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Dorothee Knees*

Abstract

In this paper we study the global regularity of the displacement and stress fields of a nonlinear elastic model of power-law type on nonsmooth domains. The proof of the regularity results relies on a difference quotient technique which we adapt from the papers by G. Savaré and C. Ebmeyer/J. Frehse to our situation. Finally, a regularity result for the stress field of the elastc, perfect plastic Hencky model is derived.

Keywords: global regularity; power-law model; Hencky model; nonsmooth domain; difference quotient technique

AMS Subject Classification: 35J70,35B65,74B20,74G40

1 Introduction

This paper is concerned with the derivation of global regularity results on nonsmooth domains for the displacement and stress fields of physically nonlinear, geometrically linearised elastic models with a constitutive relation of power-law type. Furthermore, a global regularity result for the stress fields of the elasto-plastic Hencky model is proved.

In the frame-work of deformation theory of plasticity, power-law models are frequently applied for the description of elasto-plastic materials with low proportionality limit having no extended yield plateau and which show strain hardening behaviour. Examples for such materials are stainless steel alloys or aluminium alloys. The particular model we consider here was first proposed by W. Ramberg and W.R. Osgood [39, 1943] for the description of aluminium alloys. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain, $u : \Omega \to \mathbb{R}^d$ the displacement field, $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ the linearised strain tensor, $\sigma \in \mathbb{R}^{d\times d}_{\text{sym}}$ the stress field and $\sigma^D = \sigma - \frac{1}{d}\operatorname{tr}\sigma I$ the deviator of σ. The constitutive relation introduced by W. Ramberg and W.R. Osgood is defined as

$$\varepsilon(u) = A\sigma + \alpha \left|\sigma^D\right|^{q-2} \sigma^D, \tag{1}$$

where A is the inverse of the elasticity matrix (tensor of elastic compliances), $\alpha > 0$ a material constant depending on the yield stress and $q-1 =: n$ is the strain hardening coefficient. Since typical values for n range from $4.45 - 7.9$ for steel alloys [41] up to $20 - 45$ for aluminium alloys [47, 34] we assume that $q \geq 2$. Constitutive relation (1) together with the equilibrium

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of forces and boundary conditions on $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ (equations (2)-(4) here below) form the field equations of the Ramberg/Osgood model:

$$\operatorname{div}\sigma + f = 0 \quad \text{in } \Omega, \tag{2}$$

$$\sigma\vec{n} = h \quad \text{on } \Gamma_N, \tag{3}$$

$$u = g \quad \text{on } \Gamma_D. \tag{4}$$

Here, f, h are volume and surface force densities, respectively, and g are prescribed displacements on the Dirichlet boundary Γ_D. By $\vec{n}$ we denote the exterior unit normal vector on $\partial\Omega$. This model is also known in literature as Norton/Hoff model and we refer to [41, 47, 8] for more details.

It is well known that very high stress concentrations may occur in the vicinity of re-entrant corners, cracks, edges and near those points, where the boundary conditions change. Such singularities have a strong influence on both, the strength and physical life of the body and on convergence rates of standard numerical schemes. Thus a deeper knowledge of the singular behaviour of the stress and strain fields is important. In the literature on solid mechanics stress singularities near a corner point S of a two dimensional domain Ω are studied by means of ansatzes (HRR fields, see e.g. [25, 42, 55]) of the form

$$\sigma(r,\varphi) = r^{\alpha}\sigma_0(\varphi), \tag{5}$$

where polar coordinates (r,φ) with respect to S are used. This ansatz leads to a fully nonlinear eigenvalue problem for the determination of the singular exponent α and the function $\sigma_0(\varphi)$. The worst possible singular exponent which is predicted for weak solutions by this approach is $\alpha = -\frac{1}{q}$ if S is a crack tip. But unlike the case of linear elliptic equations it is to the author's knowledge an open problem whether the singular behaviour of weak solutions of the Ramberg/Osgood model can be completely characterised by ansatzes of the form (5).

The field equations of the Ramberg/Osgood model are closely related to general systems of quasilinear elliptic partial differential equations with p-structure, see e.g. [14] for a definition. G. Savaré [44] and C. Ebmeyer and J. Frehse [15, 14] obtained independently global regularity results for weak solutions u of such systems on nonsmooth domains and proved their results with a difference quotient technique. Combining the geometrical assumptions from [44] (Lipschitz domains, non changing boundary conditions) and [15, 14] (polyhedral domains with additional constraints near those points, where the boundary conditions change) we introduce the notion of *admissible domains* on which we formulate and prove the regularity results. In particular we obtain the following global regularity in Sobolev-Slobodeckij spaces for weak solutions of the Ramberg/Osgood equations with mixed boundary conditions on admissible domains (theorem 3.4):

$$u \in W^{\frac{3}{2}-\delta,\frac{2dp}{2d-2+p}}(\Omega), \tag{6}$$

$$\sigma \in W^{\frac{1}{q}-\delta,q}(\Omega) \cap W^{\frac{1}{2}-\delta,2}(\Omega) \tag{7}$$

for every $\delta > 0$ with $q \geq 2$ from (1) and $\frac{1}{p} + \frac{1}{q} = 1$.

We give a short comparison of our results with results from literature. For $q = 2$, equations (1)-(4) reduce to the field equations for linear elastic materials. The regularity of the corresponding weak solutions is studied by numerous authors. B. Dahlberg, C. Kenig and G. Verchota [9] proved global regularity results for the elastic fields on Lipschitz domains with non changing boundary conditions and (6)-(7) coincide with the results from [9] for this case. In the linear case, the regularity of the elastic fields can also be characterised near corners and edges by means of asymptotic expansions, see e.g. [10, 23, 28, 30, 33] and the references therein. Based on such expansions, the regularity of the displacement fields of the Lamé system is investigated in [37] for polygons with mixed boundary conditions. Solutions which are less regular than (6) may occur as soon as the polygon is not any more an admissible domain in our sense. For $q \geq 2$, the Ramberg/Osgood model is closely related to quasilinear elliptic systems with p-structure and (6) coincides with the results from [44, 14] for such systems. A comparison between the worst singularities predicted by the approach with ansatzes of the form (5) and between (7) shows perfect agreement. But it remains an open problem whether asymptotic expansions like (5) describe the singular behaviour of the stress fields completely as in the linear case. Let us note that the local regularity of the displacement and stress fields of the Ramberg/Osgood model and related models is investigated in [3, 45, 17].

In the proof of (6)-(7) we adapt the ideas from [44, 15, 14] to the Ramberg/Osgood model and derive (6)-(7) with a difference quotient technique. The main difficulty is that the nonlinearity in the constitutive law (1) is "anisotropic" i.e. the power-law term depends only on σ^D and not on the full stress tensor σ. Thus we have to work with the function spaces introduced in [20] where this behaviour is taken into account. Moreover, it is a-priori not clear whether the test functions used in the proofs of the regularity results in [44, 15] are still admissible test functions in the Ramberg/Osgood case. Finally, an explicit inversion formula for the constitutive relation (1) is unknown and therefore it is convenient to work with the dual or stress-based formulation instead of the usual displacement-based formulation.

As an application of the regularity results for Ramberg/Osgood materials we deduce a global regularity result for stress fields which are solutions of the Hencky model. The Hencky model describes in the framework of deformation theory of plasticity the behaviour of linear elastic, perfect-plastic bodies being subjected to quasi-static loadings. We show that the corresponding stress field σ_H of the Hencky model satisfies

$$\sigma_H \in W^{\frac{1}{2}-\delta,2}(\Omega) \tag{8}$$

globally on Ω for arbitrary $\delta > 0$ (theorem 3.6). The key of the proof is a well known result by R. Temam [50] which states that the stress fields σ_q of the Ramberg/Osgood model converge strongly in $L^2(\Omega)$ to the stress field σ_H of the Hencky model for $q \to \infty$. Due to the regularity results for the Ramberg/Osgood model, namely $\sigma_q \in W^{\frac{1}{2}-\delta,2}(\Omega)$ for $q \geq 2$ (see (7)), it remains to prove that $\|\sigma_q\|_{W^{\frac{1}{2}-\delta,2}(\Omega)}$ is uniformly bounded for $q \geq 2$.

The paper is organised as follows: After a short description of the necessary function spaces following [20], we formulate in section 2 the weak equations of the Ramberg/Osgood and the Hencky model. In section 3 we give a definition of admissible domains and present and discuss the regularity results. Section 4 is devoted to the proof of the regularity results. The paper closes with an appendix where some useful inequalities are listed.

2 Function spaces and weak formulations

In this section the necessary function spaces and the weak formulations of the Ramberg/Osgood model and the Hencky model are introduced.

2.1 Notation

For $m \times d$ matrices $A, B \in \mathbb{R}^{m\times d}$, $m, d \geq 1$, the inner product is defined by $A : B = \operatorname{tr}(A^\top B) = \operatorname{tr}(B^\top A)$ and $|A| = \sqrt{A:A}$ denotes the corresponding Frobenius norm. $\mathbb{R}^{d\times d}_{\text{sym}}$ is the set of symmetric matrices; the deviatoric part A^D of $A \in \mathbb{R}^{d\times d}$ is defined by $A^D = A - \frac{1}{d}(\operatorname{tr} A)I$, where I is the unit matrix in $\mathbb{R}^{d\times d}$. For $R > 0$ and $x_0 \in \mathbb{R}^d$ the set $B_R(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < R\}$ is the open ball with radius R and centre x_0.

If not otherwise stated, it is assumed that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma_D} \cap \overline{\Gamma_N}$, where Γ_D and Γ_N are open and disjoint and describe the Dirichlet and Neumann boundary, respectively.

Throughout the whole paper p' is the conjugate exponent of p, $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, the dual pairing for elements u of a Banach space X and elements f of its dual X' is written as $\langle f, u\rangle = \langle f, u\rangle_X$.

2.2 Function spaces

For $p \in [1, \infty)$, $s \in \mathbb{R}$, $s > 0$, we denote by $W^{s,p}(\Omega)$ the usual Sobolev-Slobodeckij spaces [1, 22]. For a weak formulation of the boundary value problem for the Ramberg/Osgood model we need function spaces which take up the structure of the constitutive law (1), where the trace $\operatorname{tr}\sigma$ of the stress tensor appears in the linear term, only, whereas the deviator σ^D appears also in the nonlinear term. Appropriate spaces were first introduced and studied by G. Geymonat and P. Suquet [20]. Let $\Omega \subset \mathbb{R}^d$ be an open domain and $p, r \in [1, \infty)$.

$$L^{p,r}(\Omega) = \{\sigma : \Omega \to \mathbb{R}^{d\times d}_{\text{sym}} :\ \sigma^D \in L^p(\Omega),\ \operatorname{tr}\sigma \in L^r(\Omega)\}, \tag{9}$$

$$\Sigma^{p,r}(\Omega) = \{\sigma \in L^{p,r}(\Omega) :\ \operatorname{div}\sigma \in L^p(\Omega)\}, \tag{10}$$

where $\operatorname{div}\sigma(x) \in \mathbb{R}^d$ and $(\operatorname{div}\sigma(x))_i = \sum_{i=1}^d \frac{\partial\sigma_{ij}(x)}{\partial x_j}$. Furthermore,

$$U^{p,r}(\Omega) = \{u : \Omega \to \mathbb{R}^d :\ u \in L^p(\Omega),\ \varepsilon^D(u) \in L^p(\Omega),\ \operatorname{tr}\varepsilon(u) \in L^r(\Omega)\}. \tag{11}$$

These spaces are endowed with the following natural norms:

$$\|\sigma\|_{L^{p,r}(\Omega)} = \left\|\sigma^D\right\|_{L^p(\Omega)} + \|\operatorname{tr}\sigma\|_{L^r(\Omega)}\,, \quad \|\sigma\|_{\Sigma^{p,r}(\Omega)} = \|\sigma\|_{L^{p,r}(\Omega)} + \|\operatorname{div}\sigma\|_{L^p(\Omega)}\,,$$

$$\|u\|_{U^{p,r}(\Omega)} = \|u\|_{L^p(\Omega)} + \left\|\varepsilon^D(u)\right\|_{L^p(\Omega)} + \|\operatorname{tr}\varepsilon(u)\|_{L^r(\Omega)}\,.$$

The properties of these spaces are studied in detail in [20] for $p, r \in (1, \infty)$. Results for the space $LD(\Omega) \equiv U^{1,1}(\Omega)$ were derived in [50]. In particular, traces are well defined for functions u from $U^{p,r}(\Omega)$ with $r \geq p \geq 1$ and the trace operator

$$\gamma\big|_{\partial\Omega} : U^{p,r}(\Omega) \to \begin{cases} W^{1-\frac{1}{p},p}(\partial\Omega), & \text{if } r \geq p > 1, \\ L^1(\partial\Omega), & \text{if } r = p = 1, \end{cases} \tag{12}$$

with $\gamma\big|_{\partial\Omega} u = u\big|_{\partial\Omega}$ is linear, continuous and surjective. For $p > 1$, $q = p' = \frac{p}{p-1}$, $\Gamma \subset \partial\Omega$, Γ open, we need also the following trace space and its dual:

$$\tilde{W}^{1-\frac{1}{p},p}(\Gamma) = \{v \in L^p(\Gamma) : \exists \tilde{v} \in W^{1-\frac{1}{p},p}(\partial\Omega) \text{ with } \operatorname{supp}\tilde{v} \subset \overline{\Gamma},\ \tilde{v}\big|_\Gamma = v\}, \tag{13}$$

$$W^{-\frac{1}{q},q}(\Gamma) = \left(\tilde{W}^{1-\frac{1}{p},p}(\Gamma)\right)'. \tag{14}$$

It is meaningful to speak about *normal stresses* for functions $\sigma \in \Sigma^{q,s}(\Omega)$ if $q \geq s > 1$. More precisely, it holds

Lemma 2.1. *[20] Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $1 < s \leq q < \infty$ and $\Gamma \subset \partial\Omega$ open. Then there exists a linear and continuous mapping*

$$\gamma_1 : \Sigma^{q,s}(\Omega) \to W^{-\frac{1}{q},q}(\Gamma)$$

with $\gamma_1\sigma = \sigma\big|_\Gamma \vec{n}$ on Γ for every $\sigma \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^{d\times d}_{sym})$. Here, $\vec{n}$ is the exterior unit normal vector on $\partial\Omega$. γ_1 is surjective and Green's formula is valid for every $\sigma \in \Sigma^{q,s}(\Omega)$, $u \in U^{q',s'}(\Omega)$ with $u\big|_{\partial\Omega\setminus\overline{\Gamma}} = 0$:

$$\int_\Omega \sigma : \varepsilon(u)\,\mathrm{d}x + \int_\Omega u \operatorname{div}\sigma\,\mathrm{d}x = \langle \gamma_1\sigma, u\rangle_{\tilde{W}^{1-\frac{1}{q'},q'}(\Gamma)} = \langle \sigma\vec{n}, u\rangle_{\tilde{W}^{1-\frac{1}{q'},q'}(\Gamma)}. \tag{15}$$

Remark 2.2. The existence of the mapping γ_1 and Green's formula are proved in [20], we show in A.3 in the appendix that γ_1 is surjective.

For convenience and to fix the notation, we reformulate here Korn's and Poincaré-Friedrichs' inequality for $U^{p,r}(\Omega)$ and an embedding theorem for $U^{1,1}(\Omega)$.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with Lipschitz boundary.*

Korn's inequality: *[20] Let $p \in (1,\infty)$. The spaces $W^{1,p}(\Omega)$ and $U^{p,p}(\Omega)$ have the same elements and the norms are equivalent. That means that there exist constants $c_1^K, c_2^K > 0$ such that for every $v \in W^{1,p}(\Omega)$:*

$$c_1^K \|v\|_{W^{1,p}(\Omega)} \leq \|v\|_{U^{p,p}(\Omega)} \leq c_2^K \|v\|_{W^{1,p}(\Omega)}. \tag{16}$$

Poincaré/Friedrichs' inequality: *[20, 50] Let $\Gamma_D \subset \partial\Omega$ be open and not empty and $r \geq p > 1$ or $r = p = 1$. We define*

$$V^{p,r}(\Omega) = \{v \in U^{p,r}(\Omega) : v\big|_{\Gamma_D} = 0\}. \tag{17}$$

Then there exists a constant $c_{p,r}^{PF} > 0$ such that it holds for every $v \in V^{p,r}(\Omega)$:

$$c_{p,r}^{PF} \|\varepsilon(v)\|_{L^{p,r}(\Omega)} \geq \|v\|_{U^{p,r}(\Omega)}. \tag{18}$$

If $p = r > 1$, then there is a constant $c_p^{PF} > 0$ such that for every $v \in V^{p,p}(\Omega)$

$$c_p^{PF} \|\nabla v\|_{L^p(\Omega)} \geq \|v\|_{W^{1,p}(\Omega)}. \tag{19}$$

Remark 2.4. [50] Korn's inequality does not hold for $p = 1$ and $W^{1,1}(\Omega)$ is a proper subspace of $U^{1,1}(\Omega)$.

Due to an idea by M. Fuchs [17] it can be shown that the spaces $\Sigma^{q,s}(\Omega)$ and $\Sigma^{q,q}(\Omega)$ in fact are equal for fixed $q > 1$ and every $s \in (1, q]$:

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and $1 < s \leq q < \infty$. The spaces $\Sigma^{q,s}(\Omega)$ and $\Sigma^{q,q}(\Omega)$ are equal and there exists a constant $c > 0$, which is indepentent of s, such that for every $\sigma \in \Sigma^{q,s}(\Omega)$*

$$\|\operatorname{tr}\sigma\|_{L^q(\Omega)} \leq c\,\|\sigma\|_{\Sigma^{q,s}(\Omega)}. \tag{20}$$

The proof of this lemma is postponed to the appendix, section A.4. Besides the spaces introduced above we deal also with Nikolskii spaces. Nikolskii spaces are very useful for proving regularity results with a difference quotient technique since their norms are based on difference quotients. For convenience we cite here the definition of Nikolskii spaces and an embedding theorem.

Definition 2.6 (Nikolskii space). [1, 38] Let $s = m + \delta$, where $m \geq 0$ is an integer and $0 < \delta < 1$. For $1 < p < \infty$ the Nikolskii spaces are defined as

$$\mathcal{N}^{s,p}(\Omega) := \Big\{ u \in L^p(\Omega) : \ \|u\|_{\mathcal{N}^{s,p}(\Omega)} < \infty \Big\}$$

with

$$\|u\|^p_{\mathcal{N}^{s,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \sum_{|\alpha|=m} \sup_{\substack{\eta>0 \\ h\in\mathbb{R}^d \\ 0<|h|<\eta}} \int_{\Omega_\eta} \frac{|D^\alpha u(x+h) - D^\alpha u(x)|^p}{|h|^{\delta p}} \,\mathrm{d}x \tag{21}$$

and $\Omega_\eta = \{x \in \Omega : \ \operatorname{dist}(x, \partial\Omega) > \eta\}$.

Lemma 2.7. *[1, 38, 53, 54] Let s, p be as in definition 2.6 and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. The following embeddings are continuous for every $\epsilon > 0$:*

$$\mathcal{N}^{s+\epsilon,p}(\Omega) \subset W^{s,p}(\Omega) \subset \mathcal{N}^{s,p}(\Omega).$$

Lemma 2.7 is a consequence of [22, Thm. 1.4.1.3], [38, p. 381], [53, sections 1.3, 2.1.1, 2.2.9] and [54, sec. 2.3.2]. An equivalent norm is generated if the supremum in (21) is replaced by $\sup_{\substack{\eta>0, h=\eta e_j, \\ e_j\in\{e_1,\ldots,e_d\}}}$, where $\{e_1, \ldots, e_d\}$ is a basis of $\mathbb{R}^d$ [38, 31].

2.3 The Ramberg/Osgood model

Let $q \geq 2$, $\alpha_q > 0$ and $A \in \mathbb{R}^{(d\times d)\times(d\times d)}$ be the symmetric and positive definite fourth order tensor of elastic compliances, i.e.

$$A_{ijkl} = A_{klij} = A_{jikl} \ \text{ and } \ (A\tau) : \tau \geq c^A |\tau|^2 \ \text{ for every } \tau \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{22}$$

Here, $(A\tau)_{ij} = \sum_{k,l=1}^d A_{ijkl}\tau_{kl}$. The complementary energy density reads for the Ramberg/Osgood model

$$W_{c,q}(\tau) = \frac{1}{2}(A\tau) : \tau + \frac{\alpha_q}{q} \left|\tau^D\right|^q, \quad \tau \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{23}$$

Note that constitutive relation (1) can be rewritten as $\varepsilon = DW_{c,q}(\sigma)$ with $(DW_{c,q}(\sigma))_{ij} = \frac{\partial W_{c,q}(\sigma)}{\partial\sigma_{ij}}$. For $q \geq 2$, $p = q' = \frac{q}{q-1}$, $f \in L^q(\Omega)$, $h \in W^{-\frac{1}{q},q}(\Gamma_N)$, the set of admissible stress fields is defined by

$$\mathcal{K}(f,h,q) = \{\tau \in L^{q,2}(\Omega) : \int_\Omega \tau : \varepsilon(v) \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x + \langle h, v\rangle_{\tilde{W}^{1-\frac{1}{p},p}(\Gamma_N)} \ \forall v \in V^{p,2}(\Omega)\}. \tag{24}$$

Since $f \in L^q(\Omega)$, it holds

$$\tau \in \mathcal{K}(f,h,q) \ \Leftrightarrow \ \tau \in \Sigma^{q,2}(\Omega), \operatorname{div}\tau + f = 0 \text{ in } \Omega, \tau\vec{n} = h \text{ in } W^{-\frac{1}{q},q}(\Gamma_N). \tag{25}$$

If $\Gamma_D \neq \emptyset$ or if $\Gamma_D = \emptyset$ and the solvability condition $\int_\Omega f r \,\mathrm{d}x + \langle h, r\rangle = 0$, $r \in \mathcal{R} = \{r : \Omega \to \mathbb{R}^d : r(x) = a + Bx, a \in \mathbb{R}^d, B \in \mathbb{R}^{d\times d}, B + B^\top = 0\}$, is satisfied, then $\mathcal{K}(f,h,q)$ is not empty. Let finally $g_0 \in U^{p,2}(\Omega)$. The weak formulation of the Ramberg/Osgood model (1)-(4) is given by

($\mathbf{W}_q$) *Find $\sigma_q \in \mathcal{K}(f,h,q)$ and $\tilde{u}_q \in V^{p,2}(\Omega)$ such that it holds for every $\tau \in L^{q,2}(\Omega)$*

$$\int_\Omega (DW_{c,q}(\sigma_q) - \varepsilon(g_0)) : \tau \,\mathrm{d}x = \int_\Omega \tau : \varepsilon(\tilde{u}_q)\,\mathrm{d}x. \tag{26}$$

Let $u_q = \tilde{u}_q + g_0$. The pair (σ_q, u_q) are the stress and displacement fields we are looking for. In addition we consider the following minimisation problem

($\mathbf{M}_q$) *Find $\sigma_q \in \mathcal{K}(f,h,q)$ such that for every $\tau \in \mathcal{K}(f,h,q)$*

$$I_{c,q}(\sigma_q) \leq I_{c,q}(\tau) = \int_\Omega W_{c,q}(\tau)\,\mathrm{d}x - \int_\Omega \varepsilon(g_0) : \tau \,\mathrm{d}x. \tag{27}$$

The following existence theorem is due to standard arguments for minimisation problems with constraints, see e.g. [56, Prop. 43.1, 38.15] and [50, 3]

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $q \geq 2$, $p = q' = \frac{q}{q-1}$, $f \in L^q(\Omega)$, $h \in W^{-\frac{1}{q},q}(\Gamma_N)$, $g_0 \in U^{p,2}(\Omega)$ and assume that $\mathcal{K}(f,h,q) \neq \emptyset$. Then problems ($\mathbf{W}_q$) and ($\mathbf{M}_q$) are equivalent and solvable. Moreover, $\sigma_q \in \Sigma^{q,2}(\Omega)$ is unique since $W_{c,q}$ is strictly convex, and u_q is unique if $\Gamma_D \neq \emptyset$. If $\partial\Omega = \Gamma_N$, then u_q is unique up to elements from $\mathcal{R}$.*

2.4 The Hencky model

The Hencky model in its strong form reads as follows for a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, see e.g. [12, 8, 24]:

Find a displacement field $u : \Omega \to \mathbb{R}^d$ and a stress field $\sigma : \Omega \to \mathbb{R}^{d\times d}_{sym}$ such that (2)-(4) *are satisfied and in addition*

$$\varepsilon(u(x)) = \varepsilon_{el}(x) + \varepsilon_{pl}(x) \quad \textit{in } \Omega, \tag{28}$$

$$\varepsilon_{el} = A\sigma \quad \textit{in } \Omega, \tag{29}$$

$$\mathcal{F}(\sigma(x)) \leq 0 \quad \textit{in } \Omega, \tag{30}$$

$$\mathcal{F}(\tau) = \left|\tau^D\right|^2 - \sigma_y^2 \quad \textit{for } \tau \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \tag{31}$$

$$(\sigma - \tau) : \varepsilon_{pl} \geq 0 \quad \textit{for every } \tau \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \textit{ with } \mathcal{F}(\tau) \leq 0. \tag{32}$$

The constant $\sigma_y > 0$ denotes the yield stress and $\mathcal{F}$ the von Mises yield function. Minimisation problems and weak formulations of the field equations of the Hencky model are extensively studied in literature, see e.g. [50, 2, 3, 27, 49, 18] and the references cited therein. It is shown under suitable assumptions on the given data f, g, h that the primal and dual minimisation problems and the corresponding weak formulations are solvable in the spaces $\Sigma(\Omega) = \{\sigma \in L^2(\Omega) : \operatorname{div}\sigma \in L^d(\Omega), \sigma^D \in L^\infty(\Omega)\}$ for the stress field σ and in $U(\Omega) = \{u \in BD(\Omega) : \operatorname{div} u \in L^2(\Omega)\}$ for the displacement field u. Here, $BD(\Omega)$ is the space of vector fields of bounded deformation and is introduced in [32, 48, 49]. Furthermore it is proved in [2, 50] that the stress σ minimises the complementary energy. We take this complementary minimisation problem as a starting point. Let

$$\mathcal{M} = \{\tau \in L^2(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}}) : \left|\tau^D(x)\right| \le \sigma_y \text{ a.e. in } \Omega\} \tag{33}$$

be the set of all stress fields which are admissible according to flow rule (30). For $f \in L^2(\Omega)$, $h \in W^{-\frac{1}{2},2}(\Gamma_N)$ and $g_0 \in W^{1,2}(\Omega)$, the minimisation problem reads:

(MH) *Find $\sigma_H \in \mathcal{K}(f, h, 2) \cap \mathcal{M}$ such that for every $\tau \in \mathcal{K}(f, h, 2) \cap \mathcal{M}$*

$$I_H(\sigma_H) \le I_H(\tau),$$

where

$$I_H(\tau) = \int_\Omega \frac{1}{2}(A\tau) : \tau \, \mathrm{d}x - \int_\Omega \varepsilon(g_0) : \tau \, \mathrm{d}x. \tag{34}$$

The existence theorem here below is a consequence of [56, Thm. 46.A].

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^d$, $d \ge 2$, be a bounded domain with Lipschitz boundary. Let further $f \in L^2(\Omega)$, $h \in W^{-\frac{1}{2},2}(\Gamma_N)$ and $g_0 \in W^{1,2}(\Omega)$ and assume that $\mathcal{K}(f, h, 2) \cap \mathcal{M} \ne \emptyset$. Then there exists a unique stress field $\sigma_H \in \mathcal{K}(f, h, 2) \cap \mathcal{M}$ which solves* **(MH)**.

3 Regularity results

In order to get higher global regularity of weak solutions, more assumptions on the geometry and the smoothness of the right hand sides are necessary.

3.1 Admissible domains

First we give an abstract definition of what we call *admissible domain*. In the subsequent lemma 3.3 we then describe examples in 2D and 3D. The definition depends on the type of the boundary conditions.

Definition 3.1 (Cone). A set $\mathcal{K} \subset \mathbb{R}^d$ is a cone with vertex in $x_0 \in \mathbb{R}^d$ if there exists a simply connected, open and nonempty set $\mathcal{C} \subset \partial B_1(0) = \{x \in \mathbb{R}^d : |x| = 1\}$ such that $\mathcal{K} = \{x \in \mathbb{R}^d \backslash \{x_0\} : (x - x_0)/|x - x_0| \in \mathcal{C}\}$.

Definition 3.2 (Admissible domain). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ where Γ_D and Γ_N are open (possibly empty) and disjoint.

1. Case, $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$: Ω is an admissible domain if it has a Lipschitz boundary.

2. Case, $\overline{\Gamma_D} \cap \overline{\Gamma_N} \ne \emptyset$: Ω is an admissible domain if it has a Lipschitz boundary and if in addition there exists a finite number of open balls $B_{R_j}(x_j)$ with radius R_j and centre $x_j \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$ and a finite number of cones $\mathcal{K}_j \subset \mathbb{R}^d$ with vertex in 0 such that $\overline{\Gamma_D} \cap \overline{\Gamma_N} \subset \cup_{j=1}^J B_{R_j}(x_j)$. Furthermore, for every j there exist nonempty Lipschitz domains Ω_D^j, $\Omega_N^j \subset B_{R_j}(x_j)$ with $\Omega_D^j \cap \Omega_N^j = \emptyset$ and

$$\overline{B_{R_j}(x_j)\backslash\Omega} = \overline{\Omega_D^j \cup \Omega_N^j}, \;\; \Gamma_D \cap B_{R_j}(x_j) \subset \partial\Omega_D^j, \;\; \Gamma_N \cap B_{R_j}(x_j) \subset \partial\Omega_N^j, \tag{35}$$

$$\Big(\big(B_{R_j}(x_j)\backslash\overline{\Omega_N^j}\big) + \mathcal{K}_j\Big) \cap \Omega_N^j = \emptyset, \tag{36}$$

$$(\Omega_D^j + \mathcal{K}_j) \cap \Big(B_{R_j}(x_j)\backslash\overline{\Omega_D^j}\Big) = \emptyset, \tag{37}$$

Figure 1: Examples for admissible domains

see also figure 1 (left, the index j is omitted). Here, the notation $\Omega + \mathcal{K} = \{y \in \mathbb{R}^d : y = x + h, x \in \Omega, h \in \mathcal{K}\}$ is used.

The next lemma describes some examples of admissible domains for $d = 2, 3$. The proof of this lemma is technical and is given in [26].

Lemma 3.3. *1. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz-polygon. Ω is admissible if and only if the interior opening angle at those points, where $\overline{\Gamma_D}$ and $\overline{\Gamma_N}$ intersect, is strictly less than π: $\measuredangle(\Gamma_D, \Gamma_N) < \pi$.*

2. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz-polyhedron where at most three faces intersect in the neighbourhood of those points, where the type of the boundary conditions changes. Assume in addition that the interior opening angle between the Dirichlet and Neumann boundary is strictly less than π. Then Ω is an admissible domain, see figure 1 (right) for an example.

3.2 Regularity of weak solutions of the Ramberg/Osgood model

Besides the assumptions on the domain Ω we have to impose also further assumptions on the smoothness of the given data f, g, h. Let $h \in W^{-\frac{1}{q},q}(\Gamma_N)$. Due to lemma 2.1 there exists an element $H \in \Sigma^{q,q}(\Omega)$ with $H\vec{n} = h$ in $W^{-\frac{1}{q},q}(\Gamma_N)$ and it holds due to Green's formula

$$\langle h, v \rangle_{\tilde{W}^{\frac{1}{p},p}(\Gamma_N)} = \langle H\vec{n}, v \rangle_{\tilde{W}^{\frac{1}{p},p}(\Gamma_N)} = \int_\Omega H : \varepsilon(v)\,\mathrm{d}x + \int_\Omega v \operatorname{div} H \,\mathrm{d}x \tag{38}$$

for every $v \in V^{p,p}(\Omega)$ with $\frac{1}{q} + \frac{1}{p} = 1$ and $V^{p,p}(\Omega)$ from (17). We will formulate the assumptions on h via H. In particular we assume:

($\mathbf{D}_q$) Let $\hat{\Omega} \supset\supset \Omega$ be an arbitrary domain, $q \geq 2$, $p = q' \in (1, 2]$, $f \in L^q(\Omega)$, $g \in W^{2,p}(\hat{\Omega})$ with $\nabla g \in L^\infty(\hat{\Omega})$ and $H \in W^{1,q}(\hat{\Omega}, \mathbb{R}^{d\times d}_{\text{sym}}) \cap L^\infty(\hat{\Omega}, \mathbb{R}^{d\times d}_{\text{sym}})$.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an admissible domain according to definition 3.2. Let $q \geq 2$, $p = q' \in (1, 2]$ and assume $(\mathbf{D}_q)$. Let further $(u_q, \sigma_q) \in U^{p,2}(\Omega) \times \mathcal{K}(f, H\vec{n}, q)$ be a weak solution of the Ramberg/Osgood model (26) with $u_q\big|_{\Gamma_D} = g\big|_{\Gamma_D}$ and $\sigma_q\vec{n} = H\vec{n}$ in $W^{-\frac{1}{q},q}(\Gamma_N)$. Note that $\sigma_q \in \Sigma^{q,q}(\Omega)$ due to (25) and lemma 2.5. It holds for every $\delta > 0, \epsilon > 0$:*

$$u_q \in \mathcal{N}^{\frac{3}{2}, \frac{2dp}{2d-2+p} - \epsilon}(\Omega) \cap W^{\frac{3}{2}-\delta, \frac{2dp}{2d-2+p}}(\Omega), \tag{39}$$

$$\sigma_q, \operatorname{div} u_q \in \mathcal{N}^{\frac{1}{2},2}(\Omega) \cap \mathcal{N}^{\frac{1}{q},q}(\Omega). \tag{40}$$

It is $p \leq \frac{2dp}{2d-2+p} \leq 2$.

The proof of theorem 3.4 relies on a difference quotient technique and is postponed to section 4.

Corollary 3.5. *Let the assumptions be the same as in theorem 3.4 with $d = 2$. The standard embedding theorems for Sobolev-Slobodeckij spaces yield for every $\delta > 0$:*

$$u \in \mathcal{C}^{0,\frac{1}{q}-\delta}(\overline{\Omega}).$$

Note that the Ramberg/Osgood model reduces for $q = 2$ to the equations of linear elasticity. If $\partial\Omega = \Gamma_D$ or $\partial\Omega = \Gamma_N$ then theorem 3.4 reproduces the results by B. Dahlberg, C. Kenig and G. Verchota [9] for weak solutions of the equations of linear elasticity on Lipschitz domains. Moreover, it is well known that the behaviour of the displacement field u near a corner point S can be completely characterised by means of asymptotic expansions, see e.g. the books [28, 10, 36, 30]. Assume that $\Omega \subset \mathbb{R}^2$ is a polygon, $p = 2$ and that the material is isotropic. A careful study of the corresponding asymptotic expansions is carried out in [37] and shows that in this case $u \in W^{\frac{3}{2},2}(\Omega)$ if and only if Ω is an admissible polygon. This shows (up to δ) good correlation with theorem 3.4 and indicates that the geometric assumptions cannot be weakened.

In continuum mechanics, ansatzes of the form (5) are applied to study the behaviour of displacement and stress fields of nonlinear power-law materials. First investigations for Ramberg/Osgood materials in this direction were done by J. W. Hutchinson [25] and J. R. Rice /G. F. Rosengren [42] for plane stress and plane strain states of infinite bodies with a straight crack. Based on the assumption that the displacement and stress fields have an asymptotic structure like in the linear case, they derived a strongly nonlinear eigenvalue problem from which they calculated the dominant terms in the asymptotic expansion. In particular they obtained in the two dimensional case the following leading terms near a crack tip S:

$$u(r,\varphi) = r^{\frac{1}{q}} v_0(\varphi) + u_{\text{reg}}, \quad \sigma(r,\varphi) = r^{-\frac{1}{q}} \tau_0(\varphi) + \sigma_{\text{reg}}. \tag{41}$$

Here, v_0, τ_0 are eigenfunctions of an appropriate eigenvalue problem, see e.g. [55], and $u_{\text{reg}}, \sigma_{\text{reg}}$ are more regular functions. The terms $r^{\frac{1}{q}} v_0(\varphi)$ and $r^{-\frac{1}{q}} \tau_0(\varphi)$ are called HRR-fields. Expansion (41) fits well with our regularity theorem since it holds in two dimensions [43, p. 44]

$$r^{\alpha}\tilde{v}(\varphi) \in \mathcal{N}^{\frac{3}{2},\frac{4p}{2+p}-\delta}(\Omega) \text{ for every } \delta > 0 \quad \Leftrightarrow \quad \alpha \geq q^{-1},$$
$$r^{\beta}\tilde{\tau}(\varphi) \in \mathcal{N}^{\frac{1}{q},q}(\Omega) \quad \Leftrightarrow \quad \beta \geq -q^{-1}.$$

To our knowledge, however, it remains an open problem whether the behaviour of weak solutions of power-law models can be completely characterised near corners by asymptotic expansions like in the linear case. For results on asymptotic expansions of weak solutions of scalar equations of p-structure we refer to [52, 6, 11, 35] and the references therein.

3.3 Global stress regularity for the Hencky model

Before we formulate the regularity theorem for the Hencky model we have to introduce a further condition on the given force densities f and h [50, p. 262]:

(SL) Safe load condition

There exists a stress field $\tau \in \mathcal{K}(f,h,2) \cap \mathcal{M}$ and a number $\delta_0 > 0$ such that

$$\left|\tau^D(x)\right| \leq \sigma_y - \delta_0 \quad \text{for a.e. } x \in \Omega.$$

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an admissible domain and assume that $\Gamma_D \neq \emptyset$. Let furthermore $\hat{\Omega} \supset\supset \Omega$ be an arbitrary domain and $f \in L^\infty(\Omega)$, $g_0 \in W^{2,2}(\hat{\Omega})$, $H \in W^{1,\infty}(\hat{\Omega}, \mathbb{R}^{d\times d}_{sym})$ with $\left|H^D\right| \leq \sigma_y$ a.e. in $\hat{\Omega}$. As in the previous section we describe here the Neumann datum h in the form $h = H\vec{n}$ on Γ_N. Finally, the safe load condition* **(SL)** *shall be satisfied.*
Let $\sigma_H \in \mathcal{K}(f, H\vec{n}, 2) \cap \mathcal{M}$ be a solution of the minimisation problem **(MH)** *for the Hencky model. Then it holds for every $\delta > 0$*

$$\sigma_H \in W^{\frac{1}{2}-\delta,2}(\Omega). \tag{42}$$

This theorem will be proved in the next section.

Remark 3.7. In order to simplify the arguments we assume in theorem 3.6 that $\Gamma_D \neq \emptyset$. This implies that the displacement fields of the Ramberg/Osgood model are uniquely determined.

The local regularity of the stress field σ_H was investigated by M. Fuchs, G.A. Seregin [18] and by A. Bensoussan, J. Frehse [3, 4]. Under suitable assumptions on the volume force density f, the regularity

$$\sigma_H \in W^{1,2}_{\mathrm{loc}}(\Omega) \tag{43}$$

is proved. To our knowledge there are no global regularity results reported in literature and attempts to prove (43) globally for smooth domains failed, see the discussions in [46] and [16].

4 Proof of regularity theorems 3.4 and 3.6

The regularity theorem 3.4 for the Ramberg/Osgood model is proved with a difference quotient technique where we adapt arguments from [14, 44]. The main idea is to insert difference quotients of weak solutions as test functions into the weak formulation and to exploit the convexity of the complementary energy density $W_{c,q}$. Differences across the boundary $\partial\Omega$ have to be considered. This makes it necessary to extend weak solutions across the boundary in such a way that differences of the extended functions are still admissible test functions. Due to the geometrical assumptions on Ω it is possible to find such extensions.

In order to prove the result on the Hencky stress σ_H we approximate σ_H by stress fields $\{\sigma_q,\, q \geq 2\}$ of the Ramberg/Osgood model. Since $\sigma_q \in W^{\frac{1}{2}-\delta,2}(\Omega)$ for every $q \geq 2$ (see theorem 3.4), it remains to derive the uniform estimate

$$\|\sigma_q\|_{W^{\frac{1}{2}-\delta,2}(\Omega)} \leq c_\delta \tag{44}$$

for every $q \geq 2$, $\delta > 0$, where c_δ is independent of q. Since the proof of (40) and of (44) are nearly identical, we give a detailed proof of (44) with right hand sides f, g_0, h as in theorem 3.6 and indicate necessary changes for obtaining theorem 3.4 also for the more general assumption $\mathbf{(D}_q\mathbf{)}$.

The proof is split into three parts. First, we cite a result by R. Temam [50] and A. Bensoussan/J. Frehse [3] which describes uniform estimates and convergence results for the elastic fields of the Ramberg/Osgood model. Second, we prove the uniform estimate (44) and finally, we derive the remaining assertions of theorem 3.4 on the displacement field u_q.

Step 1: Approximation of the Hencky stress

Let $\sigma_y > 0$, $A \in \mathbb{R}_{\text{sym}}^{(d\times d)\times(d\times d)}$ and assume that the data f, g_0, $h = H\vec{n}$ is given according to theorem 3.6. For $q \geq 2$ and $p = q'$ we assume that $(u_q, \sigma_q) \in U^{p,2}(\Omega) \times \Sigma^{q,2}(\Omega)$ is a solution of the Ramberg/Osgood model $(\mathbf{W}_q)$ with $\alpha_q = \sigma_y^{1-q}$.

Lemma 4.1. *[5, 50] Under the same assumptions as in theorem 3.6 it holds: the sequence $\{\sigma_q,\, q \geq 2\}$, where $\sigma_q \in \Sigma^{q,2}(\Omega)$ is a solution of $(\mathbf{W}_q)$ with $\alpha_q = \sigma_y^{1-q}$, converges strongly in $L^2(\Omega)$ to the solution σ_H of $(\mathbf{MH})$:*

$$\|\sigma_q - \sigma_H\|_{L^2(\Omega)} \to 0 \text{ for } q \to \infty. \tag{45}$$

Moreover there exists a constant $c > 0$ such that for every $q \geq 2$ and $p = q' \in (1, 2]$

$$\|\sigma_q\|_{L^2(\Omega)} \leq c, \quad \frac{1}{q}\left\|\frac{\sigma^D}{\sigma_y}\right\|_{L^q(\Omega)}^q \leq c, \tag{46}$$

$$\sigma_y^{1-q}\left\|\sigma_q^D\right\|_{L^q(\Omega)}^q \leq c, \tag{47}$$

$$|\Omega|^{-\frac{1}{q}}\|\varepsilon(u_q)\|_{L^1(\Omega)} \leq \|\varepsilon(u_q)\|_{L^p(\Omega)} \leq c. \tag{48}$$

Since $\Gamma_D \neq \emptyset$ we obtain from (48) by Poincaré/Friedrichs' inequality for $U^{1,1}(\Omega)$, see (18):

$$\|u_q\|_{L^1(\Omega)} + \|\varepsilon(u_q)\|_{L^1(\Omega)} \leq c. \tag{49}$$

Remark 4.2. Estimates (46) and (47) are proved in [5, Theorem 10.8, Proposition 10.10] for the case of vanishing Dirichlet conditions, i.e. for $g_0 = 0$. The case $g_0 \neq 0$ can be treated in the same way with some simple modifications. We remark that the safe load condition **(SL)** enters in the proof of (47). Estimate (48) follows via the constitutive relation $\varepsilon(u_q) = A\sigma_q + \frac{1}{\sigma_y^{q-1}}\left|\sigma_q^D\right|^{q-2}\sigma_q^D$ from estimates (46), (47) and inequality (96). The convergence result (45) is shown in [50, Theorem III.1.2].

Step 2: Proof of the results for the stress fields

Lemma 4.3. *1. Let the assumptions of theorem 3.6 be satisfied. For every $\epsilon, \delta > 0$ there exists a constant $c_{\epsilon,\delta} \geq 0$ such that for every $q \geq d + \epsilon$ and every solution σ_q of $(\mathbf{M}_q)$*

$$\|\sigma_q\|_{\mathcal{N}^{\frac{1}{2}-\delta,2}(\Omega)} \leq c_{\epsilon,\delta}. \tag{50}$$

Together with the convergence result of lemma 4.1 this estimate implies (42).

2. Let the assumptions of theorem 3.4 be satisfied. Then $\sigma_q \in \mathcal{N}^{\frac{1}{2},2}(\Omega)$ and $\sigma_q^D \in \mathcal{N}^{\frac{1}{q},q}(\Omega)$.

Proof of lemma 4.3, part 1. We apply a difference quotient technique to deduce estimates for the stress fields in Nikolskii norms. For the derivation of these estimates the domain Ω is covered by a finite number of balls and the estimates are proved for each of these balls separately.

Let $\Omega \subset \mathbb{R}^d$ be an admissible domain. In particular, Ω is a Lipschitz domain and satisfies therefore the uniform interior and exterior cone condition [22]. It follows together with part *2.* of definition 3.2 that there exists a finite number of balls $B_{R_j}(x_j)$ and cones $\mathcal{K}_j$ with vertices in 0 such that $\overline{\Omega} \subset \cup_{j=1}^J B_{R_j}(x_j)$ and each of the pairs $(B_{R_j}(x_j), \mathcal{K}_j)$ satisfies one of the following four cases:

1. $\overline{B_{R_j}(x_j)} \subset \Omega$.
2. $\overline{(B_{R_j}(x_j) \cap \partial\Omega)} \subset \Gamma_D$ and for every $x \in B_{R_j}(x_j) \cap \Gamma_D$ it holds $((x+\mathcal{K}_j) \cap B_{R_j}(x_j)) \cap \overline{\Omega} = \emptyset$.
3. $\overline{(B_{R_j}(x_j) \cap \partial\Omega)} \subset \Gamma_N$ and for every $x \in B_{R_j}(x_j) \cap \overline{\Omega}$ it holds $((x+\mathcal{K}_j) \cap B_{R_j}(x_j)) \subset \Omega$.
4. $x_j \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$ and the pair $(B_{R_j}(x_j), \mathcal{K}_j)$ satisfies (35)-(37) of definition 3.2 with suitable domains Ω_D^j and Ω_N^j.

Note that there exists a constant $\theta > 0$ such that the balls $B_{R_j-\theta}(x_j)$ still cover $\overline{\Omega}$. We prove now for every $\epsilon, \delta > 0$ that there exists a constant $c_{\epsilon,\delta} > 0$ such that it holds for every $q \geq d+\epsilon$

$$\|\sigma_q\|_{\mathcal{N}^{\frac{1-\delta}{2},2}(\Omega \cap B_{R_j-\theta}(x_j))} \leq c_{\epsilon,\delta}. \tag{51}$$

We consider the fourth case in detail, the remaining cases can be treated similarly. In order to simplify the notation we omit the index j in the following.

Let $B_R(x_0)$ be a ball with centre $x_0 \in \partial\Omega$, $\mathcal{K}$ a cone with vertex in 0 and $\Omega_D, \Omega_N \subset B_R(x_0)$ domains such that (35)-(37) of definition 3.2 hold, see also figure 1. Let the data f, g_0 and H satisfy assumptions of theorem 3.6. Furthermore, let $(u_q, \sigma_q) \in U^{p,2}(\Omega) \times \Sigma^{q,2}(\Omega)$ be a solution of $(\mathbf{W}_q)$, i.e. $\sigma_q \in \mathcal{K}(f, H\vec{n}, q)$, $u_q\big|_{\Gamma_D} = g_0\big|_{\Gamma_D}$ and it holds for every $\tau \in L^{q,2}(\Omega)$

$$\int_\Omega (DW_{c,q}(\sigma_q) - \varepsilon(g_0)) : \tau \, \mathrm{d}x = \int_\Omega \tau : \varepsilon(u_q - g_0) \, \mathrm{d}x. \tag{52}$$

Note that $\sigma_q \in \Sigma^{q,q}(\Omega)$ due to lemma 2.5. We define the following extensions of u_q and σ_q to $B_r(x_0)$:

$$\tilde{\sigma}_q(x) = \begin{cases} \sigma_q(x) & \text{for } x \in \Omega, \\ H(x) & \text{for } x \in B_R(x_0) \backslash \Omega. \end{cases} \tag{53}$$

$$\hat{u}_q(x) = \begin{cases} u_q(x) & \text{for } x \in \Omega, \\ g_0(x) & \text{for } x \in \Omega_D. \end{cases} \tag{54}$$

It follows from $\sigma_q \in \Sigma^{q,q}(\Omega)$ and the assumptions on H that $\tilde{\sigma}_q \in L^q(\Omega \cup B_R(x_0))$. Moreover, calculating $\operatorname{div} \tilde{\sigma}_q$ in the distributional sense and taking into account that $\sigma\vec{n} = H\vec{n}$ on Γ_N yields

$$\operatorname{div} \tilde{\sigma}_q \in L^q(\Omega \cup B_R(x_0) \backslash \overline{\Omega_D}).$$

Thus, $\tilde{\sigma}_q \in \Sigma^{q,q}(\Omega \cup B_R(x_0) \backslash \overline{\Omega_D})$. Similar arguments show that $\hat{u}_q \in U^{p,2}(\Omega \cup B_R(x_0) \backslash \overline{\Omega_N}) \subset U^{1,1}(\Omega \cup B_R(x_0) \backslash \overline{\Omega_N})$. Moreover, $\hat{u}_q - g_0 = 0$ on Γ_D. By the extension theorem for elements from $U^{1,1}$ [50, Rem. II.1.3] there exists a function $\tilde{u}_q \in U^{1,1}(\Omega \cup B_R(x_0))$ with $\tilde{u}_q\big|_{\Omega \cup \backslash \overline{\Omega_N}} = \hat{u}_q$ and

$$\|\tilde{u}_q\|_{U^{1,1}(\Omega \cup B_R(x_0))} \leq c_E \|\hat{u}_q\|_{U^{1,1}(\Omega \cup B_R(x_0) \backslash \overline{\Omega_N})}. \tag{55}$$

The constant c_E is independent of q and $\hat{u}_q$. Let $\varphi \in \mathcal{C}_0^\infty(B_R(x_0))$ be a cut-off function with $\varphi\big|_{B_{R-\theta}(x_0)} = 1$. For $x \in \Omega$ and $h \in \mathcal{K}$ with $0 < |h| < h_0 = \frac{1}{3} \operatorname{dist}(\operatorname{supp} \varphi, \partial B_R(x_0))$ we define

$$\tau_q(x) = \varphi^2 \big((\tilde{\sigma}_q(x) - H(x)) - (\tilde{\sigma}_q(x-h) - H(x-h)) \big) \equiv \varphi^2(x) \triangle^h (\tilde{\sigma}_q(x) - H(x)). \tag{56}$$

Here we use the notation $\triangle^h g(x) = g(x) - g(x-h)$ for backward differences. The geometrical assumption (37) implies for $h \in \mathcal{K}$ with $|h| < h_0$ and $x \in B_{R-h_0}(x_0)$

$$x - h \in \Omega_D \;\Rightarrow\; x \in \Omega_D + h \;\Rightarrow x \notin B_R(x_0)\backslash\overline{\Omega_D}.$$

Thus, if $x \in \Omega \cap B_R(x_0)$, then $x - h \in B_R(x_0)\backslash\overline{\Omega_D}$ and, since $\tilde{\sigma}_q$ and H are elements of $\Sigma^{q,q}(\Omega \cup B_R(x_0)\backslash\overline{\Omega_D})$, the function τ_q is an element of $\Sigma^{q,q}(\Omega)$. Moreover, it follows from (36) that

$$x \in \Gamma_N \cap B_{R-h_0}(x_0) \;\Rightarrow\; x - h \in \overline{\Omega_N}$$

for every $h \in \mathcal{K}$ with $|h| < h_0$. Therefore, $\tau_q \vec{n} = 0$ on Γ_N since either $\varphi(x) = 0$ if $x \in \Gamma_N \backslash B_{R-h_0}(x_0)$ or $(\tilde{\sigma}_q(x) - H(x))\vec{n} = 0$ and $\tilde{\sigma}_q(x-h) - H(x-h) = 0$ if $x \in \Gamma_N \cap B_{R-h_0}(x_0)$.

Inserting τ_q into the weak formulation (52) and applying Green's formula (15) yields

$$\begin{aligned}\int_\Omega \varphi^2 DW_{c,q}(\tilde{\sigma}_q) : \triangle^h \tilde{\sigma}_q \,\mathrm{d}x &= \int_\Omega \varphi^2 DW_{c,q}(\tilde{\sigma}_q) : \triangle^h H \,\mathrm{d}x + \int_\Omega \varepsilon(\tilde{u}_q) : \tau_q \,\mathrm{d}x \\ &= \int_\Omega \varphi^2 DW_{c,q}(\tilde{\sigma}_q) : \triangle^h H \,\mathrm{d}x + \int_\Omega \varepsilon(g_0) : \tau_q \,\mathrm{d}x + \int_\Omega (g_0 - \tilde{u}_q) : \operatorname{div} \tau_q \,\mathrm{d}x. \qquad (57)\end{aligned}$$

Note that the boundary terms vanish since $(\tilde{u}_q - g_0)\big|_{\Gamma_D} = 0$ and $\tau_q \vec{n} = 0$ on Γ_N. Inequalities (22) and (94) with $A = \tilde{\sigma}_q(x-h)$, $B = \tilde{\sigma}_q(x)$ and $c_q = 2^{-1-2q}$ imply

$$\begin{aligned}&\alpha_q c_q \int_\Omega \varphi^2 \big(\left|\tilde{\sigma}_q^D(x)\right| + \left|\tilde{\sigma}_q^D(x-h)\right| \big)^{q-2} \left|\triangle^h \tilde{\sigma}_q^D\right|^2 \,\mathrm{d}x + \frac{c^A}{2} \int_\Omega \varphi^2 \left|\triangle^h \tilde{\sigma}_q\right|^2 \,\mathrm{d}x \\ &\quad\le \int_\Omega \varphi^2 \left(W_{c,q}(\tilde{\sigma}_q(x-h)) - W_{c,q}(\tilde{\sigma}_q(x)) - DW_{c,q}(\tilde{\sigma}_q(x)) : (\tilde{\sigma}_q(x-h) - \tilde{\sigma}_q(x))\right) \mathrm{d}x \\ &\quad\overset{(57)}{=} \int_\Omega \varphi^2 \left(-\triangle^h W_{c,q}(\tilde{\sigma}_q)\right) \mathrm{d}x + \int_\Omega \varphi^2 DW_{c,q}(\tilde{\sigma}_q) : \triangle^h H \,\mathrm{d}x \\ &\qquad + \int_\Omega \varepsilon(g_0) : \tau_q \,\mathrm{d}x + \int_\Omega \varphi^2 (g_0 - \tilde{u}_q) \operatorname{div}(\triangle^h(\tilde{\sigma}_q - H)) \,\mathrm{d}x \\ &\qquad + \int_\Omega (\triangle^h(\tilde{\sigma}_q - H)) : \left((g_0 - \tilde{u}_q) \otimes \nabla\varphi^2\right) \mathrm{d}x \\ &\quad= I_1 + \ldots + I_5. \qquad (58)\end{aligned}$$

Here, $a \otimes b \in \mathbb{R}^{d\times d}$ denotes the tensor product of $a, b \in \mathbb{R}^d$ with $(a \otimes b)_{ij} = a_i b_j$. Our next task is to derive the following estimate:

There exists for every $\epsilon, \delta > 0$ a constant $c(\epsilon, \delta)$, which is independent of q, such that it holds for every $q \ge d + \epsilon$ and $h \in \mathcal{K}$ with $|h| < h_0$

$$I_1 + \ldots + I_5 \le c(\epsilon, \delta)\, |h|^{1-\delta}. \qquad (59)$$

Estimation of I_1

By the product rule for differences, $\triangle^h(f(x)g(x)) = g(x)\triangle^h f(x) + f(x-h)\triangle^h g(x)$, we obtain

$$\begin{aligned}I_1 &= -\int_\Omega \triangle^h\big(\varphi^2 W_{c,q}(\tilde{\sigma}_q)\big) \,\mathrm{d}x + \int_\Omega \big(\triangle^h \varphi^2\big) W_{c,q}(\tilde{\sigma}_q(x-h)) \,\mathrm{d}x \\ &= I_{11} + I_{12}. \qquad (60)\end{aligned}$$

For I_{12} we get, since $\varphi \in \mathcal{C}_0^\infty(B_R(x_0))$,

$$\begin{aligned}|I_{12}| &\leq |h| \left\|\nabla\varphi^2\right\|_{L^\infty(B_R(x_0))} \|W_{c,q}(\tilde{\sigma}_q)\|_{L^1(\hat{\Omega})} \\ &= |h| \left\|\nabla\varphi^2\right\|_{L^\infty(B_R(x_0))} \big(\|W_{c,q}(\sigma_q)\|_{L^1(\Omega)} + \|W_{c,q}(H)\|_{L^1(\hat{\Omega}\setminus\Omega)} \big).\end{aligned}$$

Due to lemma 4.1, the term $\|W_{c,q}(\sigma_q)\|_{L^1(\Omega)}$ is bounded independently of q. Since $\alpha_q = \sigma_y^{1-q}$ and $\left|H^D\right| \leq \sigma_y$, the term $\|W_{c,q}(H)\|_{L^1(\hat{\Omega}\setminus\Omega)}$ is bounded independently of q, as well. Thus there exists a constant c_{12}, which is independent of h and q, such that

$$|I_{12}| \leq c_{12}\, |h|\,.$$

Let $\Omega_R = B_R(x_0) \cap \Omega$. The term I_{11} can be estimated as follows after a change of coordinates and taking into account that $(\operatorname{supp}\varphi) \pm h \subset B_R(x_0)$

$$I_{11} = -\int_\Omega \triangle^h\big(\varphi^2 W_{c,q}(\tilde{\sigma}_q)\big)\,\mathrm{d}x = -\int_{\Omega_R\setminus\Omega_R - h} \varphi^2 W_{c,q}(\sigma_q)\,\mathrm{d}x + \int_{\Omega_R - h\setminus\Omega_R} \varphi^2 W_{c,q}(\tilde{\sigma}_q)\,\mathrm{d}x.$$

Note that $\tilde{\sigma}_q = H$ for $x \in \Omega_R - h\setminus\Omega_R$. Due to the assumptions on H, we have

$$\left|\varphi^2(x) W_{c,q}(H(x))\right| \leq c(\varphi, H),$$

where the constant $c(\varphi, H)$ is independent of q, x and h. Moreover, $|\Omega_R - h\setminus\Omega_R| \leq c\,|h|$ and c is independent of h. Thus, there exists a constant $\hat{c}_1$ which is independent of h and q such that

$$I_1 = I_{11} + I_{12} \leq \hat{c}_1\,|h| - \int_{\Omega_R\setminus\Omega_R - h} \varphi^2 W_{c,q}(\sigma_q)\,\mathrm{d}x. \tag{61}$$

Estimation of I_2

Due to the assumptions on H we get again with the product rule and Hölder's inequality

$$\begin{aligned}I_2 &= \int_\Omega \varphi^2 DW_{c,q}(\sigma_q) : \triangle^h H\,\mathrm{d}x \\ &\leq \|DW_{c,q}(\sigma_q)\|_{L^p(\Omega)} \left(\left\|\triangle^h(\varphi^2 H)\right\|_{L^q(\Omega)} + \left\|H(\cdot - h)\triangle^h\varphi^2\right\|_{L^q(\Omega)} \right).\end{aligned} \tag{62}$$

Lemma 7.23 in [21] implies for the terms in the second factor

$$\begin{aligned}\left\|\triangle^h(\varphi^2 H)\right\|_{L^q(\Omega)} &\leq |h| \left\|\nabla(\varphi^2 H)\right\|_{L^q(\hat{\Omega})} \leq |h|\,|\Omega|^{\frac{1}{q}} \left\|\varphi^2 H\right\|_{W^{1,\infty}(\hat{\Omega})}, \\ \left\|H(\cdot - h)\triangle^h\varphi^2\right\|_{L^q(\Omega)} &\leq c(\varphi)\,|h|\,\|H\|_{L^q(\hat{\Omega})},\end{aligned}$$

and $c(\varphi)$ is independent of h and q. Together with $DW_{c,q}(\sigma_q) = \varepsilon(u_q)$ and lemma 4.1, we obtain for (62)

$$|I_2| \leq c_2\,|h|$$

and the constant c_2 is independent of h and q.

Estimation of I_3

Again by the product rule for differences

$$
\begin{aligned}
I_3 &= \int_{\Omega} \varphi^2 \varepsilon(g_0) : \triangle^h(\tilde{\sigma}_q - H)\,\mathrm{d}x \\
&= \int_{\Omega_R} \triangle^h\big(\varphi^2 \varepsilon(g_0) : (\tilde{\sigma}_q - H)\big)\,\mathrm{d}x - \int_{\Omega_R} \big(\triangle^h(\varphi^2 \varepsilon(g_0))\big) : (\tilde{\sigma}_q - H)\big|_{x-h}\,\mathrm{d}x \\
&= I_{31} + I_{32}. \qquad (63)
\end{aligned}
$$

By Hölder's inequality and lemma 7.23 in [21] we obtain

$$
\begin{aligned}
|I_{32}| &\leq \left\| \triangle^h(\varphi^2 \varepsilon(g_0)) \right\|_{L^2(B_R(x_0))} \|\tilde{\sigma}_q - H\|_{L^2(B_R(x_0))} \\
&\leq |h| \left\| \nabla(\varphi^2 \varepsilon(g_0)) \right\|_{L^2(B_R(x_0))} \big(\|\sigma_q\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \big) \\
&\leq c_{32}\,|h| \qquad (64)
\end{aligned}
$$

and the constant c_{32} is independent of q and h due to lemma 4.1 and the assumptions on g_0 and H. For I_{31} we obtain after a change of coordinates

$$
I_{31} = \int_{\Omega_R \setminus \Omega_R - h} \varphi^2 \varepsilon(g_0) : (\sigma_q - H)\,\mathrm{d}x - \int_{\Omega_R - h \setminus \Omega_R} \varphi^2 \varepsilon(g_0) : (\tilde{\sigma}_q - H)\,\mathrm{d}x. \qquad (65)
$$

Since $(\Omega_R - h \setminus \Omega_R) \subset \Omega_N$ and since $\tilde{\sigma}_q - H = 0$ on Ω_N, the second term vanishes. Furthermore, due to the assumptions on H, it holds together with $|\Omega_R \setminus \Omega_R - h| \leq c\,|h|$ that

$$
|I_{31}| \leq \int_{\Omega_R \setminus \Omega_R - h} \varphi^2\, |\varepsilon(g_0)|\, |\sigma_q|\;\mathrm{d}x + \hat{c}_{31}\,|h| \qquad (66)
$$

and $\hat{c}_{31}$ does not depend on h and q. Young's inequality finally implies for every $\delta > 0$

$$
\begin{aligned}
|I_{31}| &\leq \frac{\delta^2}{2} \|\varphi\, |\varepsilon(g_0)|\|^2_{L^2(\Omega_R \setminus \Omega_R - h)} + \frac{\delta^{-2}}{2} \int_{\Omega_R \setminus \Omega_R - h} \varphi^2\, |\sigma_q|^2\;\mathrm{d}x + \hat{c}_{31}\,|h| \\
&\leq \big(\frac{\delta^2}{2} + 1\big) c_{31}\,|h| + \frac{\delta^{-2}}{2} \int_{\Omega_R \setminus \Omega_R - h} \varphi^2\, |\sigma_q|^2\;\mathrm{d}x. \qquad (67)
\end{aligned}
$$

The constant c_{31} is independent of h and q. Combining (61), (64) and (67) we get

$$
I_1 + |I_3| \leq \big(\hat{c}_1 + \big(\frac{\delta^2}{2} + 1\big) c_{31} + c_{32}\big)\,|h| + \int_{\Omega_R \setminus \Omega_R - h} \varphi^2 \left(\frac{\delta^{-2}}{2} |\sigma_q|^2 - W_{c,q}(\sigma_q) \right) \mathrm{d}x. \qquad (68)
$$

Choosing $\delta^{-2} = c^A$ with c^A from (22) yields

$$
\frac{\delta^{-2}}{2} |\sigma_q(x)|^2 - W_{c,q}(\sigma_q(x)) \leq 0
$$

for every $x \in \Omega$ and thus

$$
I_1 + |I_3| \leq c_1\,|h|
$$

for a constant c_1 which is independent of h and q. The estimates of I_4 and I_5 are based on the following lemma due to L. Paris [40], see also [50], where difference quotients of functions from $U^{1,1}(\mathbb{R}^d)$ with compact support are estimated by the corresponding strain tensor:

Lemma 4.4. *[40, 50] Let $\omega \subset \mathbb{R}^d$ be a compact set. For every $s \in [1, \frac{d}{d-1})$ and every $\delta \in (0,1]$ there exists a constant $c = c(\omega, s, \delta) \geq 0$ such that it holds for every $h \in \mathbb{R}^d$ and $u \in U^{1,1}(\mathbb{R}^d)$ with* $\operatorname{supp} u \subset \omega$

$$\|u(x+h) - u(x)\|_{L^s(\mathbb{R}^d)} \leq c\,|h|^{1-\delta}\,\|u\|_{U^{1,1}(\mathbb{R}^d)}\,.$$

Remark 4.5. It follows by Hölder's inequality that $c(\omega, s_1, \delta) \leq (2\,|\omega|)^{\frac{s_2 - s_1}{s_1 s_2}}\, c(\omega, s_2, \delta)$ for $s_1 \leq s_2$.

Estimation of I_4

We define the following function for $x \in \mathbb{R}^d$:

$$F(x) = \begin{cases} -(f + \operatorname{div} H) & \text{for } x \in \Omega_R = \Omega \cap B_R(x_0), \\ 0 & \text{else.} \end{cases}$$

It holds $F(x) = \operatorname{div}(\tilde{\sigma}_q(x) - H(x))$ for $x \in B_R(x_0)\backslash\overline{\Omega}_D$. Moreover, (36) implies that $x - h \notin \Omega_R$ for $h \in \mathcal{K}$, $x \in \Omega_N$ and thus $\triangle^h F(x) = 0$ for $x \in \Omega_N$, $h \in \mathcal{K}$. Furthermore, $g_0 - \tilde{u}_q = 0$ in Ω_D. Therefore, the domain Ω in the definition of I_4 can be replaced by $B_R(x_0)$ and we get by the product rule for differences:

$$\begin{aligned} I_4 &= \int_{B_R(x_0)} \varphi^2 (g_0 - \tilde{u}_q) \triangle^h F(x)\,\mathrm{d}x \\ &= \int_{B_R(x_0)} \triangle^h \big(\varphi^2 (g_0 - \tilde{u}_q) F\big)\,\mathrm{d}x \\ &\quad - \int_{B_R(x_0)} \big(\triangle^h (\varphi^2 (g_0 - \tilde{u}_q))\big) F(x-h)\,\mathrm{d}x. \end{aligned} \tag{69}$$

The first term vanishes since $\operatorname{supp}\varphi \subset \big(B_R(x_0) \cap (B_R(x_0) - h)\big)$ for $|h| < h_0$. It follows by Hölder's inequality

$$|I_4| \leq \left\|\triangle^h (\varphi^2 (g_0 - \tilde{u}_q))\right\|_{L^p(B_R(x_0))} \|F\|_{L^q(B_R(x_0))}\,. \tag{70}$$

Due to the assumptions on f and H, the factor $\|F\|_{L^q(B_R(x_0))}$ is uniformly bounded with respect to q. Applying lemma 4.4 to the first factor in (70) yields for every $\epsilon, \delta \in (0,1)$ and $q \geq d + \epsilon$ (thus $p < \frac{d}{d-1}$)

$$\left\|\triangle^h (\varphi^2 (g_0 - \tilde{u}_q))\right\|_{L^p(B_R(x_0))} \leq c(B_R(x_0), p, \delta)\,|h|^{1-\delta}\,\left\|\varphi^2 (g_0 - \tilde{u}_q)\right\|_{U^{1,1}(B_R(x_0))}\,.$$

The definition of $\tilde{u}_q$ and inequalities (55) and (49) imply that $\left\|\varphi^2 (g_0 - \tilde{u}_q)\right\|_{U^{1,1}(B_R(x_0))}$ is uniformly bounded with respect to q. Moreover, the constant $c(B_R(x_0), p, \delta)$ is bounded independently of $p = q'$ for fixed $\delta, \epsilon \in (0,1)$ and arbitrary $p \in (1, \frac{d-\epsilon}{d-\epsilon-1}]$, see remark 4.5. Therefore, there exists for every $\epsilon, \delta \in (0,1)$ a constant $c_4(\epsilon, \delta)$ which is independent of $q \geq d + \epsilon$ and h such that

$$|I_4| \leq c_4(\epsilon, \delta)\,|h|^{1-\delta}\,.$$

Estimation of I_5

As before, the domain Ω in I_5 may be replaced by $B_R(x_0)$. Applying the product rule for differences leads to

$$I_5 = \int_{B_R(x_0)} \triangle^h \left((\tilde{\sigma}_q - H) : (g_0 - \tilde{u}_q) \otimes \nabla\varphi^2 \right) \mathrm{d}x \tag{71}$$

$$- \int_{B_R(x_0)} (\tilde{\sigma}_q - H)\big|_{x-h} : \triangle^h \left((g_0 - \tilde{u}_q) \otimes \nabla\varphi^2 \right) \mathrm{d}x. \tag{72}$$

The first term on the right hand side vanishes. For $\delta, \epsilon \in (0,1)$ and $q \geq d + \epsilon$ it follows from Hölder's inequality and lemma 4.4

$$\begin{aligned} |I_5| &\leq \|\tilde{\sigma}_q - H\|_{L^{d+\epsilon}(B_R(x_0))} \sum_{j=1}^{d} \left\| \triangle^h \left((\partial_j \varphi^2)(g_0 - \tilde{u}_q) \right) \right\|_{L^{(d+\epsilon)'}(B_R(x_0))} \\ &\leq |h|^{1-\delta} c(B_R(x_0), (d+\epsilon)', \delta) \, \|\tilde{\sigma}_q - H\|_{L^{d+\epsilon}(B_R(x_0))} \sum_{j=1}^{d} \left\| (\partial_j \varphi^2)(g_0 - \tilde{u}_q) \right\|_{U^{1,1}(B_R(x_0))}. \end{aligned}$$

The last factor can be estimated in the same way as the corresponding factor in I_4 and it remains to show that $\|\sigma_q\|_{L^{d+\epsilon}(\Omega)}$ is bounded independently of $q \geq d + \epsilon$ for fixed $\epsilon > 0$. For the trace $\operatorname{tr} \sigma_q$ we get from lemma 2.5 that

$$\|\operatorname{tr} \sigma_q\|_{L^{d+\epsilon}(\Omega)} \leq c(\epsilon) \left(\|\operatorname{tr} \sigma_q\|_{L^2(\Omega)} + \left\|\sigma_q^D\right\|_{L^{d+\epsilon}(\Omega)} + \|\operatorname{div} \sigma_q\|_{L^{d+\epsilon}(\Omega)} \right). \tag{73}$$

Moreover, by Hölder's inequality and (47) of lemma 4.1,

$$\left\|\sigma_q^D\right\|_{L^{d+\epsilon}(\Omega)} \leq |\Omega|^{\frac{q-(d+\epsilon)}{q(d+\epsilon)}} \left\|\sigma_q^D\right\|_{L^q(\Omega)} \leq c_0(\epsilon) \tag{74}$$

and $c_0(\epsilon)$ is independent of $q \geq d + \epsilon$. Inequalities (73), (74) and $\operatorname{div} \sigma_q + f = 0$ finally imply that there exists a constant $c(\epsilon)$, which is independent of $q \geq d + \epsilon$, such that

$$\|\sigma_q\|_{L^{d+\epsilon}(\Omega)} \leq \|\operatorname{tr} \sigma_q\|_{L^{d+\epsilon}(\Omega)} + \left\|\sigma_q^D\right\|_{L^{d+\epsilon}(\Omega)} \leq c(\epsilon). \tag{75}$$

We obtain finally from the previous estimates that for every $\epsilon, \delta \in (0,1)$ there exists a constant $c_5(\epsilon, \delta)$, which is independent of h and $q \geq d + \epsilon$, such that

$$|I_5| \leq c_5(\epsilon, \delta) \, |h|^{1-\delta}.$$

Collecting the estimates for $I_1, \ldots, I_5$ shows that there exists for every $\epsilon, \delta \in (0,1)$ a constant $c(\epsilon, \delta) > 0$ such that for every $q \geq d + \epsilon$ and $h \in \mathcal{K}$ with $|h| < h_0$

$$I_1 + \ldots + I_5 \leq c(\epsilon, \delta) \, |h|^{1-\delta}. \tag{76}$$

This proves (59). Since $\varphi = 1$ on $B_{R-\theta}(x_0)$, inequality (76) implies together with (58) that

$$\|\sigma_q\|_{\mathcal{N}^{\frac{1-\delta}{2},2}(B_{R-\theta}(x_0)\cap\Omega)} \leq c(\epsilon, \delta). \tag{77}$$

Since the balls $\{B_{R_j-\theta}(x_j), 1 \leq j \leq J\}$ are an open covering of $\overline{\Omega}$, we deduce from (77) the uniform global estimate (50) in lemma 4.3:

$$\|\sigma_q\|_{\mathcal{N}^{\frac{1-\delta}{2},2}(\Omega)} \leq c_{\epsilon,\delta} \tag{78}$$

and $c_{\epsilon,\delta}$ is independent of $q \geq d + \epsilon$. This finishes the proof of lemma 4.3, part 1, and of theorem 3.6. □

Proof of lemma 4.3, part 2. Assume now that $q \geq 2$ and that the functions f, q and H satisfy the weaker assumption $(\mathbf{D}_q)$ from page 9. Let $\tilde{\sigma}_q$ be defined as in (53) and choose $\tilde{u}_q \in W^{1,p}(B_R(x_0))$ with $\tilde{u}_q\big|_{B_R(x_0)\setminus\overline{\Omega}_N} = \hat{u}_q$ with $\hat{u}_q$ from (54). Inequality (58) can be deduced analogously to the previous part. We now have to show that there exists a constant c_q such that for every $h \in \mathcal{K}$, $|h| < h_0$,

$$I_1 + \ldots + I_5 \leq c_q\,|h| \tag{79}$$

and c_q is independent of h but may depend on q. The terms I_1, I_2 and I_{32} may be treated analogously to (60)–(64). For the term I_{31} from (63) we obtain analogously to (65)-(66)

$$|I_{31}| \leq \int_{\Omega_R\setminus\Omega_{R-h}} \varphi^2\left(\left|\varepsilon^D(g_0)\right|\left|\sigma_q^D\right| + |\mathrm{tr}\,\varepsilon(g_0)|\,|\mathrm{tr}\,\sigma_q|\right)\,\mathrm{d}x + \hat{c}_{31}\,|h|\,. \tag{80}$$

Young's inequality implies for every $\delta_1, \delta_2 > 0$

$$|I_{31}| \leq \hat{c}_{31}\,|h| + \int_{\Omega_R\setminus\Omega_{R-h}} \varphi^2\left(\frac{\delta_1^p}{p}\left|\varepsilon^D(g_0)\right|^p + \frac{\delta_2^2}{2}\,|\mathrm{tr}\,\varepsilon(g_0)|^2\right)\,\mathrm{d}x \tag{81}$$

$$+ \int_{\Omega_R\setminus\Omega_{R-h}} \varphi^2\left(\frac{\delta_1^{-q}}{q}\left|\sigma_q^D\right|^q + \frac{\delta_2^{-2}}{2}\,|\mathrm{tr}\,\sigma_q|^2\right)\,\mathrm{d}x. \tag{82}$$

Due to (22)-(23), δ_1 and δ_2 may be chosen in such a way that

$$\left(\frac{\delta_1^{-q}}{q}\left|\sigma_q^D\right|^q + \frac{\delta_2^{-2}}{2}\,|\mathrm{tr}\,\sigma_q|^2\right) - W_{c,q}(\sigma_q) \leq 0.$$

Taking into account that $\nabla g_0 \in L^\infty(\Omega)$, we finally get together with (61)

$$I_1 + |I_3| \leq c_1\,|h|$$

and c_1 is independent of h but may depend on q. The term I_4 can be estimated similar to the first part: applying lemma 7.23 from [21] to the first factor of the last term in (69) shows that

$$|I_4| \leq |h|\,\left\|\nabla(\varphi^2(g_0 - \tilde{u}_q))\right\|_{L^p(B_R(x_0))}\,\|F\|_{L^q(B_R(x_0))}\,.$$

The term I_5 can be treated in the same way as I_4 and (79) is proved. Combining (79) with (58) we get finally, since $\varphi\big|_{B_{R-\theta}(x_0)} = 1$,

$$\int_{\Omega\cap B_{R-\theta}(x_0)} \left(\left|\triangle^h\tilde{\sigma}_q^D\right|^q + \left|\triangle^h\tilde{\sigma}_q\right|^2\right)\,\mathrm{d}x \leq c_q\,|h| \tag{83}$$

for every $h \in \mathcal{K}$ with $|h| < h_0$. The constant c_q is independent of h but may depend on q. Arguing as subsequent to (76) yields

$$\sigma_q^D \in \mathcal{N}^{\frac{1}{q},q}(\Omega), \quad \sigma_q \in \mathcal{N}^{\frac{1}{2},2}(\Omega).$$

This finishes the proof of lemma 4.3. ◻

Our next task is to prove the following lemma on $\mathrm{tr}\,\sigma_q$:

Lemma 4.6. *Under the assumptions of theorem 3.4 it holds* $\operatorname{tr}\sigma_q \in \mathcal{N}^{\frac{1}{q},q}(\Omega)$.

Proof. The proof relies on an argument from [13] which uses Nečas' lemma ([7], see also lemma A.1 in the appendix). Note that $\operatorname{tr}\sigma^q \in L^q(\Omega)$ due to lemma 2.5. We use here the same notation as in the proof of lemma 4.3, in particular, $\tilde{\sigma}_q$ is the function defined in (53). Our goal is to show that

$$\int_{\Omega_{R-\theta}} \left|\triangle^h \operatorname{tr}\tilde{\sigma}_q\right|^q \,\mathrm{d}x \le c_q\,|h|\,.$$

For that purpose we derive uniform estimates of $\triangle^h \operatorname{tr}\tilde{\sigma}_q$ and $\nabla\triangle^h \operatorname{tr}\tilde{\sigma}_q$ in $W^{-1,q}$-norms and apply Nečas' lemma. Let

$$F(x) = \begin{cases} f(x) & x \in \Omega_R, \\ -\operatorname{div} H(x) & x \in \Omega_N, \\ 0 & \text{else.} \end{cases}$$

It holds

$$\operatorname{div}\tilde{\sigma}_q(x) + F(x) = 0 \qquad \text{for a.e. } x \in B_R(x_0)\backslash\overline{\Omega}_D.$$

Moreover, it holds for every $h \in \mathcal{K}$ with $|h| < h_0$ and for every $x \in \Omega \cap B_{R-\theta} = \Omega_{R-\theta}$

$$\operatorname{div}\tilde{\sigma}_q(x-h) + F(x-h) = 0. \tag{84}$$

Thus $\operatorname{div}\triangle^h\tilde{\sigma}_q + \triangle^h F = 0$ a.e. in $\Omega_{R-\theta}$. Multiplying (84) with $v \in \mathcal{C}_0^\infty(\Omega_{R-\theta})$ we get therefore after applying Green's formula:

$$\frac{1}{d}\int_{\Omega_{R-\theta}} \left(\triangle^h \operatorname{tr}\tilde{\sigma}_q I\right) : \varepsilon(v)\,\mathrm{d}x = \int_{\Omega_{R-\theta}} (\triangle^h F)v\,\mathrm{d}x - \int_{\Omega_{R-\theta}} (\triangle^h\tilde{\sigma}_q^D) : \varepsilon(v)\,\mathrm{d}x. \tag{85}$$

By $\nabla(\triangle^h \operatorname{tr}\tilde{\sigma}_q)$ we denote the distributional derivative of $\triangle^h \operatorname{tr}\tilde{\sigma}_q$ on $\Omega_{R-\theta}$. It holds

$$\begin{aligned} \left\|\nabla(\triangle^h \operatorname{tr}\tilde{\sigma}_q)\right\|_{W^{-1,q}(\Omega_{R-\theta})} &= \sup_{\substack{v\in\mathcal{C}_0^\infty(\Omega_{R-\theta},\mathbb{R}^d)\\ \|v\|_{W^{1,p}(\Omega)}=1}} \int_{\Omega_{R-\theta}} \left(\triangle^h \operatorname{tr}\tilde{\sigma}_q\right) \operatorname{div} v\,\mathrm{d}x \\ &\overset{(85)}{=} d \sup_{\substack{v\in\mathcal{C}_0^\infty(\Omega_{R-\theta})\\ \|v\|_{W^{1,p}(\Omega)}=1}} \int_{\Omega_{R-\theta}} (\triangle^h F)v\,\mathrm{d}x - \int_{\Omega_{R-\theta}} (\triangle^h\tilde{\sigma}_q^D) : \varepsilon(v)\,\mathrm{d}x. \end{aligned} \tag{86}$$

We prove now that the right hand side is bounded by $c\,|h|^{\frac{1}{q}}$ with a constant c which is independent of h. It holds for every $v \in \mathcal{C}_0^\infty(\Omega_{R-\theta})$ and every $h \in \mathcal{K}$, $|h| < h_0$, that

$$\int_{\Omega_{R-\theta}} (\triangle^h F)v\,\mathrm{d}x = \int_{B_R(x_0)} \triangle^h(Fv)\,\mathrm{d}x - \int_{B_R(x_0)} F(x-h)\triangle^h v\,\mathrm{d}x. \tag{87}$$

Since $\operatorname{supp} v \subset \Omega_{R-\theta} \subset B_R(x_0) \cap (B_R(x_0) - h))$, the first term vanishes. Moreover, $v \in \mathcal{C}_0^\infty(\Omega_{R-\theta})$ implies together with lemma 7.23 from [21]

$$\left|\int_{\Omega_{R-\theta}} (\triangle^h F)v\,\mathrm{d}x\right| \le \int_{B_R(x_0)} |F(x-h)|\left|\triangle^h v\right|\,\mathrm{d}x \le |h|\,\|F\|_{L^q(B_R(x_0))}\,\|v\|_{W^{1,p}(\Omega_{R-\theta})} \tag{88}$$

and therefore

$$\left\|\triangle^h F\right\|_{W^{-1,q}(\Omega_{R-\theta})} \le |h|\,\|F\|_{L^q(B_R(x_0))}\,.$$

Hölder's inequality yields for the second term in (86)

$$\left|\int_{\Omega_{R-\theta}} \varepsilon(v) : \triangle^h \tilde{\sigma}_q^D \,\mathrm{d}x\right| \le \left\|\triangle^h \tilde{\sigma}_q^D\right\|_{L^q(\Omega_{R-\theta})} \|\varepsilon(v)\|_{L^p(\Omega_{R-\theta})}$$
$$\overset{(83)}{\le} c_2^K c_q \,|h|^{\frac{1}{q}} \,\|v\|_{W^{1,p}(\Omega_{R-\theta})}\,, \tag{89}$$

where c_2^K is the constant in Korn's inequality (16), and c_q is independent of h. Inserting these estimates into (85) results in

$$\left\|\nabla(\triangle^h \operatorname{tr}\tilde{\sigma}_q)\right\|_{W^{-1,q}(\Omega_{R-\theta})} \le |h|^{\frac{1}{q}}\, d(\|F\|_{L^q(_R(x_0))} + c_2^K c_q). \tag{90}$$

Furthermore one gets analogously to (88)

$$\left\|\triangle^h \operatorname{tr}\tilde{\sigma}_q\right\|_{W^{-1,q}(\Omega_{R-\theta})} \le |h|\, \|\operatorname{tr}\tilde{\sigma}_q\|_{L^q(B_R(x_0))}\,. \tag{91}$$

Nečas' lemma A.1 applied to (90) and (91) finally implies $\left\|\triangle^h \operatorname{tr}\tilde{\sigma}_q\right\|_{L^q(\Omega_{R-\theta})} \le c\,|h|^{\frac{1}{q}}$, and the constant c is independent of h. Thus, $\operatorname{tr}\sigma_q \in \mathcal{N}^{\frac{1}{q},q}(\Omega_{R-\theta})$. ☐

Step 3: Regularity of the displacement field u_q

Lemma 4.7. *Let the assumptions of theorem 3.4 be satisfied. Then* $\operatorname{div} u_q \in \mathcal{N}^{\frac{1}{q},q}(\Omega) \cap \mathcal{N}^{\frac{1}{2},2}(\Omega)$ *and* $u_q \in \mathcal{N}^{\frac{3}{2},\frac{2dp}{2d-2+p}-\epsilon}(\Omega)$ *for every* $\epsilon > 0$.

Proof. Note first that $\operatorname{div} u_q = \operatorname{tr}\varepsilon(u_q) = \operatorname{tr}(A\sigma_q)$ in Ω and therefore $\operatorname{div} u_q$ has at least the same smoothness as σ_q.
Let be $\Omega' \subset\subset \Omega$ and $h \in \mathbb{R}^d$ with $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$. For $\epsilon > 0$ we set $r = \frac{2dp}{2d-2+p} - \epsilon$. The constitutive law $\varepsilon(u_q) = A\sigma_q + \alpha_q \left|\sigma_q^D\right|^{q-2} \sigma_q^D$ implies that there exists a constant $c > 0$ depending on r, but not on Ω' and h, such that

$$\int_{\Omega'} |\triangle_h \varepsilon(u_q)|^r \,\mathrm{d}x \le c\left(\int_{\Omega'} |\triangle_h \sigma_q|^r + \left|\triangle_h\left(\left|\sigma_q^D\right|^{q-2}\sigma_q^D\right)\right|^r \mathrm{d}x\right).$$

Hölder's inequality applied to the first term and inequality (95) applied to the second term yields

$$\|\triangle_h \varepsilon(u_q)\|_{L^r(\Omega')}^r \le c\Bigg(\|\triangle_h \sigma_q\|_{L^2(\Omega')}^r + \int_{\Omega'}\left(\left(\left|\sigma_q^D(x+h)\right| + \left|\sigma_q^D\right|\right)^{q-2}\left|\triangle_h \sigma_q^D\right|\right)^r \mathrm{d}x\Bigg).$$

Again by Hölder's inequality we get for the last term

$$\int_{\Omega'}\left(\left(\left|\sigma_q^D(x+h)\right| + \left|\sigma_q^D(x)\right|\right)^{q-2}\left|\triangle_h \sigma_q^D\right|\right)^r \mathrm{d}x$$
$$\le \left(\int_{\Omega'}\left(\left|\sigma_q^D(x+h)\right| + \left|\sigma_q^D\right|\right)^{\frac{(q-2)r}{2-r}} \mathrm{d}x\right)^{\frac{2-r}{2}}$$
$$\times\left(\int_{\Omega'}\left(\left|\sigma_q^D(x+h)\right| + \left|\sigma_q^D\right|\right)^{q-2}\left|\triangle_h \sigma_q^D\right|^2 \mathrm{d}x\right)^{\frac{r}{2}}.$$

From $\sigma_q \in \mathcal{N}^{\frac{1}{q},q}(\Omega) \subset L^{\frac{qd}{d-1}-\epsilon}(\Omega)$ and $\frac{r(q-2)}{2-r} < \frac{dq}{d-1}$ it follows that the first factor is finite and can be estimated independently of h and Ω'. Inequalities (58) and (79) now imply that there exists a constant $c > 0$ such that

$$\int_{\Omega'} |\triangle_h \varepsilon(u_q)|^r \, \mathrm{d}x \le c\,|h|^{\frac{r}{2}} \tag{92}$$

for every $\Omega' \subset\subset \Omega$ and $h \in \mathbb{R}^d$ with $|h| < \operatorname{dist}(\Omega', \partial\Omega)$. This yields $\varepsilon(u_q) \in \mathcal{N}^{\frac{1}{2},r}(\Omega)$. Since $W^{1,p}(\Omega)$ is continuously embedded in $L^r(\Omega)$, Korn's inequality applied to (92) shows that $\nabla u_q \in \mathcal{N}^{\frac{1}{2},r}(\Omega)$ as well and thus $u_q \in \mathcal{N}^{\frac{3}{2},r}(\Omega)$. This finishes the proof of theorem 3.4. □

A Inequalities and proofs for section 2

A.1 Nečas' lemma

For $1 < p < \infty$, the following expression defines a norm for $u \in L^p(\Omega, \mathbb{R})$ with $q = p'$:

$$\begin{aligned} |||u|||_p &= \|u\|_{(W_0^{1,q}(\Omega))'} + \|\nabla u\|_{(W_0^{1,q}(\Omega))'} \\ &= \sup_{\substack{v \in W_0^{1,q}(\Omega,\mathbb{R}), \\ \|v\|_{W^{1,q}(\Omega)}=1}} \left| \int_\Omega uv \, \mathrm{d}x \right| + \sup_{\substack{w \in W_0^{1,q}(\Omega,\mathbb{R}^d), \\ \|w\|_{W^{1,q}(\Omega)}=1}} \left| \int_\Omega u \operatorname{div} w \, \mathrm{d}x \right| . \end{aligned}$$

Lemma A.1 (Nečas). *[7] Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and $1 < p < \infty$. Then $|||\cdot|||_p$ is a norm on $L^p(\Omega)$ which is equivalent to the usual norm $\|\cdot\|_{L^p(\Omega)}$ on $L^p(\Omega)$.*

A.2 Some inequalities

Lemma A.2. *Let $n \in \mathbb{N}$. For $A, B \in \mathbb{R}^n$ with $|B| \ge |A|$ and $t \in [0, \frac{1}{4}]$ it holds [51, formula (2.20)]:*

$$4\,|B + t(A - B)| \ge |A| + |B| . \tag{93}$$

Let $q \ge 2$. It holds for every $A, B \in \mathbb{R}^n$

$$\frac{1}{q}|A|^q - \frac{1}{q}|B|^q - |B|^{q-2} B : (A - B) \ge 2^{-1-2q} (|A| + |B|)^{q-2} |A - B|^2 . \tag{94}$$

$$\left| |A|^{q-2} A - |B|^{q-2} B \right| \le c\,(|A| + |B|)^{q-2} |A - B| . \tag{95}$$

For $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ with $a_i \ge 0$ for $1 \le i \le n$, we have [29]:

$$\left(\sum_{i=1}^n a_i\right)^\alpha \le n^{\alpha-1} \left(\sum_{i=1}^n a_i^\alpha\right) \quad \text{if } \alpha \ge 1, \tag{96}$$

$$\left(\sum_{i=1}^n a_i\right)^\alpha \ge n^{\alpha-1} \left(\sum_{i=1}^n a_i^\alpha\right) \quad \text{if } 0 \le \alpha \le 1. \tag{97}$$

Proof of (94)**.** Let $A, B \in \mathbb{R}^n$ and $\gamma(t) = B + t(A-B)$ for $t \in \mathbb{R}$. Taylor's expansion yields

$$\frac{1}{q}|A|^q - \frac{1}{q}|B|^q - |B|^{q-2} B : (A-B) = \int_0^1 (1-t) \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\frac{1}{q} |\gamma(t)|^q \right) \mathrm{d}t$$
$$\geq \int_0^1 (1-t) |\gamma(t)|^{q-2} |A-B|^2 \, \mathrm{d}t. \tag{98}$$

Assume first that $|B| \geq |A|$. By (93) we obtain

$$(98) \geq 4^{2-q} \int_0^{\frac{1}{4}} (1-t) \, \mathrm{d}t (|A| + |B|)^{q-2} |A-B|^2 .$$

If $|A| > |B|$, then a change of coordinates leads to

$$(98) = \int_0^1 s \, |A + s(B-A)|^{q-2} |A-B|^2 \, \mathrm{d}s \overset{(93)}{\geq} 4^{2-q} \int_0^{\frac{1}{4}} s \, \mathrm{d}s (|A| + |B|)^{q-2} |A-B|^2 .$$

Proof of (95)**.** Again by Taylor's formula:

$$\left| |A|^{q-2} A - |B|^{q-2} B \right| \leq \int_0^1 \left| \frac{d}{dt} \left(|B + t(A-B)|^{q-2} (B + t(A-B)) \right) \right| \mathrm{d}t$$
$$\leq \int_0^1 (q-1) |B + t(A-B)|^{q-2} |A-B| \, \mathrm{d}t.$$

□

A.3 Proof of the surjectivity of γ_1 in lemma 2.1

The surjectivity of the mapping γ_1 in lemma 2.1 is proved by solving a boundary value problem. Let $\Gamma \subset \partial\Omega$ be open and not empty. In order to avoid solvability conditions, which would be necessary in the case $\overline{\Gamma} = \partial\Omega$, an additional boundary is introduced, where Dirichlet conditions are prescribed. Since $\Sigma^{q,q}(\Omega) \subset \Sigma^{q,s}(\Omega)$ for $s \leq q$, it suffices to consider the case $s = q$ in the sequel. Choose $x_0 \in \Omega$ and $\epsilon > 0$ small enough such that $B_{2\epsilon}(x_0) \subset\subset \Omega$. The domain $\tilde{\Omega} = \Omega \backslash \overline{B_\epsilon(x_0)}$ is a bounded domain with Lipschitz boundary $\partial\tilde{\Omega} = \partial\Omega \cup \partial B_\epsilon(x_0)$. Let $h \in W^{-\frac{1}{q},q}(\Gamma)$ and consider the following boundary value problem:

Find $u \in V(\tilde{\Omega}) = \{u : \tilde{\Omega} \to \mathbb{R}^d : u \in W^{1,q'}(\tilde{\Omega}), u\big|_{\partial\Omega\backslash\overline{\Gamma}} = 0, u\big|_{\partial B_\epsilon(x_0)} = 0\}$ *such that for every* $v \in V(\tilde{\Omega})$

$$\int_{\tilde{\Omega}} |\varepsilon(u)|^{q'-2} \varepsilon(u) : \varepsilon(v) \, \mathrm{d}x = \langle h, v \rangle_{\tilde{W}^{1-\frac{1}{q'},q'}(\Gamma)} . \tag{99}$$

Due to the main theorem on monotone operators this problem has a weak solution $u \in V(\tilde{\Omega})$. Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \eta \subset B_{2\epsilon}(x_0)$, $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $B_{\frac{3}{2}\epsilon}(x_0)$. Direct calculations show that the function

$$\sigma(x) = \begin{cases} (1-\eta) |\varepsilon(u(x))|^{q'-2} \varepsilon(u(x)) & x \in \tilde{\Omega}, \\ 0 & x \in \overline{B_\epsilon(x_0)} \end{cases}$$

is an element of $\Sigma^{q,q}(\Omega)$ and satisfies $\sigma\vec{n} = h$ in $W^{-\frac{1}{q},q}(\Gamma)$.

A.4 Proof of lemma 2.5

Let $1 < s \le q$. We have to show that $\Sigma^{q,s}(\Omega) = \Sigma^{q,q}(\Omega)$ and that estimate (20) is valid. Since $s \le q$, it follows with Hölder's inequality, that $\Sigma^{q,q}(\Omega)$ is continuously embedded in $\Sigma^{q,s}(\Omega)$. For the inverse relation it remains to prove that it holds for every $\sigma \in \Sigma^{q,s}(\Omega)$: $\operatorname{tr}\sigma \in L^q(\Omega)$ and (20) is satisfied. For the proof of (20) we use a trick by M.Fuchs, [17], which is based on Bogovskiĭ's theorem [19, Theorem 3.1]:

Let $p \in (1,\infty)$. For every $f \in L^p(\Omega)$, there exists an element $v \in W_0^{1,p}(\Omega,\mathbb{R}^d)$ with

$$\operatorname{div} v = f - \frac{1}{|\Omega|}\int_\Omega f \,\mathrm{d}x \quad \textit{and} \quad \|\nabla v\|_{L^p(\Omega)} \le c_B \|f\|_{L^p(\Omega)},$$

and $c_B > 0$ is a constant, which is independent of f and v.

For $\psi \in \mathcal{C}_0^\infty(\Omega)$, we denote by $v_\psi \in W_0^{1,p}(\Omega)$ the function which is given by Bogovskiĭ's theorem with $p = q' = \frac{q}{q-1}$, i.e.

$$\operatorname{div} v_\psi = \psi - \frac{1}{|\Omega|}\int_\Omega \psi \,\mathrm{d}x \quad \text{and} \quad \|\nabla v_\psi\|_{L^p(\Omega)} \le c_B \|\psi\|_{L^p(\Omega)}. \tag{100}$$

Note that $v_\psi \in U_0^{p,s'}(\Omega)$ since $v_\psi \in W_0^{1,p}(\Omega)$ and $\operatorname{tr}\varepsilon(v_\psi) = \operatorname{div} v_\psi \in \mathcal{C}^\infty(\overline{\Omega})$. It follows for $\sigma \in \Sigma^{q,s}(\Omega)$ by Green's formula (15):

$$\frac{1}{d}\int_\Omega \operatorname{tr}\sigma \operatorname{tr}(\varepsilon(v_\psi))\,\mathrm{d}x = -\int_\Omega \sigma^D : \varepsilon^D(v_\psi)\,\mathrm{d}x - \int_\Omega v_\psi \operatorname{div}\sigma \,\mathrm{d}x. \tag{101}$$

Using (100) we obtain for every $\psi \in \mathcal{C}_0^\infty(\Omega)$:

$$\frac{1}{d}\int_\Omega \psi \operatorname{tr}\sigma\,\mathrm{d}x = -\int_\Omega \sigma^D : \varepsilon^D(v_\psi)\,\mathrm{d}x - \int_\Omega v_\psi \operatorname{div}\sigma\,\mathrm{d}x + \frac{1}{d\,|\Omega|}\int_\Omega \psi\,\mathrm{d}x \int_\Omega \operatorname{tr}\sigma\,\mathrm{d}x.$$

By Hölder's and Poincaré/Friedrichs' inequality

$$\begin{aligned}\frac{1}{d}\left|\int_\Omega \psi \operatorname{tr}\sigma\,\mathrm{d}x\right| &\le \|\sigma^D\|_{L^q(\Omega)} \|\varepsilon^D(v_\psi)\|_{L^p(\Omega)} \\ &\quad + \|v_\psi\|_{L^p(\Omega)} \|\operatorname{div}\sigma\|_{L^q(\Omega)} + d^{-1}|\Omega|^{\frac{1}{q}+\frac{1}{s'}-1}\|\psi\|_{L^p(\Omega)}\|\operatorname{tr}\sigma\|_{L^s(\Omega)} \\ &\le \|\sigma\|_{\Sigma^{q,s}(\Omega)}\Big((1+c_{p,p}^{PF})\|\varepsilon(v_\psi)\|_{L^p(\Omega)} + c_1(\Omega)\|\psi\|_{L^p(\Omega)}\Big)\end{aligned}$$

with $c_1(\Omega) = d^{-1}|\Omega|^{-1+\frac{1}{q}+\frac{1}{s'}}$. Korn's inequality (16), Poincaré/Friedrichs' inequality (19) and estimate (100) imply

$$\|\varepsilon(v_\psi)\|_{L^p(\Omega)} \le c_2^K c_p^{PF}\|\nabla v_\psi\|_{L^p(\Omega)} \le c_2^K c_p^{PF} c_B \|\psi\|_{L^p(\Omega)}.$$

Thus, we obtain finally

$$\|\operatorname{tr}\sigma\|_{L^q(\Omega)} = \sup_{\substack{\psi\in\mathcal{C}_0^\infty(\Omega)\\ \|\psi\|_{L^p(\Omega)}=1}} \left|\int_\Omega \psi\operatorname{tr}\sigma\,\mathrm{d}x\right| \le d\big((1+c_{p,p}^{PF})c_2^K c_p^{PF} c_B + c_1(\Omega)\big)\|\sigma\|_{\Sigma^{q,s}(\Omega)}$$

which finishes the proof of lemma 2.5.

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