Universität Stuttgart

Sonderforschungsbereich 404

Mehrfeldprobleme in der Kontinuumsmechanik

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Griffith-formula and J-integral for a crack in a power-law hardening material

Bericht 2005/12

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ISSN 0949-2046

Griffith-formula and J-integral for a crack in a power-law hardening material

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Abstract

Taking into account the actual regularity of the displacement and stress fields, we derive the well-known Griffith-formula and the Eshelby-Cherepanov-Rice integral for the energy release rate of an elastic body with a straight crack. It is assumed that the constitutive relation is of power-law type (Ramberg/Osgood model).

Keywords: power-law model; fracture mechanics; derivative of the energy functional; J-integral; Eshelby-Cherepanov-Rice integral; Griffith's formula

AMS Subject Classification: 35J60, 74R10, 74G65

1 Introduction

We consider a body with a preexisting crack which is subjected to exterior loadings. The Griffith-criterion [13, 1920] is a classical and commonly applied fracture criterion to decide whether or not the crack will propagate under given forces. In Griffith's energy approach the crack is considered as stationary if the total potential energy in the actual configuration is minimal compared to the energies of all admissible neighbouring configurations. Under suitable assumptions on the crack and the applied forces, this criterion can be reformulated in terms of the energy release rate which is related to the derivative of the potential deformation energy with respect to the crack length. Simple formulas are needed to calculate this quantity.

In the case of linear elastic materials the energy release rate can be expressed by Griffith's formula, the J-integral or by stress intensity factors [15, 5, 6, 24, 18]. These formulas are rigorously proved taking into account the regularity of weak solutions and in particular the special singular stress behaviour of weak solutions near the crack tip. For nonlinear elastic models similar formulas can also be formally deduced under the assumption that weak solutions (displacement and stress fields) are smooth enough or that they admit an asymptotic expansion near the crack tip like in the linear case. But in general such regularity results have not been proved yet and it is even not clear in general whether the terms in these formally derived formulas are well defined for weak solutions.

The goal of this paper is to deduce the well-established Griffith-formula and J-integral for elastic materials with a constitutive relation of power-law type (Ramberg/Osgood model) in a mathematically rigorous way from the definition of the energy release rate taking into

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account the actual regularity of weak solutions. Suitable regularity results were derived in [2, 10, 29] and recently in [20, 19].

The paper is organised as follows: after a short description of Griffith's energy criterion, the Ramberg/Osgood model and the assumptions on the domain in section 2, we formulate our main result (theorem 3.1, Griffith-formula, J-integral) in section 3. Techniques from the linear models treated in [6, 17, 18] are adapted for the proof of our case. The paper closes with an appendix, where we provide frequently used inequalities and a generalised Greens' formula.

2 Formulation of the problem

2.1 Griffith-criterion and energy release rate

Let Ω_0 be a body with preexisting crack C_0 and assume that a loading F is applied to Ω_0. Griffith's fracture criterion reads as follows [15, 22]:

The crack C_0 is stationary with respect to the applied loading F if the total potential energy of the body in the actual configuration is minimal compared to all admissible neighbouring configurations.

The total potential energy $\Pi(\Omega, u, F)$ of an elastic body $\Omega \subset \mathbb{R}^d$ with respect to the displacement field $u : \Omega \to \mathbb{R}^d$ and the exterior loading $F = (f, h)$, where f is a volume force density and h a surface force density, is given by

$$\Pi(\Omega, u, F) = I_{\mathrm{el}}(\Omega, u) - W(\Omega, u, F) + D(\Omega).$$

Here, $I_{\mathrm{el}}(\Omega, u)$ denotes the elastic strain energy

$$I_{\mathrm{el}}(\Omega, u) = \int_\Omega W_{\mathrm{el}}(\varepsilon(u))\,\mathrm{d}x$$

with the stored strain energy density W_{el} which we specify later; $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ denotes the linearised strain tensor. Moreover,

$$W(\Omega, u, F) = \int_\Omega f u\,\mathrm{d}x + \int_{\Gamma_N} h u\,\mathrm{d}s$$

is the work of the exterior forces $F = (f, h)$ and

$$E(\Omega, u, F) = I_{\mathrm{el}}(\Omega, u) - W(\Omega, u, f)$$

denotes the potential deformation energy. The quantity $D(\Omega)$ describes a dissipative energy which in our case characterises the energy which is spent to create the new crack surface. In the simplest case it is assumed that $D(\Omega)$ is proportional to the macroscopic crack surface [15]. We impose rather restrictive assumptions on the geometry of the crack and of possible crack extensions. In particular we assume that the body is in a plane strain state, that the crack is straight and that it can propagate straight on, only. Finally we assume that the crack faces are traction free. This leads to the following condition on the domain $\Omega = \Omega_0$:

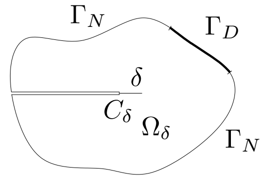

Figure 1: Domain Ω_δ with crack C_δ

H1 Let $S_\delta = \{x \in \mathbb{R}^2 : x_2 = 0, x_1 \leq \delta\}$ for $\delta > 0$. $\tilde{\Omega} \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary and there exist $l, \delta_0 > 0$ such that $\partial\tilde{\Omega} \cap S_\delta = \{(-l, 0)^\top\}$ is a single point for every $|\delta| \leq \delta_0$. We set $\Omega_\delta = \tilde{\Omega} \backslash S_\delta$ and $C_\delta = \tilde{\Omega} \cap S_\delta$ for $|\delta| \leq \delta_0$. The boundary of Ω_δ is split as follows

$$\partial\Omega_\delta = \overline{C_\delta} \cup \overline{\Gamma_N} \cup \overline{\Gamma_D},$$

where $C_\delta, \Gamma_D, \Gamma_N$ are pairwise disjoint and denote the crack with length $l + \delta$, the Dirichlet and the Neumann boundary, respectively. Moreover, $\Gamma_D \neq \emptyset$, $\overline{\Gamma_D} \cap \overline{C_\delta} = \emptyset$, Γ_D and Γ_N are open and do not depend on δ, see figure 1.

We call Ω_0 actual configuration with crack C_0. Note that the domains Ω_δ satisfy the cone condition, [23]. The dissipative energy $D(\Omega)$ takes now the form

$$D(\Omega_\delta) = 2\gamma(l + \delta),$$

where the fracture toughness γ depends on the material. With the above assumptions and notations, Griffith's fracture criterion can be reformulated as follows:

A crack C_0 in a domain Ω_0 is stationary for a given loading $F = (f, h)$, if the potential deformation energy, which would be released at a crack extension, is less than the energy which is needed to create the new surface.

In other words, if for $\delta > 0$

$$E(\Omega_0, u_0, F) - E(\Omega_\delta, u_\delta, F) \leq D(\Omega_\delta) - D(\Omega_0) = 2\gamma\delta,$$

where u_0 and u_δ are the corresponding displacement fields, then the crack C_0 is stationary. This motivates the following definition:

Definition 2.1 (Energy release rate)**.** For $\delta \geq 0$ let u_δ be the displacement field corresponding to Ω_δ and $F = (f, h)$. The energy release rate, shortly ERR, for the domain Ω_0 with crack C_0 and exterior forces $F = (f, h)$ is defined as

$$ERR(\Omega_0, F) = \lim_{\substack{\delta \to 0, \\ \delta > 0}} \frac{1}{\delta}\big(E(\Omega_0, u_0, F) - E(\Omega_\delta, u_\delta, F)\big) \tag{1}$$

$$= -\left(\frac{dE(\Omega_\delta, u_\delta, F)}{d\delta}\right)\Big|_{\delta=0}. \tag{2}$$

The question now is whether (1) is well defined and whether there exist simple formulas to calculate the energy release rate. Up to now we did not specify the underlying material model. It is shown for linear elastic materials, i.e. $W_{\mathrm{el}}(\varepsilon) = \frac{1}{2}C\varepsilon : \varepsilon$, where C denotes the fourth order, symmetric and positive definite elasticity tensor, that the energy release rate is well defined and can be expressed through Griffith's formula, the J-integral or via stress intensity factors, [6, 18, 24]. In this paper we focus on energy densities W_{el} which correspond to power-law hardening models.

2.2 Notation

The following notation is used for $m \times d$-matrices $\theta, \tau \in \mathbb{R}^{m\times d}$:

$$\theta : \tau = \mathrm{tr}(\tau^\top \theta) = \mathrm{tr}(\theta^\top \tau) = \sum_{i=1}^{m}\sum_{j=1}^{d} \theta_{ij}\tau_{ij}, \quad |\theta| = \sqrt{\theta : \theta}.$$

For $P \in \mathbb{R}^d$ and $R > 0$, the set $B_R(P) = \{x \in \mathbb{R}^d : |x - P| < R\}$ denotes the open ball with centre P and radius R. Let us note that we do not distinguish in our notation between scalars, vectors etc. In some special cases we write e.g. $L^p(\Omega, \mathbb{R}^m)$ for vector valued functions $u : \Omega \to \mathbb{R}^m$ being p-integrable.

2.3 The Ramberg/Osgood model

We consider a physically nonlinear elastic material model, where the constitutive relation is given by a power-law like relation ship. In the frame-work of deformation theory of plasticity such models are frequently applied for the description of elastic-plastic materials with low proportionality limit and which show strain hardening behaviour. Examples for such materials are stainless steel alloys or aluminium alloys. The particular model we consider here was first proposed by W. Ramberg and W.R. Osgood [26] and reads as follows for $\Omega \subset \mathbb{R}^2$:

Find a displacement field $u : \Omega \to \mathbb{R}^2$ and a stress tensor field $\sigma : \Omega \to \mathbb{R}^{2\times 2}_{sym}$ such that it holds for given volume and surface force densities f and h and for a given displacement g on Γ_D:

$$\begin{aligned} \operatorname{div}\sigma + f &= 0 \quad \text{in } \Omega, &(3)\\ \varepsilon(u) - A\sigma - \alpha \left|\sigma^D\right|^{q-2}\sigma^D &= 0 \quad \text{in } \Omega, &(4)\\ \sigma\vec{n} &= h \quad \text{on } \Gamma_N, &(5)\\ u &= g \quad \text{on } \Gamma_D. &(6) \end{aligned}$$

Here, $\sigma^D = \sigma - \frac{1}{2}\mathrm{tr}\,\sigma I$ is the deviatoric part of σ, $q \geq 2$ the strain hardening coefficient, $\alpha > 0$ a material parameter depending on the yield stress, $\vec{n}$ the exterior unit normal vector and A the inverse of the elasticity matrix (tensor of elastic compliances). It is assumed that A is symmetric and positive definite, i.e.

$$A_{ijkl} = A_{klij} = A_{jikl} \quad \text{and} \quad \sum_{i,j,k,l=1}^{2} A_{ijkl}\sigma_{ij}\sigma_{kl} = (A\sigma) : \sigma \geq c_A |\sigma|^2 . \tag{7}$$

We assume here that $q \geq 2$ since typical values for q range from $5 - 8$ for austenitic steel alloys [27] and $20 - 45$ for aluminium alloys [25, 30]. The model is known in literature also as Norton/Hoff model and we refer to [27, 30, 4] for more details on physical aspects.

The complementary energy density corresponding to constitutive relation (4) is given by

$$W_c(\sigma) = \frac{1}{2}(A\sigma) : \sigma + \frac{\alpha}{q}\left|\sigma^D\right|^q \tag{8}$$

for $\sigma \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and fixed $q \geq 2$ and the constitutive equation (4) can be rewritten as $\varepsilon(u) = DW_c(\sigma)$ with $DW_c(\sigma)_{ij} = \frac{\partial W_c(\sigma)}{\partial \sigma_{ij}}$. The complementary energy density W_c is strictly convex and the corresponding stored strain energy density W_{el} is defined as the conjugate function of W_c in the sense of convex analysis [35, 7]:

$$\forall\, \varepsilon \in \mathbb{R}^{2\times 2}_{\mathrm{sym}} \quad W_{\mathrm{el}}(\varepsilon) = \sup_{\tau \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}} \big(\tau : \varepsilon - W_c(\tau)\big). \tag{9}$$

To the author's knowledge an explicit formula for W_{el} is unknown.

Lemma 2.2. *There exist constants $c_0, \ldots, c_4 > 0$ such that for every $\varepsilon \in \mathbb{R}^{2\times 2}_{sym}$*

$$-c_0 + c_1 \left|\mathrm{tr}\,\varepsilon\right|^2 + c_2 \left|\varepsilon^D\right|^p \leq W_{el}(\varepsilon) \leq c_3 \left|\mathrm{tr}\,\varepsilon\right|^2 + c_4 \left|\varepsilon^D\right|^p \tag{10}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $q \geq 2$ is the exponent from (4)*. Moreover, W_{el} is strictly convex, continuously differentiable and it holds*

$$W_{el}(\varepsilon) + W_c(\sigma) = \sigma : \varepsilon \;\Leftrightarrow\; DW_{el}(\varepsilon) = \sigma \;\Leftrightarrow\; \varepsilon = DW_c(\sigma). \tag{11}$$

Remark 2.3. Estimate (10) follows by the same arguments as in [32, Chapter III, lemma 1.2], see also [20]. The remaining assertions follow from classical theorems in convex analysis and in particular from [35, Prop. 51.5].

As can be seen from estimate (10), the density W_{el} has different growth properties with respect to $\mathrm{tr}\,\varepsilon$ and ε^D. Therefore, function spaces are needed which take into account this behaviour. Appropriate spaces were first introduced and studied by G. Geymonat and P. Suquet, [12].

2.4 Function spaces and weak formulations

For $s > 0$, $p \in (1, \infty)$, we denote by $W^{s,p}(\Omega)$ the usual Sobolev-Slobodeckij spaces [1, 14]. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a domain and $r, s \in (1, \infty)$.

$$L^{r,s}(\Omega) = \{\sigma : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}} :\, \sigma^D \in L^r(\Omega),\, \mathrm{tr}\,\sigma \in L^s(\Omega)\},$$
$$\Sigma^{r,s}(\Omega) = \{\sigma \in L^{r,s}(\Omega) :\, \mathrm{div}\,\sigma \in L^r(\Omega)\},$$
$$U^{r,s}(\Omega) = \{u : \Omega \to \mathbb{R}^d :\, u \in L^r(\Omega), \varepsilon^D(u) \in L^r(\Omega), \mathrm{tr}\,\varepsilon(u) \in L^s(\Omega)\}.$$

These spaces are endowed with the following natural norms:

$$\left\|\sigma\right\|_{L^{r,s}(\Omega)} = \left\|\sigma^D\right\|_{L^r(\Omega)} + \left\|\mathrm{tr}\,\sigma\right\|_{L^s(\Omega)}, \quad \left\|\sigma\right\|_{\Sigma^{r,s}(\Omega)} = \left\|\sigma\right\|_{L^{r,s}(\Omega)} + \left\|\mathrm{div}\,\sigma\right\|_{L^r(\Omega)},$$
$$\left\|u\right\|_{U^{r,s}(\Omega)} = \left\|u\right\|_{L^r(\Omega)} + \left\|\varepsilon^D(u)\right\|_{L^r(\Omega)} + \left\|\mathrm{tr}\,\varepsilon(u)\right\|_{L^s(\Omega)}$$

and are reflexive and separable Banach-spaces [12]. Moreover, Korn's and Poincaré/Friedrichs' inequalities hold under suitable assumptions on r, s:

Lemma 2.4. *[12] Let $\Omega \subset \mathbb{R}^d$ be a bounded domain which satisfies the cone condition.*

Korn's inequality: *Let $r \in (1, \infty)$. The spaces $U^{r,r}(\Omega)$ and $W^{1,r}(\Omega)$ have the same elements and the norms $\|\cdot\|_{W^{1,r}(\Omega)}$ and $\|\cdot\|_{U^{r,r}(\Omega)}$ are equivalent. That means that there exist constants $c_1^K, c_2^K > 0$ such that for every $u \in W^{1,r}(\Omega)$*

$$c_1^K \|u\|_{W^{1,r}(\Omega)} \leq \|u\|_{L^r(\Omega)} + \|\varepsilon(u)\|_{L^r(\Omega)} \leq c_2^K \|u\|_{W^{1,r}(\Omega)}. \tag{12}$$

Poincaré/Friedrichs' inequality: *Let $1 < r \leq s < \infty$. If $V \subset U^{r,s}(\Omega)$ is a closed subspace with the property $u \in V, \varepsilon(u) = 0 \Rightarrow u = 0$, then there exists a constant $c^{PF} > 0$ such that for every $u \in V$*

$$c^{PF} \|\varepsilon(u)\|_{L^{r,s}(\Omega)} \geq \|u\|_{L^r(\Omega)}. \tag{13}$$

Remark 2.5. Korn's inequality is proved e.g. in [12] for bounded domains with Lipschitz boundaries. Taking into account that bounded domains which satisfy the cone condition can be written as the union of a finite number of Lipschitz domains, Korn's inequality can be carried over to that case, too. The proof of (13) is given in [12] for Lipschitz domains and is based on a Sobolev embedding theorem. Since this theorem is also valid for domains satisfying the cone property [23], the proof in [12] covers also the situation in lemma 2.4.

Finally we have the following relation between $\Sigma^{r,s}(\Omega)$ and $\Sigma^{r,r}(\Omega)$:

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain which satisfies the cone condition, $1 < s \leq r < \infty$. The spaces $\Sigma^{r,s}(\Omega)$ and $\Sigma^{r,r}(\Omega)$ are equal and the corresponding norms are equivalent.*

Remark 2.7. This lemma is based on Bogovskiĭ's theorem [11, Theorem 3.1] and an idea by M. Fuchs [10] and is proved in [19, 20] for bounded Lipschitz domains. Since Bogovskiĭ's theorem is also valid for bounded domains satisfying the cone condition, the proof from [19] applies directly to the situation in lemma 2.6.

It is convenient to work with both weak formulations, the displacement based formulation $\mathbf{P}_\delta$ and the stress based formulation $\mathbf{Q}_\delta$ here below. We define

$$\Omega_{\delta,+} = \Omega_\delta \cap \{x \in \mathbb{R}^2 : x_2 > 0\}, \quad \Omega_{\delta,-} = \Omega_\delta \cap \{x \in \mathbb{R}^2 : x_2 < 0\} \tag{14}$$

and assume **H2**:

H2 $q \geq 2$, $p = q' = \frac{q}{q-1} \in (1, 2)$, $0 \leq \delta \leq \delta_0$, $f \in L^q(\tilde{\Omega})$, $g \in U^{p,2}(\Omega_0)$, $H \in W^{1,q}(\Omega_0, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ with $H\big|_{\Omega_{\delta,\pm}} \vec{n}_\pm = 0$ on C_{δ_0}, where $\vec{n}_+ = (0, -1)^\top$ and $\vec{n}_- = (0, 1)^\top$.

The stress based weak formulation for (3)-(6) reads as follows:

$\mathbf{Q}_\delta$ *Find a stress field $\sigma_\delta \in L^{q,2}(\Omega_\delta)$ and a displacement field $u_\delta \in U^{p,2}(\Omega_\delta)$ with $u\big|_{\Gamma_D} = g\big|_{\Gamma_D}$ such that it holds for every $\tau \in L^{q,2}(\Omega_\delta)$ and $v \in U^{p,2}(\Omega_\delta)$ with $v\big|_{\Gamma_D} = 0$:*

$$\int_{\Omega_\delta} DW_c(\sigma_\delta) : \tau \,\mathrm{d}x = \int_{\Omega_\delta} \varepsilon(u_\delta) : \tau \,\mathrm{d}x, \tag{15}$$

$$\int_{\Omega_\delta} \sigma_\delta : \varepsilon(v) \,\mathrm{d}x = \int_{\Omega_\delta} (f + \operatorname{div} H) v \,\mathrm{d}x + \int_{\Omega_\delta} H : \varepsilon(v) \,\mathrm{d}x. \tag{16}$$

Note that Green's formula applied to Ω_+ and Ω_- separately implies

$$\int_{\Omega_\delta} v \operatorname{div} H \,\mathrm{d}x + \int_{\Omega_\delta} H : \varepsilon(v) \,\mathrm{d}x = \int_{\Gamma_N} v(H\vec{n}) \,\mathrm{d}s$$

for every $v \in U^{p,2}(\Omega_\delta)$ with $u\big|_{\Gamma_D} = 0$. The requirement $H\vec{n} = 0$ on C_δ realises the assumption that the crack faces are traction free. The displacement based formulation reads

$\mathbf{P}_\delta$ *Find a displacement field* $u_\delta \in U^{p,2}(\Omega_\delta)$ *with* $u_\delta\big|_{\Gamma_D} = g\big|_{\Gamma_D}$ *such that it holds for every* $v \in U^{p,2}(\Omega_\delta)$ *with* $v\big|_{\Gamma_D} = 0$

$$\int_{\Omega_\delta} DW_{\mathrm{el}}(\varepsilon(u_\delta)) : \varepsilon(v) \,\mathrm{d}x = \int_{\Omega_\delta} (f + \operatorname{div} H) v \,\mathrm{d}x + \int_{\Omega_\delta} H : \varepsilon(v) \,\mathrm{d}x. \tag{17}$$

Finally we consider the following minimisation problem for $0 \le \delta \le \delta_0$:

$\mathbf{M}_\delta$ *Let* $F = (f, H\vec{n})$. *Find a displacement field* $u_\delta \in U^{p,2}(\Omega_\delta)$ *with* $u_\delta\big|_{\Gamma_D} = g\big|_{\Gamma_D}$ *such that it holds for every* $v \in U^{p,2}(\Omega_\delta)$ *with* $v\big|_{\Gamma_D} = g\big|_{\Gamma_D}$

$$I_{\mathrm{el}}(\Omega_\delta, u_\delta) - W(\Omega_\delta, u_\delta, F) \le I_{\mathrm{el}}(\Omega_\delta, v) - W(\Omega_\delta, v, F). \tag{18}$$

Here, $I_{\mathrm{el}}(\Omega_\delta, v) = \int_{\Omega_\delta} W_{\mathrm{el}}(\varepsilon(v)) \,\mathrm{d}x$ with W_{el} from (9) and $W(\Omega_\delta, v, F) = \int_{\Omega_\delta} (f + \operatorname{div} H) v \,\mathrm{d}x + \int_{\Omega_\delta} H : \varepsilon(v) \,\mathrm{d}x$.

Theorem 2.8. *Let* **H1** *and* **H2** *be satisfied. Problems* $\mathbf{Q}_\delta$, $\mathbf{P}_\delta$ *and* $\mathbf{M}\delta$ *are uniquely solvable and equivalent and* I_{el} *is Fréchet-differentiable with* $\langle DI_{el}(\Omega_\delta, \varepsilon(v_1)), \varepsilon(v_2)\rangle = \int_{\Omega_\delta} DW_{el}(\varepsilon(v_1)) : \varepsilon(v_2) \,\mathrm{d}x$ *for every* $v_1, v_2 \in U^{p,2}(\Omega_\delta)$. *Note that* $\sigma_\delta \in \Sigma^{q,2}(\Omega_\delta)$ *due to the assumptions on* f *and thus* $\sigma_\delta \in L^q(\Omega_\delta)$ *due to lemma 2.6. Finally it holds for the weak solution* $(u_\delta, \sigma_\delta) \in U^{p,2}(\Omega_\delta) \times \Sigma^{q,q}(\Omega_\delta)$:

$$I_{el}(\Omega_\delta, u_\delta) + I_c(\Omega_\delta, \sigma_\delta) = \int_{\Omega_\delta} \sigma_\delta : \varepsilon(u_\delta) \,\mathrm{d}x.$$

Here, $I_c(\Omega_\delta, \sigma_\delta) = \int_{\Omega_\delta} W_c(\sigma_\delta) \,\mathrm{d}x$.

Proof. The theorem follows with standard arguments from convex analysis, see e.g. [35, Prop. 51.5] and from the direct method in the calculus of variations. Note that $\mathbf{P}_\delta$ is the weak Euler-Lagrange equation of $\mathbf{M}_\delta$. $\square$

2.5 A priori estimates and regularity

In the next lemma we show that weak solutions are uniformly bounded with respect to the parameter δ. Such estimates are essential in the derivation of formulas for the energy release rate.

Lemma 2.9. *Assume* **H1** *and* **H2**. *There exists a constant* $\tilde{c} > 0$ *such that it holds for every* $0 \le \delta \le \delta_0$ *and every weak solution* $(u_\delta, \sigma_\delta) \in U^{p,2}(\Omega_\delta) \times L^{q,2}(\Omega_\delta)$:

$$\|\varepsilon(u_\delta)\|_{L^{p,2}(\Omega_\delta)}, \; \|u_\delta\|_{W^{1,p}(\Omega_\delta)} \le \tilde{c}, \tag{19}$$

$$\|\sigma_\delta\|_{L^{q,2}(\Omega_\delta)}, \; \|\operatorname{tr} \sigma_\delta\|_{L^q(\Omega_\delta)} \le \tilde{c}. \tag{20}$$

Proof. Note first that the constants in Poincré/Friedrichs' inequality are uniformly bounded with respect to $\delta \in [0, \delta_0]$, i.e. there exist $c_1, c_2 > 0$ such that

$$c_1 \le c_\delta^{PF} \le c_2 \tag{21}$$

for every $\delta \in [0, \delta_0]$, and similar for the constants in Korn's inequality. This is due to $V^{p,2}(\Omega_{\delta_1}) \subset V^{p,2}(\Omega_{\delta_2})$ for $\delta_1 \le \delta_2$.

We prove now (19)-(20). The occurring numbers c may vary from line to line but they are independent of δ. Choosing $\tau = \sigma_\delta$ and $v = u_\delta - g$ as test functions for the weak formulation $\mathbf{Q}_\delta$, adding both equations and applying the Hölder and the Poincaré/Friedrichs inequality yields

$$\begin{aligned}\|\sigma_\delta\|^2_{L^2(\Omega_\delta)} + \|\sigma_\delta^D\|^q_{L^q(\Omega_\delta)} \le \frac{1}{\min\{c_A, \alpha\}} \Big(& 2(\|\sigma_\delta\|_{L^{q,2}(\Omega_\delta)} + \|H\|_{L^{q,2}(\Omega_\delta)}) \, \|\varepsilon(g)\|_{L^{p,2}(\Omega_\delta)} \\ & + (\|f\|_{L^q(\Omega_\delta)} + \|\operatorname{div} H\|_{L^q(\Omega_\delta)}) \, \|u_\delta - g\|_{L^p(\Omega_\delta)} \\ & + 2 \, \|H\|_{L^{q,2}(\Omega_\delta)} \, \|\varepsilon(u_\delta)\|_{L^{q,2}(\Omega_\delta)} \Big). \end{aligned} \tag{22}$$

By Poincaré/Friedrichs' inequality (13), (21) and the triangle inequality one obtains

$$\|u_\delta - g\|_{L^p(\Omega_\delta)} \le c(\|\varepsilon(u_\delta)\|_{L^{p,2}(\Omega_\delta)} + \|\varepsilon(g)\|_{L^{p,2}(\Omega_\delta)}) \tag{23}$$

and c is independent of δ due to (21). Thus there exists a constant c which is independent of δ such that

$$\|\sigma_\delta\|^2_{L^2(\Omega_\delta)} + \|\sigma_\delta^D\|^q_{L^q(\Omega_\delta)} \le c\big(1 + \|\sigma_\delta\|_{L^{q,2}(\Omega_\delta)} + \|\varepsilon(u_\delta)\|_{L^{p,2}(\Omega_\delta)}\big). \tag{24}$$

It follows from the constitutive law (4) and inequality (88) that

$$|\operatorname{tr} \varepsilon(u_\delta)| = |A\sigma_\delta| \le c \, |\sigma_\delta| \, , \tag{25}$$

$$|\varepsilon(u_\delta)|^p \le c_p(|A\sigma_\delta|^p + \alpha^p \, |\sigma_\delta^D|^q) \le c(|\sigma_\delta|^2 + 1 + |\sigma_\delta^D|^q). \tag{26}$$

The last estimate follows from Young's inequality with $|\sigma_\delta|^p \le \frac{p}{2} |\sigma_\delta|^2 + \frac{2-p}{2}$ for $p \in (1, 2)$. Thus

$$\begin{aligned}\|\operatorname{tr} \varepsilon(u_\delta)\|^2_{L^2(\Omega_\delta)} + \|\varepsilon^D(u_\delta)\|^p_{L^p(\Omega_\delta)} &\le c\big(\|\sigma_\delta\|^2_{L^2(\Omega_\delta)} + \|\sigma_\delta^D\|^q_{L^q(\Omega_\delta)} + 1\big) \\ &\overset{(24)}{\le} c\big(1 + \|\sigma_\delta\|_{L^{q,2}(\Omega_\delta)} + \|\varepsilon(u_\delta)\|_{L^{p,2}(\Omega_\delta)}\big). \end{aligned} \tag{27}$$

The constant c is independent of δ. Adding (24) and (27) yields

$$\begin{aligned}&\|\operatorname{tr} \varepsilon(u_\delta)\|^2_{L^2(\Omega_\delta)} + \|\varepsilon^D(u_\delta)\|^p_{L^p(\Omega_\delta)} + \|\operatorname{tr} \sigma_\delta\|^2_{L^2(\Omega_\delta)} + \|\sigma_\delta^D\|^q_{L^q(\Omega_\delta)} \\ &\quad \le c\big(1 + \|\operatorname{tr} \varepsilon(u_\delta)\|_{L^2(\Omega_\delta)} + \|\varepsilon^D(u_\delta)\|_{L^p(\Omega_\delta)} + \|\operatorname{tr} \sigma_\delta\|_{L^2(\Omega_\delta)} + \|\sigma_\delta^D\|_{L^q(\Omega_\delta)}\big) \end{aligned} \tag{28}$$

and the constant c is independent of δ. Since the left hand side of (28) grows at least with power $p > 1$ and the right hand side grows linearly, it follows that there exists a constant $\tilde{c} > 0$, which depends on c but not on δ, such that

$$\|\varepsilon(u_\delta)\|_{L^{p,2}(\Omega_\delta)} + \|\sigma_\delta\|_{L^{q,2}(\Omega_\delta)} \le \tilde{c}. \tag{29}$$

Combining (29) with (23) and (21) yields (19). Since σ_δ is a weak solution it holds $\sigma_\delta \in \Sigma^{q,q}(\Omega)$, see theorem 2.8. Furthermore, lemma 2.6 implies that there exists a constant $c_\delta > 0$ such that

$$\|\mathrm{tr}\,\sigma_\delta\|_{L^q(\Omega_\delta)} \leq c_\delta \,\|\mathrm{tr}\,\sigma_\delta\|_{L^2(\Omega_\delta)} \overset{(29)}{\leq} c_\delta \tilde{c}.$$

It follows in the same way as for the constant c_δ^{PF} of the Poincaré/Friedrichs inequality that there exist $c_1, c_2 > 0$ with $c_1 \leq c_\delta \leq c_2$ for every $\delta \in [0, \delta_0]$. This finishes the proof of (20). □

For the derivation of our main result, theorem 3.1, we need also higher differentiability of weak solutions near plane parts of the boundary and in the interior of the domain. Specialised to the two dimensional domains Ω_δ it holds [2, 20, 19]:

Theorem 2.10. *Assume* **H1**, **H2** *and let* $(u_\delta, \sigma_\delta) \in U^{p,2}(\Omega_\delta) \times \Sigma^{q,q}(\Omega_\delta)$ *be a weak solution of* $\mathbf{P}_\delta$ *and* $\mathbf{Q}_\delta$.
Local regularity: *For every* $\epsilon > 0$

$$\sigma_\delta,\ \mathrm{div}\, u_\delta \in W^{1,2}_{loc}(\Omega_\delta) \cap W^{\frac{2}{q}-\epsilon,q}_{loc}(\Omega_\delta), \tag{30}$$
$$u_\delta \in W^{2,2-\epsilon}_{loc}(\Omega_\delta). \tag{31}$$

Regularity near the crack face C_δ: *Let* $P \in C_\delta$ *and* $r > 0$ *such that* $\overline{B_r(P)} \cap \big(\partial\tilde{\Omega} \cup \{(\delta, 0)^\top\}\big) = \emptyset$. *Let furthermore* $B_r^+(P) = B_r(P) \cap \Omega_{\delta,+}$ *with* $\Omega_{\delta,+}$ *from* (14). *It holds for every* $\epsilon > 0$:

$$\sigma_\delta,\ \mathrm{div}\, u_\delta \in W^{\frac{1}{q-1}-\epsilon,q}(B_r^+(P)), \tag{32}$$
$$\left|\sigma_\delta^D\right|^{q-2} \sigma_\delta^D \in W^{1,p-\epsilon}(B_r^+(P)), \tag{33}$$
$$u_\delta \in W^{2,p-\epsilon}(B_R^+(P)), \tag{34}$$
$$\frac{\partial}{\partial x_1} u_\delta \in W^{1,p}(B_r^+(P)). \tag{35}$$

Moreover, $\|\partial_1 u_\delta\|_{W^{1,p}(B_r^+(P))} \leq c_\delta \leq \hat{c}$, *where* c_δ *depends continuously on* $c^K_{1,\delta}$, $c^K_{2,\delta}$, c^{PF}_δ, $\|f\|_{L^q(\Omega_\delta)}$, $\|\sigma_\delta\|_{L^q(\Omega_\delta)}$, $\|u_\delta\|_{W^{1,p}(\Omega_\delta)}$. *Here,* $c^K_{1,\delta}$, $c^K_{2,\delta}$, c^{PF}_δ, *are the constants in Korn's inequality* (12) *and in Poincaré/Friedrichs' inequality* (13), *respectively. Since all these quantities are bounded with respect to* $\delta \in [0, \delta_0]$, *see lemma 2.9, the constants* c_δ *are bounded as well by a new constant* $\hat{c}$.

Remark 2.11. For a proof of the local regularity results we refer to [2] and [20, Theorem 2.3]. The regularity near plane parts of the boundaries and the crack face C_δ is proved with a difference quotient technique in [20, Theorems 2.19, 3.33] based on the ideas from [9]. Let us note that local regularity results for the displacement and stress fields of a closely related material model were derived by G.A. Seregin [29] and M. Fuchs [10] and correlate with the results cited above.

3 Griffith-formula and J-integral

We introduce two further hypotheses:

H3 $\theta \in \mathcal{C}_0^\infty(\tilde{\Omega})$ with $\theta = 1$ in a neighbourhood of the crack tip $(0,0)^\top$, $f \in \mathcal{C}^1(\overline{\tilde{\Omega}})$.

H3' $f \in \mathcal{C}^1(\overline{\tilde{\Omega}})$ and there exists $R > 0$ such that $\overline{B_R(0)} \subset \tilde{\Omega}$ and $\frac{\partial}{\partial x_1} f\big|_{B_R(0)} = 0$. Furthermore, $\Gamma \subset B_{R'}(0)$, $R' < R$, is a not self-intersecting Lipschitz continuous path around the crack tip $(0,0)^\top$ with normal vector $\vec{n}$ pointing into the surrounded domain. For example, $\Gamma = \partial B_{R'/2}(0)$, $\vec{n} = -x/|x|$, $x \in \Gamma$.

Theorem 3.1. *Assume* **H1**, **H2** *and let* $(u_0, \sigma_0) \in U^{p,2}(\Omega_0) \times \Sigma^{q,q}(\Omega_0)$ *be a weak solution of* $\mathbf{P}_0$ *and* $\mathbf{Q}_0$.

1. *Assume in addition that* **H3** *is satisfied. The energy release rate is well defined for the Ramberg/Osgood model and the* **Griffith-formula** *is valid:*

$$ERR(\Omega_0, F) = \int_{\Omega_0} \sigma_0 : (\partial_1 u_0 \otimes \nabla\theta)_{sym} \, \mathrm{d}x + \int_{\Omega_0} u_0 \, \partial_1(\theta f) \, \mathrm{d}x \\ - \int_{\Omega_0} \big(\sigma_0 : \varepsilon(u_0) - W_c(\sigma_0)\big)\partial_1\theta \, \mathrm{d}x. \quad (36)$$

2. *Let* **H1**, **H2** *and* **H3'** *be satisfied. Then*

$$ERR(\Omega_0, F) = \int_\Gamma (\sigma_0 \vec{n}) \, \partial_1 u_0 \, \mathrm{d}s - \int_\Gamma \big(\sigma_0 : \varepsilon(u_0) - W_c(\sigma_0)\big) n_1 \, \mathrm{d}s + \int_\Gamma u f n_1 \, \mathrm{d}s. \quad (37)$$

   This path integral is called **J-integral**.

The integrands of (36) *and* (37) *are* L^1*-functions and the formulas are independent of the special choice of the function* θ *and the path* Γ.

Remark 3.2. The J-integral and its generalisations was first discovered by J.D. Eshelby [8, 1951] and applied in fracture mechanics by G.P. Cherepanov [5, 1967] and J. Rice [28, 1968]. In literature it is also called Cherepanov-Rice integral and is a frequently used quantity in fracture criteria for linear and nonlinear material models [3, 15, 30]. As we already mentioned in the introduction, a mathematical rigorous derivation of (36)-(37) taking into account the actual regularity of weak solutions is to the author's knowledge carried out for linear elastic materials, only: (36)-(37) is proved by P. Destuynder and M. Djaoua [6, 1981] for traction free stress faces and by A.M. Khludnev and J. Sokolowski [17, 18, 1999/2000] for mutual non-penetration conditions on the crack faces. Furthermore, V.G. Maz'ya and S.A. Nazarov [24, 1987] proved a formula for the energy release rate which is based on stress intensity factors. In this paper we transfer the arguments from the linear case [6, 17, 18] to the Ramberg/Osgood model in order to obtain (36)-(37) taking into account the actual regularity of solutions formulated in theorem 2.10.

4 Proof of theorem 3.1

The proof of theorem 3.1 is long and technical. The arguments in [6, 17, 18], where this theorem is proved for linear elastic materials, have to be transferred to our case. The main idea there is to construct a diffeomorphism $T_\delta : \Omega_\delta \to \Omega_0$ and to transform the integral expressions in the difference quotient $\delta^{-1}(E(\Omega_0, u_0, F) - E(\Omega_\delta, u_\delta, F))$ to the fixed domain Ω_0. The limes is then calculated in the transformed expressions. Our proof is split into the following steps:

Step 1: Let $(u_\delta, \sigma_\delta)$ be the weak solution of problem $\mathbf{P}_\delta$. By $(u^\delta, \sigma^\delta)$ we denote the transformed displacement and stress fields: $u^\delta = u_\delta \circ T_\delta^{-1}$, $\sigma^\delta = \sigma_\delta \circ T_\delta^{-1}$ with T_δ from (38) here below. In the first step we prove that $u^\delta - u_0$ and $\sigma^\delta - \sigma_0$ are admissible test functions for $\mathbf{P}_0$ and we show the convergence $u^\delta \to u_0$, $\sigma^\delta \to \sigma_0$ for $\delta \to 0$. For this step regularity results for u_δ and the uniform a-priori estimates for u_δ and σ_δ are needed.

Step 2: Griffith's formula is deduced based on the convergence results from the first step. The main tools are a mean value theorem for Fréchet differentiable functionals and Lebesgue's convergence theorem.

Step 3: The J-integral is derived from Griffith's formula by a generalised Green's formula. For this step the regularity results listed in theorem 2.10 are essential. Note that we do not need in our proof the actual regularity and structure of the displacement and stress fields near the crack tip. In particular we do not make any assumptions on the asymptotic behaviour of u_0 or σ_0 near the crack tip.

4.0.1 Step 1: Convergence of u^δ and σ^δ

For $\delta \in [0, \delta_0]$ the domain Ω_δ is transformed to Ω_0 in the following way: let $\theta \in \mathcal{C}_0^\infty(\tilde{\Omega})$ be a function according to **H3**. We define

$$T_\delta : \Omega_\delta \to \Omega_0, \quad x \mapsto y = T_\delta(x) = x - \delta \begin{pmatrix} \theta(x) \\ 0 \end{pmatrix}. \tag{38}$$

It is

$$\nabla_x T_\delta(x) = \begin{pmatrix} 1 - \delta\theta_{,1}(x) & -\delta\theta_{,2}(x) \\ 0 & 1 \end{pmatrix}, \quad \det{}_\delta(x) = \det \nabla_x(T_\delta(x)) = 1 - \delta\theta_{,1}(x), \tag{39}$$

where we use the abbreviation $\theta_{,i}(x) = \frac{\partial}{\partial x_i}\theta(x)$ for $i \in \{1, 2\}$. The mapping T_δ is an element of $\mathcal{C}^\infty(\overline{\Omega_\delta})$ and $\det \nabla_x(T_\delta(x)) > 0$ if δ is small enough. Therefore T_δ is a diffeomorphism [6]. For functions $v_\delta : \Omega_\delta \to \mathbb{R}^2$ we introduce the notation

$$v^\delta(y) = v_\delta(T_\delta^{-1}(y)) \quad \text{for } y \in \Omega_0.$$

Derivatives are transformed as follows for $x \in \Omega_\delta$ and $y \in \Omega_0$:

$$\nabla_x v_\delta(T_\delta^{-1}(y)) = \nabla_y v^\delta(y) - \delta \left(\frac{\partial}{\partial y_1} v^\delta(y) \otimes \nabla_x \theta(T_\delta^{-1}(y)) \right). \tag{40}$$

For a function $v : \Omega_0 \to \mathbb{R}^2$, $y \mapsto v(y)$ with $v_\delta(x) = v(T_\delta(x))$ for $x \in \Omega_\delta$ it holds

$$\nabla_y v(T_\delta(x)) = \frac{1}{\det_\delta(x)} \nabla_x v_\delta(x) + \delta \frac{1}{\det_\delta(x)} \nabla_x v_\delta(x) \begin{pmatrix} 0 & \theta_{,2}(x) \\ 0 & -\theta_{,1}(x) \end{pmatrix}. \tag{41}$$

In the sequel we use the following abbreviations for $y \in \Omega_0$

$$\nabla_x \theta^\delta(y) = \nabla_x \theta(T_\delta^{-1}(y)),\ \det{}^\delta(y) = \det{}_\delta(T_\delta^{-1}(y)) = \det(\nabla_x T_\delta(x))\big|_{x = T_\delta^{-1}(y)}.$$

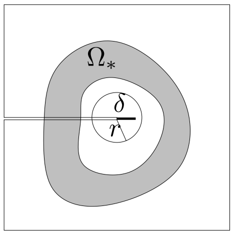

Figure 2: Example for the notation

Finally we define

$$\Omega_* = \overline{\cup_{\delta\in[0,\delta_0]} \operatorname{supp}\left(\nabla_x \theta^\delta\right)} = \overline{\cup_{\delta\in[0,\delta_0]} T_\delta(\operatorname{supp}\nabla_x\theta)}. \tag{42}$$

If δ_0 is small enough then it holds

$$\operatorname{dist}(\Omega_*, \partial\tilde{\Omega}) > 0$$

and there exists $r > 0$ such that $B_r(0) \cap \Omega_* = \emptyset$ which means that the set Ω_* has a positive distance to the boundary of $\tilde{\Omega}$ and also to the crack tip $(0, 0)$, see figure 2. Therefore the regularity results from theorem 2.10 hold on $\Omega_\delta \cap \Omega_*$.

The following lemma states that if a weak solutions $(u_\delta, \sigma_\delta)$ of $\mathbf{P}_\delta$ is transformed by T_δ then the transformed functions are in the same class of spaces as the original functions. Due to the anisotropic structure of the space $U^{p,2}$ this is not obvious and in general the set $U^{p,2}(\Omega_\delta) \circ T_\delta^{-1}$ is not contained in $U^{p,2}(\Omega_0)$. But for weak solutions the differential properties are preserved after the transformation.

Lemma 4.1. *Let $(u_\delta, \sigma_\delta) \in U^{p,2}(\Omega_\delta) \times L^{q,2}(\Omega_\delta)$ be a weak solution of $\mathbf{P}_\delta$. Then it holds for every $\delta \in [0, \delta_0]$*

$$u^\delta = u_\delta \circ T_\delta^{-1} \in U^{p,2}(\Omega_0) \;\; \textit{with} \;\; u^\delta\big|_{\Gamma_D} = g, \tag{43}$$

$$u_0 \circ T_\delta \in U^{p,2}(\Omega_\delta) \;\; \textit{with} \;\; u_0 \circ T_\delta\big|_{\Gamma_D} = g. \tag{44}$$

Furthermore T_δ induces an isomorphism between the spaces $W^{1,r}(\Omega_0)$ and $W^{1,r}(\Omega_\delta)$ in the following way for $r \in (1, \infty)$:

$$\mathbf{T}_\delta : W^{1,r}(\Omega_0) \to W^{1,r}(\Omega_\delta) : \; u \mapsto u \circ T_\delta.$$

For fixed $r \in (1, \infty)$ the operator norms of $\mathbf{T}_\delta$ and $\mathbf{T}_\delta^{-1}$ are bounded with respect to $\delta \in [0, \delta_0]$. A similar result holds for the spaces $L^{r,s}(\Omega_0)$ and $L^{r,s}(\Omega_\delta)$ with $r, s \in (1, \infty)$.

Proof. Note that the space $U^{p,2}(\Omega_\delta)$ is equal to

$$U^{p,2}(\Omega_\delta) = \{u \in W^{1,p}(\Omega_\delta) : \operatorname{tr}\varepsilon(u) = \operatorname{div} u \in L^2(\Omega_\delta)\}$$

due to Korn's inequality. By simple calculations one can see immediately that $u_\delta \circ T_\delta^{-1} \in W^{1,p}(\Omega_0)$ and $u_0 \circ T_\delta \in W^{1,p}(\Omega_\delta)$. Furthermore $T_\delta\big|_{\Gamma_D} = id$ since $\operatorname{supp}\theta \cap \Gamma_D = \emptyset$ by assumptions **H1** and **H3**. Therefore $u_\delta \circ T_\delta^{-1}\big|_{\Gamma_D} = g = u_0 \circ T_\delta\big|_{\Gamma_D}$. It remains to prove that $\operatorname{tr}\varepsilon_x(u_0 \circ T_\delta) \in L^2(\Omega_\delta)$ and $\operatorname{tr}\varepsilon_y(u^\delta) \in L^2(\Omega_0)$. After a change of coordinates we obtain

$$\begin{aligned}\int_{\Omega_\delta} |\operatorname{tr}\varepsilon_x(u_0 \circ T_\delta(x))|^2 \,\mathrm{d}x &= \int_{\Omega_0} \frac{1}{\det^\delta(y)} \left|\operatorname{tr}\varepsilon_x(u_0(y))\right|^2 \,\mathrm{d}y \\ &\overset{(40)}{\leq} c\left(\|\operatorname{tr}\varepsilon_y(u_0)\|^2_{L^2(\Omega_0)} + \delta^2 \int_{\Omega_0} \left|\frac{\partial}{\partial y_1} u_0(y)\right|^2 \left|\nabla_x \theta^\delta\right|^2 \,\mathrm{d}y\right). \end{aligned} \tag{45}$$

The first term is finite since $u_0 \in U^{p,2}(\Omega_0)$. Furthermore it is $\operatorname{supp}(\nabla_x\theta^\delta) \subset \Omega_*$ and by (31), (35) and the Sobolev embedding theorems $\frac{\partial}{\partial y_1} u_0\big|_{\Omega_*} \in W^{1,p}(\Omega_*) \subset L^2(\Omega_*)$. Therefore the second term in (45) is finite as well and relation (44) is proved. In a similar way one can show relation (43). The mapping properties of $\mathbf{T}_\delta$ follow by straight forward calculations. □

The following corollary is an immediate consequence of lemmata 2.9, 4.1 and of theorem 2.10.

Corollary 4.2. *There exists a constant $C > 0$ which is independent of δ such that it holds for every $\delta \in [0, \delta_0]$ and every weak solution $(u_\delta, \sigma_\delta) \in U^{p,2}(\Omega_\delta) \times L^{q,2}(\Omega_\delta)$ of $\mathbf{P}_\delta$:*

$$\left\|u^\delta\right\|_{W^{1,p}(\Omega_0)}, \left\|\sigma^\delta\right\|_{L^{q,2}(\Omega_0)}, \left\|\operatorname{tr}\sigma^\delta\right\|_{L^{q,2}(\Omega_0)} \leq c, \tag{46}$$

$$\left\|\frac{\partial}{\partial x_1} u_\delta\right\|_{L^2(\Omega_* \cap \Omega_\delta)} \leq c. \tag{47}$$

As before, $u^\delta = u_\delta \circ T_\delta^{-1}$ etc.

We are now ready to formulate the convergence properties of the transformed solutions u^δ and σ^δ for $\delta \to 0$.

Lemma 4.3. *For $\delta \in [0, \delta_0]$ let $(u_\delta, \sigma_\delta) \in U^{p,2}(\Omega_\delta) \times L^{q,2}(\Omega_\delta)$ be the weak solution of $\mathbf{P}_\delta$. There exists a constant $c > 0$ such that for every $\delta \in [0, \delta_0]$*

$$\left\|\sigma^\delta - \sigma_0\right\|_{L^2(\Omega_0)} \leq c\delta^{\frac{1}{2}}, \tag{48}$$

$$\left\|\sigma^{\delta,D} - \sigma_0^D\right\|_{L^q(\Omega_0)} \leq c\delta^{\frac{1}{q}}, \tag{49}$$

$$\left\|\varepsilon_y(u^\delta) - \varepsilon_y(u_0)\right\|_{L^{p,2}(\Omega_0)} \leq c\delta^{\frac{1}{2}}, \tag{50}$$

$$\left\|\tilde{\varepsilon}_\delta(u^\delta) - \varepsilon_y(u_0)\right\|_{L^{p,2}(\Omega_0)} \leq c\delta^{\frac{1}{2}}. \tag{51}$$

Here we have set for $y \in \Omega_0$

$$\tilde{\varepsilon}_\delta(u^\delta)(y) = \varepsilon_y(u^\delta(y)) - \delta\left(\frac{\partial}{\partial y_1} u^\delta(y) \otimes \nabla_x\theta^\delta(y)\right)_{sym}. \tag{52}$$

Remark 4.4. It is not clear whether the exponents in (48)-(51) are optimal. In [18], the sharper result $\left\|u^\delta - u_0\right\|_{W^{1,2}(\Omega_0)} \leq c\delta$ is shown for linear elasticity. For the proof of the main theorem on Griffith's formula it is sufficient to have convergence of $u^\delta \to u_0$, the particular estimate (50) is not relevant.

Proof. The lemma is proved by transforming problem $\mathbf{P}_\delta$ to the domain Ω_0 and by inserting $\sigma^\delta - \sigma_0$ and $u^\delta - u_0$ as test functions. The a-priori estimates from lemma 2.9 and corollary 4.2 play an essential role.

Note first that the function H in (16) may be replaced by

$$\tilde{H} = \eta H, \tag{53}$$

where $\eta \in \mathcal{C}^\infty(\mathbb{R}^{2\times 2})$ with $\eta\big|_{\partial\tilde{\Omega}} = 1$ and $\operatorname{supp}\eta \cap \operatorname{supp}\theta = \emptyset$, since it holds due to Green's formula for every $v \in U^{p,2}(\Omega_\delta)$ with $v\big|_{\Gamma_D} = 0$:

$$\int_{\Omega_\delta} \eta H : \varepsilon(v)\,\mathrm{d}x + \int_{\Omega_\delta} v \operatorname{div}(\eta H)\,\mathrm{d}x = \int_{\Omega_\delta} H : \varepsilon(v)\,\mathrm{d}x + \int_{\Omega_\delta} v \operatorname{div} H\,\mathrm{d}x.$$

Let $(u_\delta, \sigma_\delta) \in U^{p,2}(\Omega_\delta) \times L^{q,2}(\Omega_\delta)$ be the weak solution of $\mathbf{P}_\delta$ and $\sigma^\delta = \sigma_\delta \circ T_\delta^{-1}$, $u^\delta = u_\delta \circ T_\delta^{-1}$. Since $\mathbf{T}_\delta$ is an isomorphism between the spaces $L^{q,2}(\Omega_0)$ and $L^{q,2}(\Omega_\delta)$ it follows after a change of coordinates in equation (15) that u^δ and σ^δ satisfy for every $\tau \in L^{q,2}(\Omega_0)$

$$\int_{\Omega_0} \frac{1}{\det^\delta} DW_c(\sigma^\delta) : \tau \mathrm{d}y = \int_{\Omega_0} \frac{1}{\det^\delta} \left(\varepsilon_y(u^\delta) - \delta \left(\frac{\partial}{\partial y_1} u^\delta \otimes \nabla_x \theta^\delta \right)_{\text{sym}} \right) : \tau \mathrm{d}y. \tag{54}$$

Moreover it follows from the weak formulation (16) that σ^δ satisfies the following equation for every $v \in W^{1,p}(\Omega_0)$ with $v \circ T_\delta \in V^{p,2}(\Omega_\delta) = \{w \in U^{p,2}(\Omega_\delta) : w\big|_{\Gamma_D} = 0\}$

$$\begin{aligned}\int_{\Omega_0} \frac{1}{\det^\delta} \sigma^\delta : \left(\varepsilon_y(v) - \delta \left(\frac{\partial}{\partial y_1} v \otimes \nabla_x \theta^\delta \right)_{\text{sym}} \right) \mathrm{d}y \\ = \int_{\Omega_0} \frac{v}{\det^\delta} \left(f^\delta + \operatorname{div}_x \tilde{H}^\delta \right) \mathrm{d}y + \int_{\Omega_0} \frac{1}{\det^\delta} \tilde{H}^\delta : \tilde{\varepsilon}_\delta(v)\,\mathrm{d}y.\end{aligned} \tag{55}$$

As before, $f^\delta = f \circ T_\delta^{-1}$ etc.; $\tilde{\varepsilon}_\delta$ is defined in (52). Due to lemma 4.1 it holds

- $\sigma^\delta - \sigma_0 \in L^{q,2}(\Omega_0)$ is an admissible test function for $\mathbf{P}_0$ and for (54),
- $u^\delta - u_0 \in V^{p,2}(\Omega_0)$ is an admissible test function for $\mathbf{P}_0$,
- $(u^\delta - u_0) \circ T_\delta \in V^{p,2}(\Omega_\delta)$ and therefore $u^\delta - u_0$ is admissible for (55).

Testing $\mathbf{P}_0$ and equations (54)-(55) with $\sigma^\delta - \sigma_0$ and $u^\delta - u_0$ and subtracting corresponding equations leads to

$$\begin{aligned}\int_{\Omega_0} \left(\frac{1}{\det^\delta} DW_c(\sigma^\delta) - DW_c(\sigma_0) \right) : (\sigma^\delta - \sigma_0)\,\mathrm{d}x \\ = \int_{\Omega_0} \left(\frac{1}{\det^\delta} \varepsilon_y(u^\delta) - \varepsilon_y(u_0) - \frac{\delta}{\det^\delta} \left(\frac{\partial}{\partial y_1} u^\delta \otimes \nabla_x \theta^\delta \right) \right) : (\sigma^\delta - \sigma_0)\,\mathrm{d}y.\end{aligned} \tag{56}$$

For (55) we obtain

$$
\begin{aligned}
&\int_{\Omega_0}\Big(\frac{1}{\det^\delta}\sigma^\delta-\sigma_0\Big):(\varepsilon_y(u^\delta)-\varepsilon_y(u_0))\,\mathrm{d}y\\
&\qquad=\delta\int_{\Omega_0}\frac{1}{\det^\delta}\left(\frac{\partial}{\partial y_1}(u^\delta-u_0)\otimes\nabla_x\theta^\delta\right):\sigma^\delta\,\mathrm{d}y\\
&\qquad\quad+\int_{\Omega_0}\left(\frac{1}{\det^\delta}(f^\delta+(\operatorname{div}_x\tilde H)^\delta)-f-\operatorname{div}_y\tilde H\right)(u^\delta-u_0)\,\mathrm{d}y\\
&\qquad\quad+\int_{\Omega_0}\left(\frac{1}{\det^\delta}\tilde H^\delta-\tilde H\right):(\varepsilon_y(u^\delta)-\varepsilon_y(u_0))\,\mathrm{d}x\\
&\qquad\quad-\delta\int_{\Omega_0}\frac{1}{\det^\delta}\tilde H^\delta:\left(\frac{\partial}{\partial y_1}(u^\delta-u_0)\otimes\nabla_x\theta^\delta\right)\,\mathrm{d}y
\end{aligned}\tag{57}
$$

By the definition of $\tilde H$ we have $\operatorname{supp}\tilde H\cap\operatorname{supp}\theta=\emptyset$ and therefore $T_\delta\big|_{\operatorname{supp}\tilde H}=id$ which implies $\operatorname{supp}(\tilde H^\delta)=T^\delta(\operatorname{supp}\tilde H)=\operatorname{supp}\tilde H$. Moreover $\tilde H^\delta(y)=\tilde H(y)$ for every $y\in\Omega_0$ and thus $\frac{1}{\det^\delta}\tilde H^\delta-\tilde H=0$ for every $y\in\Omega_0$ and also $\frac{1}{\det^\delta}(\operatorname{div}_x\tilde H)^\delta-\operatorname{div}_y\tilde H=0$ in Ω_0. These considerations show that the terms with $\tilde H$ vanish in (57). Note further that

$$
\begin{aligned}
&\Big(\frac{1}{\det^\delta}DW_c(\sigma^\delta)-DW_c(\sigma_0)\Big):(\sigma^\delta-\sigma_0)\\
&\quad=\Big(DW_c(\sigma^\delta)-DW_c(\sigma_0)\Big):(\sigma^\delta-\sigma_0)+\delta\frac{\theta^\delta_{,1}}{\det^\delta}DW_c(\sigma^\delta):(\sigma^\delta-\sigma_0),
\end{aligned}\tag{58}
$$

$$
\begin{aligned}
&\Big(\frac{1}{\det^\delta}\varepsilon_y(u^\delta)-\varepsilon_y(u_0)\Big):(\sigma^\delta-\sigma_0)\\
&\quad=\left(\frac{1}{\det^\delta}\sigma^\delta-\sigma_0\right):(\varepsilon_y(u^\delta)-\varepsilon_y(u_0))+\delta\frac{\theta^\delta_{,1}}{\det^\delta}\Big(\varepsilon_y(u_0):\sigma^\delta-\sigma_0:\varepsilon_y(u^\delta)\Big),
\end{aligned}\tag{59}
$$

where $\theta^\delta_{,i}=\theta_{,i}(T_\delta^{-1}(\cdot))$, $i=1,2$. Combining equations (56)-(59) results in

$$
\begin{aligned}
&\int_{\Omega_0}\big(DW_c(\sigma^\delta)-DW_c(\sigma_0)\big):(\sigma^\delta-\sigma_0)\,\mathrm{d}y\\
&\quad=-\delta\int_{\Omega_0}\left(\frac{\theta^\delta_{,1}}{\det^\delta}DW_c(\sigma^\delta)+\frac{1}{\det^\delta}\left(\frac{\partial}{\partial y_1}u^\delta\otimes\nabla_x\theta^\delta\right)_{\mathrm{sym}}\right):(\sigma^\delta-\sigma_0)\,\mathrm{d}y\\
&\qquad+\delta\int_{\Omega_0}\frac{\theta^\delta_{,1}}{\det^\delta}\Big(\varepsilon_y(u_0):\sigma^\delta-\sigma_0:\varepsilon_y(u^\delta)\Big)\,\mathrm{d}y\\
&\qquad+\delta\int_{\Omega_0}\frac{1}{\det^\delta}\left(\frac{\partial}{\partial y_1}(u^\delta-u_0)\otimes\nabla_x\theta^\delta\right):\sigma^\delta\,\mathrm{d}y\\
&\qquad+\int_{\Omega_0}\left(\frac{1}{\det^\delta}f^\delta-f\right)(u^\delta-u_0)\,\mathrm{d}y\\
&\quad=\delta(I_1+I_2+I_3)+I_4(\delta).
\end{aligned}\tag{60}
$$

Hölder's inequality and the uniform a-priori estimates of corollary 4.2 imply the existence of a constant $c_1\geq 0$ such that for every $\delta\in[0,\delta_0]$

$$|I_1+I_2+I_3|\leq c_1.\tag{61}$$

Moreover it follows from $f \in \mathcal{C}^1(\overline{\widetilde{\Omega}})$ and the uniform a-priori estimates for u^δ (corollary 4.2) that there exists a constant $c_2 \geq 0$ such that

$$|I_4(\delta)| \leq c_2\delta. \tag{62}$$

Inequality (92) from the appendix applied to the left hand side of (60) finally leads to

$$\begin{aligned}
&\big\|\sigma^\delta - \sigma_0\big\|^2_{L^2(\Omega_0)} + \big\|\sigma^{\delta,D} - \sigma_0^D\big\|^q_{L^q(\Omega_0)} \\
&\qquad\leq \big\|\sigma^\delta - \sigma_0\big\|^2_{L^2(\Omega_0)} + \int_{\Omega_0} \big(\big|\sigma^{\delta,D}\big| + \big|\sigma_0^D\big|\big)^{q-2}\big|\sigma^{\delta,D} - \sigma_0^D\big|^2 \,\mathrm{d}x \leq c\delta
\end{aligned} \tag{63}$$

for $\delta \in [0, \delta_0]$ and the constant c is independent of δ. This implies estimates (48) and (49).

For the proof of (51) we use the relation $\tilde{\varepsilon}_\delta(u^\delta(y)) = DW_c(\sigma^\delta(y))$ for $y \in \Omega_0$. It follows by Hölder's inequality, inequality (87) and the uniform a-priori estimates of corollary 4.2 that

$$\begin{aligned}
\int_{\Omega_0} \big|\tilde{\varepsilon}_\delta(u^\delta) - \varepsilon_y(u_0)\big|^p \,\mathrm{d}y &= \int_{\Omega_0} \Big|DW_c(\sigma^\delta) - DW_c(\sigma_0)\Big|^p \,\mathrm{d}y \\
&\overset{(87)}{\leq} c\big\|\sigma^\delta - \sigma_0\big\|^p_{L^2(\Omega_0)} \\
&\quad + c\int_{\Omega_0} \big(\big|\sigma^{D,\delta}\big| + \big|\sigma_0^D\big|\big)^{\frac{p(q-2)}{2}}\Big(\big(\big|\sigma^{D,\delta}\big| + \big|\sigma_0^D\big|\big)^{\frac{q-2}{2}}\big|\sigma^{D,\delta} - \sigma_0^D\big|\Big)^p \,\mathrm{d}y \\
&\overset{(48)}{\leq} c\delta^{\frac{p}{2}} + c\Big\|\big|\sigma^{D,\delta}\big| + \big|\sigma_0^D\big|\Big\|^{\frac{q(2-p)}{2}}_{L^q(\Omega_0)}\bigg(\int_{\Omega_0}\Big(\big|\sigma^{D,\delta}\big| + \big|\sigma_0^D\big|\Big)^{q-2}\Big|\sigma^{D,\delta} - \sigma_0^D\Big|^2 \mathrm{d}y\bigg)^{\frac{p}{2}} \\
&\overset{(63)}{\leq} c\delta^{\frac{p}{2}},
\end{aligned} \tag{64}$$

where the constant c is independent of δ. Moreover, again due to $\tilde{\varepsilon}_\delta(u^\delta) = DW_c(\sigma^\delta)$, it holds

$$\big\|\mathrm{tr}(\tilde{\varepsilon}_\delta(u^\delta)) - \mathrm{tr}\,\varepsilon_y(u_0)\big\|_{L^2(\Omega_0)} \leq c\big\|\sigma^\delta - \sigma_0\big\|_{L^2(\Omega_0)} \overset{(48)}{\leq} c\delta^{\frac{1}{2}}. \tag{65}$$

Estimates (64) and (65) imply (51). Finally, (50) follows from (51) and (52) taking into account (47) and the definition of Ω_* in (42). $\square$

Step 2: Energy release rate and Griffith's formula

We will now prove that the limes in the definition of the energy release rate, definition 2.1, exists and that it can be expressed through Griffith's formula. The following notation is used for $\delta \in [0, \delta_0]$ and $v \in U^{p,2}(\Omega_\delta)$

$$E(\Omega_\delta, v) = \int_{\Omega_\delta} W_{\mathrm{el}}(\varepsilon(v)) \,\mathrm{d}x - \int_{\Omega_\delta} (f + \operatorname{div} \tilde{H})v \,\mathrm{d}x - \int_{\Omega_\delta} \tilde{H} : \varepsilon(v) \,\mathrm{d}x. \tag{66}$$

Here, $W_{\mathrm{el}}(\varepsilon) = \sup_{\tau \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}} (\tau : \varepsilon - W_c(\tau))$ is the stored strain energy density for Ramberg/Osgood materials from (9) and $\tilde{H} = \eta H$, see (53). For $v \in W^{1,p}(\Omega_0)$ with $\tilde{\varepsilon}_\delta(v) \in L^{p,2}(\Omega_0)$ (see (52)) we define

$$\begin{aligned}
E_\delta(\Omega_0, v) = &\int_{\Omega_0} \frac{1}{\det^\delta} W_{\mathrm{el}}(\tilde{\varepsilon}_\delta(v)) \,\mathrm{d}y - \int_{\Omega_0} \frac{1}{\det^\delta}(f^\delta + \operatorname{div}_x \tilde{H}^\delta)v \,\mathrm{d}y \\
&- \int_{\Omega_0} \frac{1}{\det^\delta} \tilde{H}^\delta : \tilde{\varepsilon}_\delta(v) \,\mathrm{d}y.
\end{aligned} \tag{67}$$

Note that it holds for every weak solution u_δ of $\mathbf{P}_\delta$:

$$E_\delta(\Omega_0, u^\delta) = E(\Omega_\delta, u_\delta), \quad E_\delta(\Omega_0, u_0) = E(\Omega_\delta, u_0 \circ T_\delta).$$

Taking into account that u_0 is a minimiser of $E(\Omega_0, \cdot)$ and that u_δ is a minimiser of $E(\Omega_\delta, \cdot)$ and noting that $u_\delta \circ T_\delta^{-1}$ is admissible for $\mathbf{M}_0$ and $u_0 \circ T_\delta$ is admissible for $\mathbf{M}_\delta$ we obtain for every $\delta \in (0, \delta_0]$:

$$\begin{aligned} \delta^{-1}(E(\Omega_0, u_0) - E_\delta(\Omega_0, u_0)) &= \delta^{-1}(E(\Omega_0, u_0) - E(\Omega_\delta, u_0 \circ T_\delta)) \\ &\leq \delta^{-1}(E(\Omega_0, u_0) - E(\Omega_\delta, u_\delta)) \\ \leq \delta^{-1}(E(\Omega_0, u_\delta \circ T_\delta^{-1}) - E(\Omega_\delta, u_\delta)) &= \delta^{-1}(E(\Omega_0, u^\delta) - E_\delta(\Omega_0, u^\delta)). \end{aligned} \tag{68}$$

In order to show that the energy release rate $\lim_{\delta\to 0} \delta^{-1}(E(\Omega_0, u_0) - E(\Omega_\delta, u_\delta))$ is well defined, we calculate the limes superior of the right hand side in (68) and the limes inferior of the left hand side and show that they are finite and coincide. We begin with the limes superior. For $\delta \in (0, \delta_0]$ it is

$$\begin{aligned} E(\Omega_0, u^\delta) - E_\delta(\Omega_0, u^\delta) &= \int_{\Omega_0} W_{\mathrm{el}}(\varepsilon_y(u^\delta)) - W_{\mathrm{el}}(\tilde{\varepsilon}_\delta(u^\delta))\,\mathrm{d}y \\ &\quad - \delta \int_{\Omega_0} \frac{\theta^\delta_{,1}}{\det^\delta} W_{\mathrm{el}}(\tilde{\varepsilon}_\delta(u^\delta))\,\mathrm{d}y - \int_{\Omega_0} (f - \frac{1}{\det^\delta} f^\delta) u^\delta\,\mathrm{d}y \\ &\quad - \int_{\Omega_0} \left(\operatorname{div}_y \tilde{H} - \frac{1}{\det^\delta} \operatorname{div}_x \tilde{H}^\delta \right) u^\delta\,\mathrm{d}y \\ &\quad - \int_{\Omega_0} \left(\tilde{H} - \frac{1}{\det^\delta} \tilde{H}^\delta \right) : \varepsilon_y(u^\delta)\,\mathrm{d}y \\ &\quad - \delta \int_{\Omega_0} \frac{1}{\det^\delta} \tilde{H}^\delta : \left(\frac{\partial}{\partial y_1} u^\delta \otimes \nabla_x \theta^\delta \right)_{\mathrm{sym}} \mathrm{d}y \\ &= I_1 + \ldots + I_6 \end{aligned} \tag{69}$$

From the definition of $\tilde{H}$ it follows $\operatorname{supp} \tilde{H} \cap \operatorname{supp} \theta = \emptyset$ and $T_\delta\big|_{\operatorname{supp} \tilde{H}} = id$. Thus, the terms I_4, I_5 and I_6 vanish.

Convergence of $\frac{1}{\delta} I_1$: The first term on the right hand side of (69) can be rewritten as follows

$$\frac{1}{\delta} I_1 = \frac{1}{\delta} \left(I_{\mathrm{el}}(\varepsilon_y(u^\delta)) - I_{\mathrm{el}}(\tilde{\varepsilon}(u^\delta)) \right)$$

where $I_{\mathrm{el}} : L^{p,2}(\Omega_0) \to \mathbb{R}$, $\varepsilon \mapsto \int_{\Omega_0} W_{\mathrm{el}}(\varepsilon)\,\mathrm{d}y$ is the stored strain energy for Ramberg/Osgood materials. Due to theorem 2.8, I_{el} is Fréchet-differentiable with derivative

$$\langle DI_{\mathrm{el}}(\varepsilon_1), \varepsilon_2 \rangle_{L^{p,2}(\Omega_0)} = \int_{\Omega_0} DW_{\mathrm{el}}(\varepsilon_1) : \varepsilon_2\,\mathrm{d}x$$

for every $\varepsilon_1, \varepsilon_2 \in L^{p,2}(\Omega_0)$. By the mean value theorem for Fréchet-differentiable functionals (lemma A.3 in the appendix) there exists a constant $t_\delta \in [0, 1]$ such that

$$\frac{1}{\delta} I_1 = \int_{\Omega_0} DW_{\mathrm{el}} \left(\varepsilon_y(u^\delta) - \delta t_\delta \left(\frac{\partial}{\partial y_1} u^\delta \otimes \nabla_x \theta^\delta \right)_{\mathrm{sym}} \right) : \left(\frac{\partial}{\partial y_1} u^\delta \otimes \nabla_x \theta^\delta \right)_{\mathrm{sym}} \mathrm{d}y.$$

Note that the term $\left(\frac{\partial}{\partial y_1}u^\delta \otimes \nabla_x \theta^\delta\right)_{\mathrm{sym}}$ is an element of $L^{p,2}(\Omega_0)$ due to theorem 2.10. From the convergence results (50)-(51) it follows for $\delta \to 0$

$$\begin{aligned}
\left(\frac{\partial}{\partial y_1}u^\delta \otimes \nabla_x \theta^\delta\right)_{\mathrm{sym}} &\longrightarrow \left(\frac{\partial}{\partial y_1}u_0 \otimes \nabla_y \theta\right)_{\mathrm{sym}} && \text{in } L^{p,2}(\Omega_0),\\
\varepsilon_y(u^\delta) - \delta t_\delta \left(\frac{\partial}{\partial y_1}u^\delta \otimes \nabla_x \theta^\delta\right)_{\mathrm{sym}} &\longrightarrow \varepsilon_y(u_0) && \text{in } L^{p,2}(\Omega_0).
\end{aligned}$$

From the continuity of the Fréchet-derivative DI_{el} of I_{el} we obtain for $\delta \to 0$

$$\frac{1}{\delta}I_1 \to \int_{\Omega_0} DW_{\mathrm{el}}(\varepsilon_y(u_0)) : \left(\frac{\partial}{\partial y_1}u_0 \otimes \nabla_y \theta\right)_{\mathrm{sym}} \mathrm{d}y. \tag{70}$$

Convergence of $\frac{1}{\delta}I_2$: Since $\theta \in \mathcal{C}_0^\infty(\tilde{\Omega})$, it holds for $\delta \to 0$

$$\frac{\partial}{\partial x_1}\theta^\delta \to \frac{\partial}{\partial y_1}\theta \quad \text{in } L^\infty(\Omega_0). \tag{71}$$

Moreover the mapping $W_{\mathrm{el}} : L^{p,2}(\Omega_0) \to L^1(\Omega_0)$, $\varepsilon \mapsto W_{\mathrm{el}}(\varepsilon)$ is continuous due to estimate (10) and proposition 26.6 in [36] on Nemytskii-operators. Combining (71) and and the convergence result (51) leads to

$$\frac{1}{\delta}I_2 = -\int_{\Omega_0} \frac{\theta^\delta_{,1}}{\det^\delta} W_{\mathrm{el}}(\tilde{\varepsilon}_\delta(u^\delta))\,\mathrm{d}y \to -\int_{\Omega_0} W_{\mathrm{el}}(\varepsilon_y(u_0))\frac{\partial}{\partial y_1}\theta\,\mathrm{d}y \quad \text{for } \delta \to 0. \tag{72}$$

Convergence of $\frac{1}{\delta}I_3$: The following identities are valid for $y \in \Omega_0$:

$$-\frac{1}{\delta}\left(f(y) - \frac{1}{\det^\delta}f^\delta(y)\right) = -\frac{1}{\delta}\left(f(y) - f^\delta(y)\right) + \frac{\theta^\delta_{,1}}{\det^\delta}f^\delta(y), \tag{73}$$

$$y - T_\delta^{-1}(y) \overset{y=T_\delta(x)}{=} T_\delta(x) - x = -\delta\begin{pmatrix}\theta^\delta(y)\\ 0\end{pmatrix}. \tag{74}$$

From (71) and assumption **H3** we obtain for the second term on the right hand side of (73)

$$\frac{\theta^\delta_{,1}}{\det^\delta}f^\delta \to f\frac{\partial}{\partial y_1}\theta \quad \text{in } L^\infty(\Omega_0). \tag{75}$$

The mean value theorem and (74) imply that there exists a constant $t_{(\delta,y)} \in [0,1]$ such that it holds for the first term on the right hand side of (73)

$$\frac{1}{\delta}\left(f(y) - f^\delta(y)\right) = -\nabla_y f(\xi(\delta,y))\begin{pmatrix}\theta^\delta(y)\\ 0\end{pmatrix},$$

where $\xi(\delta,y) = y - \delta t_{(\delta,y)}\begin{pmatrix}\theta^\delta(y)\\ 0\end{pmatrix}$. Note that $\xi(\delta,y)$ converges uniformly on Ω_0 to y for $\delta \to 0$. Due to assumption **H3**, the functions f and $\nabla_y f$ are uniformly continuous on $\overline{\tilde{\Omega}}$ and therefore

$$\frac{1}{\delta}(f - f^\delta) = -\nabla_y f(\xi(\delta,\cdot))\begin{pmatrix}\theta^\delta\\ 0\end{pmatrix} \to -\theta\frac{\partial}{\partial y_1}f \quad \text{in } L^\infty(\Omega_0). \tag{76}$$

Combining (75) and (76) we finally arrive at

$$\frac{1}{\delta}I_3 = -\frac{1}{\delta}\int_{\Omega_0} u^\delta \left(f - \frac{1}{\det^\delta} f^\delta \right) \mathrm{d}y \to \int_{\Omega_0} u_0 \frac{\partial}{\partial y_1}(\theta f)\,\mathrm{d}y \tag{77}$$

for $\delta \to 0$. Summing up (70), (72) and (77) we get

$$\begin{aligned}
\limsup_{\delta\to 0} \delta^{-1}(E(\Omega_0, u_0) - E(\Omega_\delta, u_\delta)) &\leq \lim_{\delta\to 0} \delta^{-1}(E(\Omega_0, u^\delta) - E_\delta(\Omega_0, u^\delta)) \\
&= \int_{\Omega_0} DW_{\mathrm{el}}(\varepsilon_y(u_0)) : \left(\frac{\partial}{\partial y_1} u_0 \otimes \nabla_y \theta\right)_{\mathrm{sym}} \mathrm{d}y \\
&\quad - \int_{\Omega_0} W_{\mathrm{el}}(\varepsilon_y(u_0)) \frac{\partial}{\partial y_1}\theta\,\mathrm{d}y + \int_{\Omega_0} u_0 \frac{\partial}{\partial y_1}(\theta f)\,\mathrm{d}y.
\end{aligned} \tag{78}$$

Due to lemma 2.2 the terms $W_{\mathrm{el}}(\varepsilon_y(u_0))$ and $DW_{\mathrm{el}}(\varepsilon_y(u_0))$ can be replaced by $\sigma_0 : \varepsilon_y(u_0) - W_c(\sigma_0)$ and σ_0, respectively.

The limes inferior of the left hand side in (68) can be calculated similarly and it coincides with (78). This shows that the energy release rate is well defined and that relation (36) holds.

Step 3: J-integral

The representation of the energy release rate by the J-integral is deduced from Griffith's formula with the help of a generalised Green's formula which we formulate and prove in the appendix, lemma A.2. The regularity results of theorem 2.10 are fundamental for this step.

Let assumptions **H1**, **H2** and **H3'** be satisfied and let $\theta \in \mathcal{C}_0^\infty(B_R(0))$ with $\theta\big|_{B_{R'}(0)} = 1$. Here we use the notation from **H3'**. The functions f and θ satisfy **H3** as well and therefore Griffith's formula is valid:

$$\begin{aligned}
ERR(\Omega_0, F) &= \int_{\Omega_0} (\sigma_0 \partial_1 u_0)\nabla\theta\,\mathrm{d}x + \int_{\Omega_0} u_0 f \partial_1\theta\,\mathrm{d}x + \int_{\Omega_0} \theta u_0 \partial_1 f\,\mathrm{d}x \\
&\quad - \int_{\Omega_0} \sigma_0 : \varepsilon(u_0)\partial_1\theta\,\mathrm{d}x + \int_{\Omega_0} W_c(\sigma_0)\partial_1\theta\,\mathrm{d}x \\
&= I_1 + \ldots + I_5.
\end{aligned} \tag{79}$$

Using the generalised Green's formula of lemma A.2 and the regularity results of theorem 2.10 we show that it is possible to integrate (79) by parts. In order to simplify the notation we assume that the path Γ is a circular path around the crack tip: $\Gamma = \partial B_{R_0}(0)$ for some $R_0 < R'$, see figure 3. Let us note that all arguments here below can easily be carried over to the case of general non intersecting Lipschitz paths Γ satisfying **H3'**.

We set $\Omega_* = B_R(0)\backslash(B_{R_0}(0) \cup C_0)$. It holds $\operatorname{supp}\nabla\theta \subset \overline{\Omega_*}$ and therefore the integration domains in I_1, I_2, I_4 and I_5 can be replaced by Ω_*. Since $\overline{\Omega_*}$ has a positive distance to the exterior boundary $\partial\tilde{\Omega}$ and also to the crack tip, the regularity results of theorem 2.10 are valid on Ω_*. It follows from these regularity results that

$$\begin{aligned}
\theta\partial_1 u_0 &\in W^{1,p}(\Omega_*), \\
\sigma_0 &\in B_{q,\frac{1}{q-1}-\epsilon,q}(\Omega_*)
\end{aligned}$$

for $\epsilon > 0$, where

$$B_{\delta,s,\gamma} = \{v : \Omega_* \to \mathbb{R}^d : \, v \in W^{s,\delta}(\Omega_*),\, \operatorname{div} v \in L^\gamma(\Omega_*)\}$$

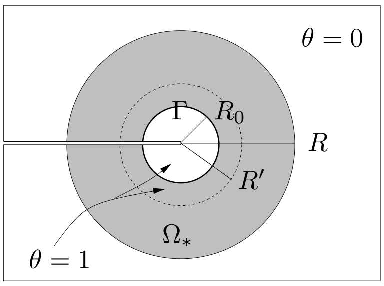

Figure 3: Circular path Γ

is the space defined in lemma A.2. The parameters $\delta = q$, $s = \frac{1}{q-1} - \epsilon$ and $\gamma = q$ satisfy (b) in lemma A.2 with $d = 2$. Thus we obtain for I_1 by Green's formula

$$I_1 = \int_{\partial\Omega_*} \theta\partial_1 u_0(\sigma_0\vec{n})\,\mathrm{d}s - \int_{\Omega_*} \theta\sigma_0 : \varepsilon(\partial_1 u_0)\,\mathrm{d}x - \int_{\Omega_*} \theta\partial_1 u_0 \operatorname{div}\sigma_0\,\mathrm{d}x \tag{80}$$

and the integrands are L^1-functions. Theorem 2.10 and assumption **H3'** imply

$$u_0 f \in W^{1,p-\epsilon}(\Omega_*)$$

for every $\epsilon > 0$ and therefore, again by lemma A.2,

$$I_2 = -\int_{\Omega_*} \theta(f\partial_1 u_0 + u_0\partial_1 f)\,\mathrm{d}x + \int_{\partial\Omega_*} \theta u_0 f n_1\,\mathrm{d}s. \tag{81}$$

The term I_4 can be rewritten as follows with the product rule

$$I_4 = -\sum_{i,j=1}^{2} \int_{\Omega_*} \begin{pmatrix}\sigma_{0,ij}\\ 0\end{pmatrix} \nabla(\theta\varepsilon_{ij}(u_0))\,\mathrm{d}x + \int_{\Omega_*} \theta\sigma_0 : \partial_1\varepsilon(u_0)\,\mathrm{d}x.$$

Regularity theorem 2.10 implies

$$\begin{aligned} \theta\varepsilon_{ij}(u_0) &\in W^{1,p-\epsilon}(\Omega_*) && \text{for } \epsilon > 0,\\ (\sigma_{0,ij}, 0)^T &\in B_{(p-\epsilon)', \frac{1}{q-1}-\epsilon_1(\epsilon), 2}(\Omega_*) && \text{with } \epsilon_1(\epsilon) \searrow 0 \text{ for } \epsilon \to 0. \end{aligned}$$

Assumption (b) of lemma A.2 is satisfied for $\delta = (p-\epsilon)'$, $s = \frac{1}{q-1} - \epsilon_1(\epsilon)$, $\gamma = 2$ and $\epsilon > 0$, small. Therefore

$$I_4 = -\int_{\partial\Omega_*} \theta\varepsilon(u_0) : \sigma_0 n_1\,\mathrm{d}s + \int_{\Omega_*} \theta\varepsilon(u_0) : \partial_1\sigma_0\,\mathrm{d}x + \int_{\Omega_*} \theta\sigma_0 : \partial_1\varepsilon(u_0)\,\mathrm{d}x. \tag{82}$$

Using that $W_c(\sigma_0) = (\frac{1}{2}\varepsilon(u_0) + \frac{\alpha(2-q)}{2q}\left|\sigma_0^D\right|^{q-2}\sigma_0^D) : \sigma_0 \equiv \varepsilon_* : \sigma_0$, the term I_5 is equal to

$$I_5 = \int_{\Omega_0} \varepsilon_* : \sigma_0\partial_1\theta\,\mathrm{d}y$$

Due to theorem 2.10 it is

$$\theta\varepsilon_* = \theta\left(\frac{1}{2}\varepsilon(u_0) + \frac{\alpha(2-q)}{2q}\left|\sigma_0^D\right|^{q-2}\sigma_0^D\right) \in W^{1,p-\epsilon}(\Omega_*) \quad \text{for } \epsilon > 0.$$

In the same way as for I_4 we obtain

$$\begin{aligned} I_5 &= \int_{\partial\Omega_*} \theta\varepsilon_* : \sigma_0 n_1 \,\mathrm{d}s - \int_{\Omega_*} \theta\varepsilon_* : \partial_1\sigma_0 \,\mathrm{d}x - \int_{\Omega_*} \theta\sigma_0 : \partial_1\varepsilon_* \,\mathrm{d}x \\ &= \int_{\partial\Omega_*} \theta W_c(\sigma_0) n_1 \,\mathrm{d}s - \int_{\Omega_*} \theta DW_c(\sigma_0) : \partial_1\sigma_0 \,\mathrm{d}y. \end{aligned} \tag{83}$$

Summing up (80), (81), (82), (83) and I_3 we arrive at

$$\begin{aligned} ERR(\Omega_0, F) &= I_1 + \ldots + I_5 \\ &= \int_{\partial\Omega_*} \theta\partial_1 u_0 \sigma_0 \vec{n} \,\mathrm{d}s + \int_{\partial\Omega_*} \theta u_0 f n_1 \,\mathrm{d}s - \int_{\partial\Omega_*} \theta(\varepsilon(u_0) : \sigma_0 - W_c(\sigma_0)) n_1 \,\mathrm{d}s. \end{aligned} \tag{84}$$

By the assumptions it is $\theta\big|_{\partial B_R(0)} = 0$, $\theta\big|_{\partial B_{R_0}(0)} = 1$ and on the crack face C_0 we have $\sigma_0\vec{n}\big|_{C_0} = 0$ and $n_1\big|_{C_0} = 0$. Therefore the path $\partial\Omega_*$ in (84) can be replaced by $\Gamma = \partial B_{R_0}(0)$ and the proof of theorem 3.1 is finished.

A Appendix

A.1 Some inequalities

Lemma A.1. *Let $n \in \mathbb{N}$. For $A, B \in \mathbb{R}^n$ with $|B| \geq |A|$ and $t \in [0, \frac{1}{4}]$ it holds [33, formula (2.20)]:*

$$4\,|B + t(A - B)| \geq |A| + |B|\,. \tag{85}$$

Let $q \geq 2$. It holds for every $A, B \in \mathbb{R}^n$

$$\frac{1}{q}\,|A|^q - \frac{1}{q}\,|B|^q - |B|^{q-2}\,B : (A - B) \geq 2^{-1-2q}\,(|A| + |B|)^{q-2}\,|A - B|^2\,. \tag{86}$$

$$\Big||A|^{q-2}\,A - |B|^{q-2}\,B\Big| \leq c\,(|A| + |B|)^{q-2}\,|A - B|\,. \tag{87}$$

For $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ with $a_i \geq 0$ for $1 \leq i \leq n$, we have [21]:

$$\left(\sum_{i=1}^n a_i\right)^\alpha \leq n^{\alpha-1}\left(\sum_{i=1}^n a_i^\alpha\right) \quad \text{if } \alpha \geq 1, \tag{88}$$

$$\left(\sum_{i=1}^n a_i\right)^\alpha \geq n^{\alpha-1}\left(\sum_{i=1}^n a_i^\alpha\right) \quad \text{if } 0 \leq \alpha \leq 1. \tag{89}$$

Proof of (86)**.** Let $A, B \in \mathbb{R}^n$ and $\gamma(t) = B + t(A - B)$ for $t \in \mathbb{R}$. Taylor's expansion yields

$$\begin{aligned} \frac{1}{q}\,|A|^q - \frac{1}{q}\,|B|^q - |B|^{q-2}\,B : (A - B) &= \int_0^1 (1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{q}\,|\gamma(t)|^q\right)\,\mathrm{d}t \\ &\geq \int_0^1 (1-t)\,|\gamma(t)|^{q-2}\,|A - B|^2\,\mathrm{d}t. \end{aligned} \tag{90}$$

Assume first that $|B| \geq |A|$. By (85) we obtain

$$(90) \geq 4^{2-q} \int_0^{\frac{1}{4}} (1-t)\,\mathrm{d}t (|A| + |B|)^{q-2} |A - B|^2 .$$

If $|A| > |B|$, then a change of coordinates leads to

$$(90) = \int_0^1 s \left|A + s(B-A)\right|^{q-2} |A-B|^2 \;\mathrm{d}s \overset{(85)}{\geq} 4^{2-q} \int_0^{\frac{1}{4}} s\,\mathrm{d}s (|A| + |B|)^{q-2} |A - B|^2 .$$

Proof of (87)**.** Again by Taylor's formula:

$$\begin{aligned}
\left| |A|^{q-2} A - |B|^{q-2} B \right| &\leq \int_0^1 \left| \frac{d}{dt} \left(|B + t(A-B)|^{q-2} (B + t(A-B)) \right) \right| \;\mathrm{d}t \\
&\leq \int_0^1 (q-1) \left|B + t(A-B)\right|^{q-2} |A - B| \;\mathrm{d}t.
\end{aligned}$$

□

Let $q \geq 2$. The following convexity and monotonicity inequalities hold for every $\sigma_1, \sigma_2 \in \mathbb{R}^{2\times 2}$ due to the previous lemma:

$$\begin{aligned}
&W_c(\sigma_1) - W_c(\sigma_2) - DW_c(\sigma_2) : (\sigma_1 - \sigma_2) \\
&\qquad \geq c \left(|\sigma_1 - \sigma_2|^2 + \left(|\sigma_1^D| + |\sigma_2^D| \right)^{q-2} |\sigma_1^D - \sigma_2^D|^2 \right), \qquad (91) \\
&\left(DW_c(\sigma_1) - DW_c(\sigma_2) \right) : (\sigma_1 - \sigma_2) \\
&\qquad \geq c \left(|\sigma_1 - \sigma_2|^2 + \left(|\sigma_1^D| + |\sigma_2^D| \right)^{q-2} |\sigma_1^D - \sigma_2^D|^2 \right). \qquad (92)
\end{aligned}$$

A.2 Some lemmata

Lemma A.2 (Green's formula)**.** *Let $\Omega \subset \mathbb{R}^d, d \geq 2$, be a bounded domain with Lipschitz boundary. For $s \in [0,1]$ and $1 < \gamma \leq \delta < \infty$ we define*

$$\begin{aligned}
B_{\delta,s,\gamma}(\Omega) &= \left\{ v : \Omega \to \mathbb{R}^d : \; v \in W^{s,\delta}(\Omega),\, \operatorname{div} v \in L^\gamma(\Omega) \right\}, \\
\|v\|_{B_{\delta,s,\gamma}(\Omega)} &= \|v\|_{W^{s,\delta}(\Omega)} + \|\operatorname{div} v\|_{L^\gamma(\Omega)} .
\end{aligned}$$

This space endowed with the above defined norm is a Banach space. Moreover $\left(\mathcal{C}^\infty(\overline{\Omega})\right)^d$ is dense in $B_{\delta,s,\gamma}(\Omega)$.
Let in addition $s - \frac{1}{\delta} > 0$ and let either (a) or (b) here below be satisfied:

- *(a)* $1 < \gamma \leq \delta \leq \frac{d}{d-1}$,
- *(b)* $\delta > \frac{d}{d-1}$ *and* $1 < \frac{d\delta}{d+\delta} \leq \gamma \leq \delta$.

Then the following Green's formula is valid for every $v \in B_{\delta,s,\gamma}(\Omega)$ and every $w \in W^{1,\delta'}(\Omega)$ with $\frac{1}{\delta'} + \frac{1}{\delta} = 1$

$$\int_\Omega v \nabla w \,\mathrm{d}x + \int_\Omega w \operatorname{div} v \,\mathrm{d}x = \int_{\partial\Omega} w\big|_{\partial\Omega} \left(v\big|_{\partial\Omega} \vec{n} \right) \;\mathrm{d}s. \qquad (93)$$

The vector $\vec{n}$ is the exterior unit normal vector on $\partial\Omega$. Note that the integrands are elements of $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively. Note further that the space $W^{1,\delta'}(\Omega)$ is continuously embedded in $L^{\gamma'}(\Omega)$ for parameters γ, δ satisfying (a) or (b). Therefore the left hand side of (93) is well defined.

Proof. For the proof of the density result we follow the standard arguments in [12, 31]. Since Ω is a bounded domain with Lipschitz boundary there exists a finite number of open sets Ω_j, $1 \leq j \leq J$, such that $\overline{\Omega} \subset \cup_{j=1}^{J} \Omega_j$ and $\Omega_j \subset\subset \Omega$ or $\Omega_j \cap \Omega$ is star-shaped with respect to an element $z_j \in \Omega_j \cap \Omega$, see [31, 23]. Moreover there exist open sets $\tilde{\Omega}_j \subset\subset \Omega_j$ with $\overline{\Omega} \subset \cup_{j=1}^{J} \tilde{\Omega}_j$ and $\tilde{\Omega}_j \subset\subset \Omega$ or $\tilde{\Omega}_j \cap \Omega$ is star-shaped with respect to an element $\tilde{z}_j \in \tilde{\Omega}_j \cap \Omega$. Let $\{\alpha_j, 1 \leq j \leq J\}$ be a partition of unity with respect to Ω subordinate to the covering $\{\tilde{\Omega}_j, 1 \leq j \leq J\}$, i.e. $\alpha_j \in \mathcal{C}_0^\infty(\tilde{\Omega}_j)$, $\alpha_j \geq 0$ and $\sum_{j=1}^{J} \alpha_j(x) = 1$ for every $x \in \Omega$. Note that $\alpha_j u \in B_{\delta,s,\gamma}(\Omega)$ for $u \in B_{\delta,s,\gamma}(\Omega)$. We will prove now for $u \in B_{\delta,s,\gamma}(\Omega)$

$$\text{For every } \epsilon > 0 \text{ there exists a function } \varphi_{j,\epsilon} \in \mathcal{C}_0^\infty(\Omega_j) \text{ such that}$$
$$\|\varphi_{j,\epsilon} - \alpha_j u\|_{B_{\delta,s,\gamma}(\Omega\cap\Omega_j)} \leq \frac{\varepsilon}{J}. \tag{94}$$

If (94) holds, then the function $\psi_\epsilon = \sum_{j=1}^{J} \varphi_{j,\epsilon}$ is an element of $\mathcal{C}^\infty(\overline{\Omega})$ and

$$\|\psi_\epsilon - u\|_{B_{\delta,s,\gamma}(\Omega)} \leq \sum_{j=1}^{J} \|\alpha_j u - \varphi_{j,\epsilon}\|_{B_{\delta,s,\gamma}(\Omega\cap\Omega_j)} \leq \varepsilon.$$

For the proof of (94) we distinguish two cases.

Case 1: For $\Omega_j \subset\subset \Omega$ we apply the standard regularising procedure [12, 31]. Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ with $\rho \geq 0$, $\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1$ and $\operatorname{supp}\rho \subset B_1(0)$. Straight forward calculations show that the function

$$\varphi_{j,\nu}(x) = \frac{1}{\nu^d} \int_\Omega \rho\left(\frac{x-y}{\nu}\right) \alpha_j(y) u(y)\,\mathrm{d}y$$

is an element of $\mathcal{C}_0^\infty(\Omega_j)$ and satisfies (94) if $\nu > 0$ is small enough.

Case 2: Let $\tilde{\Omega}_j \cap \Omega$ be star-shaped with respect to $\tilde{z}_j \in \tilde{\Omega}_j \cap \Omega$ where we assume without loss of generality that $\tilde{z}_j = 0$. For $\lambda > 1$ and $y \in \lambda(\tilde{\Omega}_j \cap \Omega)$ we define

$$S_\lambda(\alpha_j u)(y) = \alpha_j\left(\frac{y}{\lambda}\right) u\left(\frac{y}{\lambda}\right).$$

Straight forward calculations show that $S_\lambda(\alpha_j u)$ is an element of $B_{\delta,s,\gamma}(\lambda(\tilde{\Omega}_j \cap \Omega))$. If $\lambda > 1$ is small enough then

$$\tilde{\Omega}_j \cap \Omega \subset\subset \lambda(\tilde{\Omega}_j \cap \Omega) \subset\subset \Omega_j.$$

Let $\eta^\lambda \in \mathcal{C}_0^\infty(\lambda(\tilde{\Omega}_j \cap \Omega))$ be a cut-off function with $\eta^\lambda\big|_{\frac{1+\lambda}{2}(\tilde{\Omega}_j\cap\Omega)} = 1$. It holds

$$\eta^\lambda S_\lambda(\alpha_j u) \in B_{\delta,s,\gamma}(\Omega_j) \quad \text{and} \quad \operatorname{supp}(\eta^\lambda S_\lambda(\alpha_j u)) \subset \Omega_j.$$

For $\lambda \to 1$ we have the following convergence

$$\eta^\lambda S_\lambda(\alpha_j u)\big|_{\tilde{\Omega}_j\cap\Omega} = S_\lambda(\alpha_j u)\big|_{\tilde{\Omega}_j\cap\Omega} \to \alpha_j u \quad \text{in } B_{\delta,s,\gamma}(\tilde{\Omega}_j \cap \Omega). \tag{95}$$

This assertion is a consequence of lemma 1.1 in [31] which states that

$$\left\| S_\lambda(v)\big|_{\tilde{\Omega}_j\cap\Omega} - v \right\|_{L^\delta(\tilde{\Omega}_j\cap\Omega)} \to 0$$

for every $v \in L^\delta(\tilde{\Omega}_j \cap \Omega)$. Moreover it is $\operatorname{div}(S_\lambda(\alpha_j u))\big|_{\tilde{\Omega}_j\cap\Omega} = \frac{1}{\lambda} S_\lambda(\operatorname{div}(\alpha_j u))$ and therefore we have, again by lemma 1.1 in [31], $\|\operatorname{div}(S_\lambda(\alpha_j u) - \alpha_j u\|_{L^\gamma(\Omega\cap\tilde{\Omega}_j)} \to 0$. In a similar way the convergence of $S_\lambda(\alpha_j u)$ in $W^{s,\delta}(\Omega)$ is proved.
The functions $\eta^\lambda S_\lambda(\alpha_j u)$ can be approximated in $B_{\delta,s,\gamma}(\Omega_j)$ due to case 1 for fixed $\lambda > 1$ by functions $\{\varphi_n^\lambda, n \in \mathbb{N}\} \subset \mathcal{C}_0^\infty(\Omega_j)$. This together with (95) proves (94) for the star-shaped domain $\tilde{\Omega}_j \cap \Omega$.
For the proof of Green's formula (93) we define the following bilinear forms for $v \in B_{\delta,s,\gamma}(\Omega)$ and $w \in W^{1,\delta'}(\Omega)$

$$L_1(v,w) = \int_\Omega v \nabla w \,\mathrm{d}x + \int_\Omega w \operatorname{div} v \,\mathrm{d}x, \tag{96}$$

$$L_2(v,w) = \int_{\partial\Omega} \left(v\big|_{\partial\Omega}\right) \vec{n} w\big|_{\partial\Omega} \,\mathrm{d}s. \tag{97}$$

Due to the assumptions (a) and (b) the space $W^{1,\delta'}(\Omega)$ is continuously embedded in $L^{\gamma'}(\Omega)$ and therefore L_1 is well defined and continuous. Moreover it follows from $s - \frac{1}{\delta} > 0$ and trace theorem [14, Theorem 1.5.1.2] that $v\big|_{\partial\Omega} \in L^\delta(\partial\Omega)$ and $w\big|_{\partial\Omega} \in L^{\delta'}(\partial\Omega)$ and therefore L_2 is well defined and continuous as well. From the classical Green's formula we obtain that L_1 and L_2 coincide on the set $(\mathcal{C}^\infty(\Omega))^d \times \mathcal{C}^\infty(\Omega)$. Since this set is dense in $B_{\delta,s,\gamma}(\Omega) \times W^{1,\delta'}(\Omega)$ and since the bilinear forms are continuous we get $L_1(v,w) = L_2(v,w)$ for every $v \in B_{\delta,s,\gamma}(\Omega)$ and every $w \in W^{1,\delta'}(\Omega)$. □

Lemma A.3. *Let X be a Banach space and $I : X \to \mathbb{R}$ a functional which is Fréchet-differentiable with derivative $DI \in X'$. For every u and $h \in X$ there exists a constant $t_0 = t_0(u,h) \in [0,1]$ such that*

$$I(u+h) - I(u) = \langle DI(u + t_0 h), h\rangle_{(X',X)}.$$

Proof. The functional I admits the following Taylor expansion, see [37, 34],

$$I(u+h) - I(u) = \int_0^1 \langle DI(u+th), h\rangle_{(X',X)} \,\mathrm{d}t.$$

Since I is Fréchet-differentiable, the function $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto \langle DI(u+th), h\rangle_{(X',X)}$ is continuous. The lemma now follows from the mean value theorem for integrals of continuous functions [16]. □

Acknowledgement

The author was supported by the German Research Foundation (DFG), Sonderforschungsbereich 404 *Multifield Problems in Continuum Mechanics*, University of Stuttgart.

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