \,\mathrm{d}x$ for $v \in W(\mathcal{T}_\omega) := \{\, \widetilde{v} \in H^1(\mathcal{T}_\omega)\,;\, \widetilde{v}\big|_{\partial\mathcal{T}_\omega \backslash C_s} = 0 \,\}$ has a unique solution $w \in H^2(\mathcal{T}_\omega) \cap W(\mathcal{T}_\omega)$ provided that $f \in L^2(\mathcal{T}_\omega)$. Such an angle ω exists and depends on the material tensor $\mathbf{C}$, see eg. [Gri89, Paragraph 6.2], where the isotropic case is studied. The regularity properties on E together with the regularity properties with respect to $\mathcal{T}_\omega$ now should be used instead of [NS74, Lemma 1.1] in the derivation of [NS74, Theorem 5.1].

A first consequence of the above two corollaries is the following approximation result for the energy release rate:

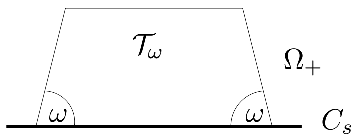


Figure 2: Example for $\mathcal{T}_\omega$.

Corollary 4.10. *Assume conditions* (4.19), (4.31) *and* (4.32) *are valid. Let $\epsilon > 0$ be arbitrary. Let further $\theta \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function with $\theta\big|_{E_{4\rho_\epsilon^2}(0)} = 1$ and $\theta\big|_{\mathbb{R}^2\setminus E_{49\rho_\epsilon^2}(0)} = 0$. Define $\theta_s(x) = \theta(|x - x_s|^2)$ with x_s the crack tip of Ω_s. Then there exists a constant $c_\epsilon > 0$ such that in the notation of Theorem 2.1 it holds for every $t \in [0, T]$, $N \in \mathbb{N}$ and $s \in Z^N \cap [\epsilon, L - \epsilon]$:*

$$|(-\partial_s \mathcal{I}(t, s)) - G(s, u_N(t, s))| = |G(s, u(t, s)) - G(s, u_N(t, s))| \leq c_\epsilon h_N.$$

Proof. Observe that $\operatorname{supp} \nabla\theta_s \subset A_{2\rho_\epsilon, 7\rho_\epsilon}(x_s)$. Hence, the assertion follows from the formula for $G(s, v)$ from Theorem 2.1 in combination with Corollary 4.8. □

Remark 4.11. Corollary 4.10 shows that a good approximation of the energy release rate can be obtained by inserting the discrete solution (i.e. u_N) into the Griffith formula provided in Theorem 2.1. The examples in Section 5.1 indicate that the order of convergence predicted in Corollary 4.10 is optimal.

The final goal of this section is to derive an analog of estimate (4.29) and hence to verify condition (4.1). The idea is to imitate Corollary 4.10 for the discrete energy release rate defined by finite differences. For this we need a further compatibility condition for the spaces V_s^N associated with different crack lengths. In general, spatial transformations, which map Ω_{s_1} onto Ω_{s_2} do not induce isomorphisms between the discrete spaces $V_{s_1}^N$ and $V_{s_2}^N$. Roughly speaking we assume that there exists a family of spatial transformations such that elements from $V_{s_1}^N$ with support outside a certain annulus around x_{s_1} are mapped on elements of $V_{s_2}^N$.

To be more precise let $T_{s,\delta} : \mathbb{R}^2 \to \mathbb{R}^2$ be a family of mappings with the following properties: For every $\epsilon > 0$ exists $\delta_\epsilon > 0$ such that for all $s \in [\epsilon, L - \epsilon]$ and $|\delta| \leq \delta_\epsilon$ the mapping $T_{s,\delta} : \Omega_s \to \Omega_{s+\delta}$ is a diffeomorphism with

$$T_{s,\delta}(x_s) = x_{s+\delta}, \quad T_{s,\delta}(C_s) = C_{s+\delta} \quad \text{and} \quad T_{s,\delta}(x) = x \text{ for } x \in \partial\Omega.$$

Moreover, $T : [\epsilon, L - \epsilon] \times [-\delta_\epsilon, \delta_\epsilon] \times \mathbb{R}^2$, $(s, \delta, x) \to T_{s,\delta}(x)$, and $\widetilde{T} : [\epsilon, L - \epsilon] \times [-\delta_\epsilon, \delta_\epsilon] \times \mathbb{R}^2$, $(s, \delta, x) \to T_{s,\delta}^{-1}(x)$, belong to $C^2([\epsilon, L - \epsilon] \times [-\delta_\epsilon, \delta_\epsilon] \times \mathbb{R}^2)$. Finally it is assumed that for ρ_ϵ from above (cf. (4.35)) we have

$$\operatorname{supp}_x \partial_\delta \nabla_x T_{s,\delta} \subset A_{3\rho_\epsilon, 6\rho_\epsilon}(x_s) \quad \text{for all } |\delta| \leq \delta_\epsilon.$$

Such a family of mappings can be constructed like in the proof of Theorem 2.2. Observe that the mappings $T_{s,\delta}$ induce isomorphisms between the spaces V_s and $V_{s+\delta}$. However, in general they do not map the discrete spaces V_s^N and $V_{s+\delta}^N$ onto each other. In the sequel the next compatibility condition is needed, which relates the mappings $T_{s,\delta}$ with the operators Q_s^N from condition (4.32):

There exists a family of linear operators $Q_s^N : V_s \to V_s^N$ with (4.32) and a family of diffeomorphisms $(T_{s,\delta})_{s,\delta}$ as described above which satisfy the following: For every $\epsilon > 0$ exists $N_\epsilon \in \mathbb{N}$ such that for all $N \geq N_\epsilon$ and $s \in Z^N \cap [\epsilon, L-\epsilon]$ it holds

(a) $Q_s^N : V_s \to V_s^N$ is a projection with $Q_s^N(v) = v$ for all $v \in V_s^N$.

(b) $$v \in H^2(A_{\rho_\epsilon,8\rho_\epsilon}(x_s)) \quad \Rightarrow \quad \left\|v - Q_s^N(v)\right\|_{H^1(A_{2\rho_\epsilon,7\rho_\epsilon}(x_s))} \leq c_\epsilon h_N \left\|v\right\|_{H^2(A_{\rho_\epsilon,8\rho_\epsilon}(x_s))}, \tag{4.37}$$

(c) $v \in V_{s+\sigma_N}^N \Rightarrow (v \circ T_{s,\sigma_N})\big|_{\Omega_s \setminus A_{2\rho_\epsilon,7\rho_\epsilon}(x_s)} = \big(Q_s^N(v \circ T_{s,\sigma_N})\big)\big|_{\Omega_s \setminus A_{2\rho_\epsilon,7\rho_\epsilon}(x_s)}$,

(d) $v \in V_s^N$
$\Rightarrow \quad \big(v \circ T_{s,\sigma_N}^{-1}\big)\big|_{\Omega_{s+\sigma_N} \setminus A_{2\rho_\epsilon,7\rho_\epsilon}(x_{s+\sigma_N})} = \big(Q_{s+\sigma_N}^N(v \circ T_{s,\sigma_N}^{-1})\big)\big|_{\Omega_{s+\sigma_N} \setminus A_{2\rho_\epsilon,7\rho_\epsilon}(x_{s+\sigma_N})}$.

The main result of this section is the following theorem on the convergence of finite difference quotients of the energy to the energy release rate:

Theorem 4.12. *Assume that conditions* (4.19), (4.31), (4.32) *and* (4.37) *are valid. Then for every $\epsilon > 0$ there exist constants $c_\epsilon > 0$ and $N_\epsilon \in \mathbb{N}$ such that for all $t \in [0,T]$, $N \geq N_\epsilon$ and $s \in Z^N \cap [\epsilon, L-\epsilon]$ it holds*

$$\left|\frac{1}{\sigma_N}\big(\mathcal{I}_N(t, s \pm \sigma_N) - \mathcal{I}_N(t,s)\big) \mp \partial_s \mathcal{I}(t,s)\right| \leq c_\epsilon(\sigma_N + h_N + h_N^2 \sigma_N^{-1}). \tag{4.38}$$

Hence, condition (4.1) *is satisfied provided that the right hand side in* (4.38) *tends to zero for $N \to \infty$.*

In view of estimate (4.38) the optimal relation is $\sigma_N \approx h_N$, which gives the error estimate

$$\big|\tfrac{1}{\sigma_N}\big(\mathcal{I}_N(t, s \pm \sigma_N) - \mathcal{I}_N(t,s)\big) \mp \partial_s \mathcal{I}(t,s)\big| \leq c_\epsilon h_N.$$

This rate of convergence is also observed in the numerical examples in Section 5.1.

Proof. Let $\mathbf{B}_s(\rho, y)$ be the coefficient tensor introduced in (2.7) on the basis of the family $T_{s,\delta}$ from assumption (4.37). The energy $\widetilde{\mathcal{E}}_s : [-\delta_\epsilon, \delta_\epsilon] \times [0,T] \times V_s \to \mathbb{R}$ is defined by

$$\widetilde{\mathcal{E}}_s(\delta, t, v) = \int_{\Omega_s} \frac{1}{2}\mathbf{B}_s(\delta, y)\nabla v : \nabla v \,\mathrm{d}y - \int_{\Gamma_N} \ell(t) \cdot v \,\mathrm{d}\Gamma.$$

It holds $\widetilde{\mathcal{E}}_s(0,t,v) = \mathcal{E}(t,v,s)$ for $v \in V_s$ and $\widetilde{\mathcal{E}}_s(\delta, t, w \circ T_{s,\delta}) = \mathcal{E}(t, w, s+\delta)$ for $w \in V_{s+\delta}$. Let $w^N := u_N(t, s+\sigma_N) \circ T_{s,\sigma_N}$ and $u^N := u_N(t,s)$. It follows for $s \in Z^N \cap [\epsilon, L-\epsilon]$ and $N \geq N_\epsilon$, where $N_\epsilon \in \mathbb{N}$ is chosen such that $\sigma_{N_\epsilon} \leq \min\{\delta_\epsilon, \rho_\epsilon\}$,

$$\begin{aligned}
\frac{1}{\sigma_N}\left(\mathcal{I}_N(t, s+\sigma_N) - \mathcal{I}(t,s)\right) &= \frac{1}{\sigma_N}\left(\widetilde{\mathcal{E}}_s(\sigma_N, t, w^N) - \widetilde{\mathcal{E}}_s(0,t,u^N)\right) \\
&= \frac{1}{\sigma_N}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\delta}\widetilde{\mathcal{E}}_s(\delta\sigma_N, t, u^N + \delta(w^N - u^N))\,\mathrm{d}\delta \\
&= \int_0^1\int_{\Omega_s} \frac{1}{2}\mathbf{B}_s'(\delta\sigma_N, y)\nabla w_\delta^N : \nabla w_\delta^N \,\mathrm{d}y\,\mathrm{d}\delta + \frac{1}{\sigma_N}\int_0^1 \mathrm{D}_u\widetilde{\mathcal{E}}_s(\delta\sigma_N, t, w_\delta^N)[w^N - u^N]\,\mathrm{d}\delta \\
&= S_1^N + S_2^N,
\end{aligned}$$

with $w_\delta^N = u^N + \delta(w^N - u^N)$. The goal is to show that S_1^N approximates the energy release rate and that the error S_2^N tends to zero:

$$\left|S_1^N - \partial_s\mathcal{I}(t,s)\right| \leq c_\epsilon(\sigma_N + h_N), \tag{4.39}$$

$$\left|S_2^N\right| \leq c_\epsilon(\sigma_N + h_N + h_N^2\sigma_N^{-1}). \tag{4.40}$$

We first discuss (4.39). In view of the representation formula for $\partial_s\mathcal{I}$ provided in Theorem 2.1, see also the proof of this formula in [KMZ08, Section 3], it holds with $u := u(t,s)$

$$\partial_s\mathcal{I}(t,s) = -G(s, u(t,s)) = \int_{\Omega_s}\frac{1}{2}\mathbf{B}_s'(0,y)\nabla u : \nabla u\,\mathrm{d}y.$$

Hence,

$$\begin{aligned}
S_1^N - \partial_s\mathcal{I}(t,s) &= \frac{1}{2}\int_0^1\int_{\Omega_s}\mathbf{B}_s'(\delta\sigma_N, y)\nabla w_\delta^N : \nabla w_\delta^N - \mathbf{B}_s'(0,y)\nabla u : \nabla u\,\mathrm{d}y\,\mathrm{d}\delta \\
&= \frac{1}{2}\int_0^1\int_{\Omega_s}(\mathbf{B}_s'(\delta\sigma_N, y) - \mathbf{B}_s'(0,y))\nabla w_\delta^N : \nabla w_\delta^N\,\mathrm{d}y\,\mathrm{d}\delta \\
&\quad + \frac{1}{2}\int_0^1\int_{\Omega_s}\mathbf{B}_s'(0,y)\nabla(w_\delta^N + u) : \nabla(w_\delta^N - u)\,\mathrm{d}y\,\mathrm{d}\delta \\
&= S_{11}^N + S_{12}^N.
\end{aligned}$$

It follows from the definition of $\mathbf{B}_s(\delta, y)$, the assumptions on the family $T_{s,\delta}$ and the uniform estimate (4.33) that $\left|S_{11}^N\right| \leq c_\epsilon\sigma_N$. The term S_{12}^N can be treated as follows. Note first that $\operatorname{supp}_x \mathbf{B}_s'(0,\cdot) \subset A_{3\rho_\epsilon, 6\rho_\epsilon}(x_s)$. Hence, with $w := u(t, s+\sigma_N)\circ T_{s,\sigma_N}$ we find

$$\left|S_{12}^N\right| \leq c_\epsilon\left(\left\|u^N - u\right\|_{H^1(A_{3\rho_\epsilon,6\rho_\epsilon}(x_s)\cap\Omega_s)} + \left\|w^N - w\right\|_{H^1(A_{3\rho_\epsilon,6\rho_\epsilon}(x_s)\cap\Omega_s)} + \left\|w - u\right\|_{H^1(A_{3\rho_\epsilon,6\rho_\epsilon}(x_s)\cap\Omega_s)}\right). \tag{4.41}$$

The local error estimates from Corollary 4.8 imply that

$$\left\|u^N - u\right\|_{H^1(A_{3\rho_\epsilon,6\rho_\epsilon}(x_s)\cap\Omega_s)} + \left\|w^N - w\right\|_{H^1(A_{3\rho_\epsilon,6\rho_\epsilon}(x_s)\cap\Omega_s)} \leq c_\epsilon h_N.$$

Further, in the same way as in the derivation of estimate (2.9) in the proof of Proposition 2.2, we conclude that $\|w-u\|_{H^1(A_{3\rho_\epsilon,6\rho_\epsilon}(x_s)\cap\Omega_s)} \leq c_\epsilon\sigma_N$. Collecting the estimates, inequality (4.39) is shown.

In order to prove estimate (4.40) we split S_2^N into a part which vanishes due the fact that minimizers satisfy the Euler-Lagrange equations and into a part where the integration in fact is taken with respect to an annulus, only. On this part, the local error estimates due to Nitsche and Schatz (Corollary 4.8) and assumption (4.37) are applied. From the linearity of $\mathrm{D}_u\widetilde{\mathcal{E}}_s$ with respect to the displacements we deduce

$$\begin{aligned} S_2^N &= \frac{1}{\sigma_N}\int_0^1 (1-\rho)\mathrm{D}_u\widetilde{\mathcal{E}}_s(\rho\sigma_N,t,u^N)[w^N-u^N]\,\mathrm{d}\rho + \frac{1}{\sigma_N}\int_0^1 \rho\mathrm{D}_u\widetilde{\mathcal{E}}_s(\rho\sigma_N,t,w^N)[w^N-u^N]\,\mathrm{d}\rho \\ &= S_{21}^N + S_{22}^N. \end{aligned}$$

In the following we discuss the term S_{21}^N. The term S_{22}^N can be treated similarly.

$$\begin{aligned} S_{21}^N &= \frac{1}{\sigma_N}\int_0^1 (1-\rho)\mathrm{D}_u\widetilde{\mathcal{E}}_s(0,t,u^N)[w^N-u^N]\,\mathrm{d}\rho \\ &\quad + \frac{1}{\sigma_N}\int_0^1 (1-\rho)\underbrace{\Big(\mathrm{D}_u\widetilde{\mathcal{E}}_s(\rho\sigma_N,t,u^N) - \mathrm{D}_u\widetilde{\mathcal{E}}_s(0,t,u^N)\Big)[w^N-u^N]}_{=\int_{\Omega_s}(\mathbf{B}(\rho\sigma_N,y)-\mathbf{B}(0,y))\nabla u^N:\nabla(w^N-u^N)\,\mathrm{d}y}\,\mathrm{d}\rho \\ &= S_{211}^N + S_{212}^N. \end{aligned}$$

In order to estimate S_{212}^N we use again that $\operatorname{supp}\mathbf{B}'(\rho\sigma,\cdot)\subset A_{3\rho_\epsilon,6\rho_\epsilon}(x_s)$ and that $\mathbf{B}$ is Lipschitz continuous with uniform bounds with respect to its first argument (see assumption (4.37)). Hence, together with the uniform bound (4.33) we obtain in the same way as in (4.41)

$$\left|S_{212}^N\right| \le c_\epsilon \left\|u^N\right\|_{H^1(\Omega_s)} \left\|w^N-u^N\right\|_{H^1(A_{3\rho_\epsilon,6\rho_\epsilon}(x_s)\cap\Omega_s)} \le c_\epsilon(\sigma_N+h_N).$$

It remains to estimate S_{211}^N. Let Q_s^N be the projection operator introduced in condition (4.37).

$$S_{211}^N = \frac{1}{2\sigma_N}\mathrm{D}_u\widetilde{\mathcal{E}}_s(0,t,u^N)[Q_s^N(w^N)-u^N] + \frac{1}{2\sigma_N}\mathrm{D}_u\widetilde{\mathcal{E}}_s(0,t,u^N-u)[w^N-Q_s^N(w^N)]. \quad (4.42)$$

In the last term we used again the linearity of $\mathrm{D}_u\widetilde{\mathcal{E}}_s$ in the displacements and the fact that u is the minimizer of $\widetilde{\mathcal{E}}_s(0,t,\cdot)$ with respect to V_s and hence satisfies the Euler Lagrange equation $\mathrm{D}_u\widetilde{\mathcal{E}}_s(0,t,u)[v]=0$ for every $v\in V_s$. In view of assumption (4.37) it follows that

$$\begin{aligned} &\frac{1}{2\sigma_N}\left|\mathrm{D}_u\widetilde{\mathcal{E}}_s(0,t,u^N-u)[w^N-Q_s^N(w^N)]\right| \\ &\qquad \le c_\epsilon\frac{1}{\sigma_N}\left\|u^N-u\right\|_{H^1(A_{2\rho_\epsilon,7\rho_\epsilon}(x_s))}\left\|w^N-Q_s^N(w^N)\right\|_{H^1(A_{2\rho_\epsilon,7\rho_\epsilon}(x_s))} \end{aligned}$$

The first factor can be estimated by $c_\epsilon h_N$ using Corollary 4.8. To the second factor we apply also Corollary 4.8 and assumption (4.37):

$$\begin{aligned} &\left\|w^N-Q_s^N(w^N)\right\|_{H^1(A_{2\rho_\epsilon,7\rho_\epsilon}(x_s))} \\ &\qquad \le \left\|(\mathbb{I}-Q_s^N)(w^N-w)\right\|_{H^1(A_{2\rho_\epsilon,7\rho_\epsilon}(x_s))} + \left\|(\mathbb{I}-Q_s^N)(w)\right\|_{H^1(A_{2\rho_\epsilon,7\rho_\epsilon}(x_s))} \\ &\qquad \le c_\epsilon(h^N+h^N). \end{aligned}$$

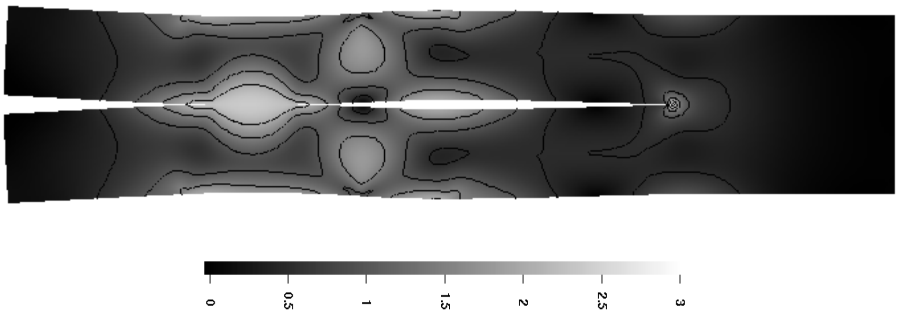


Figure 3: Static solution (deformation multiplied by factor 20), von Mises equivalent stress.

The first term on the right hand side of (4.42) is equal to 0 since $Q_s^N(w^N) - u^N \in V_s^N$ and since u^N is a minimizer of $\widetilde{\mathcal{E}}_s(0, t, \cdot)$ with respect to V_s^N. Hence, $|S_{211}^N| \leq c_\epsilon h_N^2 \sigma_N^{-1}$ and we have finally proved estimate (4.40). □

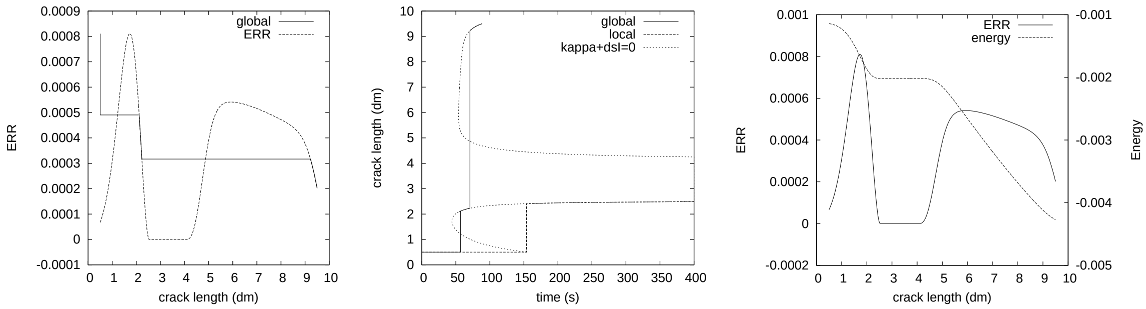
5 Numerical results

5.1 Numerical approximation global energetic and BV-solutions

In this Section, we study some numerical experiments to confirm the convergence results of Sections 3 and 4. For this purpose, we define the domain $\Omega := (-1, 1) \times (0, 10)$ with a maximum crack length $L := 9.5$ as introduced in Section 2.1. We assume homogenous Dirichlet boundary conditions on the boundary part $\Gamma_0 := \{10\} \times (-1, 1)$ and a monotone surface load $h(t, x) := tg(x)$ on the boundary parts $\Gamma_{1,\pm} := (0, 2) \times \{\pm 1\}$, $\Gamma_{2,\pm} := (2, 4) \times \{\pm 1\}$ and $\Gamma_{3,\pm} := (4, 5) \times \{\pm 1\}$. The function g is defined as $g(x) := \pm 0.15$ if $x \in \Gamma_{1,\pm}$, $g(x) := \mp 1$ if $x \in \Gamma_{2,\pm}$ and $g(x) := \pm 1$ if $x \in \Gamma_{3,\pm}$, cf. Figure 3. In our experiments, we use Hooke's law with modulus of elasticity $E := 210\,\mathrm{kN/mm^2}$ and Poisson's number $\nu := 0.28$ with fracture toughness $\kappa := 50\,\mathrm{MPa\,m^{1/2}}$. These material parameters correspond to steel. The end time is set to $T := 400\,\mathrm{s}$ and the initial crack length is chosen as $\mathfrak{s}_0 = 0.5$. In this section, we only consider the case with contact as introduced in Section 4.2.1 as this case seems to be more realistic than the case without contact where in principle only traction loads are physically reasonable. Note, the contact conditions (2.1) describing self contact can be simplified to unilateral one-body contact conditions under the assumption of symmetric surface loads.

To discretize the variational inequality (2.2), we apply a finite element discretization with continuous, piecewise bilinear ansatz functions on a quadrilateral finite element mesh with mesh size h_N. We assume that the crack is partitioned by the edges of the finite element mesh so that the mesh of Ω_{s_i} with $s_i := ih_N$, $i = 1, 2, \ldots$, can easily be constructed from the mesh of $\Omega_{s_{i-1}}$ via the doubling of edges.

Due to the monotone load, the reduced energy and the energy release rate are determined by $\mathcal{I}(t, s) = t^2 \mathcal{I}(1, s)$ and $-\partial_s \mathcal{I}(t, s) = -t^2 \partial_s \mathcal{I}(1, s)$, respectively. In Figure 4(a) and (c), the



(a) (b) (c)

Figure 4: **(a)** Energy release rate, **(b)** global energetic solution and BV-solution, **(c)** energy release rate and energy.

approximated energy release rate $s \mapsto G_N(1,s)$ with $G_N(t,s) := G(s, u_N(1,s))$ and the approximated reduced energy $s \mapsto \mathcal{I}_N(1,s)$ are shown. We use a uniform mesh with mesh size $h_N = 1/64$ to calculate these approximations. To obtain a rough guess, one may construct the global energetic solution and the BV-solution via the mapping $s \mapsto G(s, u(1,s))$ and the level set $\mathcal{L} := \{(t,s) \in [0,T] \times (0,L) \mid \kappa + \partial_s \mathcal{I}(t,s) = 0\}$, cf. [KMZ08]. This is done in Figure 4(b) where $s \mapsto G_N(1,s)$ is used.

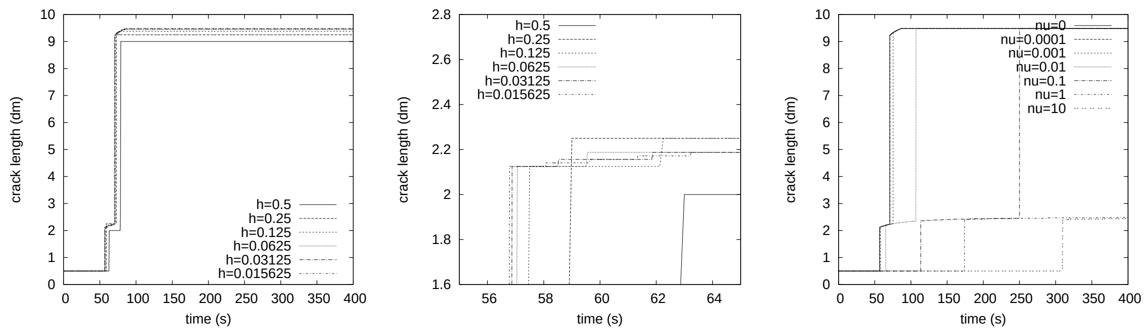
To implement the minimization problem (4.3), we define the piecewise affine interpolant I_N on the data set $(s_i, \mathcal{I}_N(1,s_i))_{0\le i<n}$, where n is the number of edges partitioning the crack. Thanks to the monotone load we have $\mathcal{I}(t,s) \approx t^2 I_N(s)$. Note that the data set $(s_i, \mathcal{I}_N(1,s_i))_{0\le i<n}$ can be computed in a preprocessing step. The input data of the following algorithm consists of the initial crack length $\mathfrak{s}_0 \in (0,L)$, the crack increment $\sigma_N > 0$, the viscosity parameter $\nu_N > 0$ and the time-step size $\tau_N > 0$. The output is the set of incremental solutions $(t_N^k, s_N^k)_{1\le k\le N}$, where t_N^k is the time-step and s_N^k the crack length at the corresponding time-step. Defining $Z^N := \{\mathfrak{s}_0 + k\sigma_N \mid k \in \mathbb{N}, 0 \le k \le M_N\}$ and

$$F_N^k(s) := (t_N^k)^2 I_N(s) + \tau_N \mathcal{R}_{\nu_N}(s_N^{k-1}; (s - s_N^{k-1})/\tau_N),$$

`Algorithm I` is given as follows:

(1) $k = 0$, $t_N^k = 0$, $s_N^k := \mathfrak{s}_0$.
(2) $k := k+1$. If $k\tau_N > T$, stop.
(3) $t_N^k := k\tau_N$, $s_N^k := \mathrm{Argmin}\{F_N^k(s) \mid s \in Z^N,\ s \ge s_N^{k-1}\}$.
(4) back to (2)

Clearly, the minimization problem (4.3) is exactly solved, if $\mathfrak{s}_0$ corresponds to a node of the finite element mesh and σ_N is a multiple of the mesh size h_N. For $\nu_N := 0$, `Algorithm I`, determines incremental solutions approximating the global energetic solution. In Figures 5(a),(b) the convergence of these incremental solutions is depicted with $\sigma_N := h_N$ and h_N tending to 0.



(a) (b) (c)

Figure 5: **(a)** Approximation of the global energetic solution, **(b)** zoom at the first jump, **(c)** viscous solutions.

The first jump is predicted for $t \in (56, 58)$ which corresponds to the global energetic solution as depicted in Figure 4(b). We conclude that `Algorithm I` is applicable to compute approximations for the global energetic solution. Indeed, this confirms the assertion in Theorem 3.2. In Figure 5(c), some approximations for viscous solutions are shown, where the viscosity parameter varies from 10 to zero. The incremental solutions have two jumps along the level set $\mathcal{L}$ as shown in Figure 4(b). We observe that the approximative viscous solutions converge from the right to the global energetic solution.

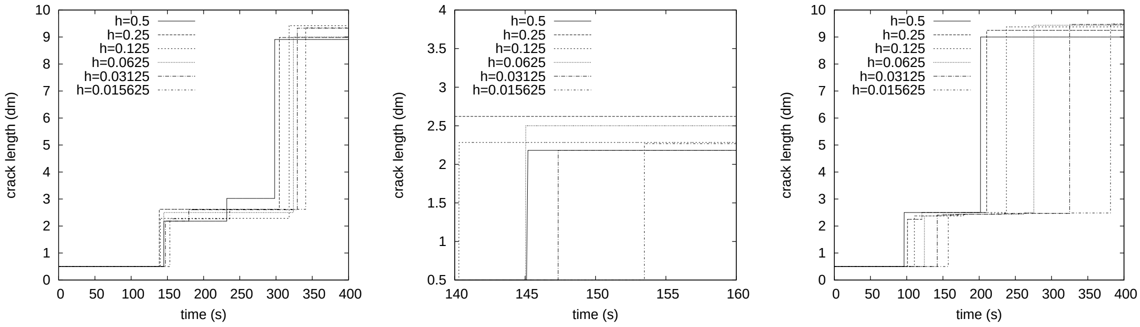
Using `Algorithm I` to approximate the BV-solution, we have to balance the parameters σ_N, ν_N and τ_N in dependence of the mesh size h_N and according to conditions (4.1) and (4.4) in Theorem 4.1 and (4.29) in Theorem 4.5. The parameters α, β occurring in (4.29) are based on the interpolation estimates formulated in condition (4.22). In our case, since $u(t, s_N) \in B^{\frac{3}{2}}_{2,\infty}(\Omega_{s_N})$, it follows, choosing $\widetilde{u}^N_{t,s_N} \in K^N_{s_N}$ as the Lagrange interpolant on bilinear elements, that $\alpha = \frac{1}{2} - \mu$ and $\beta = \frac{3}{2} - \mu$ for arbitrary (small) $\mu > 0$. Indeed, this choice is justified as follows, where the arguments should be done for the Lipschitz domains Ω_+ and Ω_- separately: Complex interpolation theory implies that for $s \in (1, 2)$ it holds $H^s(\Omega_+) = (W^{1,r}(\Omega_+), W^{2,\rho}(\Omega_+))_\theta$ provided that $\theta = s - 1$ and $\frac{1}{2} = \frac{1-\theta}{r} + \frac{\theta}{\rho}$, [Tri83]. Since Ω_+ is two-dimensional, for $r > 2$ the Lagrange interpolation operator L_h is well defined and uniformly continuous on $W^{1,r}(\Omega_+)$. Moreover, for all $v \in W^{2,\rho}(\Omega_+)$ the estimate

$$\|v - L_h v\|_{W^{1,r}(\Omega_+)} \leq c_{r,\rho} h^{1+\frac{2}{r}-\frac{2}{\rho}} \|v\|_{W^{2,\rho}(\Omega_+)}$$

is valid with a constant $c_{r,\rho}$ that is independent of h. Hence, by interpolation (cf. [LM72]), we find for $v \in H^s(\Omega_+)$ with $s \in (1, 2)$ and arbitrary (small) $r > 2$ that

$$\|v - L_h v\|_{H^1(\Omega)} \leq c_r \|v - L_h v\|_{W^{1,r}(\Omega)} \leq c_{r,\rho} h^{\theta(1+2(\frac{1}{r}-\frac{1}{\rho}))} \|v\|_{H^s(\Omega_+)} . \qquad (5.1)$$

Thus, with $s = \frac{3}{2} - \delta_1$, $\theta = s - 1$, $r = 2 + \delta_2$, where δ_1, δ_2 are positive, but can be chosen arbitrary small, it follows that $\rho = 2 - \delta_3$ with a suitable (small) $\delta_3 > 0$. Hence, the exponent in (5.1)



(a) (b) (c)

Figure 6: Approximation of the BV-solution: **(a)** $\sigma_N = h_N^{0.25}$, $\nu_N = 0.8h_N^{0.875}$, $\tau_N = 0.1h_N$, **(b)** zoom at the jump, **(c)** $\sigma_N = h_N$, $\nu_N = 0.2h_N^{0.5}$, $\tau_N = 0.1h_N$.

behaves like $\frac{1}{2} - \mu$ with some suitable $\mu > 0$ depending on δ_1, δ_2, and $\mu \to 0$ for $\delta_1, \delta_2 \to 0$. Similar arguments show that $\beta = \frac{3}{2} - \mu$.

In the following, we ignore the constant μ. Hence, setting $\sigma_N := \delta h_N^{\gamma}$ with $\gamma > 0$ and using the exponents $\alpha = 1/2$ and $\beta = 3/2$, the right side of condition (4.29) becomes $h_N^{\gamma} + h_N^{1/2-\gamma}$. For $\gamma = 1/4$, both terms uniformly tend to zero. For $\nu_N := \delta_1 h_N^{\gamma_1}$ and $\tau_N := \delta_2 h_N^{\gamma_2}$, condition (4.4) becomes $h_N^{\gamma+\gamma_1+\gamma_2} \to 0$ and $h_N^{\gamma_2-\gamma_1} \to 0$ so that $\gamma_1 = \gamma_2 - 1/8$. To link the time-step size to the crack increment, we have to choose $\gamma_1 := 1/8$ and $\gamma_2 := 1/4$. Ensuring the time-step size to be equal to the mesh size, we may take $\gamma_1 := 7/8$ and $\gamma_2 := 1$. However, in view of the experiments shown in Figure 11, condition (4.29) seems to be too pessimistic, so that also $\sigma_N := h_N$ and, therefore, $\nu_N := h_N^{1/2}$ as well as $\tau_N := h_N$ may be a reasonable choice. We expect that the position of the first jump should be between 153 and 154 as shown in Figure 4(b) and, moreover, the smaller h_N is, the more to the right the second jump is located

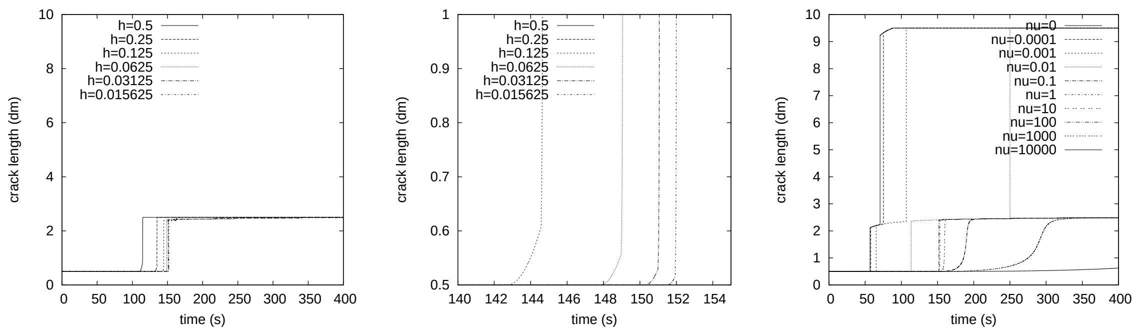
In Figure 6, the output of `Algorithm I` describing the approximation of the BV-solution is depicted. In our experiments, we observed a high sensitivity of the algorithm with respect to the parameters δ_1 and δ_2. Improperly chosen parameters lead to jumps far from the predicted jump so that convergence is not visible for large mesh sizes h_N. See also the discussion to Figure 8(a).

To overcome these difficulties, we extend `Algorithm I` using some derivative information of the interpolant I_N and the function $\mathcal{R}_\nu$. The input and output data of `Algorithm II` are the same as for `Algorithm I` except for the crack increment σ_N, where we assume $\sigma_N := h_N$. Furthermore, step (3) is replaced by

`(3)` $t_N^k := k\tau_N$, $s_N^k := \mathrm{Argmin}\{F_N^k(s) \mid s \in Z^N \cup \tilde{Z}_k^N,\ s \geq s_N^{k-1}\}$

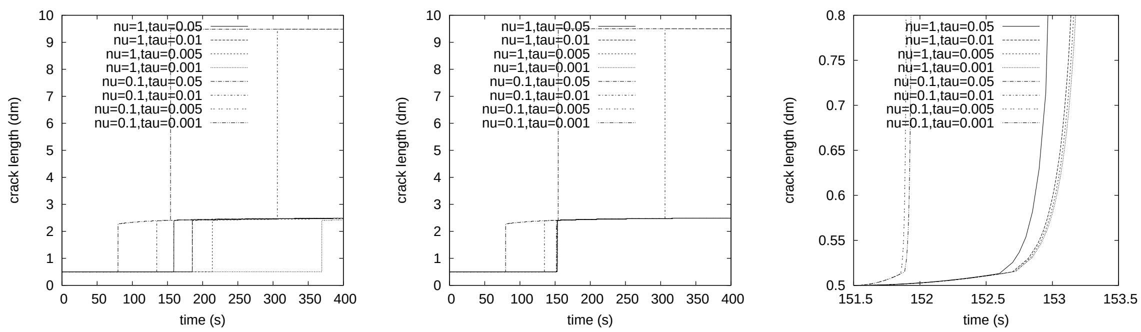
where

$$\tilde{Z}_k^N := \{s \in (s_{i-1}, s_i) \mid (t_N^i)^2 I_N'(s) + \tau\mathcal{R}_\nu'(s_N^{i-1}; s) = 0,\ 1 \leq i < n\}.$$



(a) (b) (c)

Figure 7: Approximation of the BV-solution: **(a)** $\nu_N = h_N^{0.5}$, $\tau_N = 0.1h_N$, **(b)** zoom at the jump, **(c)** viscous solutions.



(a) (b) (c)

Figure 8: Approximation of the BV-solution: **(a)** two fixed viscosity parameters ν_N, time step $\tau_N \to 0$ (`Algorithm I`), **(b)** two fixed viscosity parameters ν_N, time step $\tau_N \to 0$ (`Algorithm II`), **(c)** zoom at the jump (`Algorithm II`).

In Figure 7, some approximative BV-solutions are depicted which are obtained on the basis of `Algorithm II`. In our experiments we observed that the sensitivity of the algorithm with respect to the parameters is essentially smaller. In Figure 7(c), viscous solutions are shown with time-step size and mesh size $\tau_N = h_N = 1/64$ and viscosity parameter ν tending to 0. For large viscosity parameters we observe smooth viscous incremental solutions, whereas for small viscosity parameters the solutions have steep slopes which move to the first jump of the energetic solution.

In Figure 8, we study the influence of the time-step size τ_N on the approximation of the BV-solution using `Algorithm I` and `Algorithm II`. In Figure 8(a) and (b), we fix the viscosity parameter ν_N and the mesh size h_N. Using `Algorithm I`, we observe that the first jump of the approximated BV-solution moves to the right as τ_N tending to 0, see Figure 8(a). Moreover, we see some dependencies of the parameters ν_N and τ_N which may be explained by the assumptions

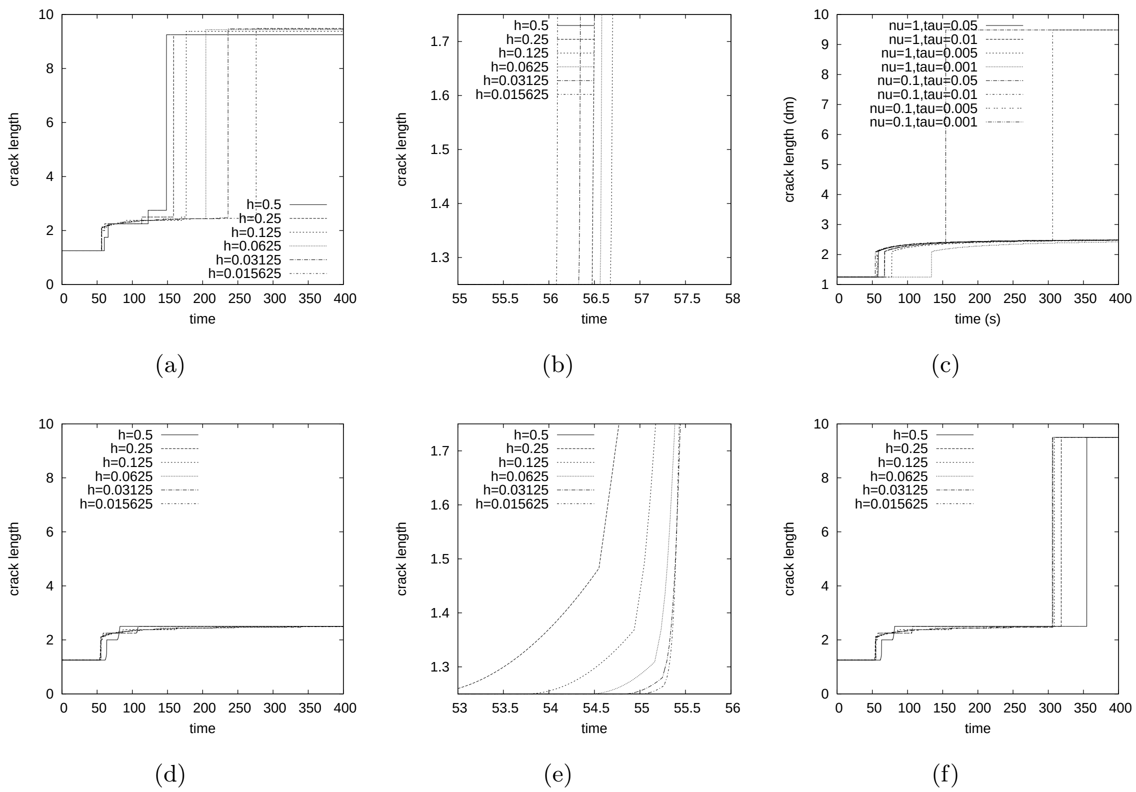


Figure 9: Approximation of the BV-solution for initial crack $\mathfrak{s}_0 = 1.25$: **(a)** $\sigma_N = h_N$, $\nu_N = 0.1h_N^{0.5}$ $\tau_N = 0.1h_N$ (`Algorithm I`), **(b)** zoom at the jump (`Algorithm I`), **(c)** fixed ν_N, $\tau_N \to 0$ (`Algorithm I`), **(d)** $\nu_N = h_N^{0.5}$, $\tau_N = 0.1h_N$ (`Algorithm II`), **(e)** zoom at the jump (`Algorithm II`), **(f)** $\nu_N = \tau_N = h_N$ (`Algorithm II`).

in Theorem 4.1. In the case of `Algorithm II`, however, the viscous solutions with fixed ν_N converge as $\tau_N \to 0$, cf. Figure 8(b) and (c). This means that τ_N can be chosen arbitrary small. Thus, τ_N and ν_N are independent of each other. A further observation is that small viscosity parameters ν_N lead to steep slopes, which is, of course, expectable. However, they also lead to a less accurate approximation of the jump of the BV-solution (which should approximatively be between 153 and 154). On the other hand, large viscosity parameters result in less steep and 'rounded' curves, cf. Figure 8(c). This effect can also be observed for `Algorithm I`. Due to the dependence of τ_N and ν_N the time-step size has to be increased in this case which may lead to a rough approximation.

In Figure 9, we study the same experiments, but with the longer initial crack length $\mathfrak{s}_0 = 1.25$. At first sight, the sensitivity of `Algorithm I` with respect to the parameters seems to be smaller than in the previous experiments. In particular, the convergence of the approximative BV-solutions seems to be more clear, cf. Figure 9(a). However, we have the same set of problems using `Algorithm I`, in particular, if we want to balance the parameters ν_N and τ_N with δ_1 and

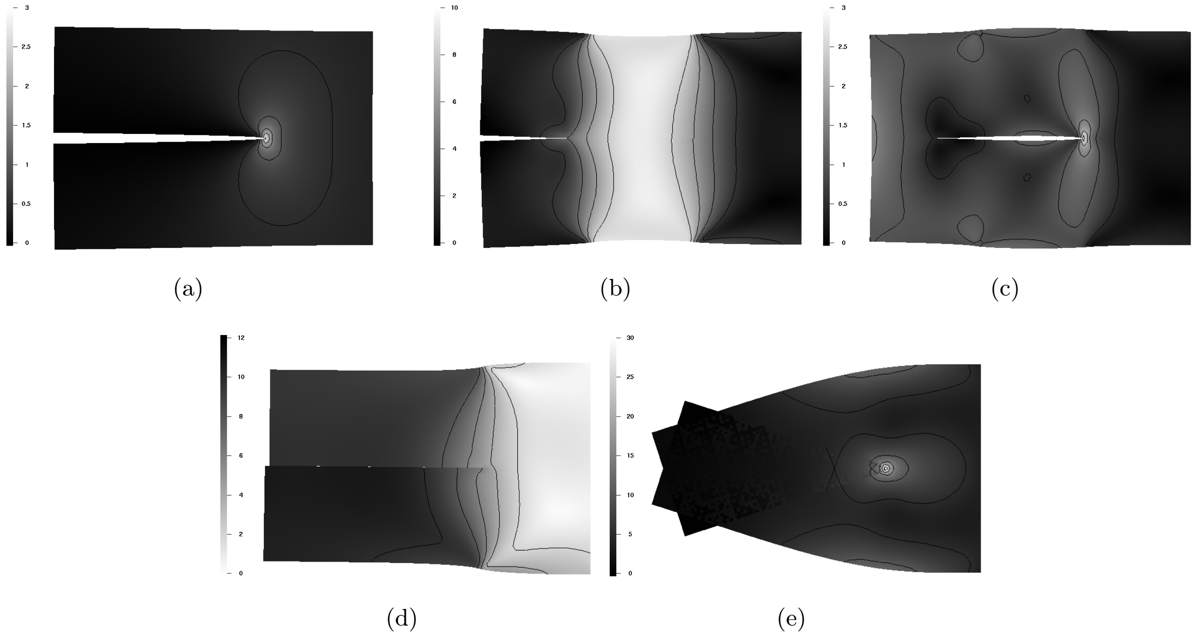


Figure 10: Several examples (von Mises equivalent stress): **(a)** mode I (multiplied by factor 50), **(b),(c)** contact solution with symmetric load (factor 10,50), **(d)** mode II with self-contact (factor 20), **(e)** solution with symmetric load without contact (factor 20).

δ_2, cf. Figure 9(c). Again, `Algorithm II` produces approximations of the BV-solution which inspire much more confidence.

Finally, we advert to the dependence of the viscosity parameter ν_N from the mesh size h_N and time step size τ_N in the application of `Algorithm II`. As we can see in Figure 9(f), the second jumps of the approximative BV-solutions converges to some final time as $h_N \to 0$ for $\nu_N = h_N = \tau_N$. This means that the approximative BV-solutions do not converge to the BV-solution. Choosing $\nu_N = h_N^{0.5} = \tau_N^{0.5}$, we observe that the second jumps move to the right as desired. This highlights that the assumption $\tau_N/\nu_N \to 0$ is, in fact, needed.

5.2 Convergence of the energy release rates

At last, we study the convergence rates predicted in (4.29) and (4.38). In Fig. 10, several examples for contact and non-contact problems are depicted. The maximum crack length is $L = 2$ and the domain is defined by $\Omega := [0, 3] \times [-1, 1]$. The example (a) is a mode I function with non-homogenous Dirichlet boundary conditions on $\Gamma_0 := \{3\} \times (-1, -1)$, cf. [Gro96].

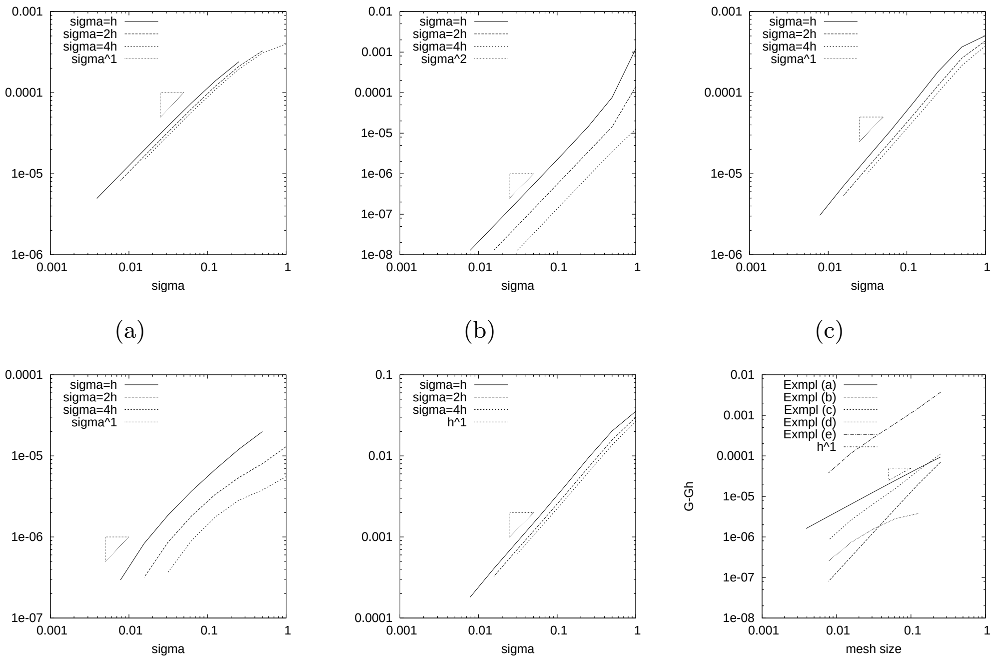
The functions in the Examples (b)-(d) are solutions of the variational inequality (2.2), where homogenous Dirichlet boundary conditions on Γ_0 and surface loads on the boundary parts $\Gamma_{1,\pm} := (0, 1) \times \{\pm 1\}$, $\Gamma_{2,\pm} := (1, 2) \times \{\pm 1\}$, $\Gamma_3 := \{0\} \times (-1, 0)$, $\Gamma_3 := \{0\} \times (0, 1)$ are assumed. The surface loads are given in Table 1. In Example (e), contact conditions on the crack are not

Example	$\Gamma_{1,+}$	$\Gamma_{1,-}$	$\Gamma_{2,+}$	$\Gamma_{2,-}$	Γ_3	Γ_4
(b)	1	-1	-10	10	0	0
(c)	-1	1	1	-1	0	0
(d)	-10	-10	-10	-10	1	-1
(e)	-1	1	1	-1	0	0

Table 1: Surface loads.

enforced so that self-penetration occurs which is, of course, physically unreasonable.

In Figure 11, the convergence rates for the terms $|\sigma_N^{-1}(\mathcal{I}_N(t,s-\sigma_N)-\mathcal{I}_N(t,s))+\partial_s\mathcal{I}(t,s)|$ and $|G(s,u(t,s))-G(s,u_N(t,s))|$ with $s=L$ are shown. We observe that the convergence rate is at least $\mathcal{O}(h_N)$ for both terms, where $\sigma_N\in\{h_N,2h_N,4h_N\}$. Indeed, the rate $\mathcal{O}(h_N)$ is predicted in Theorem 4.12 for non-contact as given in Example (a). In the case of contact, the estimations (4.29) seem to be too pessimistic. Provided that the surface loads act orthogonally and the crack is closed, we even obtain quadratic rates for $|G(s,u(t,s))-G(s,u_N(t,s))|$, cf. Figure 11(b). For surface loads leading to shear strains and, moreover, to a closing crack, the rates may not be quadratic, but seem to be better than linear. Also, the absence of contact conditions could lead to higher convergence rates, cf. Figure 11(e).



(a) (b) (c) (d) (e) (f)

Figure 11: **(a)-(e)**: $|\sigma_N^{-1}(\mathcal{I}_N(t,s-\sigma_N)-\mathcal{I}_N(t,s))+\partial_s\mathcal{I}(t,s)|$, **(f)** $|G(s,u(t,s))-G(s,u_N(t,s))|$.

We finally remark that condition (4.37) is satisfied if one chooses meshes that are locally invariant with respect to a translation of length h (mesh width) parallel to the crack. Then, an appropriate choice for Q_s^N is the Zhang/Scott interpolation operator [SZ96].

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