Universität Stuttgart

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Mehrfeldprobleme in der Kontinuumsmechanik

Dorothee Knees

On the regularity of weak solutions of a shear thinning fluid of power-law type

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Contents

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1 Introduction

In this paper we investigate the solvability and regularity of the velocity and pressure fields of a class of fluids with shear dependent viscosity, where the constitutive relation is of power-law type. The corresponding field equations are given by a quasilinear elliptic system of partial differential equations, which include as a special case the stationary, linear Stokes system.
Besides the presentation of known results on local regularity of the velocity fields in appropriate Sobolev-Slobodeckij-spaces, we derive some new aspects concerning the local and global regularity of the pressure, stress and velocity fields on polygonal or polyhedral domains and include the case of mixed non-vanishing boundary conditions. In the whole paper we will focus our attention on higher regularity in Sobolev-Slobodeckij-spaces.
Local regularity results, i.e. higher regularity on subsets $\Omega' \subset\subset \Omega$, for quasilinear degenerated elliptic systems of $p-$structure were derived e.g. by P. Tolksdorf, [25], F. de Thélin, [8], and J.-P. Raymond, [24]. They used a difference quotient technique in order to obtain a better regularity in Sobolev spaces of integral order. In contrast to the systems they considered, the equations of our fluid model also contain a pressure term π ($n \geqslant 1$):

$$\begin{aligned} \operatorname{div}\left(\alpha\left|\varepsilon^D(u)\right|^{\frac{1}{n}-1}\varepsilon^D(u)\right) - \nabla\pi &= -f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= g && \text{on } \Gamma_D, \\ \sigma\vec{n} &= h && \text{on } \Gamma_N. \end{aligned}$$

J. Naumann proved in [22] on the basis of Tolksdorf's and de Thélin's results local regularity of the velocity field u in three dimensions for this equation. We reformulate his result also for the two-dimensional case and investigate in addition the regularity of the shear stress and pressure field. For this we apply techniques which were developed by C. Ebmeyer in [11]. We will also use these techniques to obtain higher regularity for tangential derivatives at a flat part of the boundary. Let us finally note, that M. Fuchs proves local regularity results in Hölder spaces for a fluid model, which is a modification of our model, [15].
Global regularity results will be derived for a class of polyhedral domains with mixed and non-vanishing boundary conditions. Here we combine Ebmeyer's considerations from [11] and [13]. In [11], Ebmeyer proved global results for non-Newtonian flows where the equations contain the convection term $(u \cdot \nabla)u$. Since our model has no such term we can carry over the investigations from [13] to our problem which leads to a higher regularity than in [11]. Again, the proofs are based on a difference quotient technique to get estimates in appropriate Nikolskii-spaces, which are closely related by embedding theorems to the usual Sobolev-Slobodeckij-spaces.
Since the linear Stokes system is a special case of our model, we will compare the obtained results to those which are well known for linear elliptic equations. This indicates some optimality of the results.
The paper is organized as follows:
In section 2 we will shortly prove existence of solutions of finite energy. This can be done by well known arguments in the framework of the theory of monotone operators. We also describe the connection between the weak formulation and the minimization problem for the corresponding energy functional.

Section 3 is devoted to the study of local regularity of the velocity, pressure and stress fields. While the regularity of the velocity field is proved in [22], we deduce the regularity of the stress and pressure field by applying the techniques from [11].
In section 4, we study the regularity properties of higher tangential derivatives of the fields near a flat part of the boundary. Thereby we admit non-vanishing Dirichlet- or Neumann-data.
In section 5 we state and prove global regularity results on polyhedral domains. As already mentioned, these results are a combination of Ebmeyer's in [11] and [13] and therefore, the proofs are also based on Ebmeyer's ideas.
This paper closes with an appendix, where some functional analytic tools are collected. The appendix contains some essential inequalities, mapping properties of the divergence operator, a solvability theorem for nonlinear saddle point problems and a simplified variant of Ljusternik's Theorem which describes the Euler-Lagrange equations for a minimization problem with constraints.

2 Existence and uniqueness results

2.1 Field equations for a class of shear thinning fluids

By equations (3)-(6) here below we describe the velocity and pressure fields of the steady motion of a class of incompressible, shear thinning fluids, where the constitutive relation is of power-law type.
Let $\Omega \subset \mathbb{R}^d, d = 2,3$ be a bounded domain, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$. By $u : \Omega \to \mathbb{R}^d$ we denote the velocity field of the fluid, $\varepsilon(u) := \frac{1}{2}(\nabla u + \nabla u^T)$ is the strain rate tensor, $\varepsilon^D(u) := \varepsilon(u) - \frac{1}{d}\operatorname{tr}(\varepsilon(u))I$ is the deviatoric part of $\varepsilon(u)$ and describes the shear velocity. Furthermore, σ is the stress tensor and is decomposed in the following way:

$$\sigma = -\pi I + T = -\pi I + \sigma^D, \tag{1}$$

where π can be interpreted as hydrostatic pressure and $T = \sigma^D = \sigma - \frac{1}{d}\operatorname{tr}\sigma I$ is the tensor of viscous stresses. Note, that this splitting of σ into a pressure term and the viscous stresses is not stringent for an incompressible fluid, in contrast to the case of compressible fluids, where π stands for the thermodynamic pressure.
We assume that the fluid satisfies the following constitutive relation between the shear rate ε^D and the shear stress σ^D:

$$\sigma^D = \alpha \left|\varepsilon^D(u)\right|^{\frac{1}{n}-1} \varepsilon^D(u), \tag{2}$$

where $\alpha > 0$ and $n \geqslant 1$ are some material parameters which can be fixed by experimental data. One can interpret the constitutive law as follows:
Consider as a special case a steady plane parallel flow where the velocity is of the form $\vec{u}(\vec{x}) = (u_1(x_2), 0, 0)$, see Figure 1. In this case

$$\varepsilon(\vec{u}) = \varepsilon^D(\vec{u}) = \begin{pmatrix} 0 & \varepsilon_{12} & 0 \\ \varepsilon_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the constitutive law reduces to

$$\sigma_{12} = \alpha \left|\varepsilon_{12}(u)\right|^{\frac{1}{n}-1} \varepsilon_{12}(u).$$

This relation is plotted in Figure 1. The quantity $\eta(\varepsilon_{12}(u)) := \alpha\,|\varepsilon_{12}(u)|^{\frac{1}{n}-1}$ can be interpreted as the shear viscosity and is of *Ostwald-de Waele* type. For fixed $n > 1$, the shear viscosity η decreases as the shear rate ε_{12} increases and therefore this model describes a shear thinning fluid. It should be mentioned, that for $|\varepsilon_{12}| \to 0$, the shear viscosity η tends to infinity and one should be careful when applying this constitutive model to flows with very small shear rates $|\varepsilon_{12}|$. Examples for shear thinning fluids are molten plastics and polymer solutions. For more details we refer e.g. to [4].
The problem we are interested in is the following: Find a velocity field u and a pressure field π such that for given volume and surface forces f and h and for a given surface velocity g there holds:

$$
\begin{aligned}
\operatorname{div}\sigma + f &= 0 && \text{in } \Omega && \text{equations of motion,}\\
\sigma^D - \alpha\left|\varepsilon^D(u)\right|^{\frac{1}{n}-1}\varepsilon^D(u) &= 0 && \text{in } \Omega && \text{constitutive law,}\\
\operatorname{div} u &= 0 && \text{in } \Omega && \text{incompressibility condition,}\\
u &= g && \text{on } \Gamma_D, &&\\
\sigma\vec{n} &= h && \text{on } \Gamma_N. &&
\end{aligned}
$$

The vector $\vec{n}$ is the exterior normal vector on the boundary Γ_N. These equations can be shortly written as

$$
\begin{aligned}
\operatorname{div}\left(\alpha\left|\varepsilon^D(u)\right|^{\frac{1}{n}-1}\varepsilon^D(u)\right) - \nabla\pi &= -f && \text{in } \Omega, && (3)\\
\operatorname{div} u &= 0 && \text{in } \Omega, && (4)\\
u &= g && \text{on } \Gamma_D, && (5)\\
\sigma\vec{n} &= h && \text{on } \Gamma_N. && (6)
\end{aligned}
$$

Note, that in the case $n = 1$, this system reduces to the well known linear Stokes system.

Remark 2.1. For $n \geqslant 1$ the function $F : \mathbb{R}^s\backslash\{0\} \to \mathbb{R}^s : \vec{x} \to |\vec{x}|^{\frac{1}{n}-1}\,\vec{x}$ can be extended continuously to $\vec{x} = 0$ by setting $F(0) = 0$. We interpret the term in the brackets of equation (3) in this sense.

Before we describe in which sense we solve these equations we have to introduce some appropriate function spaces.

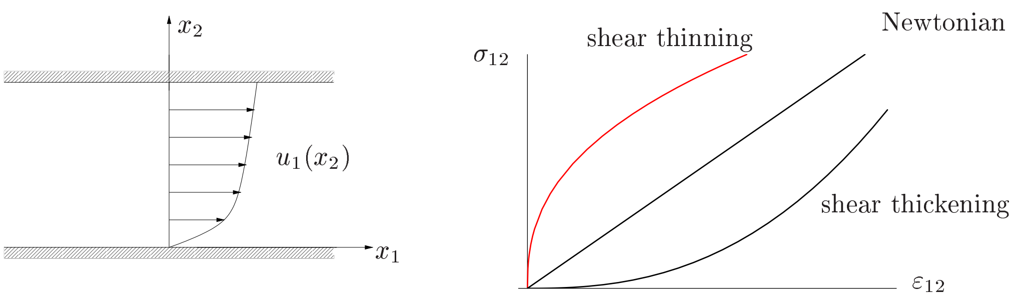

Figure 1: Typical constitutive behavior for shear thinning, Newtonian and shear thickening fluids for a steady plane parallel flow $\vec{u} = (u_1(x_2), 0, 0)$

2.2 The spaces

Throughout the whole paper we do not distinguish in our notation between scalars, vectors and tensors since in general it is clear from the context of which type a variable or function is. Moreover we use the same notation for function spaces of scalar valued, vector valued or tensorial valued functions. Only in some special cases we will write e.g. $L^p(\Omega, \mathbb{R}^d)$ which is the space of vector valued functions $\{u = (u_1, \ldots, u_d) : \Omega \to \mathbb{R}^d,\ u_i \in L^p(\Omega), 1 \leqslant i \leqslant d\}$.
For open subsets $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, we introduce the following Sobolev-Slobodeckij-spaces:
Let $s = m + \sigma$, where $m \in \mathbb{N}_0$, $0 \leqslant \sigma < 1$ and $1 < p < \infty$. The space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) :\ D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leqslant m \text{ and } \|u\|_{W^{s,p}(\Omega)} < \infty\},$$

see also [1, 26]. In this definition we make use of the usual multi-index notation, D^α denotes the distributional derivative of order α and the norm is given by

$$\|u\|^p_{W^{s,p}(\Omega)} = \sum_{|\alpha| \leqslant m} \|D^\alpha u\|^p_{L^p(\Omega)} + \sum_{|\alpha| \leqslant m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{d+p\sigma}} \, \mathrm{d}x \, \mathrm{d}y.$$

Furthermore, the corresponding trace spaces on $\Gamma \subset \partial\Omega$, Γ open, are defined in the sense of Sobolev-Slobodeckij-spaces on compact manifolds, see [18]. Here we need $C^{k,1}$-smoothness of $\partial\Omega$ for the definition of $W^{s,p}(\Gamma)$, where s and k are related as follows: $k \in \mathbb{N}_0, |s| \leqslant k + 1$.
As a special case of [18, Thm. 1.5.2.1] we have for $s = 1$ and $k = 0$ the following trace theorem:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $\Gamma \subseteq \partial\Omega$ open and $1 < p < \infty$. Then the mapping*

$$\gamma\big|_\Gamma : u \to u\big|_\Gamma,$$

which is defined for $u \in C^\infty(\overline{\Omega})$, has a unique continuous extension denoted by the same operator:

$$\gamma\big|_\Gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\Gamma).$$

Furthermore, the mapping $\gamma\big|_\Gamma$ is surjective.

With this theorem the following definition is meaningful:

Definition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where Γ_D and Γ_N are open and disjoint; $1 < p < \infty$. For $g \in W^{1-\frac{1}{p},p}(\Gamma_D)$ we set*

$$V_p(g) := \{u \in W^{1,p}(\Omega) :\ \ u\big|_{\Gamma_D} = g\}.$$

We will shortly write V_p instead of $V_p(0)$.
Furthermore,

$$\tilde{W}^{1-\frac{1}{p},p}(\Gamma_N) := \left\{u :\ u = \tilde{u}\big|_{\Gamma_N},\ \textit{where } \tilde{u} \in W^{1-\frac{1}{p},p}(\partial\Omega) \textit{ with } \operatorname{supp}\tilde{u} \subset \overline{\Gamma_N}\right\},$$

which is endowed with the norm $\|u\|_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_N)} := \|\tilde{u}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}$.

Remark 2.2. By the linearity and surjectivity of the trace operator, there exists for every $g \in W^{1-\frac{1}{p},p}(\Gamma_D)$ an element $\tilde{g} \in W^{1,p}(\Omega)$ with $\tilde{g}|_{\Gamma_D} = g$. Thus $V_p(g) = \tilde{g} + V_p(0)$.

Important tools in the proof of existence of solutions are Korn's inequality and a generalized version of Poincaré-Friedrichs' inequality:

Theorem 2.2 (Korn's inequality). *[16] Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. For $1 < p < \infty$ we have the following estimate: There exist $c_1, c_2 > 0$ such that for all $u \in W^{1,p}(\Omega, \mathbb{R}^d)$*

$$c_1 \|u\|_{W^{1,p}(\Omega)} \leqslant \|u\|_{L^p(\Omega)} + \left\|\varepsilon^D(u)\right\|_{L^p(\Omega)} + \|\operatorname{tr}\varepsilon(u)\|_{L^p(\Omega)} \leqslant c_2 \|u\|_{W^{1,p}(\Omega)}.$$

Thus, the expression $|||u|||_p := \|u\|_{L^p(\Omega)} + \left\|\varepsilon^D(u)\right\|_{L^p(\Omega)} + \|\operatorname{tr}\varepsilon(u)\|_{L^p(\Omega)}$ defines an equivalent norm in $W^{1,p}(\Omega)$. Furthermore, the spaces $W^{1,p}(\Omega, \mathbb{R}^d)$ and $U^{1,p}(\Omega, \mathbb{R}^d) := \{u \in L^p(\Omega, \mathbb{R}^d) : |||u|||_p < \infty\}$ are equal.

Theorem 2.3 (Poincaré-Friedrichs' inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and $1 < p < \infty$.*

1. *If $V \subset W^{1,p}(\Omega)$ is a closed, convex subset with the property*

$$u \in V, \nabla u = 0 \implies u = 0.$$

Then there exists a constant $c > 0$ such that for every $u \in V$:

$$\|u\|_{L^p(\Omega)} \leqslant c \|\nabla u\|_{L^p(\Omega)}.$$

2. *[16] If $V \subset W^{1,p}(\Omega, \mathbb{R}^d)$ is a closed, convex subset with the property*

$$u \in V, \|\varepsilon(u)\|_{L^p(\Omega)} = 0 \implies u = 0,$$

then there exists a constant $c > 0$ such that for every $u \in V$:

$$\|u\|_{W^{1,p}(\Omega)} \leqslant c \|\varepsilon(u)\|_{L^p(\Omega)}.$$

We will prove the regularity results by estimating difference quotients of weak solutions. Suitable spaces, where the norms take into account difference quotients in an explicit way, are the Nikolskii-spaces.

Definition 2.2 (Nikolskii-space). *[1] Let $s = m + \sigma$ where $m \geqslant 0$ is an integer and $0 < \sigma < 1$. For $1 \leqslant p < \infty$*

$$\mathcal{N}^{s,p}(\Omega) := \left\{u \in L^p(\Omega) : \|u\|_{\mathcal{N}^{s,p}(\Omega)} < \infty\right\}, \tag{7}$$

where

$$\|u\|^p_{\mathcal{N}^{s,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \sum_{|\alpha|=m} \sup_{\substack{\eta>0 \\ h\in\mathbb{R}^d \\ 0<|h|<\eta}} \int_{\Omega_\eta} \frac{|D^\alpha u(x+h) - D^\alpha u(x)|^p}{|h|^{\sigma p}} \,\mathrm{d}x \tag{8}$$

and $\Omega_\eta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \eta\}$.

The relation between Nikolskii-spaces and Sobolev-Slobodeckij-spaces is described in the next lemma:

Lemma 2.1. *[1] Let s, p be as in Definition 2.2. The following embeddings are continuous:*

$$\text{for every } \varepsilon > 0: \qquad \mathcal{N}^{s+\varepsilon,p}(\Omega) \subset W^{s,p}(\Omega) \subset \mathcal{N}^{s,p}(\Omega).$$

In the definition of Nikolskii-spaces we have to take into account difference quotients with respect to every direction $h \in \mathbb{R}^d$. It is also possible to define a space, where the difference quotients are formed with respect to a fixed basis of $\mathbb{R}^d$, only. If Ω is a bounded Lipschitzian domain, then these two definitions coincide. More precisely:
Let $\xi_1, \ldots, \xi_d$ be a basis of $\mathbb{R}^d$ with $|\xi_i| = 1$. For s, p as in Definition 2.2 we define

$$\begin{aligned} \tilde{\mathcal{N}}^{s,p}(\Omega) &:= \{u \in L^p(\Omega) : \ \mathcal{N}_{\alpha,i}(u) < \infty \text{ for } 1 \leqslant i \leqslant d \text{ and } |\alpha| = m\}, \\ \|u\|_{\tilde{\mathcal{N}}^{s,p}(\Omega)} &:= \|u\|_{L^p(\Omega)} + \sum_{\substack{1 \leqslant i \leqslant d \\ |\alpha| = m}} \mathcal{N}_{\alpha,i}(u), \end{aligned}$$

where

$$\mathcal{N}_{\alpha,i}(u) := \sup_{h>0} \left(\int_{\Omega_h} \frac{|D^\alpha u(x + h\xi_i) - D^\alpha u(x))|^p}{h^{\sigma p}} \, \mathrm{d}x \right)^{\frac{1}{p}} \qquad \text{for } 1 \leqslant i \leqslant d.$$

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz-boundary and $\xi_1, \ldots, \xi_d \subset \mathbb{R}^d$ a normed basis of $\mathbb{R}^d$. Then*

$$\mathcal{N}^{s,p}(\Omega) = \tilde{\mathcal{N}}^{s,p}(\Omega)$$

and the norms are equivalent. The constants in the equivalence relation of the norms depend on the choice of the basis.

Proof. Since Ω is a bounded Lipschitzian domain, functions from $\mathcal{N}^{s,p}(\Omega)$ and $\tilde{\mathcal{N}}^{s,p}(\Omega)$ can be extended to $\mathbb{R}^d$ with preservation of the norm, [23, Thm. 1, Thm. 2, pp. 381]. Furthermore, $\tilde{\mathcal{N}}^{s,p}(\mathbb{R}^d) = \mathcal{N}^{s,p}(\mathbb{R}^d)$ and the norms are equivalent, [20]. □

2.3 Existence results

We are now able to describe in which sense we want to solve equations (3)-(6).

Definition 2.3 (Weak Solution). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ where Γ_D and Γ_N are disjoint open sets. Let further be $n \geqslant 1$, $p = n + 1$ and $q = p' = 1 + \frac{1}{n}$. We assume that the given data satisfy $f \in V_q'$, $g \in W^{1-\frac{1}{q},q}(\Gamma_D)$ and $h \in (\tilde{W}^{1-\frac{1}{q},q}(\Gamma_N))'$.*
A pair $(u_0, \pi) \in V_q(g) \times L^p(\Omega)$ is a weak solution of the nonlinear field equations (3)-(6) if for every $v \in V_q(0)$, $r \in L^p(\Omega)$:

$$\int_\Omega \alpha \left|\varepsilon^D(u_0)\right|^{q-2} \varepsilon^D(u_0) : \varepsilon^D(v) \, \mathrm{d}x - \int_\Omega \pi \operatorname{div} v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x + \int_{\Gamma_N} h v \, \mathrm{d}s, \tag{9}$$

$$\int_\Omega r \operatorname{div} u_0 \, \mathrm{d}x = 0. \tag{10}$$

The integrals on the right hand side are to be understood in the sense of the dual pairing between V_q, V_q' and $(\tilde{W}^{1-\frac{1}{q},q}(\Gamma_N))'$, $\tilde{W}^{1-\frac{1}{q},q}(\Gamma_N)$, respectively.

The weak formulation can formally be obtained by multiplying the field equations (3)-(6) with v and integration by parts.
The weak formulation has the structure of a nonlinear saddlepoint problem. To make this more evident we introduce the following forms for p, q, n as in Definition 2.3:

$$a(\cdot,\cdot): \qquad W^{1,q}(\Omega)\times W^{1,q}(\Omega)\to\mathbb{R}: \ a(u,v)=\int_\Omega \alpha\left|\varepsilon^D(u)\right|^{q-2}\varepsilon^D(u):\varepsilon^D(v)\,\mathrm{d}x, \tag{11}$$

$$b(\cdot,\cdot): \qquad L^p(\Omega)\times V_q\to\mathbb{R}: \ b(\pi,v)=-\int_\Omega \pi\operatorname{div}v\,\mathrm{d}x. \tag{12}$$

The next lemmata collect some properties of the forms a and b which we will need in the sequel.

Lemma 2.3. *Let p, q, n be as in Definition 2.3. For $(u,v)\in W^{1,q}(\Omega)\times W^{1,q}(\Omega)$ the expression $a(u,v)$ is well defined and by Hölder's inequality the following estimate holds: There exists a constant $c>0$ such that for every $(u,v)\in W^{1,q}(\Omega)\times W^{1,q}(\Omega)$:*

$$|a(u,v)|\leqslant\alpha\left\|\varepsilon^D(u)\right\|_{L^q(\Omega)}^{q-1}\left\|\varepsilon^D(v)\right\|_{L^q(\Omega)}\leqslant c\left\|u\right\|_{W^{1,q}(\Omega)}^{q-1}\left\|v\right\|_{W^{1,q}(\Omega)}. \tag{13}$$

Thus we can associate to every fixed $u\in W^{1,q}(\Omega)$ a unique operator $A(u)\in(W^{1,q}(\Omega))'=W_0^{-1,p}(\Omega)$ such that for every $u,v\in W^{1,q}(\Omega)$:

$$\langle A(u),v\rangle_{(W_0^{-1,p}(\Omega),W^{1,q}(\Omega))}=a(u,v).$$

Consider now the mapping

$$A:W^{1,q}(\Omega)\to W_0^{-1,p}(\Omega): \ u\mapsto A(u).$$

The properties of this nonlinear operator are described in the next lemma.

Lemma 2.4. *Let p, q, n be as in Definition 2.3.*
$A: W^{1,q}(\Omega)\to W_0^{-1,p}(\Omega)$ is a continuous operator. There exists $c>0$ such that for every $u,v\in W^{1,q}(\Omega)$:

$$\langle A(u)-A(v),u-v\rangle_{(W_0^{-1,p}(\Omega),W^{1,q}(\Omega))}\geqslant c\int_\Omega G(\varepsilon(u)(x),\varepsilon(v)(x))\,\mathrm{d}x, \tag{14}$$

where we have set for $\varepsilon_1,\varepsilon_2\in\mathbb{R}^{d\times d}$:

$$G(\varepsilon_1,\varepsilon_2)=\begin{cases}(|\varepsilon_1|+|\varepsilon_2|)^{q-2}\left|\varepsilon_1-\varepsilon_2\right|^2 & \textit{if } (\varepsilon_1,\varepsilon_2)\neq(0,0),\\ 0 & \textit{else.}\end{cases}$$

Thus, A is a monotone operator on $W^{1,q}(\Omega)$. Note, that $\langle A(u)-A(v),u-v\rangle=0$ if and only if $\varepsilon(u)=\varepsilon(v)$.

Remark 2.3. The above defined function G is continuous on $\mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}$.

Proof. The continuity of A is a direct consequence of the continuity of the operator $W^{1,q}(\Omega)\to L^q(\Omega)$: $u\mapsto\left|\varepsilon^D(u)\right|^{q-2}\varepsilon^D(u)$, which follows with the help of [29, Prop. 26.6] where the continuity for a class of Nemickij-operators is shown.

Inequality (14) can be derived by a pointwise application of the following inequality, see also Lemma A.1:
For every $1 < q < 2$, $s \in \mathbb{N}$ exists $c > 0$ such that for every $x, y \in \mathbb{R}^s$ with $(x, y) \neq (0, 0)$:

$$(|x|^{q-2}\, x - |y|^{q-2}\, y) \cdot (x - y) \geqslant c(|x| + |y|)^{q-2}\, |x - y|^2 .$$

$\square$

Obviously, the form $b(\cdot,\cdot) : L^p(\Omega) \times V_q \to \mathbb{R}$ is a continuous bilinear form, i.e. there exists a constant $c > 0$ such that for all $\pi \in L^p(\Omega)$ and $u \in V_q$ there holds:

$$|b(\pi, u)| \leqslant c\, \|\pi\|_{L^p(\Omega)}\, \|u\|_{W^{1,q}(\Omega)} .$$

Thus we can associate in a unique way the following linear and continuous operators with $b(\cdot,\cdot)$:

$$B : L^p(\Omega) \to V_q',\ \pi \mapsto B(\pi) = -\int_\Omega \pi \mathrm{div}\,(\cdot)\, \mathrm{d}x, \tag{15}$$

$$B^* : V_q \to L^q(\Omega),\ u \mapsto -\mathrm{div}\, u. \tag{16}$$

B and B^* are connected via the relation

$$\langle B^*(u), \pi\rangle_{((L^p(\Omega))', L^p(\Omega))} = -\int_\Omega \pi \mathrm{div}\, u\, \mathrm{d}x = \langle B(\pi), u\rangle_{(V_q', V_q)} \quad \text{for every } u \in V_q, \pi \in L^p(\Omega).$$

For $1 < q, p < \infty, q = p'$ the spaces V_q and $L^p(\Omega)$ are reflexive, thus the operators B and B^* are adjoint. In appendix B, the mapping properties of the operators B and B^* are investigated in detail.

We now reformulate equations (9)-(10) in terms of the the operators A, B, B^*:
Let $\Gamma_D \subset \partial\Omega$ be open, p, q, n as in Definition 2.3. Let further $f \in V_q'$, $h \in (\tilde{W}^{1-\frac{1}{q},q}(\Gamma_N))'$, $g \in W^{1-\frac{1}{q},q}(\Gamma_D)$. By Theorem 2.1 and remark 2.2 we have $V_q(g) = g_0 + V_q(0)$, where $g_0 \in W^{1,q}(\Omega)$ with $g_0|_{\Gamma_D} = g$.
We define

$$\tilde{A} : V_q \to V_q' : u \mapsto \tilde{A}(u) := A(g_0 + u).$$

Furthermore, we can associate to f and h an element $F \in V_q'$ in a unique way. With these notations, the weak formulation (9)-(10) is equivalent to the following problem:
Find $(\tilde{u}, \pi) \in V_q \times L^p(\Omega)$ such that

$$\tilde{A}(\tilde{u}) + B\pi = F \qquad \text{in } V_q', \tag{17}$$

$$B^*\tilde{u} = \mathrm{div}\, g_0 \quad \text{in } L^q(\Omega). \tag{18}$$

There holds: (u_0, π) is a solution of (9)-(10) if and only if $(\tilde{u}, \pi) = (u_0 - g_0, \pi)$ is a solution of (17)-(18).
We are now ready to state the main theorem of this section:

Theorem 2.4 (Existence and uniqueness of weak solutions). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where Γ_D and Γ_N are open and disjoint. Let further be $n \geqslant 1, p = n + 1$, $q = p' = 1 + \frac{1}{n}$. We assume that $f \in V_q'$, $h \in (\tilde{W}^{1-\frac{1}{q},q}(\Gamma_N))'$, $g \in W^{1-\frac{1}{q},q}(\Gamma_D)$.*

1. *Assume that* $\operatorname{mes}\Gamma_D > 0$. *If* $\Gamma_D = \partial\Omega$, *we further assume that the Dirichlet-data* g *satisfies the following solvability condition:*

$$\text{there axists } g_0 \in W^{1,q}(\Omega) \text{ such that } g_0|_{\partial\Omega} = g \text{ and } \int_\Omega \operatorname{div} g_0 \,\mathrm{d}x = 0. \tag{19}$$

Then there exists a pair $(u,\pi) \in V_q(g) \times L^p(\Omega)$ *which is solution of the weak problem* (9)-(10). u *is unique,* π *is unique if* $\Gamma_D \neq \partial\Omega$. *If* $\Gamma_D = \partial\Omega$ *then* π *is unique up to a constant. The solvability condition* (19) *is necessary and sufficient.*

2. *Assume that* $\Gamma_D = \emptyset$. *In this case we have a pure Neumann problem and* $V_q = W^{1,q}(\Omega)$. *We assume further that the data* f, h *satisfy the following solvability condition:* *For every* $v \in \ker(\varepsilon) \subset W^{1,q}(\Omega)$

$$\langle f, v\rangle_{(W_0^{-1,p}(\Omega), W^{1,q}(\Omega))} + \langle h, v\rangle_{((W^{1-\frac{1}{q},q}(\partial\Omega))', W^{1-\frac{1}{q},q}(\partial\Omega))} = 0. \tag{20}$$

Then there exists a pair $(u,\pi) \in W^{1,q}(\Omega) \times L^p(\Omega)$ *which solves the weak formulation. Furthermore,* u *is unique up to the addition of elements in* $\ker(\varepsilon)$, π *is unique. The solvability condition* (20) *is necessary and sufficient.*

Note, that $\ker(\varepsilon)$ is the finite dimensional space of rigid motions.

Remark 2.4. Condition (19) is equivalent to: $g \in W^{1-\frac{1}{q},q}(\partial\Omega)$ with $\int_{\partial\Omega} g\vec{n}\,\mathrm{d}s = 0$. Note, that this condition is well known in the case $n = 1$, i.e. in the case of the Stokes system with pure Dirichlet conditions.

Proof. We first prove the theorem for the case $\operatorname{mes}\Gamma_D > 0$. Here we make use of formulation (17)-(18). In the proof we apply Lemma 2.4 and Theorem B.1 where we collected some properties of the operators A, B, B^* and Lemma C.1 on the solvability of nonlinear saddlepoint problems.
By Lemma 2.4, $\tilde{A} : V_q \to V_q'$ is continuous and strongly monotone, $B : L^p(\Omega) \to V_q'$ is continuous and linear, $B^* : V_q \to L^q(\Omega)$ is the adjoint of B and by Theorem B.1, $\operatorname{Im} B^*$ and therefore also $\operatorname{Im} B$ are closed. Furthermore, it follows by Theorem B.1 and the solvability condition that $\operatorname{div} g_0 \in \operatorname{Im} B^*$. In order to apply Lemma C.1 to our equation we only have to verify the coercitivity of $\tilde{A}$ on $M := \{v \in V_q : B^*v = \operatorname{div} g_0\}$.
Let $(u_n)_{n\in\mathbb{N}} \subset M$ with $\|u_n\|_{W^{1,q}(\Omega)} \to \infty$ as $n \to \infty$. Then

$$\langle \tilde{A}(u_n), u_n\rangle = \alpha \left\|\varepsilon^D(u_n)\right\|^q_{L^q(\Omega)}.$$

By Lemma 2.3, there exists $c > 0$ such that for every $u \in M$:

$$\begin{aligned}\|u\|_{W^{1,q}(\Omega)} &\leqslant c\left(\left\|\varepsilon^D(u)\right\|_{L^q(\Omega)} + \|\operatorname{div} u\|_{L^q(\Omega)}\right)\\ &= c\left(\left\|\varepsilon^D(u)\right\|_{L^q(\Omega)} + \|\operatorname{div} g_0\|_{L^q(\Omega)}\right)\end{aligned}$$

and therefore $\|u_n\|_{W^{1,q}(\Omega)} \to \infty$ if and only if $\left\|\varepsilon^D(u_n)\right\|_{L^q(\Omega)} \to \infty$. Thus

$$\frac{\langle \tilde{A}(u_n), u_n\rangle}{\|u_n\|_{W^{1,q}(\Omega)}} \geqslant \tilde{c}\frac{\left\|\varepsilon^D(u_n)\right\|^q_{L^q(\Omega)}}{\|\varepsilon^D(u_n)\|_{L^q(\Omega)} + \|\operatorname{div} g_0\|_{L^q(\Omega)}} \to \infty \text{ as } n \to \infty,$$

which shows the coercitivity of $\tilde{A}$ on M. The first part of the theorem follows by Lemma C.1.

Now let $\Gamma_D = \emptyset$. To prove the assertions for that case we also would like to apply Lemma C.1. Since we cannot prove the coercitivity of A on $\ker B^* \subset W^{1,q}(\Omega)$ we have to split our problem. We decompose $W^{1,q}(\Omega)$ into two closed subspaces and solve the problem only on one of these subspaces. In a second step we show that the solution we found there is already a solution for the whole problem.
Since $\ker \varepsilon \subset W^{1,q}(\Omega)$ is a finite dimensional subspace there exists a closed subspace $V \subset W^{1,q}(\Omega)$ such that

$$W^{1,q}(\Omega) = \ker(\varepsilon) \oplus V,$$

see [2, Satz 7.16]. Again by Lemma 2.4, $A : V \to V'$ is continuous and strongly monotone. Furthermore we can prove as before with Lemma 2.3 the coercitivity of A on $\ker B^* \cap V$. Consider now the following problem: Find $(u, \pi) \in V \times L^p(\Omega)$ such that for all $v \in V, r \in L^p(\Omega)$:

$$\langle Au, v\rangle_{(V',V)} + \langle B\pi, v\rangle_{(V',V)} = \langle f, v\rangle + \langle h, v\rangle, \tag{21}$$

$$\langle B^* u, r\rangle_{(L^q(\Omega),L^p(\Omega))} = 0. \tag{22}$$

Before we can apply Lemma C.1 to this problem we have to check that Im B is closed in V' for $B : L^p(\Omega) \to V'$, or, what is equivalent, Im B^* is closed in $L^q(\Omega)$ for $B^* : V \to L^q(\Omega)$.
By the splitting of $W^{1,q}(\Omega) = \ker \varepsilon \oplus V$ we have the following representation for $u \in W^{1,q}(\Omega)$: $u = r + v$, where $r \in \ker \varepsilon$ and $v \in V$ are uniquely determined. Consider now $B^* u = B^* r + B^* v = -\operatorname{div} r - \operatorname{div} v$. Direct calculations show that for $r \in \ker \varepsilon$ there holds $\operatorname{div} r = 0$. Thus, $B^* u = B^* v$ and therefore by Theorem B.1: $B^*(V) = L^q(\Omega)$.
Lemma C.1 now implies that (21)-(22) has a solution $(u, \pi) \in V \times L^p(\Omega)$. Moreover, equations (21)-(22) are true not only for $v \in V$ but also for arbitrary $r \in \ker \varepsilon$. This is due to the solvability conditions on the data f and h, see equation (20). Thus (u, π) is a solution of problem (9)-(10).
The uniqueness properties follow by considerations which are similar to those in the proof of Lemma C.1. □

2.4 A minimization problem

In this section we show that one can also associate a minimization problem with constraint to (3)-(6) and describe how this minimization problem is related to the weak formulation.
For $n \geqslant 1, q = 1 + \frac{1}{n}$, $u \in W^{1,q}(\Omega)$, $f \in (W^{1,q}(\Omega))'$, $h \in (W^{1-\frac{1}{q},q}(\Gamma_N))'$ and $g \in W^{1-\frac{1}{q},q}(\Gamma_D)$ we set

$$I(u) := \int_\Omega \frac{\alpha n}{n+1} \left|\varepsilon^D(u)\right|^q \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x - \int_{\Gamma_N} h u \,\mathrm{d}s \tag{23}$$

and

$$\mathcal{M}_g := \{u \in V_q(g) : \operatorname{div} u = 0\}.$$

Definition 2.4 (Minimization Problem). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where Γ_D and Γ_N are disjoint open sets. Let further $n \geqslant 1$, $p = n + 1$, $q = p' = 1 + \frac{1}{n}$, $f \in (W^{1,q}(\Omega))'$, $h \in (W^{1-\frac{1}{q},q}(\Gamma_N))'$ and $g \in W^{1-\frac{1}{q},q}(\Gamma_D)$.*
The minimization problem is the following:

$$\text{Find } u \in \mathcal{M}_g \text{ such that } I(u) \leqslant I(v) \text{ for all } v \in \mathcal{M}_g. \tag{24}$$

Theorem 2.5 (Existence of minimizers). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where Γ_D and Γ_N are disjoint open sets. Let further $n \geqslant 1$, $p = n+1$, $q = p' = 1 + \frac{1}{n}$, $f \in (W^{1,q}(\Omega))'$, $h \in (W^{1-\frac{1}{q},q}(\Gamma_N))'$ and $g \in W^{1-\frac{1}{q},q}(\Gamma_D)$. In the case $\Gamma_D = \emptyset$ we further require that solvability condition* (20) *is satisfied, in the case $\Gamma_D = \partial\Omega$ we require that condition* (19) *is fulfilled.*
Then there exists a solution $u \in \mathcal{M}_g$ of problem (24)*. In the case $\Gamma_D \neq \emptyset$, u is unique and if $\Gamma_D = \emptyset$, then u is unique up to the addition of elements from* $\ker(\varepsilon)$.

Proof. The assertion follows by a standard argument for the minimization of functionals, see [30, Prop. 38.15]. To apply this Proposition we have to verify that $\mathcal{M}_g \subset W^{1,q}(\Omega)$ is convex and closed and that I is continuous, convex and coercive on $\mathcal{M}_g$, i.e. for any sequence $\{v_k, k \in \mathbb{N}\} \subset \mathcal{M}_g$ with $\|v_k\|_{W^{1,q}(\Omega)} \to \infty$ as $k \to \infty$ there holds $I(v_k) \to \infty$ as $k \to \infty$.
The continuity and convexity of I follow by considerations which are similar to those of Lemma 2.4. Theorem B.1 and the solvability condition for the case $\Gamma_D = \partial\Omega$ guarantee that $\mathcal{M}_g \neq \emptyset$.
If $\Gamma_D \neq \emptyset$, the coercitivity of I follows by the same arguments as in the first part of the proof of Theorem 2.4 which yields the assertion.
In the case $\Gamma_D = \emptyset$, we have $\mathcal{M} = \{u \in W^{1,q}(\Omega) : \operatorname{div} u = 0\}$. Note, that $\ker\varepsilon \subset \mathcal{M}$ is a finite dimensional subspace and therefore we can split $\mathcal{M} = \ker\varepsilon \oplus V$, where $V \subset \mathcal{M}$ is a closed subspace. The restriction of I to V is coercive and therefore there exists a solution for problem (24) with $\mathcal{M}$ replaced by V. By the solvability condition this solution is also a minimizer of the whole problem. □

The next theorem shows that equations (9)-(10) describe the weak Euler-Lagrange equations for the minimization problem (24). The pressure π appears as a Lagrange-parameter.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where Γ_D and Γ_N are disjoint open sets. Let further be $n \geqslant 1$, $p = n+1$, $q = p' = 1 + \frac{1}{n}$, $f \in (W^{1,q}(\Omega))'$, $h \in (W^{1-\frac{1}{q},q}(\Gamma_N))'$ and $g \in W^{1-\frac{1}{q},q}(\Gamma_D)$. We assume that $u_0 \in \mathcal{M}_g \subset V_q(g)$ is a solution of the minimization problem* (24).
Then there exists a function $\pi \in L^p(\Omega)$ such that the pair (u_0, π) satisfies equations (9)-(10).

Proof. The proof consists in applying a variant of Ljusternik's Theorem, Theorem D.1, to the minimization problem. In our case, we have to verify the following conditions:

a.) $I : V_q(g) \to \mathbb{R}$ is Fréchet-differentiable,

b.) the constraint operator $\operatorname{div} : V_q(g) \to L^q(\Omega)$ has a closed image.

The Gâteaux-differentiability of I can be proved with arguments which are similar to those in [6, Appendix A.8] and we get as Gâteaux-derivative:

$$DI : V_q(g) \to V_q' : u \to DI(u)[\,\cdot\,] = \int_\Omega \alpha \left|\varepsilon^D(u)\right|^{q-2} \varepsilon^D(u) : \varepsilon^D(\cdot)\, \mathrm{d}x.$$

The continuity of the Gâteaux-derivative $DI : V_q(g) \to V_q' : u \to DI(u)$ with respect to u follows by Lemma 2.4. Therefore, I is Fréchet-differentiable and a.) is proved.
The properties of the div-operator are discussed in Theorem B.1. Now, Theorem D.1 yields the assertion. □

3 Interior regularity of weak solutions

Before we state and prove higher interior regularity results for the velocity, stress and pressure fields, we recall the definitions of σ and σ^D:

$$\begin{aligned}\sigma^D &= \alpha \left|\varepsilon^D(u)\right|^{q-2} \varepsilon^D(u), \text{ where } \varepsilon^D(u) = \varepsilon(u) - \frac{1}{d}\operatorname{tr}(\varepsilon(u))I,\\ \sigma &= -\pi I + \sigma^D.\end{aligned}$$

Note, that $\varepsilon^D(u) = \varepsilon(u)$ since $\operatorname{tr}\varepsilon(u) = \operatorname{div} u = 0$.

Theorem 3.1 (Interior regularity). *Let $n \geqslant 1, p = n+1, q = p' = 1 + \frac{1}{n}$ and $f \in L^p(\Omega)$. For a weak solution $(u, \pi) \in W^{1,q}(\Omega) \times L^p(\Omega)$ there holds for every $\delta > 0$, $\epsilon > 0$:*

$$u \in W^{2,\tau}_{loc}(\Omega), \;\; \sigma^D \in W^{q-1-\delta,\frac{\tau}{q-1}}_{loc}(\Omega), \;\; \pi \in W^{q-1-\delta,p}_{loc}(\Omega), \tag{25}$$

where

$$\tau = \begin{cases} 2-\epsilon & \text{if } d = 2,\\ \frac{3q}{1+q} & \text{if } d = 3.\end{cases} \tag{26}$$

Note, that $1 \leqslant q \leqslant \tau \leqslant 2$ and $p \leqslant \frac{\tau}{q-1}$. Furthermore, $\pi \in W^{q-1-\delta,\frac{\tau}{q-1}}_{loc}(\Omega)$ if $f \in L^{\frac{\tau}{q-1}}(\Omega)$.

Remark 3.1. If we choose $n = 1$, then $p = q = 2$ and equations (9)-(10) reduce to the linear Stokes system. By Theorem 3.1 we get $u \in W^{2,2-\epsilon}_{\text{loc}}(\Omega)$ which is (up to ϵ) exactly the well known result for linear elliptic equations, see e.g. [28].
For $d = 3$ and $n \to \infty$ the regularity of u is decreasing as n grows: $\frac{3q}{1+q} = \frac{3n+3}{2n+1} \searrow \frac{3}{2}$.

Remark 3.2. In the case $d = 2$ the result coincides with a result for the $p-$Laplacian:
Let $\Omega \subset \mathbb{R}^2$ be a convex domain, $1 < q \leqslant 2$, $p = q'$, $f \in L^p(\Omega)$ and assume that $u \in W^{1,q}_0(\Omega)$ satisfies

$$\int_\Omega |\nabla u|^{q-2} \nabla u \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \text{ for every } v \in W^{1,q}_0(\Omega),$$

then $u \in W^{2,2}(\Omega)$, [3].

Proof (of Theorem 3.1). The result for the velocity field u was proved by Naumann in [22] for the three dimensional case. In a first step, he applied a difference quotient technique, which is based on Tolksdorf's ideas, [25], in order to prove $u \in W^{2,q}_{\text{loc}}(\Omega)$. He used essentially the monotonicity properties of the nonlinear differential operator, see also Lemma 2.4. The arguments of the first step are independent of the dimension of the domain Ω. In a second step he derived with the help of embedding theorems for Sobolev-Slobodeckij-spaces the better result $u \in W^{2,\tau}_{\text{loc}}(\Omega)$, τ as in (26). Since the dimension of Ω plays a role in the embedding theorems, the quantity τ in (26) depends on d. We remark, that $\tau = \frac{dq}{d+q-2}$ if $d \geqslant 3$.

Regularity of σ^D:
For the proof of the regularity of $\sigma^D = \alpha \left|\varepsilon^D(u)\right|^{q-2} \varepsilon^D(u)$ we follow the ideas in [11]. Let $\Omega' \subset \subset \Omega'' \subset\subset \Omega$ be open subsets of Ω with smooth boundaries, $h_0 := \min\{\operatorname{dist}(\Omega', \partial\Omega''), \operatorname{dist}(\Omega'', \partial\Omega)\}$.
Let further be

$$\tau = \begin{cases} 2-\epsilon & \text{if } d = 2,\\ \frac{3q}{1+q} & \text{if } d = 3.\end{cases}$$

Since $u \in W^{2,\tau}_{\text{loc}}(\Omega)$ we have $\varepsilon(u) \in W^{1,\tau}_{\text{loc}}(\Omega)$ and therefore for $h \in \mathbb{R}^d$ with $0 < |h| < h_0$:

$$\int_{\Omega'} \left|\sigma^D(x+h) - \sigma^D(x)\right|^{\frac{\tau}{q-1}} \,\mathrm{d}x \overset{(61)}{\leqslant} c \int_{\Omega'} \left|\varepsilon^D(u(x+h)) - \varepsilon^D(u(x))\right|^{\tau} \,\mathrm{d}x$$
$$\overset{[17,\text{Lemma 7.23}]}{\leqslant} c\,|h|^{\tau} \left\|\nabla\varepsilon^D(u)\right\|^{\tau}_{L^{\tau}(\Omega'')}.$$

Thus, with $\Omega'_\delta = \{x \in \Omega' : \operatorname{dist}(x, \partial\Omega') > \delta\}$

$$\sup_{\substack{\delta>0\\0<|h|<\delta}} \int_{\Omega'_\delta} \left|\frac{\sigma^D(x+h) - \sigma^D(x)}{|h|^{q-1}}\right|^{\frac{\tau}{q-1}} \,\mathrm{d}x < \infty \tag{27}$$

and therefore $\sigma^D \in \mathcal{N}^{q-1,\frac{\tau}{q-1}}(\Omega')$, see also Definition 2.2. Now, the assertion follows with Lemma 2.1.

Regularity of π:
We follow again the ideas from [11].
Let $P \in \Omega$ and choose $R' > 0$ such that with $\Omega' := B_{R'}(P)$, $\Omega'' := B_{2R'}(P)$ there holds $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Let further $h_0 = \frac{1}{2}\min\{R', \operatorname{dist}(\partial\Omega'', \partial\Omega)\}$, $\delta < h_0$ and $\Omega'_\delta = \{x \in \Omega' : \operatorname{dist}(x, \partial\Omega') > \delta\}$. For any $h \in \mathbb{R}^d$ with $0 < |h| < h_0$ we get from equation (9) in the distributional sense:

$$\triangle_h \nabla\pi = \triangle_h f + \triangle_h \operatorname{div}\sigma^D, \tag{28}$$

where $\triangle_h u = u(x+h) - u(x)$. Next, we estimate the right hand side of this equation in the $W^{-1,p}(\Omega')$-norm:

$$\begin{aligned}
\|\triangle_h f\|_{W^{-1,p}(\Omega')} &= \sup_{\substack{\psi\in W^{1,q}_0(\Omega')\\ \|\psi\|_{W^{1,q}(\Omega')}=1}} \left|\int_{\Omega'} \triangle_h f\psi \,\mathrm{d}x\right| \\
&= \sup_{\substack{\psi\in W^{1,q}_0(\Omega')\\ \|\psi\|_{W^{1,q}(\Omega')}=1}} \left|\int_{\Omega'+h} f(x)\psi(x-h)\,\mathrm{d}x - \int_{\Omega'} f(x)\psi(x)\,\mathrm{d}x\right| \\
&= \sup_{\substack{\psi\in W^{1,q}_0(\Omega')\\ \|\psi\|_{W^{1,q}(\Omega')}=1}} \left|\int_{\Omega'\cup\Omega'+h} f(x)(\triangle_{-h}\psi(x))\,\mathrm{d}x\right| \\
&\leqslant \|f\|_{L^p(\Omega'\cup\Omega'+h)} \sup_{\substack{\psi\in W^{1,q}_0(\Omega')\\ \|\psi\|_{W^{1,q}(\Omega')}=1}} \|\triangle_{-h}\psi\|_{L^q(\Omega'\cup\Omega'+h)}.
\end{aligned} \tag{29}$$

Here we have set $\Omega' + h = \{x \in \mathbb{R}^d : x = y + h,\, y \in \Omega'\} \subset \Omega$. We can extend $\psi \in W^{1,q}_0(\Omega')$ to $\psi \in W^{1,q}(\mathbb{R}^d)$ by setting $\psi(x) = 0$ for $x \notin \Omega'$. By [17, Lemma 7.23], we then get for arbitrary $\psi \in W^{1,q}_0(\Omega')$:

$$\|\triangle_h\psi\|_{L^q(\Omega'\cup\Omega'+h)} \leqslant |h|\,\|\nabla\psi\|_{L^q(\mathbb{R}^d)} = |h|\,\|\nabla\psi\|_{L^q(\Omega')}, \tag{30}$$

and therefore

$$\|\triangle_h f\|_{W^{-1,p}(\Omega')} \leqslant |h|\,\|f\|_{L^p(\Omega)}. \tag{31}$$

Furthermore, since $\sigma^D \in \mathcal{N}^{q-1,\frac{\tau}{q-1}}_{\mathrm{loc}}(\Omega) \subset \mathcal{N}^{q-1,p}_{\mathrm{loc}}(\Omega)$,

$$\begin{aligned}\left\|\triangle_h \operatorname{div} \sigma^D\right\|_{W^{-1,p}(\Omega')} &= \sup_{\substack{v\in\mathcal{C}_0^\infty(\Omega')\\ \|v\|_{W^{1,q}(\Omega')}=1}} \left|\int_{\Omega'} \triangle_h \sigma^D : \nabla v \, \mathrm{d}x\right| \\ &\leqslant \left\|\triangle_h \sigma^D\right\|_{L^p(\Omega')} \sup_{\substack{v\in\mathcal{C}_0^\infty(\Omega')\\ \|v\|_{W^{1,q}(\Omega')}=1}} \|\nabla v\|_{L^q(\Omega')} \leqslant |h|^{q-1} \left\|\sigma^D\right\|_{\mathcal{N}^{q-1,p}(\Omega'')} .\end{aligned} \tag{32}$$

Equation (28) and inequalities (31), (32) show that there is a constant $c > 0$ such that for every $h \in \mathbb{R}^d$ with $0 < |h| < h_0$

$$\left\|\nabla\left(\frac{\triangle_h \pi}{|h|^{q-1}}\right)\right\|_{W^{-1,p}(\Omega')} \leqslant c. \tag{33}$$

In addition, there exists $c > 0$, such that for every $h \in \mathbb{R}^d$ with $0 < |h| < h_0$

$$\left\|\frac{\triangle_h \pi}{|h|^{q-1}}\right\|_{W^{-1,p}(\Omega')} \leqslant c \tag{34}$$

By Nečas' Lemma, see Lemma B.2, we conclude

$$\left\|\frac{\triangle_h \pi}{|h|^{q-1}}\right\|_{L^p(\Omega')} \leqslant c \tag{35}$$

and therefore

$$\sup_{\substack{\delta>0\\ 0<|h|<\delta}} \int_{\Omega'_\delta} \left|\frac{\triangle_h \pi}{|h|^{q-1}}\right|^p \mathrm{d}x \leqslant c < \infty. \tag{36}$$

Thus, $\pi \in \mathcal{N}^{q-1,p}(\Omega')$ and the theorem is proved.
Note, that if we assume $f \in L^{\frac{\tau}{q-1}}(\Omega)$, we can derive for the pressure by the same arguments as above the regularity $\pi \in W^{q-1-\delta,\frac{\tau}{q-1}}_{\mathrm{loc}}(\Omega)$, which coincides with the regularity of σ^D. $\square$

4 Higher tangential regularity at plane parts of the boundary

One can also prove a higher regularity for derivatives which are tangential to a plane part of the Dirichlet- or Neumann-boundary.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $n \geqslant 1$, $p = n+1$, $q = p' = 1 + \frac{1}{n}$ and $f \in L^p(\Omega)$. We assume that the boundary conditions are given in the following special form:*

$$\sigma\vec{n} = H\vec{n} \quad \text{on } \Gamma_N,$$

where $H \in W^{1,p}(\Omega, \mathbb{R}^{d\times d})$ and $H = H^T$. Furthermore

$$u\big|_{\Gamma_D} = g\big|_{\Gamma_D} \quad \text{on } \Gamma_D$$

for a given $g \in W^{3,q}(\Omega)$.

Choose $\tilde{\Omega} \subset \Omega$ in such a way that $\mathcal{S} := \mathrm{int}(\partial\tilde{\Omega} \cap \partial\Omega)$ is an open subset of a hyper-plane L. We further require that there exists an open set $U \subset \partial\Omega$ with $\overline{\mathcal{S}} \subset U \subset L$ and that the boundary conditions do not change on U, see Figure 2.
Then there holds for every $\vec{t}$ which is tangential to L:

$$\partial_{\vec{t}} \nabla u \in L^q(\tilde{\Omega}).$$

Here, $\partial_{\vec{t}}$ denotes the derivative towards $\vec{t}$.

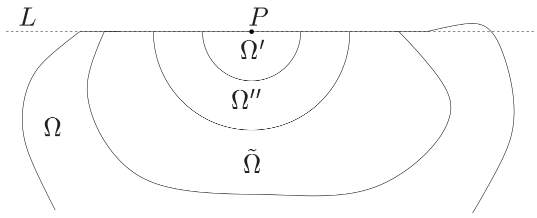

Figure 2: An example for the domain in Theorem 4.1

Proof. The proof is a modification of Naumann's proof in [22] for interior regularity and uses a difference quotient technique. We distinguish two cases according to the different boundary conditions.
Let $\tilde{\Omega} \subset \Omega$ be a domain as described in Theorem 4.1 and $P \in \mathcal{S} = \mathrm{int}(\partial\tilde{\Omega} \cap \partial\Omega)$. Choose $0 < R'$ in such a way that $\Omega' := (B_{R'}(P) \cap \Omega) \subset \tilde{\Omega}$ and $\Omega'' := (B_{2R'}(P) \cap \Omega) \subset \tilde{\Omega}$, see Figure 2.

1. Case, Neumann-conditions on $\partial\tilde{\Omega} \cap \partial\Omega$:
Due to the special structure of the Neumann data, equation (9) of the weak formulation is equivalent to

$$\int_\Omega \alpha \left|\varepsilon^D(u)\right|^{q-2} \varepsilon^D(u) : \varepsilon^D(v)\,\mathrm{d}x - \int_\Omega \pi \mathrm{div}\, v\,\mathrm{d}x = \int_\Omega (f + \mathrm{div}\, H)v\,\mathrm{d}x + \int_\Omega H : \varepsilon(v)\,\mathrm{d}x. \quad (37)$$

Let $(u, \pi) \in V_q(g) \times L^p(\Omega)$ be a weak solution. Choose $\varphi \in \mathcal{C}_0^\infty(B_{2R'}(P))$ with $\varphi\big|_{B_{R'}(P)} = 1$, $\vec{t}$ tangential to $\partial\tilde{\Omega} \cap \partial\Omega$ with $\left|\vec{t}\right| = 1$ and $h_0 := \frac{1}{2}\min\{R', \mathrm{dist}(\partial\Omega'', \partial\tilde{\Omega}\backslash\partial\Omega)\}$. Then for $h \in \mathbb{R}$ with $0 < |h| < h_0$, the function

$$\xi(x) = \varphi^2(x)\left(u(x + h\vec{t}) - u(x)\right) = \varphi^2(x)\triangle_h u(x)$$

as well as the function $\tilde{\xi}(x) := \xi(x - h\vec{t})$ are admissible test functions in $V_q(0)$. After inserting ξ and $\tilde{\xi}$ into the weak formulation, changing the variables of $\tilde{\xi}$ and subtracting the resulting equations, we obtain

$$\int_{\Omega''} \alpha\triangle_h \left(\left|\varepsilon^D(u)\right|^{q-2} \varepsilon^D(u)\right) : \varepsilon^D(\xi)\,\mathrm{d}x = \int_{\Omega''} \triangle_h\pi\,\mathrm{div}\,\xi\,\mathrm{d}x$$
$$+ \int_{\Omega''} \triangle_h\left(f + \mathrm{div}\, H\right)\xi\,\mathrm{d}x + \int_{\Omega''} \triangle_h H : \varepsilon(\xi)\,\mathrm{d}x. \quad (38)$$

In order to simplify the notation, we define $G(\varepsilon) := \alpha\left|\varepsilon\right|^{q-2}\varepsilon$. Note, that

$$\varepsilon^D(\varphi^2(\triangle_h u)) = ((\triangle_h u) \otimes \nabla\varphi^2)^D_{\mathrm{sym}} + \varphi^2\varepsilon^D(\triangle_h u), \qquad \mathrm{tr}\,\varepsilon(\varphi^2\triangle_h u) = \nabla\varphi^2 \cdot \triangle_h u,$$

here we have used that $\operatorname{div} u = 0$; $a \otimes b \in \mathbb{R}^{d\times d}$ denotes the tensor product of the vectors $a, b \in \mathbb{R}^d$ with $(a \otimes b)_{ij} = a_i b_j$; A_{sym} is the symmetric part of tensor A. Equation (38) can be transformed into

$$\begin{aligned}\int_{\Omega''} \varphi^2 (\triangle_h G(\varepsilon^D(u))) : \varepsilon^D(\triangle_h u)\,\mathrm{d}x = &-\int_{\Omega''} \triangle_h (G(\varepsilon^D(u)) + H) : (\triangle_h u \otimes \nabla\varphi^2)_{\mathrm{sym}}\,\mathrm{d}x \\ &+ \int_{\Omega''} (\triangle_h \pi) 2\varphi\nabla\varphi \cdot \triangle_h u\,\mathrm{d}x + \int_{\Omega''} \varphi^2 \triangle_h (f + \operatorname{div} H) \triangle_h u\,\mathrm{d}x \\ &+ \int_{\Omega''} \varphi^2 \triangle_h H : \triangle_h \varepsilon^D(u)\,\mathrm{d}x \\ &= I_1 + \cdots + I_4.\end{aligned} \tag{39}$$

The integrals on the right hand side can be estimated as follows: We set $V_q(\Omega'') := \{u \in W^{1,q}(\Omega'') : u\big|_{\partial\Omega''\setminus\partial\Omega} = 0\}$. Then

$$I_3 \leqslant \|\varphi\triangle_h (f + \operatorname{div} H)\|_{(V_q(\Omega''))'} \|\varphi\triangle_h u\|_{W^{1,q}(\Omega'')}$$

and by arguments, which are similar to those in (29), we obtain

$$\|\varphi\triangle_h (f + \operatorname{div} H)\|_{(V_q(\Omega''))'} \leqslant c\,|h|\,,$$

where the constant c is independent of h. The same considerations can be carried out for I_1 and I_2, which leads to

$$I_1 + I_2 + I_3 \leqslant c\,|h|\,\|\varphi\triangle_h u\|_{W^{1,q}(\Omega'')}$$

with a constant c which is independent of h. Using the product rule, Poincaré-Friedrichs inequality and Korn's inequality, see Theorems 2.2 and 2.3, and the fact, that $\operatorname{tr}\varepsilon(u) = 0$, we get:

$$\|\varphi\triangle_h u\|_{W^{1,q}(\Omega'')} \leqslant c\left(|h| + \left\|\varphi\triangle_h \varepsilon^D(u)\right\|_{L^q(\Omega'')}\right).$$

Furthermore, by Hölder's inequality and since $H \in W^{1,p}(\Omega)$, we have

$$I_4 \leqslant \|\varphi\triangle_h H\|_{L^p(\Omega'')} \left\|\varphi\triangle_h \varepsilon^D(u)\right\|_{L^q(\Omega'')} \leqslant c\,|h|\,\|H\|_{W^{1,p}(\Omega)} \left\|\varphi\triangle_h \varepsilon^D(u)\right\|_{L^q(\Omega'')}.$$

Equality (39) and the above estimates yield

$$\int_{\Omega''} \varphi^2 \left(\triangle_h G(\varepsilon^D(u))\right) : \triangle_h \varepsilon^D(u)\,\mathrm{d}x \leqslant c\,|h|^2 + c\,|h|\left\|\varphi\triangle_h \varepsilon^D(u)\right\|_{L^q(\Omega'')}. \tag{40}$$

From now on, the proof follows exactly the arguments in [22], one only has to estimate the left hand side of (40) from below: By Hölder's inequality we get for $M_h := \{x \in \Omega'' : \varepsilon(u(x)) =$

$\varepsilon(u(x+h))=0\}$ and $s=\frac{q(q-2)}{2}<0$:

$$\begin{aligned}
\left\|\varphi\triangle_h\varepsilon^D(u)\right\|_{L^q(\Omega'')} &= \Big(\int_{\Omega''\setminus M_h}\left(\left|\varepsilon^D(u(x))\right|+\left|\varepsilon^D(u(x+h))\right|\right)^{-s} \\
&\qquad\qquad \left(\left|\varepsilon^D(u(x))\right|+\left|\varepsilon^D(u(x+h))\right|\right)^{s}\left|\varphi\triangle_h\varepsilon^D(u)\right|^q\,\mathrm{d}x\Big)^{\frac{1}{q}} \\
&\leqslant \left\|\left|\varepsilon^D(u(\cdot))\right|+\left|\varepsilon^D(u(\cdot+h))\right|\right\|_{L^q(\Omega'')}^{\frac{(2-q)}{2}} \\
&\qquad\qquad \left(\int_{\Omega''}\varphi^2\left|\triangle_h\varepsilon^D(u)\right|^2\left(\left|\varepsilon^D(u(x))\right|+\left|\varepsilon^D(u(x+h))\right|\right)^{q-2}\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leqslant c_1\left(\int_{\Omega''}\varphi^2\left|\triangle_h\varepsilon^D(u)\right|^2\left(\left|\varepsilon^D(u(x))\right|+\left|\varepsilon^D(u(x+h))\right|\right)^{q-2}\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\overset{(58)}{\leqslant} c_2\left(\int_{\Omega''}\varphi^2\triangle_h(G(\varepsilon^D(u))):\triangle_h\varepsilon^D(u)\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\overset{(40),(63)}{\leqslant} c_3\left|h\right|+c_3\left|h\right|^{\frac{1}{2}}\left\|\varphi\triangle_h\varepsilon^D(u)\right\|_{L^q(\Omega'')}^{\frac{1}{2}},
\end{aligned}\tag{41}$$

where the constants c_i are independent of h. Since inequality (41) is true for every $0<|h|<h_0$, and since $\varphi\big|_{\Omega'}=1$ and $\operatorname{tr}\varepsilon(u)=0$, it follows

$$\sup_{0<|h|<h_0}\left\|\frac{\triangle_h\varepsilon(u)}{h}\right\|_{L^q(\Omega')}\leqslant c<\infty,\tag{42}$$

and thus $\partial_{\vec{t}}\varepsilon(u)\in L^q(\Omega')$. Finally we obtain with Korn's inequality that $\partial_{\vec{t}}\nabla u\in L^q(\Omega')$.

<u>2. Case, Dirichlet-conditions on $\partial\tilde{\Omega}\cap\partial\Omega$:</u> Let $(u,\pi)\in V_q(g)\times L^p(\Omega)$ be a weak solution. As before we choose $\varphi\in\mathcal{C}_0^\infty(B_{2R'}(P))$ with $\varphi\big|_{B_{R'}(P)}=1$ and $\vec{t}$ tangential to $\partial\tilde{\Omega}\cap\partial\Omega$ with $\left|\vec{t}\right|=1$. Let $h_0:=\frac{1}{2}\min\{R',\operatorname{dist}(\partial\Omega'',\partial\tilde{\Omega}\setminus\partial\Omega)\}$ and $g\in W^{3,q}(\Omega)$ with $u\big|_{\Gamma_D}=g\big|_{\Gamma_D}$. For $h\in\mathbb{R}$ with $0<|h|<h_0$, the functions

$$\xi(x):=\varphi^2((u(x+h\vec{t})-g(x+h\vec{t}))-(u(x)-g(x)))=\varphi^2(x)\triangle_h(u-g)(x)$$

and $\tilde{\xi}(x)=\xi(x-h\vec{t})$ are admissible test functions in $W_0^{1,q}(\Omega)$. After inserting these functions into the weak formulation (9), we obtain by some calculations

$$\begin{aligned}
\int_{\Omega''}\varphi^2\triangle_h G(\varepsilon^D(u)):\triangle_h\varepsilon^D(u)\,\mathrm{d}x &= -\int_{\Omega''}\triangle_h G(\varepsilon^D(u)):\left(\triangle_h(u-g)\otimes\nabla\varphi^2\right)_{\mathrm{sym}}^D\,\mathrm{d}x \\
&\quad+\int_{\Omega''}\varphi^2\triangle_h G(\varepsilon^D(u)):\triangle_h\varepsilon^D(g)\,\mathrm{d}x \\
&\quad-\int_{\Omega''}\varphi^2\triangle_h\pi\triangle_h\operatorname{div}g\,\mathrm{d}x \\
&\quad+\int_{\Omega''}\triangle_h\pi(\triangle_h(u-g))\cdot\nabla\varphi^2\,\mathrm{d}x \\
&\quad+\int_{\Omega''}\varphi^2\triangle_h f\triangle_h(u-g)\,\mathrm{d}x \\
&= I_1+\cdots+I_5.
\end{aligned}$$

The integrals I_1, I_4, I_5 can be estimated similar to the corresponding integrals in the Neumann case. For I_2 and I_3 we use the fact, that $g \in W^{3,q}(\Omega)$ in order to obtain

$$I_2 \leqslant c \left\| \triangle_h G(\varepsilon^D(u)) \right\|_{(V_q(\Omega''))'} \left\| \varphi \triangle_h \varepsilon^D(g) \right\|_{V_q(\Omega'')} \leqslant c \, |h|^2 \left\| G(\varepsilon^D(u)) \right\|_{L^p(\Omega)} \left\| \nabla \varepsilon^D(g) \right\|_{W^{1,q}(\Omega)}$$

and

$$I_3 \leqslant c \, |h|^2 \, \|\pi\|_{L^p(\Omega)} \, \|\nabla \mathrm{div}\,(g)\|_{W^{1,q}(\Omega)} \,.$$

This shows, that inequality (40) also holds in the case of Dirichlet conditions and we can proceed analogous to the case of Neumann conditions. $\square$

5 Global regularity of weak solutions

Global regularity for systems of nonlinear elliptic PDE on a class of polyhedral domains in the setting of Nikolskii- and Sobolev-Slobodeckii-spaces was first studied by C. Ebmeyer and J. Frehse in [10, 12, 13]. Later, they extended these results to the stationary Navier-Stokes equation,where they also study fluids with shear thinning viscosities, [14, 11]. Since the equation in our paper is a simplified version of the Navier-Stokes equation (no convection term), the results from [11] can be applied also to this equation and can be improved by using the ideas from [13]. This will be described in this section in detail.

5.1 Geometrical assumptions

In order to prove global regularity results, one needs some assumptions on the geometry. These assumptions arise mainly for technical reasons in the proof of global regularity, where one has to construct special extensions of the solutions across the boundary of the domain. Let us note that at least in the two dimensional case, the assumptions are optimal in comparison to those which one needs to prove similar results for linear elliptic equations.

In 2D we consider bounded Lipschitzian polygons, where the only restriction on the geometry is that if there are changing boundary conditions in a point $P \in \partial\Omega$, then the interior opening angle of the domain at P is less than π.
In the three-dimensional case we consider Lipschitzian polyhedrons where we require that at most three faces come together at points on the boundary where the boundary conditions change and that the interior angle between neighbored faces with different boundary conditions is less than π. At vertices, where the boundary conditions do not change, there is no restriction on the number of faces or the geometry.
More precisely we have the following assumptions on the geometry, see also [13]:

<u>Two-dimensional case:</u> $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitzian polygon with $\partial\Omega = \overline{\Gamma_D} \cap \overline{\Gamma_N}$, Γ_D, Γ_N open and disjoint, where Γ_D and Γ_N denote the Dirichlet- and Neumann-boundary, respectively. We further assume, that $\partial\Omega$ has the following structure:

$$\partial\Omega = \bigcup_{i=1}^{k} \overline{\Gamma_i},$$

where Γ_i are open subsets of straight lines, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$ and $\Gamma_i \subset \Gamma_D$ or $\Gamma_i \subset \Gamma_N$. If $\overline{\Gamma_i} \cap \overline{\Gamma_j} \neq \emptyset$ and $\Gamma_i \subset \Gamma_D$, $\Gamma_j \subset \Gamma_N$, then $\measuredangle(\Gamma_i, \Gamma_j) < \pi$ (here we consider the interior angle).

<u>Three-dimensional case:</u> $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitzian polyhedron with $\partial\Omega = \overline{\Gamma_D} \cap \overline{\Gamma_N}$, Γ_D, Γ_N open and disjoint. Furthermore $\partial\Omega = \cup_{i=1}^{k} \overline{\Gamma_i}$, where $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$ and $\Gamma_i \subset \Gamma_D$ or $\Gamma_i \subset \Gamma_N$. We assume that every Γ_i is an open subset of a suitable plane and has a polygonal Lipschitzian boundary. If $\Gamma_{i_1} \subset \Gamma_D$ and $\Gamma_{i_2} \subset \Gamma_N$ and $\overline{\Gamma_{i_1}} \cap \overline{\Gamma_{i_2}} \neq \emptyset$, then $\measuredangle(\Gamma_{i_1}, \Gamma_{i_2}) < \pi$, where we consider the interior opening angle. Finally, if $\Gamma_{i_1} \subset \Gamma_D$ and $\Gamma_{i_2} \subset \Gamma_N$, then $\overline{\Gamma_{i_1}} \cap \overline{\Gamma_{i_2}} \cap \overline{\Gamma_{i_3}} \cap \overline{\Gamma_{i_4}} = \emptyset$ for all $i_3 \neq i_4$ and $i_3, i_4 \notin \{i_1, i_2\}$.

5.2 Global regularity

Theorem 5.1 (Global Regularity). *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polyhedral domain which satisfies the geometrical assumptions introduced in section 5.1. Let further be $n \geqslant 1, p = n+1, q = 1 + \frac{1}{n}$, $f \in L^p(\Omega)$ and $s = \frac{2d}{2d-2+q} \geqslant 1$.*
We assume that the Dirichlet-data is given by a function $g \in W^{2,sq}(\hat{\Omega})$ with $\nabla g \in L^\infty(\hat{\Omega})$ for some domain $\hat{\Omega} \supset\supset \Omega$ and

$$u\big|_{\Gamma_D} = g\big|_{\Gamma_D}.$$

Furthermore, we assume that the Neumann-data is of the following form: There exists a function $H \in W^{1,p}(\Omega, \mathbb{R}^{d\times d}) \cap L^\infty(\Omega, \mathbb{R}^{d\times d})$, $H = H^T$, such that

$$\sigma\vec{n} = H\vec{n} \text{ on } \Gamma_N.$$

Then for a weak solution $(u, \pi) \in W^{1,q}(\Omega) \times L^p(\Omega)$ there holds for every $\delta > 0$:

$$\begin{aligned} u &\in W^{\frac{3}{2}-\delta,qs}(\Omega), \\ \sigma^D &\in W^{\frac{q-1}{2}-\delta,ps}(\Omega), \\ \pi &\in W^{\frac{q-1}{2}-\delta,p}(\Omega), \\ \pi &\in W^{\frac{q-1}{2}-\delta,ps}(\Omega) \text{ if } f \in L^{ps}(\Omega). \end{aligned}$$

Here, $\sigma^D = \alpha \left|\varepsilon^D(u)\right|^{q-2} \varepsilon^D(u)$.

Remark 5.1. If $d < p$ and $H \in W^{1,p}(\Omega)$, then the embedding theorems state that $H \in L^\infty(\Omega)$. Furthermore, one can choose any function $g \in W^{2,d}(\hat{\Omega})$ in order to describe the Dirichlet-data.

Remark 5.2. If we choose $n = 1$ in the previous theorem, i.e. the equations reduce to the linear Stokes-system, then Theorem 5.1 predicts $u \in W^{\frac{3}{2}-\delta,2}(\Omega)$. This result is well known (up to δ) from linear theory.

Proof. The proof is divided into several steps. First we cover Ω with a finite number of subdomains which can be considered as model problems. For each subdomain we then prove the result separately. Here we use the difference quotient technique developed by Ebmeyer/Frehse in [13, 10, 14]. We choose for each subdomain a suitable basis of $\mathbb{R}^d$ and show that u is contained in one of the "tilde"-Nikolskii-spaces introduced in section 2.2. Here, the geometrical assumptions on the domain play a crucial role since, e.g. in the case of pure Dirichlet conditions, we have to define extensions of u across the boundary such that functions of the type $\varphi^2(x)(u(x+h) - u(x))$ are admissible test functions.
We make use of the following notation: For $P \in \mathbb{R}^d$ and $R > 0$ we set $B_R(P) := \{x \in \mathbb{R}^d : |x - P| < R\}$.

1. Case: $P \in \partial\Omega$ and pure Dirichlet conditions in a neighborhood of P:

Let $\hat{\Omega}$ be the domain described in Theorem 5.1. Choose $P \in \partial\Omega$ such that there exists a neighborhood $U(P) \subset \hat{\Omega}$ with the following properties:

1. $\partial\Omega \cap U(P) \subset \Gamma_D$ and
2. if $P \notin \overline{\Gamma_i}$ then $\overline{\Gamma_i} \cap U(P) = \emptyset$.

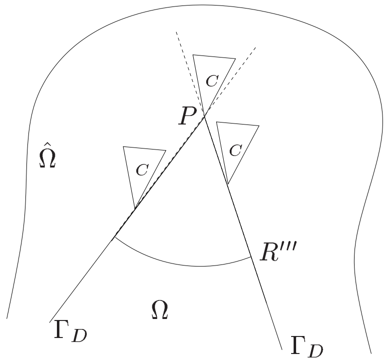

Figure 3: Example for the notation in Case 1

The second condition implies that there is is at most one vertex in $\Omega \cap U(P)$.
Since Ω is a bounded Lipschitzian domain, the uniform cone property holds, [28]. Therefore we can find a normed basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$ and numbers $R''' > R'' > R' > 0$ with the property that $B_{R'''}(P) \subset U(P)$ and that the cone

$$C := \left\{ x \in \mathbb{R}^d : \ x = h_0 \sum_{i=1}^{d} \lambda_i e_i, \ \ \lambda_i \geqslant 0, \sum_{i=1}^{d} \lambda_i \leqslant 1 \right\} \quad \text{with } h_0 := (R''' - R'')/2$$

satisfies for every $x_0 \in \partial\Omega \cap B_{R'''}(P)$:

$$(x_0 + C) \cap \Omega = \emptyset,$$

see also Figure 3.
Choose $\varphi \in \mathcal{C}_0^\infty(B_{R''}(P))$ with $\varphi\big|_{B_{R'}(P)} = 1$. For a weak solution $u \in W^{1,q}(\Omega)$ we define the following extension (and use the same symbol for the extended function):

$$u(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ g(x) & \text{if } x \in (\hat{\Omega} \backslash \Omega) \cap B_{R'''}(P). \end{cases}$$

Note, that the extended function is an element of $W^{1,q}(\Omega \cup B_{R'''}(P))$.
For $1 \leqslant i \leqslant d$, $0 < h < h_0$ and $x \in \Omega$ let

$$\xi_i(x) := \varphi^2(x)\left((u-g)(x + he_i) - (u-g)(x)\right) = \varphi^2(x) \triangle_h^i (u-g)(x).$$

The functions ξ_i are elements of $W^{1,q}(\Omega)$ with $\xi_i\big|_{\partial\Omega} = 0$. To see the second assertion, let be $x \in \partial\Omega \cap B_{R'''}(P) \subset \Gamma_D$. Then $x + he_i \in B_{R'''}(P) \backslash \Omega$ and therefore $u(x+he_i) - g(x+he_i) = 0$ as well as $u(x) - g(x) = 0$ due to the definition of the extension of u and to the Dirichlet-conditions on $\partial\Omega \cap B_{R'''}(P)$. On the remaining part of $\partial\Omega$, φ vanishes. Thus, for $1 \leqslant i \leqslant d$ and $0 < h < h_0$ the functions ξ_i are admissible test functions for the weak formulation.
Inserting ξ_i into the weak formulation (9) yields after some simple calculations where we use

that $\operatorname{div} u = tr\varepsilon(u) = 0$:

$$\int_\Omega \varphi^2 G(\varepsilon^D(u)) : \triangle_h^i \varepsilon^D(u) dx = \int_\Omega \varphi^2 G(\varepsilon^D(u)) : \triangle_h^i \varepsilon^D(g)\, \mathrm{d}x$$
$$- \int_\Omega G(\varepsilon^D(u)) : \left(\triangle_h^i (u-g) \otimes \nabla\varphi^2\right)^D_{\mathrm{sym}}\, \mathrm{d}x - \int_\Omega \varphi^2 \pi \triangle_h^i \operatorname{div} g\, \mathrm{d}x$$
$$+ \int_\Omega \pi \triangle_h^i (u-g) \cdot \nabla\varphi^2\, \mathrm{d}x + \int_\Omega \varphi^2 f \triangle_h^i (u-g)\, \mathrm{d}x. \tag{43}$$

For shortness we have set $G(\varepsilon) = \alpha |\varepsilon|^{q-2} \varepsilon$. By inequality (60) we obtain with a constant $c > 0$ which is independent of h:

$$c \int_\Omega \varphi^2 \left(\left|\varepsilon^D(u)\right| + \left|\varepsilon^D(u(x+he_i))\right| \right)^{q-2} \left|\triangle_h^i \varepsilon^D(u)\right|^2\, \mathrm{d}x$$
$$\overset{(60)}{\leqslant} - \int_\Omega \varphi^2 G(\varepsilon^D(u)) : \triangle_h^i \varepsilon^D(u)\, \mathrm{d}x + \int_\Omega \varphi^2 \triangle_h^i \left(\left|\varepsilon^D(u)\right|^q \right)\, \mathrm{d}x$$
$$\overset{(43)}{=} - \int_\Omega \varphi^2 G(\varepsilon^D(u)) : \triangle_h^i \varepsilon^D(g)\, \mathrm{d}x$$
$$+ \int_\Omega G(\varepsilon^D(u)) : \left(\triangle_h^i (u-g) \otimes \nabla\varphi^2\right)^D_{\mathrm{sym}}\, \mathrm{d}x + \int_\Omega \varphi^2 \pi \triangle_h^i \operatorname{div} g\, \mathrm{d}x$$
$$- \int_\Omega \pi \triangle_h^i (u-g) \cdot \nabla\varphi^2\, \mathrm{d}x - \int_\Omega \varphi^2 f \triangle_h^i (u-g)\, \mathrm{d}x$$
$$+ \int_\Omega \varphi^2 \triangle_h^i \left(\left|\varepsilon^D(u)\right|^q \right)\, \mathrm{d}x \tag{44}$$
$$=: I_1 + \ldots + I_6.$$

Next, we estimate the integrals $I_1, \ldots, I_6$. By Hölder's inequality and the fact that $u - g \in W^{1,q}(\Omega \cup B_{R'''}(P))$:

$$|I_5| \leqslant \|\varphi f\|_{L^p(\Omega)} \left\|\varphi \triangle_h^i (u-g)\right\|_{L^q(\Omega)} \leqslant ch \|\varphi f\|_{L^p(\Omega)} \|\nabla(u-g)\|_{L^q(\Omega \cup B_{R'''}(P))}.$$

Note, that $u = g$ on $B_{R'''}(P) \backslash \Omega$ and therefore $\|\nabla(u-g)\|_{L^q(\Omega \cup B_{R'''}(P))} = \|\nabla(u-g)\|_{L^q(\Omega)}$. Similarly

$$|I_2| \leqslant ch \left\|G(\varepsilon^D(u))\right\|_{L^p(\Omega)} \|\nabla(u-g)\|_{L^q(\Omega)},$$
$$|I_4| \leqslant ch \|\pi\|_{L^p(\Omega)} \|\nabla(u-g)\|_{L^q(\Omega)}.$$

Furthermore, since $g \in W^{2,qs}(\hat\Omega)$:

$$|I_1| \leqslant ch \left\|G(\varepsilon^D(u))\right\|_{L^p(\Omega)} \left\|\nabla \varepsilon^D(g)\right\|_{L^q(\Omega \cup B_{R'''}(P))},$$
$$|I_3| \leqslant ch \|\pi\|_{L^p(\Omega)} \|\nabla \operatorname{div} g\|_{L^q(\Omega \cup B_{R'''}(P))}.$$

In all these estimates, the constant c is independent of h. In order to estimate I_6, we use the relation $\triangle_h^i (fg) = (\triangle_h^i f) g + f(x + he_i) \triangle_h^i g$:

$$I_6 = \int_\Omega \triangle_h^i \left(\varphi^2 \left|\varepsilon^D(u)\right|^q \right)\, \mathrm{d}x - \int_\Omega (\triangle_h^i \varphi^2) \left|\varepsilon^D(u(x+he_i))\right|^q\, \mathrm{d}x =: I_{61} + I_{62}.$$

As before,
$$|I_{62}| \leqslant ch \left\| \varepsilon^D(u) \right\|_{L^q(\Omega)}^q \left\| \nabla\varphi^2 \right\|_{L^\infty(\Omega)}.$$
Taking into account the properties of the extension of u and the properties of $\operatorname{supp}\varphi$ we obtain after a change of variables:
$$I_{61} = \int_{\Omega+he_i} \varphi^2 \left|\varepsilon^D(u)\right|^q \,\mathrm{d}x - \int_\Omega \varphi^2 \left|\varepsilon^D(u)\right|^q \,\mathrm{d}x = \int_{(\Omega+he_i\backslash\Omega)\cap B_{R'''}(P)} \varphi^2 \left|\varepsilon^D(g)\right|^q \,\mathrm{d}x,$$
where $\Omega + he_i = \{x \in \mathbb{R}^d : x = y + he_i,\, y \in \Omega\}$. Due to the assumptions in Theorem 5.1 we have $\varepsilon^D(g) \in L^\infty(\hat{\Omega})$ and therefore
$$I_{61} \leqslant \left\| \varphi^2 \left|\varepsilon^D(g)\right|^q \right\|_{L^\infty((\Omega+he_i\backslash\Omega)\cap B_{R'''}(P))} \operatorname{mes}\left((\Omega + he_i\backslash\Omega) \cap B_{R'''}(p)\right) \leqslant ch,$$
where c is independent of h. Collecting all these estimates, we get with a constant c which is independent of h:
$$\int_\Omega \varphi^2 \left(\left|\varepsilon^D(u(x))\right| + \left|\varepsilon^D(u(x+he_i)\right|\right)^{q-2} \left|\triangle_h^i \varepsilon^D(u(x))\right|^2 \,\mathrm{d}x \leqslant ch. \tag{45}$$
Applying inequality (59) to the left hand side of equation (45) leads to
$$\int_\Omega \varphi^2 \left|\left|\varepsilon^D(u(x+he_i))\right|^{\frac{q}{2}} - \left|\varepsilon^D(u(x))\right|^{\frac{q}{2}}\right|^2 \,\mathrm{d}x \leqslant ch,$$
where c is independent of h. Since $\varphi(x) = 1$ for $x \in B_{R'}(P)$ we obtain for all $1 \leqslant i \leqslant d$:
$$\sup_{0<h<h_0} \left\| \frac{\triangle_h^i \left|\varepsilon^D(u)\right|^{\frac{q}{2}}}{h^{\frac{1}{2}}} \right\|_{L^2(\Omega\cap B_{R'}(P))} \leqslant c < \infty \tag{46}$$
and therefore by Lemma 2.2
$$|\varepsilon(u)|^{\frac{q}{2}} \in \mathcal{N}^{\frac{1}{2},2}(\Omega \cap B_{R'}(P)).$$
Here we have used $\operatorname{tr}\varepsilon(u) = \operatorname{div} u = 0$.
The embedding theorem of Nikolskii-spaces in Sobolev-Slobodeckii-spaces, see Lemma 2.1, yields
$$\forall\ \delta > 0 : \quad |\varepsilon(u)|^{\frac{q}{2}} \in W^{\frac{1}{2}-\delta,2}(\Omega \cap B_{R'}(P))$$
and finally by standard embedding theorems,
$$\varepsilon(u) \in L^{\frac{dq}{d-1}-\delta}(\Omega \cap B_{R'}(P)) \quad \text{for all } \delta > 0. \tag{47}$$
In the next step we prove the regularity result for a weak solution u and proceed analogous to [13]. As abbreviation we define $\Omega' := \Omega \cap B_{R'}(P)$.
By standard embedding theorems the space $W^{1,q}(\Omega)$ is continuously embedded in the space $L^{\frac{dq}{d-1}}(\Omega)$. Therefore, relation (47) and Korn's inequality, see Theorem 2.2, lead to $u \in W^{1,\frac{dq}{d-1}-\delta}(\Omega')$ for all $\delta > 0$. For arbitrary $\delta > 0$ let $\sigma := \frac{2dq}{2d-2+q} - \delta = qs - \delta$ with s from Theorem 5.1. For $1 < q < 2$ we have $1 < \sigma < 2$ (if δ is small enough) and

$\sigma < \frac{dq}{d-1}$ and therefore $u \in W^{1,\sigma}(\Omega')$. Furthermore the same is true for the extended function: $u \in W^{1,\sigma}(\Omega' \cup (B_{R'''}(P)\backslash\Omega))$. Thus for $0 < h < h_0$, $1 \leqslant i \leqslant d$ and $M_h = \{x \in \Omega' : \varepsilon^D(u(x+he_i)) = \varepsilon^D(u(x)) = 0\}$:

$$\begin{aligned}
\int_{\Omega'} \left| h^{-\frac{1}{2}} \triangle_h^i \nabla u \right|^\sigma \,\mathrm{d}x &\leqslant c \int_{\Omega'} \left| h^{-\frac{1}{2}} \triangle_h^i \varepsilon(u) \right|^\sigma \,\mathrm{d}x + c \int_{\Omega'} \left| h^{-\frac{1}{2}} \triangle_h^i u \right|^\sigma \,\mathrm{d}x \\
&\leqslant c \int_{\Omega' \backslash M_h} \left(\left|\varepsilon^D(u(x+he_i))\right| + \left|\varepsilon^D(u(x))\right| \right)^{-\frac{\sigma}{2}(q-2)} \\
&\qquad \left(\left|\varepsilon^D(u(x+he_i))\right| + \left|\varepsilon^D(u(x))\right| \right)^{\frac{\sigma}{2}(q-2)} \left| h^{-\frac{1}{2}} \triangle_h^i \varepsilon^D(u) \right|^\sigma \,\mathrm{d}x \\
&\quad + c h^{\frac{\sigma}{2}} \|u\|^\sigma_{W^{1,\sigma}(\Omega')} \\
&= I_5 + c h^{\frac{\sigma}{2}} \|u\|^\sigma_{W^{1,\sigma}(\Omega')} .
\end{aligned}$$

Here we have used $\operatorname{div} u = 0, u \in W^{1,\sigma}(\Omega')$ and Korn's inequality, see Theorem 2.2. By Hölder's inequality we further get

$$\begin{aligned}
|I_5| \leqslant & \left\| \left(\left|\varepsilon^D(u(\cdot + he_i))\right| + \left|\varepsilon^D(u)\right| \right) \right\|^{\frac{2}{2-\sigma}}_{L^{\frac{\sigma(2-q)}{2-\sigma}}(\Omega')} \\
& \left(\int_{\Omega \backslash M_h} \left(\left|\varepsilon^D(u(x+he_i))\right| + \left|\varepsilon^D(u(x))\right| \right)^{q-2} \left| h^{-\frac{1}{2}} \triangle_h^i \varepsilon^D(u) \right|^2 \,\mathrm{d}x \right)^{\frac{\sigma}{2}} .
\end{aligned} \tag{48}$$

By equation (45), the second term is bounded independently of h, furthermore, $\frac{\sigma(2-q)}{2-\sigma} < \frac{dq}{d-1}$ and thus the first term is bounded independently of h as well, see also (47). We finally obtain that there exists a constant $c > 0$ such that for all $0 < h < h_0$:

$$\int_{\Omega'} \left| h^{-\frac{1}{2}} \triangle_h^i \nabla u \right|^\sigma \,\mathrm{d}x \leqslant c \tag{49}$$

and therefore for every $\epsilon, \tilde{\epsilon} > 0$:

$$u \in \mathcal{N}^{\frac{3}{2},\sigma}(\Omega') \subset W^{\frac{3}{2}-\epsilon, qs-\delta}(\Omega') \subset W^{\frac{3}{2}-\tilde{\epsilon}, qs}(\Omega').$$

Here we have applied Lemma 2.1 for the first inclusion and the embedding theorems for Sobolev-Slobodeckii spaces for the second one. The regularity results for σ^D can be derived in the same way as in the proof of Theorem 3.1. For the case $f \in L^p(\Omega)$, the regularity of π can be shown as in the proof of Theorem 3.1. If $f \in L^{ps}(\Omega) \subset L^p(\Omega)$, then one can prove in a first step $\pi \in W^{\frac{q-1}{2}-\delta,p}(\Omega')$. This space is embedded in $L^{ps}(\Omega)$ and therefore one can achieve in a second step by applying the same arguments as in the proof of Theorem 3.1 the higher regularity $\pi \in W^{\frac{q-1}{2}-\delta,ps}(\Omega')$.

2. Case: $P \in \partial\Omega$ and pure Neumann conditions in a neighborhood of P:
Choose $P \in \partial\Omega$ such that there exists a neighborhood $U(P)$ with the following properties:

1. $\partial\Omega \cap U(P) \subset \Gamma_N$ and
2. if $P \notin \overline{\Gamma_i}$ then $\overline{\Gamma_i} \cap U(P) = \emptyset$.

Since Ω is a bounded Lipschitzian domain, the uniform cone property holds, [28]. Therefore we can choose a basis $e_1, \dots e_d$ of $\mathbb{R}^d$ with $|e_i| = 1$ and numbers $R''' > R'' > R' > 0$ in such a way that $B_{R'''}(P) \subset U(P)$ and that the cone

$$C := \left\{ x \in \mathbb{R}^d \, : \, x = h_0 \sum_{i=1}^{d} \lambda_i e_i, \;\; \lambda_i \geqslant 0, \sum_{i=1}^{d} \lambda_i \leqslant 1 \right\} \quad \text{with } h_0 := (R''' - R'')/2$$

satisfies for every $x_0 \in \overline{\Omega} \cap B_{R'''}(P)$:

$$x_0 + C \, \subset \overline{\Omega}.$$

Now let $\varphi \in \mathcal{C}_0^\infty(B_{R''}(P))$ with $\varphi|_{B_{R'}(P)} = 1$. For $1 \leqslant i \leqslant d$ and $0 < h < h_0$ we define

$$\xi_i(x) = \varphi^2(x) \left(u(x + he_i) - u(x) \right) = \varphi^2(x) \triangle_h^i u(x), \quad x \in \Omega.$$

Note, that ξ_i is well defined and that no extension of u across the boundaries is needed for the definition. Furthermore, $\xi_i \in W^{1,q}(\Omega)$ and $\xi_i|_{\Gamma_D} = 0$, which shows that ξ_i is an admissible test function.
Due to the special structure of the Neumann data, the weak formulation (9) is equivalent to:

$$\forall v \in V_q(0) : \int_\Omega G(\varepsilon^D(u)) : \varepsilon^D(v) \, \mathrm{d}x = \int_\Omega \pi \mathrm{div}\, v \, \mathrm{d}x + \int_\Omega v(f + \mathrm{div}\, H) \, \mathrm{d}x + \int_\Omega H : \varepsilon(v) \, \mathrm{d}x,$$

where H is described in Theorem 5.1. We now choose $v = \xi_i$ and proceed similar to the case of pure Dirichlet-conditions: Inserting ξ_i into the weak formulation and using inequality (60) we get:

$$\begin{aligned} c \int_{\Omega \setminus M_h} \varphi^2 \left(\left| \varepsilon^D(u(x \,+\, h)) \right| + \left| \varepsilon^D(u(x)) \right| \right)^{q-2} \left| \triangle_h^i \varepsilon^D(u) \right|^2 \, \mathrm{d}x & \\ \leqslant \int_\Omega \varphi^2 \left(\triangle_h^i \left| \varepsilon^D(u(x)) \right|^q \right) \mathrm{d}x + \int_\Omega G(\varepsilon^D(u)) : \left(\triangle_h^i u \otimes \nabla \varphi^2 \right)_{\mathrm{sym}}^D \mathrm{d}x & \\ - \int_\Omega \pi \nabla \varphi^2(x) \triangle_h^i u(x) dx - \int_\Omega \varphi^2 (f + \mathrm{div}\, H) \triangle_h^i u \, \mathrm{d}x & \\ - \int_\Omega H : \varepsilon(\varphi^2 \triangle_h u) \, \mathrm{d}x & \\ =: I_1 + I_2 + I_3 + I_4 + I_5. & \qquad (50) \end{aligned}$$

The integrals I_2, I_3, I_4 can be treated as in the Dirichlet problem:

$$|I_2| + |I_3| + |I_4| \leqslant ch,$$

where $c > 0$ is independent of h. By the product rule for finite differences

$$I_1 = \int_\Omega \triangle_h^i \left(\varphi^2 \left| \varepsilon^D(u) \right|^q \right) \mathrm{d}x - \int_\Omega \left| \varepsilon^D(u(x + he_i)) \right|^q \triangle_h^i \varphi^2 \, \mathrm{d}x = I_{11} + I_{12}. \qquad (51)$$

As before

$$|I_{12}| \leqslant ch \left\| \nabla \varphi \right\|_\infty \left\| \varepsilon^D(u) \right\|_{L^q(\Omega)}^q.$$

By a change of variables

$$I_{11} = -\int_{\Omega\setminus(\Omega+he_i)} \varphi^2 \left|\varepsilon^D(u)\right|^q \,\mathrm{d}x,$$

where $\Omega + he_i = \{x \in \mathbb{R}^d : x = y + he_i,\, y \in \Omega\}$. For I_5 we use again the product rule and the fact that $\operatorname{div} u = 0$:

$$\begin{aligned} I_5 = &-\int_\Omega H : \left(\triangle_h^i u \otimes \nabla\varphi^2\right)_{\mathrm{sym}} \,\mathrm{d}x + \int_\Omega \triangle_h^i \left(\varphi^2 H\right) : \varepsilon^D(u(x+he_i))\,\mathrm{d}x \\ &-\int_\Omega \triangle_h^i \left(\varphi^2 H : \varepsilon^D(u)\right) \mathrm{d}x \\ = &\; I_{51} + I_{52} + I_{53}. \end{aligned} \tag{52}$$

By the usual arguments

$$\begin{aligned} |I_{51}| &\leqslant ch \|H\|_{L^p(\Omega)} \|\nabla u\|_{L^q(\Omega)}, \\ |I_{52}| &\leqslant ch \|H\|_{W^{1,p}(\Omega)} \left\|\varepsilon^D(u)\right\|_{L^q(\Omega)}. \end{aligned}$$

Furthermore by Hölder's and Young's inequality and since $H \in L^\infty(\Omega)$ we get for all $\delta > 0$:

$$\begin{aligned} |I_{53}| &= \left|\int_{\Omega\setminus\Omega+he_i} \varphi^2 H : \varepsilon^D(u)\,\mathrm{d}x\right| \\ &\leqslant \delta^{-1} \left\|\varphi^{\frac{2}{p}} H\right\|_{L^p(\Omega\setminus\Omega+he_i)} \delta \left\|\varphi^{\frac{2}{q}} \varepsilon^D(u)\right\|_{L^q(\Omega\setminus\Omega+he_i)} \\ &\leqslant c_0\delta^{-p} \int_{\Omega\setminus\Omega+he_i} \varphi^2 |H|^p \,\mathrm{d}x + c_0\delta^q \int_{\Omega\setminus\Omega+he_i} \varphi^2 \left|\varepsilon^D(u)\right|^q \,\mathrm{d}x \\ &\leqslant c_0\delta^{-p} \left\|\varphi^2 |H|^p\right\|_{L^\infty(\Omega)} |\Omega\setminus(\Omega+he_i)| + c_0\delta^q \int_{\Omega\setminus(\Omega+he_i)} \varphi^2 \left|\varepsilon^D(u)\right|^q \,\mathrm{d}x \\ &\leqslant \delta^{-p} c_1 h + c_0\delta^q \int_{\Omega\setminus(\Omega+he_i)} \varphi^2 \left|\varepsilon^D(u)\right|^q \,\mathrm{d}x. \end{aligned} \tag{53}$$

Here, c_0, c_1 are independent of h. We now choose $\delta = c_0^{-\frac{1}{q}}$ and obtain

$$I_{11} + I_{53} \leqslant c_1^{1-\frac{1}{q}} h + 0 = ch, \tag{54}$$

where c is independent of h. Collecting all estimates yields: There exists $c > 0$ such that for all $0 < h < h_0$:

$$\int_{\Omega\setminus M_h} \varphi^2 \left(\left|\varepsilon^D(u(x+he_i))\right| + \left|\varepsilon^D(u(x))\right|\right)^{q-2} \left|\triangle_h^i \varepsilon^D(u(x))\right|^2 \,\mathrm{d}x \leqslant ch.$$

The remaining part of the proof for the Neumann-boundary is completely analogous to the considerations in the Dirichlet-case, see (45) and below.

3. Case: $P \in \partial\Omega$ and mixed conditions in a neighborhood of P:
We remind that d denotes the dimension of the domain Ω. We consider a vertex $P \in \partial\Omega$ and a neighborhood $U(P)$ with the following properties:

1. There exist $i_1 < \ldots < i_d$ such that $P \in \overline{\Gamma_{i_j}}$ for $1 \leqslant j \leqslant d$, and $\Gamma_{i_1} \subset \Gamma_D, \Gamma_{i_d} \subset \Gamma_N$.
2. If $P \notin \overline{\Gamma_i}$ then $\overline{\Gamma_i} \cap U(P) = \emptyset$.

As in the case of pure Dirichlet or pure Neumann conditions, we have to find a suitable basis of $\mathbb{R}^d$ for which we can prove an estimate like in (49).
If $d = 2$ choose $e_1 \parallel \Gamma_{i_1} \subset \Gamma_D$ and $e_2 \parallel \Gamma_{i_2} \subset \Gamma_N$ with the following orientation: There exists $R > 0$, such that $P + he_1 \notin \overline{\Omega}$ for every $0 < h < R$ and $P + he_2 \in \Gamma_N$ for $0 < h < R$.
In the three dimensional case we assume $P \in \overline{\Gamma_{i_1}} \cap \overline{\Gamma_{i_2}} \cap \overline{\Gamma_{i_3}}$ where $\Gamma_{i_1} \subset \Gamma_D$ and $\Gamma_{i_3} \subset \Gamma_N$. Choose $e_1 \parallel \overline{\Gamma_{i_1}} \cap \overline{\Gamma_{i_2}}$, $e_2 \parallel \overline{\Gamma_{i_2}} \cap \overline{\Gamma_{i_3}}$ and $e_3 \parallel \overline{\Gamma_{i_3}} \cap \overline{\Gamma_{i_1}}$ and assume that the vectors e_i are oriented in such a way that for a suitable $R > 0$ there holds:

1. Case: Let $\Gamma_{i_2} \subset \Gamma_D$. Then for all $0 < h < R$: $P + he_1 \in \overline{\Gamma_{i_1}} \cap \overline{\Gamma_{i_2}}$ and

> **a)** If $\measuredangle(\Gamma_{i_1}, \Gamma_{i_2}) < \pi \Longrightarrow P + he_2 \notin \overline{\Gamma}_{i_2} \cap \overline{\Gamma_{i_3}}$, $P + he_3 \notin \overline{\Gamma}_{i_3} \cap \overline{\Gamma_{i_1}}$.
> **b)** If $\measuredangle(\Gamma_{i_1}, \Gamma_{i_2}) > \pi \Longrightarrow P + he_2 \in \overline{\Gamma}_{i_2} \cap \overline{\Gamma_{i_3}}$, $P + he_3 \in \overline{\Gamma}_{i_3} \cap \overline{\Gamma_{i_1}}$.

2. Case: Let $\Gamma_{i_2} \subset \Gamma_N$. Then for all $0 < h < R$: $P + he_2 \notin \overline{\Gamma_{i_2}} \cap \overline{\Gamma_{i_3}}$ and

> **a)** If $\measuredangle(\Gamma_{i_2}, \Gamma_{i_3}) < \pi \Longrightarrow P + he_1 \in \overline{\Gamma}_{i_1} \cap \overline{\Gamma_{i_2}}$, $P + he_3 \in \overline{\Gamma}_{i_3} \cap \overline{\Gamma_{i_1}}$.
> **b)** If $\measuredangle(\Gamma_{i_2}, \Gamma_{i_3}) > \pi \Longrightarrow P + he_1 \notin \overline{\Gamma}_{i_1} \cap \overline{\Gamma_{i_2}}$, $P + he_3 \notin \overline{\Gamma}_{i_3} \cap \overline{\Gamma_{i_1}}$.

Due to the geometric assumptions described in section 5.1, it is always possible to find such a basis. It follows that every e_i satisfies either **(P1)** or **(P2)**, where

(P1) For every $x_0 \in \partial\Omega \cap B_{R/2}(P)$ there holds: $x_0 + he_i \in \overline{\Omega}$ for $0 < h < \frac{R}{2}$.

(P2) For every $x_0 \in \partial\Omega \cap B_{R/2}(P)$ there holds: $x_0 + he_i \notin \Omega$ for $0 < h < \frac{R}{2}$.

Note, that in the threedimensional case, **(P1)** is satisfied in case 1 for e_1 and in case 2 for e_1 and e_3; **(P2)** is satisfied in case 1 for e_2 and e_3 and in case 2 for e_2.
Now choose $R''' = \frac{1}{2}R, R'' = \frac{1}{3}R, R' = \frac{R}{6}$, $h_0 = \frac{R}{6}$, $\varphi \in \mathcal{C}_0^\infty(B_{R''}(P))$ with $\varphi|_{B_{R'}(P)} = 1$ and assume that $(u, \pi) \in W^{1,q}(\Omega) \times L^p(\Omega)$ is a weak solution. For $1 \leqslant i \leqslant d$ we define the following test functions:
Assume that e_i satisfies **(P1)**. Then

$$\xi_i(x) := \varphi^2(x)\,(u(x + he_i) - u(x)) = \varphi^2(x)\triangle_h^i u(x) \quad \text{for } x \in \Omega.$$

Note, that $\xi_i \in W^{1,q}(\Omega)$ with $\xi_i|_{\Gamma_D} = 0$, and therefore ξ_i is an admissible test function. Note also, that we do not need any extension of u across the boundary in this case.
If e_i satisfies **(P2)**, let be

$$\mathcal{M}_i := \{x \in \mathbb{R}^d : \ x = x_0 + he_i, 0 \leqslant h < h_0, x_0 \in \partial\Omega \cap B_{R'''}(P)\} \setminus \overline{\Omega}.$$

We define the following extension of u on $\Omega \cup \overline{\mathcal{M}_i}$ across the boundary $(\partial\Omega \cap \partial\mathcal{M}_i)$ which is a subset of Γ_D:

$$u(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ g(x) & \text{if } x \in \overline{\mathcal{M}_i}. \end{cases}$$

The extended function is an element of $W^{1,q}(\Omega \cup \overline{\mathcal{M}_i})$. We set

$$\xi_i(x) := \varphi^2(x)\,((u(x + he_i) - g(x + he_i)) - (u(x) - g(x))) = \varphi^2(x)\triangle_h^i(u - g)(x) \quad \text{for } x \in \Omega.$$

There holds $\xi_i \in W^{1,q}(\Omega)$ with $\xi_i|_{\Gamma_D} = 0$, and therefore ξ_i is an admissible test function. We now proceed analogous to the cases of pure Dirichlet or pure Neumann conditions on $\partial\Omega \cap U(P)$: Inserting the test function into the weak formulation (9) results either in (50) if e_i satisfies **(P1)** or in (44) if e_i satisfies **(P2)**. By the same arguments as subsequent to (50) and (44), respectively, we finally obtain that $u \in \mathcal{N}^{\frac{3}{2},s}(\Omega \cap B_{R'}(P))$, where s is the number in Theorem 5.1, and that the corresponding results for σ^D and π hold also.

To prove the global regularity result in Theorem 5.1 we cover $\overline{\Omega}$ with a finite number of open balls B_l, where for every l, $\Omega \cap B_l$ fits in one of the above cases or is completely contained in Ω. The regularity results now are valid not only for each $\Omega \cap B_l$ but also for the whole domain Ω. $\square$

A Some essential inequalities

We collect some basic inequalities which all deal with the following function: Let $1 < q$,

$$F : \mathbb{R}^s \to \mathbb{R} : x \to |x|^q .$$

F is continuously differentiable with

$$DF(x) = \begin{cases} q\,|x|^{q-2}\,x & \text{if } x \neq 0, \\ 0 & \text{else.} \end{cases} \tag{55}$$

Moreover $D^2F(x) = q(q-2)\,|x|^{q-4}\,x \otimes x + q\,|x|^{q-2}\,I \quad$ if $x \neq 0$. Here, $a \otimes b \in \mathbb{R}^{d\times d}$ denotes the tensor product of the vectors $a, b \in \mathbb{R}^d$ with $(a \otimes b)_{ij} = a_i b_j$.

Lemma A.1. *Let F be the function from above, $1 < q < \infty$. Then there exists a constant $c > 0$ such that*

$$\forall x \in \mathbb{R}^s\backslash\{0\} : \qquad \left|D^2F(x)\right| \leqslant c\,|x|^{q-2}\,, \tag{56}$$

$$\forall x \in \mathbb{R}^s\backslash\{0\}, \forall \xi \in \mathbb{R}^s : \qquad \left(D^2F(x)\xi\right)\cdot \xi \geqslant c\,|x|^{q-2}\,|\xi|^2\,, \tag{57}$$

$$\forall x, y \in \mathbb{R}^s, (x,y) \neq (0,0) : \quad \left(|x|^{q-2}\,x - |y|^{q-2}\,y\right)\cdot (x-y) \geqslant c\,(|x|+|y|)^{q-2}\,|x-y|^2\,, \tag{58}$$

$$\forall x, y \in \mathbb{R}^s, 0 < \alpha : \qquad \left||x|^\alpha - |y|^\alpha\right| \leqslant c\,(|x|+|y|)^{\alpha-1}\,|x-y|\,, \tag{59}$$

and there exist constants $c_1, c_2 > 0$ such that for all $x, y \in \mathbb{R}^s$, $(x,y) \neq (0,0)$:

$$|y|^q - |x|^q \geqslant c_1\,|x|^{q-2}\,x\cdot(y-x) + c_2\,(|x|+|y|)^{q-2}\,|y-x|^2\,. \tag{60}$$

For $1 < q \leqslant 2$ there exists $c > 0$ such that for all $x, y \in \mathbb{R}^s$:

$$\left||x|^{q-2}\,x - |y|^{q-2}\,y\right| \leqslant c\,|x-y|^{q-1}\,. \tag{61}$$

For $2 \leqslant q$ there exists $c > 0$ such that for all $x, y \in \mathbb{R}^s$:

$$\left||x|^{q-2}\,x - |y|^{q-2}\,y\right| \leqslant c\,(|x|+|y|)^{q-2}\,|x-y|\,. \tag{62}$$

Note, that c, c_1, c_2 may depend on q, α, s.
For $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ with $a_i \geqslant 0$, $1 \leqslant i \leqslant n$, we have [19]:

$$\left(\sum_{i=1}^n a_i\right)^\alpha \leqslant n^{\alpha-1}\left(\sum_{i=1}^n a_i^\alpha\right) \qquad \text{if } \alpha \geqslant 1, \tag{63}$$

$$\left(\sum_{i=1}^n a_i\right)^\alpha \geqslant n^{\alpha-1}\left(\sum_{i=1}^n a_i^\alpha\right) \qquad \text{if } 0 \leqslant \alpha \leqslant 1. \tag{64}$$

Proof. Inequalities (56), (57) follow by direct calculations. Inequalities (58) and (62) can be found in [25, Lemma 2.3] in a more general setting. Inequality (61) is proved in [11, formula (4.29) and below]. Inequality (60) is based on Clarkson's inequality and can be found in

[9, 21]. Finally we prove inequality (59).
1. Case, $\alpha > 1$: Let $x, y \in \mathbb{R}^s$ with $|x| > |y| \geqslant 0$. Then

$$
\begin{aligned}
|x|^\alpha - |y|^\alpha &= \alpha \int_0^1 |y + t(x-y)|^{\alpha-2} (y + t(x-y))(x-y) \mathrm{d}t \\
&\leqslant \alpha \int_0^1 |(1-t)|y| + t|x||^{\alpha-1} \mathrm{d}t \, |x-y| \\
&\leqslant \alpha \left(\int_0^{\frac{1}{2}} (1-t)^{\alpha-1} \mathrm{d}t + \int_{\frac{1}{2}}^1 t^{\alpha-1} \mathrm{d}t \right) (|x| + |y|)^{\alpha-1} |x-y| \\
&= (2 - 2^{1-\alpha}) (|x| + |y|)^{\alpha-1} |x-y| .
\end{aligned}
$$

2. Case, $0 < \alpha < 1$: Let $x, y \in \mathbb{R}^s$ with $|x| > |y| \geqslant 0$. Then

$$
0 \leqslant (|x|^\alpha - |y|^\alpha)(|x| + |y|) \leqslant |x|^{\alpha+1} - |y|^{\alpha+1} \overset{\text{1. case}}{\leqslant} c(|x| + |y|)^\alpha |x-y| .
$$

□

Note, that for $1 < q < 2$ the function $x \to |x|^{q-2} x$ can be continuously extended to $x = 0$ by 0.

B Properties of the div operator

In this section we collect and prove some properties of the div operator which are difficult to find in literature. The main tools for the proof of the main theorem are Peetre's Lemma, Nečas' Lemma and an embedding theorem for L^p into Sobolev-spaces of negative order. The proof of the main theorem follows exactly the ideas of the proof of Theorem 3 in [16], but there only Lipschitz domains with $\Gamma_D = \partial\Omega$ are considered.
Throughout the whole section we assume:
$\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz-boundary, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where Γ_D and Γ_N are open and disjoint. We first cite some essential lemmata:

Lemma B.1 (Peetre's Lemma). *[7] Let E_0, E_1, E_2 be Banach spaces, let A_1 and A_2 be two continuous linear mappings, respectively from E_0 to E_1 and from E_0 to E_2, with*

i) A_2 is a compact mapping;

ii) there exists a constant $c > 0$ such that:

$$
\|v\|_{E_0} \leqslant c \left(\|A_1 v\|_{E_1} + \|A_2 v\|_{E_2} \right) \quad \text{for all } v \in E_0. \tag{65}
$$

Then

i) $\ker A_1$ has finite dimension and $\operatorname{Im} A_1$ is closed;

ii) there exists a constant $c_0 > 0$ such that:

$$
\inf_{w \in \ker A_1} \|v + w\|_{E_0} \leqslant c_0 \|A_1 v\|_{E_1} .
$$

For $1 < p < \infty$ we define the following norm for $\pi \in L^p(\Omega, \mathbb{R})$ with $q = p' = \frac{p}{p-1}$:

$$
\begin{aligned}
|||\pi|||_p &:= \|\pi\|_{W^{-1,p}(\Omega,\mathbb{R})} + \|\nabla\pi\|_{W^{-1,p}(\Omega,\mathbb{R}^d)} \\
&= \sup_{\substack{v \in W_0^{1,q}(\Omega,\mathbb{R}) \\ \|v\|_{W^{1,q}(\Omega)}=1}} \left| \int_\Omega \pi v \, \mathrm{d}x \right| + \sup_{\substack{w \in W_0^{1,q}(\Omega,\mathbb{R}^d) \\ \|w\|_{W^{1,q}(\Omega)}=1}} \left| \int_\Omega \pi \operatorname{div} w \, \mathrm{d}x \right|
\end{aligned}
$$

Lemma B.2 (Nečas' Lemma). *[5] Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz-boundary. Then $|||\cdot|||_p$ is a norm on $L^p(\Omega)$ which is equivalent to the usual norm on $L^p(\Omega)$.*

Lemma B.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz-boundary, $1 < p < \infty, q = p'$. Then the embedding $L^p(\Omega) \to W^{-1,p}(\Omega) = (W_0^{1,q}(\Omega))'$ is compact.*

Proof. The adjoint operator to $id_1 : L^p(\Omega) \to W^{-1,p}(\Omega)$ is given by $id_2 : W_0^{1,q}(\Omega) \to L^q(\Omega)$. The Sobolev-embedding theorems state that the embedding $W_0^{1,q}(\Omega) \to L^q(\Omega)$ is compact. By Schauder's Theorem [27, Satz III.4.4, p.111] this is also true for the adjoint operator. □

We are now ready to state the main theorem of this section:

Theorem B.1 (Properties of the div operator). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz-boundary, $1 < p < \infty$ and $q = p' = \frac{p}{p-1}$. Let further $\Gamma_D \subset \partial\Omega$ be open and $V_q := \{u \in W^{1,q}(\Omega, \mathbb{R}^d) : u|_{\Gamma_D} = 0\}$. Consider the mapping $\operatorname{div} : V_q \to L^q(\Omega)$, $u \mapsto \operatorname{div} u$.*

1. *The adjoint operator of div is given by the operator $B : L^p(\Omega) \to V_q' : \pi \mapsto \int_\Omega \pi \operatorname{div}(\cdot)\, \mathrm{d}x$. If $\Gamma_D = \partial\Omega$, then $B(\pi) = -\nabla\pi$ in the distributional sense.*

2. *The image of div is closed in $L^q(\Omega)$. More exactly,*

$$\begin{aligned} &\operatorname{Im}(\operatorname{div}) = \left\{ r \in L^q(\Omega) : \int_\Omega r \, \mathrm{d}x = 0 \right\} && \text{if } \Gamma_D = \partial\Omega, \\ &\operatorname{Im}(\operatorname{div}) = L^q(\Omega) && \text{else.} \end{aligned}$$

3. *There exists $c > 0$ such that for all $\pi \in L^p(\Omega)$: $\|\pi\|_{L^p(\Omega)/\mathbb{R}} \leqslant c \|\nabla\pi\|_{W^{-1,p}(\Omega)}$.*

4. *The kernel of B (= adjoint operator of div) has the following structure:*

$$\begin{aligned} &\ker(B) = \{ \text{ constant functions } \} && \text{if } \Gamma_D = \partial\Omega, \\ &\ker(B) = \{0\} && \text{else.} \end{aligned}$$

Proof. The first assertion follows by direct calculations. By the closed image theorem [27, p.143] the following is true: $\operatorname{Im}(\operatorname{div})$ is closed if and only if $\operatorname{Im}(B)$ is closed. Therefore we prove that $\operatorname{Im}(B)$ is closed. For this we apply Peetre's Lemma to $E_0 = L^p(\Omega)$, $E_1 = V_q'$, $E_2 = W^{-1,p}(\Omega)$, $A_1 : L^p(\Omega) \to V_q', \pi \to A_1(\pi) = B(\pi)$ and $A_2 : L^p(\Omega) \to W^{-1,p}(\Omega), \pi \to \pi$. The compactness of A_2 follows by Lemma B.3 and we only have to verify inequality (65). For every $\pi \in L^p(\Omega)$ there holds $\nabla\pi \in W^{-1,p}(\Omega)$ with

$$\begin{aligned} \|\nabla\pi\|_{W^{-1,p}(\Omega)} &= \sup_{\substack{v \in W_0^{1,q}(\Omega) \\ \|v\|_{W^{1,q}(\Omega)}=1}} \left| \int_\Omega \pi \operatorname{div} v \, \mathrm{d}x \right| \\ &\leqslant \sup_{\substack{v \in V_q \\ \|v\|_{W^{1,q}(\Omega)}=1}} \left| \int_\Omega \pi \operatorname{div} v \, \mathrm{d}x \right| = \|B(\pi)\|_{V_q'} . \end{aligned}$$

Thus

$$\|\pi\|_{L^p(\Omega)} \overset{\text{Lemma B.2}}{\leqslant} c(\|\pi\|_{W^{-1,p}(\Omega)} + \|\nabla\pi\|_{W^{-1,p}(\Omega)}) \leqslant c(\|B(\pi)\|_{V_q'} + \|\pi\|_{W^{-1,p}(\Omega)}).$$

Therefore we can apply Peetre's Lemma and get the second and third assertion. To get the exact description of $\mathrm{Im}\,(\mathrm{div}\,)$ we first calculate $\ker(B)$:
For an arbitrary Dirichlet boundary we get from $B(\pi) = 0$ by testing with functions in $\mathcal{C}_0^\infty(\Omega)$ that $\nabla\pi = 0$ in the distributional sense and therefore (Ω is connected) $\pi = const.$ If $\mathrm{mes}(\partial\Omega\backslash\Gamma_D) \neq 0$ we may further conclude by testing with $\varphi \in V_q$: $0 = \int_\Omega \pi \mathrm{div}\,\varphi\,\mathrm{d}x = \pi\int_\Omega \mathrm{div}\,\varphi\,\mathrm{d}x = -\pi\int_{\partial\Omega\backslash\Gamma_D} \varphi\vec{n}\,\mathrm{d}s$ and therefore $\pi = 0$ in Ω. This leads to assertion 4. The remaining part of the second assertion follows by the following equality (theorem of the closed image [27, p.143])

$$\mathrm{Im}\,(\mathrm{div}\,) = \left\{ r \in L^q(\Omega) : \int_\Omega \pi r\,\mathrm{d}x = 0 \quad \text{for all } \pi \in L^p(\Omega) \text{ with } B(\pi) = 0 \right\}.$$

□

C An abstract theorem on nonlinear saddle point problems

Let V, W be reflexive, separable Banach spaces, V', W' their duals. Consider the following operators

$$\begin{aligned} A : V \to V' &\quad \text{continuous and monotone,} \\ B : W \to V' &\quad \text{linear and continuous,} \\ B^* : V \to W' &\quad \text{adjoint operator to } B. \end{aligned}$$

We want to solve the following problem: For given $f \in V', g \in W'$ find $(u, \pi) \in V \times W$ for which

$$A(u) + B\pi = f, \tag{66}$$

$$B^* u = g. \tag{67}$$

Lemma C.1. *Let V, W be reflexive, separable Banach spaces, $A : V \to V'$ continuous and monotone, $B : W \to V'$ linear and continuous and $B^* : V \to W'$ the adjoint operator of B. Let further be $f \in V'$, $g \in \mathrm{Im}\, B^* \subset W'$. If*

a) A is coercive on $M_g := \{v \in V : B^* v = g\}$, i.e. if $\{u_n, n \in \mathbb{N}\} \subset M_g$ with $\|u_n\| \to \infty$ as $n \to \infty$, then $\frac{\langle Au_n, u_n\rangle}{\|u_n\|_V} \to \infty$,

b) $\mathrm{Im}\,(B)$ is closed in V',

then there exists a pair $(u, \pi) \in V \times W$ which solves (66)-(67). Moreover, if A is strongly monotone, then u is unique and π is unique up to the addition of elements from $\ker B$.

Proof. Existence: In a first step we prove the lemma with $g = 0$:
Let $f \in V'$. We set $V_0 := \ker B^*$. Since $V_0 \subset V$, the converse relation holds for the duals and thus $f \in V_0'$. We now solve the following problem:
Find $u \in V_0$ such that $Au = f$ is satisfied in V_0', that means: Find $u \in V_0$ such that

$$\forall\, v \in V_0 : \ \langle Au - f, v\rangle_{(V_0', V_0)} = 0.$$

By the main theorem on monotone operators [29] this equation has a solution $u \in V_0 = \ker B^*$. Next we solve the following equation in W: Find $\pi \in W$ such that $B\pi = f - Au$ in V', that means: Find $\pi \in W$ such that

$$\forall\ v \in V : \langle \pi, v\rangle_{(V', V)} = \langle f - Au, v\rangle_{(V', V)}. \tag{68}$$

Note, that $u \in V_0 \subset V$ and therefore, by the mapping properties of A, $Au \in V'$ and not only in V_0'. Obviously problem (68) has a solution if and only if $f - Au \in \mathrm{Im}\,(B)$. Since Im B is closed, we have the following characterization of Im B, [27]:

$$\mathrm{Im}\,(B) = \left\{ v \in V' : \langle v, w \rangle_{(V',V)} = 0 \quad \text{for all } w \in \ker B^* \right\}.$$

Since $f - Au \in V'$ and since for any $w \in \ker B^* = V_0$ we have $\langle f - Au, w \rangle_{(V_0',V_0)} = 0$ we conclude that $f - Au \in \mathrm{Im}\,(B)$. Thus, the pair (u, π) solves the equations (66)-(67) with $g = 0$.

Now let $f \in V'$ and $g \in$ Im B^* be arbitrary. Since B^* is linear, there exists $u_0 \in V$ such that $M_g = u_0 + \ker B^*$. For $w \in V$ we set $G(w) := A(u_0 + w)$. Then problem (66)-(67) is equivalent to the following: Find $w \in V, \pi \in W$ such that

$$G(w) + B\pi = f, \tag{69}$$

$$B^* w = 0. \tag{70}$$

From the assumptions on operator A we deduce that G is continuous, (strongly) monotone and coercive on $\ker B^*$. Thus, we can apply the results from the first step to (69)-(70).

Uniqueness: Assume now that A is strongly monotone and that (u_1, π_1), $(u_2, \pi_2) \in V \times W$ are solutions of (66)-(67) with the same right hand side f. Then $u_1 - u_2 \in V$ and we get from equations (66),(67):

$$\langle Au_1, u_1 - u_2 \rangle + \langle B\pi_1, u_1 - u_2 \rangle = \langle f, u_1 - u_2 \rangle, \tag{71}$$

$$\langle Au_2, u_1 - u_2 \rangle + \langle B\pi_2, u_1 - u_2 \rangle = \langle f, u_1 - u_2 \rangle, \tag{72}$$

$$\langle B^* u_1, \pi_1 - \pi_2 \rangle = \langle g, \pi_1 - \pi_2 \rangle, \tag{73}$$

$$\langle B^* u_2, \pi_1 - \pi_2 \rangle = \langle g, \pi_1 - \pi_2 \rangle. \tag{74}$$

Subtracting (71) and (72) resp. (73) and (74) and using that B^* is the adjoint of B we obtain

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle = 0$$

and by the strong monotonicity of A: $u_1 - u_2 = 0$.
Now we assume that $(u, \pi_1), (u, \pi_2)$ are two solutions of (66)-(67) with the same right hand sides. Testing the equations with an arbitrary $v \in V$ we obtain:

$$\langle Au, v \rangle + \langle B\pi_1, v \rangle = \langle f, v \rangle,$$

$$\langle Au, v \rangle + \langle B\pi_2, v \rangle = \langle f, v \rangle.$$

Subtracting these equations we get for every $v \in V$: $\langle B(\pi_1 - \pi_2), v \rangle_{(V',V)} = 0$ and therefore $\pi_1 - \pi_2 \in \ker B$. $\square$

D Variant of Ljusternik's Theorem

In this section we give a simplified variant of Ljusternik's Theorem, see e.g. [30, Thm. 43.D, Prop. 43.19].

Theorem D.1. *Let X,Y be real Banach spaces. We assume, that*

(1) $F : U(u_0) \subseteq X \to \mathbb{R}$ *is Fréchet-differentiable with Fréchet-derivative* DF*,*

(2) $G : U(u_0) \subseteq X \to Y$ *is of the form* $G(u) = G_0 u + f$*, where* $G_0 : X \to Y$ *is linear and continuous and* $f \in Y$*.*

(3) $\mathrm{Im}\,(G_0)$ *is closed in* Y*.*

If u_0 *is a local Minimizer of* F *under the constraint* $u_0 \in \mathcal{M} := \{u \in X : G(u) = 0\}$*, then there exists* $\pi \in Y'$ *for which*

$$\langle DF(u_0), k\rangle_{(X',X)} - \langle \pi, G_0(k)\rangle_{(Y',Y)} = 0 \quad \textit{for every } k \in X.$$

If $\mathrm{Im}\,(G_0) = Y$*, then* π *is unique.*

Proof. To prove the assertion we apply [30, Prop. 43.1] to our problem. Therefore, we have to show that the following is true for u_0:

$$\forall\ k \in X :\ \text{if } G_0(k) = 0 \text{ then } \langle DF(u_0), k\rangle_{(X',X)} = 0.$$

Let $k \in \ker G_0$. For $t \in \mathbb{R}$ we set $c_k(t) := u_0 + tk$. Obviously $c_k(t) \in \mathcal{M}$ for all $t \in \mathbb{R}$ and $c_k'(0) = k$. Now let $f(t) := F(c_k(t))$. u_0 is a local minimum of $F|_{c_k}$, and therefore $\langle DF(u_0), k\rangle_{(X',X)} = f'(0) = 0$. [30, Prop. 43.1] yields the assertion. □

References

[1] Adams, R. A. *Sobolev Spaces*, vol. 65 of *Pure and Applied Mathematics.* Academic Press, Inc.,, Boston, 1992.

[2] Alt, H. *Lineare Funktionalanalysis*, 3. ed. Springer Verlag, Berlin, Heidelberg, 1999.

[3] Barrett, J. W., and Liu, W. B. A remark on the regularity of the solutions of the p-Laplacian and its application to their finite element approximation. *Journal of Mathematical Analysis and Applications 178*, 2 (1993), 470–487.

[4] Böhme, G. *Non-Newtonian fluid mechanics*, vol. 31 of *North-Holland Series in Applied Mathematics and Mechanics.* North-Holland, Amsterdam, 1987.

[5] Carroll, R., Duff, G., Friberg, J., Gobert, J., Grisvard, P., Necas, J., and Seeley, R. *Equations aux derivées partielles.* (Séminaire de mathématiques supérieures. 19.) Montréal: Les Presses de l'Université de Montréal, 1966.

[6] Chabrowski, J. *Variational methods for potential operator equations: with applications to nonlinear elliptic equations.* de Gruyter Studies in Mathematics. 24, Berlin, 1997.

[7] Dautray, R., and Lions, J.-L. *Mathematical analysis and numerical methods for sciences and technology*, vol. 3 Spectral theory and applications. Springer-Verlag, Berlin Heidelberg, 1990.

[8] de Thélin, F. Local regularity properties for the solutions of a nonlinear partial differential equation. *Nonlinear Analysis, Theory, Methods & Applications 8*, 8 (1982), 839–844.

[9] Drabek, P., Kufner, A., and Nicolosi, F. *Quasilinear elliptic equations with degenerations and singularities.* No. 5 in Series in nonlinear analysis and applications. De Gruyter, 1997.

[10] Ebmeyer, C. Mixed boundary value problems for nonlinear elliptic systems in n-dimensional Lipschitzian domains. *Zeitschrift für Analysis und ihre Anwendungen 18*, 3 (1999), 539–555.

[11] Ebmeyer, C. Steady flow of fluids with shear-dependent viscosity under mixed boundary value conditions in polyhedral domains. *Mathematical Models and Methods in Applied Sciences 10*, 5 (2000), 629–650.

[12] Ebmeyer, C. Nonlinear elliptic problems with p-structure under mixed boundary value conditions in polyhedral domains. *Advances in Differential Equations 6*, 7 (2001), 873–895.

[13] Ebmeyer, C. Mixed boundary value problems for nonlinear elliptic systems with p-structure in polyhedral domains. *Mathematische Nachrichten 236* (2002), 91–108.

[14] Ebmeyer, C., and Frehse, J. Steady Navier-Stokes equations with mixed boundary value conditions in three-dimensional Lipschitzian domains. *Mathematische Annalen 319*, 2 (2000), 349–381.

[15] Fuchs, M. On quasistatic non-newtonian fluids with power law. *Mathematical Methods in the Applied Sciences 19* (1996), 1225–1232.

[16] Geymonat, G., and Suquet, P. Functional spaces for Norton-Hoff materials. *Mathematical Methods in the Applied Sciences 8* (1986), 206–222.

[17] Gilbarg, D., and Trudinger, N. S. *Elliptic partial differential equations of second order.* No. 224 in Grundlehren der mathematischen Wissenschaften. Springer Verlag, 1977.

[18] Grisvard, P. *Elliptic Problems in Nonsmooth Domains.* Pitman Press, Bath, Avon, 1985.

[19] Korowkin, P. P. *Ungleichungen.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1966.

[20] Kufner, A., John, O., and Fučik, S. *Function Spaces.* Noordhof International Publishing and Academia, Leyden and Prag, 1977.

[21] Lindqvist, P. On the equation $\operatorname{div}(\nabla u^{p-2}\nabla u) + \lambda u^{p-2}u = 0$. *Proceedings of the American Mathematical Society 109* (1990), 157–164.

[22] Naumann, J. On the differentiability of weak solutions of a degenerate system of PDE's in fluid mechanics. *Annali di Matematica Pura ed Applicata 151* (1988), 225–238.

[23] Nikol'skii, S. M. *Approximation of functions of several variables and imbedding theorems.* Springer-Verlag, 1975.

[24] RAYMOND, J.-P. Théorèmes de régularité locale pour des systèmes elliptiques dégénérés et des problèmes non différentiables. *Annales Faculté des Sciences de Toulouse 9*, 3 (1988), 381–411.

[25] TOLKSDORF, P. Everywhere-regularity for some quasilinear systems with a lack of ellipticity. *Annali di Matematica Pura ed Applicata 134* (1983), 241–266.

[26] TRIEBEL, H. *Interpolation Theory, Function Spaces, Differential Operators.* North Holland Publishing Company, Amsterdam, New York, Oxford, 1978.

[27] WERNER, D. *Funktionalanalysis.* Springer Verlag, 1995.

[28] WLOKA, J. *Partielle Differentialgleichungen.* Teubner Verlag, Stuttgart, 1982.

[29] ZEIDLER, E. *Nonlinear Functional Analysis and its Applications*, vol. 2/B Nonlinear Monotone Operators. Springer Verlag Inc., New York, 1986.

[30] ZEIDLER, E. *Nonlinear Functional Analysis and its Applications*, vol. 3 Variational Methods and Optimization. Springer Verlag Inc., New York, 1986.

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