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Homogenization of elliptic systems with non-periodic, state dependent coefficients

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Abstract

In this paper, a homogenization problem for an elliptic system with non-periodic, state-dependent coefficients representing microstructure is investigated. The state functions defining the tensor of coefficients are assumed to have an intrinsic length scale denoted by $\varepsilon > 0$. The aim is the derivation of an effective model by investigating the limit process $\varepsilon \to 0$ of the state functions rigorously. The effective model is independent of the parameter $\varepsilon > 0$ but preserves the microscopic structure of the state functions ($\varepsilon > 0$), meaning that the effective tensor is given by a unit cell problem prescribed by a suitable microscopic tensor. Due to the non-periodic structure of the state functions and the corresponding microstructure, the effective tensor turns out to vary from point to point (in contrast to a periodic microscopic model).

In a forthcoming paper, these states will be solutions of an additional evolution law describing changes of the microstructure. Such changes could be the consequences of temperature changes, phase separation or damage progression, for instance. Here, in addition to the above and as a preparation for an application to time-dependent damage models (discussed in a future paper), we provide a Γ-convergence result of sequences of functionals being related to the previous microscopic models with state dependent coefficients. This requires a penalization term for piecewise constant state functions that allows us to extract from bounded sequences those sequences converging to a Sobolev function in some sense. The construction of the penalization term is inspired by techniques for Discontinuous Galerkin methods and is of own interest. A compactness and a density result are provided.

1 Introduction

In this paper, microstructure is understood as the heterogeneity of a material occupied body $\Omega \subset \mathbb{R}^d$. The heterogeneity is modeled by a forth order tensor $\mathbb{C}$ and either arises from one material in different phases or from several materials that may appear in different phases, too. In experiments, it is observed that microstructures often have an intrinsic length scale. Descriptively this length scale is related to the smallest homogeneous set of material being part of the microstructure. According to the huge variety of heterogeneity appearing in nature, modeling of microstructure in this general setting is hopeless and some approximation is needed.

One very common kind of such an approximative microstructure is the periodic one. Here, the intrinsic length scale, denoted by $\varepsilon > 0$, is associated to the size of cells $\varepsilon(\lambda+Y)$ occupying a bounded open domain $\Omega \subset \mathbb{R}^d$, where λ is an element of a given periodic lattice Λ and Y is the so called unit cell (for instance $Y = [0,1)^d$). All cells with $\varepsilon(\lambda+Y) \cap \Omega \neq \emptyset$ contain the same specific distribution of the appearing materials and their phases.

Naturally, the size of the intrinsic length scale is very small compared to the size of the considered body Ω. Together with the possibly complicated shape of the microstructure this leads for instance to problems in the numerical investigation of such microstructures. Moreover, typically the main interest is in macroscopic quantities instead of microscopic ones. Thus, looking for effective descriptions capturing the macroscopic behavior of such microstructures is a meaningful task. We are interested in the homogenization of the following elliptic boundary value problem:

$$\begin{cases} -\mathrm{div}(\mathbb{C}_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ \mathbb{C}_\varepsilon \nabla u_\varepsilon \vec{n} = h & \text{on } \Gamma_{\mathrm{N}} := \partial\Omega \setminus \Gamma_{\mathrm{Dir}}, \qquad u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n, \end{cases} \tag{1.1}$$

with volume forces f, surface forces h, and where $\vec{n}$ denotes the unit normal vector on the Neumann boundary Γ_{N}. The tensor $\mathbb{C}_\varepsilon$ reflects possibly non-periodic microstructure on the length-scale ε. The task is the performance of the limit passage $\varepsilon \to 0$ in a rigorous way and to identify the limit tensor $\mathbb{C}_0$ such that the sequence of solutions $(u_\varepsilon)_{\varepsilon>0}$ of (1.1) converges in a suitable sense to the solution u_0 of the following elliptic boundary value problem:

$$\begin{cases} -\operatorname{div}(\mathbb{C}_0 \nabla u_0) = f & \text{in } \Omega, \\ \mathbb{C}_0 \nabla u_0 \vec{n} = h & \text{on } \Gamma_{\mathrm{N}}, \qquad u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n. \end{cases} \tag{1.2}$$

To allow for a larger amount of applications fitting into this theory, we assume the existence of a linear projection $\mathbb{B} : \mathbb{R}^{n\times d} \to \mathbb{R}^{n\times d}$ satisfying for all $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ and some positive constant $C_{\mathbb{B}}$ the inequality

$$\|\mathbb{B}\nabla u\|_{\mathrm{L}^2(\Omega)^{n\times d}} \geq C_{\mathbb{B}} \|u\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n}. \tag{1.3}$$

For example, in the case of linear elasticity one sets $n = d$, $\mathbb{B} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is chosen as $\mathbb{B}(\xi) = \frac{1}{2}(\xi+\xi^T)$ and (1.3) is guaranteed by Korn's inequality. Throughout the whole paper let $0 < \alpha < \beta$ denote fixed constants. We define

$$\mathbb{M}(\alpha,\beta) := \{\mathbb{A} \in \mathrm{Lin}_{\mathrm{sym}}(\mathrm{Im}(\mathbb{B}); \mathrm{Im}(\mathbb{B})) \,|\, \forall \zeta \in \mathrm{Im}(\mathbb{B}) : \alpha|\zeta|^2_{n\times d} \leq \langle \mathbb{A}\zeta, \zeta\rangle_{n\times d} \leq \beta|\zeta|^2_{n\times d}\}, \tag{1.4}$$

where $\mathrm{Im}(\mathbb{B})$ denotes the image of the operator $\mathbb{B} : \mathbb{R}^{n\times d} \to \mathbb{R}^{n\times d}$ (linear elasticity: $\mathrm{Im}(\mathbb{B}) = \mathbb{R}^{d\times d}_{\mathrm{sym}}$). For an open set $\mathcal{O} \subset \mathbb{R}^d$ the tensors considered in this paper are elements of the space $\mathcal{M}(\mathcal{O};\alpha,\beta) := \mathrm{L}^\infty(\mathcal{O}; \mathbb{M}_{\mathbb{B}}(\alpha,\beta))$, where $\mathbb{M}_{\mathbb{B}}(\alpha,\beta)$ is the subset of $\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{n\times d}; \mathbb{R}^{n\times d})$ satisfying the following condition:

$$\forall\, \mathbb{D} \in \mathbb{M}_{\mathbb{B}}(\alpha,\beta)\, \exists\, \mathbb{A} \in \mathbb{M}(\alpha,\beta) : \forall \xi, \eta \in \mathbb{R}^{n\times d} \quad \langle \mathbb{D}\xi, \eta\rangle_{n\times d} = \langle \mathbb{A}\mathbb{B}\xi, \mathbb{B}\eta\rangle_{n\times d}.$$

Regarding the classical homogenization considering periodic coefficients, a rigorous result is gained via the two-scale convergence introduced by G. Nguetseng in [17]. This result was generalized by G. Allaire in [1] to a special non-periodic case which is stated in the following theorem:

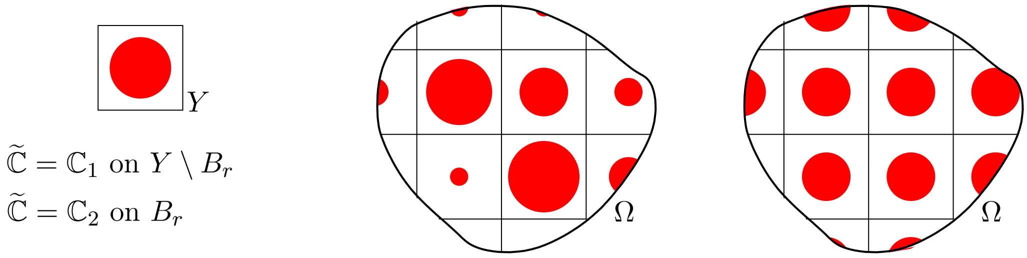
Theorem 1.1. *Given a tensor $\mathbb{C} \in \mathcal{M}(\Omega\times Y;\alpha,\beta)$ being continuous (in some sense) with respect to the first variable and being periodic with respect to the second variable let the sequence $(\mathbb{C}_\varepsilon)_{\varepsilon>0} \subset \mathcal{M}(\Omega;\alpha,\beta)$ of tensors for almost every $x \in \Omega$ be defined via $\mathbb{C}_\varepsilon(x) := \mathbb{C}(x, \frac{x}{\varepsilon})$. If $u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ is the weak solution of* (1.1)*, then there exists a function $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ such that*

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n, \\ \mathbb{C}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{C}_0 \nabla u_0 & \text{in } \mathrm{L}^2(\Omega; \mathbb{R}^{n\times d}), \end{cases}$$

and $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ is the weak solution of (1.2)*. Moreover, the tensor $\mathbb{C}_0 \in \mathcal{M}(\Omega,\alpha,\beta)$ is given by*

$$\langle \mathbb{C}_0(x)\xi, \xi\rangle_{n\times d} = \min_{v\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^n} \int_Y \langle \mathbb{C}(x,y)(\xi + \nabla_y v(y)), \xi + \nabla_y v(y)\rangle_{n\times d} \mathrm{d}y. \tag{1.5}$$

We refer to Section 2 for a definition of the space $\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^n$.



$\widetilde{\mathbb{C}}$ consists of two constant tensors $\mathbb{C}_1$ and $\mathbb{C}_2$ | $\mathbb{C}_\varepsilon$ for parameters changing from cell to cell | $\mathbb{C}_\varepsilon$ for the same parameter in every cell

Figure 1: Example for $d = 2$ where $m = 1$ and the parameter $r(z_\varepsilon)$ describes the radius of the Ball $B_{r(z_\varepsilon)}$ having the same center as Y

Besides providing the convergence of the ε-dependent solutions of (1.1) to the solution of the effective problem (1.2), this result yields an explicit structure of the macroscopic tensor $\mathbb{C}_0$. This is different in the more general theory of G-convergence allowing for arbitrary microstructures. The homogenization result of this general theory only states the existence of an effective problem.

Theorem 1.2. *[7, Theorem 6.3] Given a sequence $(\mathbb{C}_\varepsilon)_{\varepsilon>0} \subset \mathcal{M}(\Omega; \alpha, \beta)$ let $(u_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ be the weak solution of (1.1). Then there exist a subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$, a function $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ and a tensor $\mathbb{C}_0 \in \mathcal{M}(\Omega; \alpha, \beta)$ such that*

$$\begin{cases} u_{\varepsilon'} \rightharpoonup u_0 & \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n, \\ \mathbb{C}_{\varepsilon'} \nabla u_{\varepsilon'} \rightharpoonup \mathbb{C}_0 \nabla u_0 & \text{in } \mathrm{L}^2(\Omega; \mathbb{R}^{n \times d}), \end{cases}$$

where $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ is the weak solution of (1.2).

The result we are going to present in this paper is in between these two extreme cases (Theorem 1.1 and Theorem 1.2). Starting with a sequence of non-periodic tensors $(\mathbb{C}_\varepsilon)_{\varepsilon>0}$ the limit is performed in the sense of G-convergence. Under suitable continuity assumptions on the structure of $\mathbb{C}_\varepsilon$ being more general than in Theorem 1.1 we identify the limiting effective tensor $\mathbb{C}_0$ and show that it is given by a cell formula similar to (1.5).

To be more precise, let $z_\varepsilon : \Omega \to \mathbb{R}^m$ be a function that is piecewise constant with respect to the grid $\varepsilon\Lambda \cap \Omega$ and defines the microscopic states of the system. Given $\widetilde{\mathbb{C}} : \mathbb{R}^m \to \mathcal{M}(Y; \alpha, \beta)$, we define $\mathbb{C}_\varepsilon \in \mathcal{M}(\Omega; \alpha, \beta)$ by

$$\mathbb{C}_\varepsilon(x) = \widetilde{\mathbb{C}}(z_\varepsilon(x))(\{\tfrac{x}{\varepsilon}\}_Y), \tag{1.6}$$

where $\{\cdot\}_Y : \mathbb{R}^d \to Y$ is given by $\{x\}_Y := x - \lambda$ for $x \in (\lambda + Y)$. Considering a sequence of state functions $(z_\varepsilon)_{\varepsilon>0}$, we are interested in the effective behavior of the system (1.1) as ε tends to zero (see Figure 2) and obtain:

Theorem 1.3. *Let $\widetilde{\mathbb{C}} : \mathbb{R}^m \to \mathcal{M}(Y; \alpha, \beta)$ be continuous with respect to the strong L^1-topology and let $(z_\varepsilon)_{\varepsilon>0}$ be given such that $z_\varepsilon : \Omega \to \mathbb{R}^m$ is piecewise constant with respect to the grid $\varepsilon\Lambda \cap \Omega$ and $z_\varepsilon \to z_0$ in $\mathrm{L}^1(\Omega)^m$ for some function $z_0 \in \mathrm{L}^1(\Omega)^m$. Moreover, let*

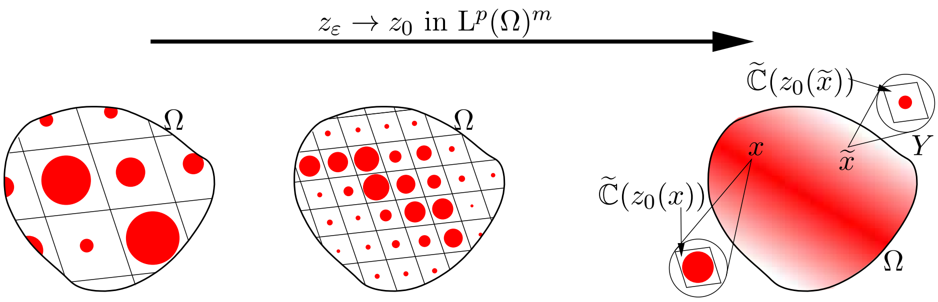


Figure 2: Schematic representation of the limit passage of the microscopic model to the effective model, where $\widetilde{\mathbb{C}}$ is assumed to be as in Figure 1

$\mathbb{C}_\varepsilon \in \mathcal{M}(\Omega;\alpha,\beta)$ be defined as explained in (1.6)*. If $u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ is the weak solution of* (1.1)*, then there exists a function $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ such that*

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$$

and $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ is the weak solution of (1.2)*. Moreover, the tensor $\mathbb{C}_0 = \widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0) \in \mathcal{M}(\Omega,\alpha,\beta)$ for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^{n\times d}$ is given by*

$$\langle \widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0)(x)\xi,\xi\rangle_{n\times d} = \min_{v\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^n} \int_Y \langle \widetilde{\mathbb{C}}(z_0(x))(y)(\xi+\nabla_y v(y)),\xi+\nabla_y v(y)\rangle_{n\times d}\mathrm{d}y. \tag{1.7}$$

This paper is the basis for the homogenization of an evolutionary problem studied in a forthcoming paper [10], where for fixed $\varepsilon > 0$ the piecewise constant function z_ε (collecting the parameters of all cells $\varepsilon(\lambda+Y)\cap\Omega \neq \emptyset$) is given by an evolution law, i.e. the state functions z_ε have to be considered as unknown and the microstructure is described by the z_ε-dependent tensor $\mathbb{C}_\varepsilon = \widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)$. For this reason, the second part of this paper is devoted to the Γ-convergence of a sequence of energy functionals $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ defined via

$$\mathcal{E}_\varepsilon(u_\varepsilon,z_\varepsilon) = \tfrac{1}{2}\langle \widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)\nabla u_\varepsilon,\nabla u_\varepsilon\rangle_{\mathrm{L}^2(\Omega)^{n\times d}} + \|R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^d} - \langle \ell, u_\varepsilon\rangle. \tag{1.8}$$

Here, $\|R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^d}$ for $p \in (1,\infty)$ denotes a penalty term (being a discrete gradient for piecewise constant functions), where $\Omega^+_\varepsilon \supset \Omega$ is a slightly larger domain than Ω. This penalty term fixes the topology used to gain the Γ-limit $\mathcal{E}_0$, which is

$$\mathcal{E}_0(u_0,z_0) = \tfrac{1}{2}\langle \widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0)\nabla u_0,\nabla u_0\rangle_{\mathrm{L}^2(\Omega)^{n\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d} - \langle \ell, u_0\rangle$$

with $\widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0)$ from (1.7). Since the Γ-convergence is investigated with respect to the two variables u_ε and z_ε, the penalty term $\|R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|^p_{\mathrm{L}^p(\Omega)^d}$ is introduced to enforce compactness that is strong enough to keep track of the simple geometry of the microstructure.

Basically, the introduction of the penalty term is motivated by the aim of an explicit formula for the limit tensor of the sequences $(\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon))_{\varepsilon>0}$, and the following observation shows that the natural candidate of topology (neglecting the penalty term) seems to be too weak, in general. Assume that $\widetilde{\mathbb{C}}(z)$ is a mixture of two constant tensors $\mathbb{C}_1$ and $\mathbb{C}_2$ for any $z \in \mathbb{R}^m$ (see Figure 1 for example), which is defined as follows: For a given piecewise constant function z_ε, let $\mathbb{1}(z_\varepsilon) \in \mathrm{L}^\infty(\Omega;\{0,1\})$ denote the geometry of the mixture $\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)$, i.e. $\mathbb{1}(z_\varepsilon)(x) = 1$ if $\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)(x) = \mathbb{C}_1$ and $\mathbb{1}(z_\varepsilon)(x) = 0$ otherwise. Assuming $⨍_\Omega \mathbb{1}(z_\varepsilon)(x)\mathrm{d}x = \theta$ for all $\varepsilon > 0$, the limit tensor of $(\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon))_{\varepsilon>0}$ is an element of the so called G-closure of $\{\mathbb{C}_1, \mathbb{C}_2\}$ with fixed volume fraction θ. In general, the determination of this G-closure is very difficult and no explicit formula is available (see [14, 16, 19, 20] for details). In particular, information on the original geometry of the microstructure in general will be lost, see also the discussion in [8].

The paper is structured as follows: Section 2 is devoted to the theory of two-scale convergence developed by G. Nguetseng in [17] and states the notations, the definitions and the results needed in the following. Note, although this theory was introduced to gain homogenization results for periodic problems, it is possible to apply this theory in our particular non-periodic case. Here, in this paper we use the so called unfolding technique introduced in [4], which is a dual formulation of the two-scale convergence theory.

The types of microstructure we are searching homogenized descriptions for are introduced in Section 3. They give rise to non-periodic coefficients entering into an ε-dependent boundary value problem. Then the limit passage $\varepsilon \to 0$ is preformed in a rigorous way. The main techniques used to identify the homogenized problem are the calculus of variation and the theory of two-scale convergence.

In Section 4, a discrete gradient for piecewise constant functions on lattices is introduced relying on the theory for broken Sobolev spaces, see for instance [3]. The aim is to construct the discrete gradient in such a way that from sequences of piecewise constant functions on finer and finer lattices, for which the discrete gradient is bounded in $\mathrm{L}^p(\Omega)$, one can extract a subsequence that converges strongly in $\mathrm{L}^p(\Omega)$ to a limit function in $\mathrm{W}^{1,p}(\Omega)$ and where the corresponding discrete gradients converge weakly to the gradient of the limit function. For that purpose, the original definition of a discrete gradient from [3] had to be modified, see also the example at the beginning of Section 4.

Section 5 is basically in preparation for the evolution model mentioned above. It is devoted to the Γ-convergence of the sequences of functionals $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ from (1.8). Thanks to the compactness enforced by the discrete gradient we are able to identify the Γ-limit $\mathcal{E}_0$ preserving the information captured in the microstructure. This compactness also motivates the assumptions made on the sequence $(z_\varepsilon)_{\varepsilon>0}$ describing the microstructure in Section 3.

2 Notation and two-scale convergence

This section introduces everything needed in the following sections concerning the notation and the theory of folding/unfolding and two-scale convergence and does not claim completeness. For further details we recommend to [1, 4, 5].

Let $d \in \mathbb{N}$ be the space dimension and $\{b_1, b_2, \ldots, b_d\}$ an arbitrary basis of $\mathbb{R}^d$, with no need of orthonormality. Furthermore, let

$$\Lambda = \left\{\lambda \in \mathbb{R}^d \,:\, \lambda = \sum_{i=1}^d k_i b_i,\, k_i \in \mathbb{Z}\right\} \tag{2.1}$$

be a periodic lattice and

$$Y = \left\{ x \in \mathbb{R}^d \, : \, x = \sum_{i=1}^{d} l_i b_i, \, l_i \in [-\tfrac{1}{2}, \tfrac{1}{2}) \right\}$$

the associated unit cell. In particular, the unit cell Y is the d-parallelotope whose axis are the basis vectors $\{b_1, b_2, \ldots, b_d\}$. The only restriction on the basis $\{b_1, b_2, \ldots, b_d\}$ is that

$$\mathrm{vol}(Y) = 1$$

is satisfied to make the following statements valid without any normalization coefficients. Due to this definition there is only one vertex contained in $\varepsilon(\lambda{+}Y)$ so that each of these cells is uniquely determined by $\varepsilon > 0$ and the associated vertex $\varepsilon\lambda$.

Finally, for an open set $\Omega \subset \mathbb{R}^d$ the set of piecewise constant functions is given by

$$\mathrm{K}_{\varepsilon\Lambda}(\Omega) := \{ v \in \mathrm{L}^1(\Omega) \,|\, \exists \, \widetilde{v} \in \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d) : \widetilde{v}|_\Omega = v \},$$

where

$$\mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d) := \{ \widetilde{v} \in \mathrm{L}^1(\mathbb{R}^d) \,|\, \forall \, \lambda \in \Lambda : \widetilde{v}|_{\varepsilon(\lambda+Y)} = const \}.$$

As already mentioned in Section 1 we are interested in microstructures varying from cell to cell. For this purpose Ω is decomposed in small cells $\varepsilon(\lambda{+}Y)$ and we introduce the subsets

$$\Lambda_\varepsilon^- := \{ \lambda \in \Lambda : \varepsilon(\lambda{+}\overline{Y}) \subset \Omega \} \qquad \text{and} \qquad \Lambda_\varepsilon^+ := \{ \lambda \in \Lambda : \varepsilon(\lambda{+}Y) \cap \Omega \neq \emptyset \}$$

of Λ to define the sets Ω_ε^- and Ω_ε^+ via

$$\Omega_\varepsilon^\pm := \bigcup_{\lambda \in \Lambda_\varepsilon^\pm} \varepsilon(\lambda{+}Y). \tag{2.2}$$

Observe that $\overline{\Omega_\varepsilon^-}$ is a compact subset of Ω. The set Ω_ε^+ is introduced in order to avoid problems with cells having a non empty intersection with Ω but which are not completely contained in it, i.e. all cells containing a part of the boundary $\partial\Omega$. From now on we will assume that

$$\Omega \text{ is an open and bounded subset of } \mathbb{R}^d \text{ having a Lipschitz boundary } \partial\Omega. \tag{2.3}$$

This guarantees that $\mathrm{vol}(\partial\Omega) = 0$ and that $\mathrm{vol}(\Omega_\varepsilon^+ \backslash \Omega) + \mathrm{vol}(\Omega \backslash \Omega_\varepsilon^-) \to 0$ for $\varepsilon \to 0$, which will be used later. In particular, this is crucial when introducing the two-scale convergence with the help of the so called periodic unfolding operator (see [15] Section 2).

Before defining the two-scale convergence with the help of the so called periodic unfolding operator we start by introducing the mappings $[\cdot]_\Lambda$ and $\{\cdot\}_Y$ on $\mathbb{R}^d$.

$$[\cdot]_\Lambda : \mathbb{R}^d \to \Lambda, \quad \{\cdot\}_Y : \mathbb{R}^d \to Y, \quad \text{and} \quad x = [x]_\Lambda + \{x\}_Y \quad \text{for all } x \in \mathbb{R}^d$$

Let $\lambda \in \Lambda$ and let $x \in \mathbb{R}^d$ be in the cell $\lambda{+}Y$, then $[x]_\Lambda = \lambda$ and $\{x\}_Y$ is determinable as $\{x\}_Y = x - [x]_\Lambda$. For $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we have the following decomposition:

$$x = \mathcal{N}_\varepsilon(x) + \varepsilon \mathcal{V}_\varepsilon(x), \quad \text{with } \mathcal{N}_\varepsilon(x) = \varepsilon \left[\frac{x}{\varepsilon} \right]_\Lambda \text{ and } \mathcal{V}_\varepsilon(x) = \left\{ \frac{x}{\varepsilon} \right\}_Y,$$

where $\mathcal{N}_\varepsilon(x)$ denotes the macroscopic center of the cell $\mathcal{N}_\varepsilon(x)+\varepsilon Y$ that contains x and $\mathcal{V}_\varepsilon(x)$ is the microscopic part of x in Y. At last, we want to distinguish the unit cell Y from the periodicity cell $\mathcal{Y} := \mathbb{R}^d/_\Lambda$. Following Ref. [22], we introduce the mappings $\mathcal{D}_\varepsilon$ and $\mathcal{S}_\varepsilon$ as follows:

$$\mathcal{D}_\varepsilon : \begin{cases} \mathbb{R}^d & \to \quad \mathbb{R}^d \times \mathcal{Y}, \\ x & \mapsto \quad (\mathcal{N}_\varepsilon(x), \mathcal{V}_\varepsilon(x)), \end{cases} \qquad \mathcal{S}_\varepsilon : \begin{cases} \mathbb{R}^d \times \mathcal{Y} & \to \quad \mathbb{R}^d, \\ (x,y) & \mapsto \quad \mathcal{N}_\varepsilon(x)+\varepsilon y, \end{cases}$$

where in the last sum $y \in \mathcal{Y}$ is identified with $y \in Y \subset \mathbb{R}^d$.

Two-scale convergence is linked to a suitable two-scale embedding of $\mathrm{L}^p(\Omega)$ in the two-scale space $\mathrm{L}^p(\mathbb{R}^d \times Y)$. Such an embedding is called periodic unfolding operator. The following definition of a periodic unfolding operator was given in Ref. [4].

Definition 2.1. (Ref. [4]) Let $\Omega \subset \mathbb{R}^d$ be open, $\varepsilon > 0$ and $p \in [1,\infty]$. Then the periodic unfolding operator $\mathcal{T}_\varepsilon$ is defined via:

$$\mathcal{T}_\varepsilon : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\mathbb{R}^d \times Y);\ v \mapsto v^{\mathrm{ex}} \circ \mathcal{S}_\varepsilon,$$

where $v^{\mathrm{ex}} \in \mathrm{L}^p(\mathbb{R}^d)$ is the extension of the function v by 0 to all of $\mathbb{R}^d$.

With this definition the following product rule is valid: Let $p, q, r \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Then

$$v_1 \in \mathrm{L}^p(\Omega), v_2 \in \mathrm{L}^q(\Omega) \Longrightarrow \mathcal{T}_\varepsilon(v_1 v_2) = (\mathcal{T}_\varepsilon v_1)(\mathcal{T}_\varepsilon v_2) \in \mathrm{L}^r(\mathbb{R}^d \times Y).$$

Note that $\overline{[\Omega \times Y]}_\varepsilon := \overline{\mathcal{S}_\varepsilon^{-1}(\Omega)} = \overline{\{(x,y) | \mathcal{S}_\varepsilon(x,y) \in \Omega\}}$ is the support of $\mathcal{T}_\varepsilon v$, and this is not contained in $\Omega \times Y$, in general.

Following the lines in Ref. [15] we now will use this periodic unfolding operator to introduce the kind of two-scale convergence, which is used here; the strong and weak two-scale convergence, respectively. But before that, we define the folding operator $\mathcal{F}_\varepsilon$. For details see [15].

Definition 2.2. (Ref. [15]) Let $\Omega \subset \mathbb{R}^d$ be open, $\varepsilon > 0$ and $p \in [1,\infty)$. Then the folding operator $\mathcal{F}_\varepsilon$ is defined via:

$$\mathcal{F}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d \times Y) \to \mathrm{L}^p(\Omega);\ V \mapsto \big(P_\varepsilon(\mathbb{1}_{[\Omega \times Y]_\varepsilon} V) \circ \mathcal{D}_\varepsilon\big)|_\Omega,$$

where $(P_\varepsilon V)(x,y) := \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} V(\zeta,y)\mathrm{d}\zeta$.

Definition 2.3. (Ref. [15]) Let $p \in (1,\infty)$ and let $(v_\varepsilon)_{\varepsilon>0}$ be a sequence in $\mathrm{L}^p(\Omega)$. Then

(a) v_ε converges strongly two-scale to $V \in \mathrm{L}^p(\Omega \times Y)$ in $\mathrm{L}^p(\Omega \times Y)$, $v_\varepsilon \xrightarrow{s} V$ in $\mathrm{L}^p(\Omega \times Y)$, if $\mathcal{T}_\varepsilon v_\varepsilon \to V^{\mathrm{ex}}$ in $\mathrm{L}^p(\mathbb{R}^d \times Y)$.

(b) v_ε converges weakly two-scale to $V \in \mathrm{L}^p(\Omega \times Y)$ in $\mathrm{L}^p(\Omega \times Y)$, $v_\varepsilon \xrightarrow{w} V$ in $\mathrm{L}^p(\Omega \times Y)$, if $\mathcal{T}_\varepsilon v_\varepsilon \rightharpoonup V^{\mathrm{ex}}$ in $\mathrm{L}^p(\mathbb{R}^d \times Y)$.

Referring to (2.2) we have that for all $\varepsilon > 0$ the support of the function $\mathcal{T}_\varepsilon v_\varepsilon$ is contained in $\overline{[\Omega \times Y]}_\varepsilon \subset \overline{\Omega}_\varepsilon^+ \times Y$ which results in the fact that the support of a possible accumulation point U of the sequence $(\mathcal{T}_\varepsilon v_\varepsilon)_{\varepsilon>0}$ has to be in $\overline{\Omega} \times Y$, since $\mathrm{vol}(\Omega_\varepsilon^+ \backslash \Omega) \to 0$. Due to $\mathrm{vol}(\partial\Omega) = 0$ we also have $\mathrm{L}^p(\Omega \times Y) = \mathrm{L}^p(\overline{\Omega} \times Y)$ and so every accumulation point of $(\mathcal{T}_\varepsilon v_\varepsilon)_{\varepsilon>0}$ can be uniquely identified with an element of $\mathrm{L}^p(\Omega\times Y)$. But notice that it is important to determine the convergence in $\mathrm{L}^p(\mathbb{R}^d\times Y)$ and not in $\mathrm{L}^p(\Omega\times Y)$. We refer to Ref. [15], where it is shown in Example 2.3 that convergence in $\mathrm{L}^p(\Omega\times Y)$ is not sufficient.

Note, that according to the definition of the two-scale convergence in $\mathrm{L}^p(\Omega\times Y)$ via the convergence of the unfolded sequence in $\mathrm{L}^p(\mathbb{R}^d\times Y)$ all convergence properties known for L^p-convergence are transmitted. For a summary of those properties we refer to Proposition 2.4 in [15]. For the convenience of the reader we state here only those properties used in the following.

Proposition 2.4 ([15]). *Let $p\in(1,\infty)$ and set $p' := \frac{p}{p-1}$. Furthermore, let $V_0\in\mathrm{L}^p(\Omega\times Y)$, $W_0\in\mathrm{L}^{p'}(\Omega\times Y)$ and $M_0\in\mathrm{L}^1(\Omega\times Y)$ be given. Then for sequences $(v_\varepsilon)_{\varepsilon>0}\subset\mathrm{L}^p(\Omega)$ and $(w_\varepsilon)_{\varepsilon>0}\subset\mathrm{L}^{p'}(\Omega)$ the following conditions hold.*

(a) *If $v_\varepsilon\xrightarrow{w}V_0$ in $\mathrm{L}^p(\Omega\times Y)$ and $w_\varepsilon\xrightarrow{s}W_0$ in $\mathrm{L}^{p'}(\Omega\times Y)$ then $\langle v_\varepsilon,w_\varepsilon\rangle_{\mathrm{L}^2(\Omega)}\to\langle V_0,W_0\rangle_{\mathrm{L}^2(\Omega\times Y)}$.*

(b) *If $v_\varepsilon\to v_0$ in $\mathrm{L}^p(\Omega)$ then $v_\varepsilon\xrightarrow{s}Ev_0$ in $\mathrm{L}^p(\Omega\times Y)$, where $E:\mathrm{L}^p(\Omega)\to\mathrm{L}^p(\Omega\times Y)$ for $v\in\mathrm{L}^p(\Omega)$ and $(x,y)\in\Omega\times Y$ is defined via $Ev(x,y):=v(x)$.*

(c) *If $v_\varepsilon\xrightarrow{s}V_0$ in $\mathrm{L}^p(\Omega\times Y)$ and if $(m_\varepsilon)_{\varepsilon>0}$ is a bounded sequence of $\mathrm{L}^\infty(\Omega)$ such that $\mathcal{T}_\varepsilon m_\varepsilon(x,y)\to M_0(x,y)$ for almost every $(x,y)\in\Omega\times Y$. Then $m_\varepsilon v_\varepsilon\xrightarrow{s}M_0V_0$ in $\mathrm{L}^p(\Omega\times Y)$.*

The following corollary extends property (c) of Proposition 2.4 to a special case appearing when applying the two-scale theory to (1.1) for a tensor $\mathbb{C}_\varepsilon$ given by (1.6). The proof is done via a standard contradiction argument.

Corollary 2.5. *For $p\in(1,\infty)$ let $(v_\varepsilon)_{\varepsilon>0}\subset\mathrm{L}^p(\Omega)$ and $V_0\in\mathrm{L}^p(\Omega\times Y)$ be given such that $v_\varepsilon\xrightarrow{s}V_0$ in $\mathrm{L}^p(\Omega\times Y)$. Moreover, let $(m_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $\mathrm{L}^\infty(\Omega)$ satisfying $m_\varepsilon\xrightarrow{s}M_0$ of $\mathrm{L}^1(\Omega\times Y)$ for some function $M_0\in\mathrm{L}^1(\Omega\times Y)$. Then $m_\varepsilon v_\varepsilon\xrightarrow{s}M_0V_0$ in $\mathrm{L}^p(\Omega\times Y)$.*

In Section 5, we are going to prove Γ-convergence results with respect to the weak two-scale topology for functionals being related to the boundary value problems mentioned in Section 1. There, the following integral identity for $v\in\mathrm{L}^1(\Omega)$ will be central.

$$\int_\Omega v(x)\mathrm{d}x=\int_{[\Omega\times Y]_\varepsilon}\mathcal{T}_\varepsilon v(x,y)\mathrm{d}y\mathrm{d}x \tag{2.4}$$

Moreover, this identity immediately gives us the norm-preservation of the periodic unfolding operator $\mathcal{T}_\varepsilon$ and it is proved by decomposing $\mathbb{R}^d$ into cells $\varepsilon(\lambda+Y)$ for $\lambda\in\Lambda$.

Since the models introduced in Section 1 contain gradients we now will consider bounded sequences of $\mathrm{W}^{1,p}(\Omega)$ and state the main two-scale convergence results for these. In particular we will need the function space

$$\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y})=\left\{v\in\mathrm{W}^{1,p}_{\mathrm{per}}(\overline{Y})\,\middle|\,\int_{\mathcal{Y}}v(y)\mathrm{d}y=0\right\}.$$

To describe the weak two-scale convergence of gradients we introduce the function space $\mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$, which is the space of functions $V\in\mathrm{L}^p(\Omega\times Y)=\mathrm{L}^p(\Omega;\mathrm{L}^p(Y))$, having the same traces on opposite faces of Y and satisfying $\int_Y V(x,y)\mathrm{d}y = 0$ for almost every $x \in \Omega$ and $\nabla_y V \in \mathrm{L}^p(\Omega{\times}Y)^d$ in the sense of distributions. We equip this space with the norm $\|V\|_{\mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))} := \|\nabla_y V\|_{\mathrm{L}^p(\Omega\times Y)^d}$.

With this, we have the following compactness result used for the convergence of the displacement component of the microscopic models in Section 3, cf. [18, Theorem 3.1.4]:

Proposition 2.6. *Let $(v_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $\mathrm{W}^{1,p}(\Omega)$. Then there exists a subsequence $(v_{\varepsilon'})_{\varepsilon'>0}$ of $(v_\varepsilon)_{\varepsilon>0}$ and functions $v_0 \in \mathrm{W}^{1,p}(\Omega)$ and $V_1 \in \mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ so that:*

$$\begin{aligned} v_{\varepsilon'} &\rightharpoonup v_0 && \text{in } \mathrm{W}^{1,p}(\Omega),\\ v_{\varepsilon'} &\xrightarrow{s} Ev_0 && \text{in } \mathrm{L}^p(\Omega{\times}Y),\\ \nabla v_{\varepsilon'} &\xrightharpoonup{w} \nabla_x Ev_0 + \nabla_y V_1 && \text{in } \mathrm{L}^p(\Omega{\times}Y)^d, \end{aligned}$$

where $E : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\Omega{\times}Y)$ is defined via $Ev(x,y) := v(x)$.

For the construction of the displacement component of the joint recovery sequence the following density result is important, cf. [9, Proposition 2.11].

Proposition 2.7. *Let $(w_0, W_1) \in \mathrm{W}^{1,p}_0(\Omega){\times}\mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ be given. Moreover, for every $\varepsilon > 0$ let $w_\varepsilon \in \mathrm{W}^{1,p}_0(\Omega)$ be the solution of the following elliptic problem:*

$$\int_\Omega ((w_\varepsilon - \mathcal{F}_\varepsilon(Ew_0)^{\mathrm{ex}})w + \langle \nabla w_\varepsilon - \mathcal{F}_\varepsilon(\nabla_x Ew_0 + \nabla_y W_1)^{\mathrm{ex}}, \nabla v\rangle_d)\mathrm{d}x = 0 \quad \forall\, v \in \mathrm{W}^{1,p'}_0(\Omega).$$

Then

$$\begin{aligned} w_\varepsilon &\rightharpoonup w_0 && \text{in } \mathrm{W}^{1,p}_0(\Omega),\\ w_\varepsilon &\xrightarrow{s} Ew_0 && \text{in } \mathrm{L}^p(\Omega{\times}Y),\\ \nabla w_\varepsilon &\xrightarrow{s} \nabla_x Ew_0 + \nabla_y W_1 && \text{in } \mathrm{L}^p(\Omega{\times}Y)^d, \end{aligned}$$

3 Homogenization of non-periodic coefficients

In this section, the non-periodic microstructures having some intrinsic length scale denoted by $\varepsilon > 0$ are introduced. These microstructures are modeled by non-periodic coefficients of an elliptic boundary value problem. The aim is to find a homogenized description of this boundary value problem preserving the microstructure in some sense but being independent of the small parameter $\varepsilon > 0$.

The microstructure is based on a tensor $\widetilde{\mathbb{C}} : \mathbb{R}^m \to \mathcal{M}(Y;\alpha,\beta)$ where α and β are positive constants independent of the parameters $z \in \mathbb{R}^m$ ($m \in \mathbb{N}$ fixed). In contrast to the periodic case, this tensor is allowed to vary with respect to the parameters $z \in \mathbb{R}^m$. The crucial assumptions on $\widetilde{\mathbb{C}} : \mathbb{R}^m \to \mathcal{M}(Y;\alpha,\beta)$ are the following:

Measurability: For every measurable function $z : \mathbb{R}^d \to \mathbb{R}^m$ the mapping

$$\widetilde{\mathbb{C}}(z(\cdot))(\cdot) : \begin{cases} \mathbb{R}^d{\times}Y \to \mathbb{M}_{\mathbb{B}}(\alpha,\beta), \\ (x,y) \mapsto \widetilde{\mathbb{C}}(z(x))(y) \end{cases} \quad \text{is measurable on } \mathbb{R}^d{\times}Y. \tag{3.1}$$

Continuity: For every sequence $(z_\delta)_{\delta>0} \subset \mathbb{R}^m$ satisfying $\lim_{\delta\to 0} z_\delta = z$ for $z \in \mathbb{R}^m$ we have

$$\lim_{\delta\to 0} \|\widetilde{\mathbb{C}}(z_\delta) - \widetilde{\mathbb{C}}(z)\|_{\mathrm{L}^1(Y;\mathbb{M}_{\mathbb{B}}(\alpha,\beta))} = 0 \tag{3.2}$$

Given $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$, the tensor $\mathbb{C}_\varepsilon := \widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon) \in \mathcal{M}(\Omega;\alpha,\beta)$ for almost every $x \in \Omega$ is defined by

$$\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)(x) := \widetilde{\mathbb{C}}(z_\varepsilon(x))(\{\tfrac{x}{\varepsilon}\}_Y). \tag{3.3}$$

Having such microstructures in mind we are interested in the limit passage $\varepsilon \to 0$ in the following elliptic boundary value problem:

Given a function $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ let $u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ be the weak solution of

$$\langle \widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)\nabla u_\varepsilon, \nabla v\rangle_{\mathrm{L}^2(\Omega)^{n\times d}} = \langle \ell, v\rangle \quad \text{for all } v \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n, \tag{3.4}$$

where $\ell \in (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n)^*$.

Our first result is the following theorem, where we study the limit passage of solutions of (3.4) for converging sequences $(z_\varepsilon)_{\varepsilon>0}$:

Theorem 3.1. *Let $\widetilde{\mathbb{C}} : \mathbb{R}^m \to \mathcal{M}(Y;\alpha,\beta)$ satisfy the conditions* (3.1) *and* (3.2) *and let $(z_\varepsilon)_{\varepsilon>0}$ be given such that $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and $z_\varepsilon \to z_0$ in $\mathrm{L}^1(\Omega)^m$ for $\varepsilon \searrow 0$ with some function $z_0 \in \mathrm{L}^1(\Omega)^m$. Moreover, let $\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon) \in \mathcal{M}(\Omega;\alpha,\beta)$ be defined by* (3.3)*. If $u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ is the solution of* (3.4)*, then there exists a function $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ such that*

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n,$$

where $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ satisfies

$$\langle \widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0)\nabla u_0, \nabla v\rangle_{\mathrm{L}^2(\Omega)^{n\times d}} = \langle \ell, v\rangle \quad \text{for all } v \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n. \tag{3.5}$$

The tensor $\widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0) \in \mathcal{M}(\Omega,\alpha,\beta)$ for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^{n\times d}$ is given by

$$\langle \widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0)(x)\xi, \xi\rangle_{n\times d} = \min_{v\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^n} \int_Y \langle \widetilde{\mathbb{C}}(z_0(x))(y)(\xi + \nabla_y v(y)), \xi + \nabla_y v(y)\rangle_{n\times d} \mathrm{d}y. \tag{3.6}$$

The proof of this theorem is split into the following three propositions. Observe first, that by standard arguments (cf. for instance [9]) it follows that the minimization problem in (3.6) indeed defines a quadratic expression in ξ. We summarize this in

Proposition 3.2. *For every $z \in \mathbb{R}^m$ there exists $\mathbb{C}_{\mathrm{eff}}(z) \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{n\times d}, \mathbb{R}^{n\times d})$ such that*

$$\forall\, \xi \in \mathbb{R}^{n\times d}: \qquad \langle \mathbb{C}_{\mathrm{eff}}(z)\xi, \xi\rangle_{n\times d} = \min_{v\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^n} \int_Y \langle \widetilde{\mathbb{C}}(z)(y)(\xi + \nabla_y v(y)), \xi + \nabla_y v(y)\rangle_{n\times d} \mathrm{d}y.$$

In Proposition 3.3 below the convergence of the sequence $(u_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ of solutions of (3.4) to the unique solution of the following two-scale problem is proven:

For a given function $z_0 \in \mathrm{L}^1(\Omega)^m$ let $(u_0, U_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ be the unique solution of the two-scale equation

$$\langle \widetilde{\mathbb{C}}_0(z_0)(\nabla_x E u_0 + \nabla_y U_1), \nabla_x E v + \nabla_y V\rangle_{\mathrm{L}^2(\Omega\times Y)^{n\times d}} = \langle \ell, v\rangle, \tag{3.7}$$

where

$$\widetilde{\mathbb{C}}_0(z_0)(x,y) := \widetilde{\mathbb{C}}(z_0(x))(y) \quad \text{for almost every } (x,y) \in \Omega\times Y. \tag{3.8}$$

Proposition 3.3. *Let $\widetilde{\mathbb{C}} : \mathbb{R}^m \to \mathcal{M}(Y;\alpha,\beta)$ satisfy the conditions* (3.1) *and* (3.2) *and let $(z_\varepsilon)_{\varepsilon>0}$ be given such that $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and $z_\varepsilon \to z_0$ in $\mathrm{L}^1(\Omega)^m$ for $\varepsilon \searrow 0$ with some function $z_0 \in \mathrm{L}^p(\Omega)^m$. Moreover, let $\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon) \in \mathcal{M}(\Omega;\alpha,\beta)$ be defined by* (3.3)*. If $u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ is the solution of* (3.4)*, then there exists a function $(u_0, U_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ such that*

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n, \\ u_\varepsilon &\overset{s}{\to} Eu_0 && \text{in } \mathrm{L}^2(\Omega\times Y)^n, \\ \nabla u_\varepsilon &\overset{w}{\rightharpoonup} \nabla_x Eu_0 + \nabla_y U_1 && \text{in } \mathrm{L}^p(\Omega\times Y)^{n\times d}. \end{aligned}$$

Moreover, $(u_0, U_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ is the unique solution of (3.7)*.*

Proof. Let $(z_\varepsilon)_{\varepsilon>0}$ be given such that $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and $z_\varepsilon \to z_0$ in $\mathrm{L}^1(\Omega)^m$ for some function $z_0 \in \mathrm{L}^p(\Omega)^m$ and $\varepsilon \searrow 0$.

1. Since $\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon) \in \mathcal{M}(\Omega;\alpha,\beta)$ for all $\varepsilon > 0$ according to assumption (1.3), we have the following a priori estimate for the solutions $u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ of (3.4)

$$\|u_\varepsilon\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n} \le C,$$

where $C = C(\alpha, C_{\mathbb{B}}, \ell) > 0$ is independent of $\varepsilon > 0$. Hence, according to Proposition 2.6 there exist a subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$ and functions $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ and $U_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ such that

$$\begin{aligned} u_{\varepsilon'} &\rightharpoonup u_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n, \\ u_{\varepsilon'} &\overset{s}{\to} Eu_0 && \text{in } \mathrm{L}^2(\Omega\times Y)^n, \\ \nabla u_{\varepsilon'} &\overset{w}{\rightharpoonup} \nabla_x Eu_0 + \nabla_y U_1 && \text{in } \mathrm{L}^2(\Omega\times Y)^{n\times d}. \end{aligned}$$

2. We now investigate the convergence of the coefficient tensor $\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)$ and prove $\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon) \overset{s}{\to} \widetilde{\mathbb{C}}_0(z_0)$ in $\mathrm{L}^1(\Omega\times Y;\mathbb{M}_{\mathbb{B}}(\alpha,\beta))$. For this purpose, we rewrite $\mathcal{T}_\varepsilon\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)$ according to Definition 2.1.

The case $x \in \mathbb{R}^d \setminus \overline{\Omega}$:

For fixed $x \in \mathbb{R}^d\setminus\overline{\Omega}$ due to (2.3) there exists $\varepsilon_0 > 0$ such that $x \in \mathbb{R}^d\setminus\Omega^+_\varepsilon$ for all $\varepsilon \in (0,\varepsilon_0)$. Hence, $\mathcal{T}_\varepsilon\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)(x,\cdot) \equiv 0$ on Y for all $\varepsilon \in (0,\varepsilon_0)$. Moreover, the extension $\widetilde{\mathbb{C}}^{\mathrm{ex}}_0(z_0)$ trivially fulfills $\widetilde{\mathbb{C}}^{\mathrm{ex}}_0(z_0)(x,\cdot) \equiv 0$ for all $x \in \mathbb{R}^d \setminus \overline{\Omega}$ by definition. Altogether, this shows that for all $x \in \mathbb{R}^d \setminus \overline{\Omega}$ we have

$$\mathcal{T}_\varepsilon\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)(x,\cdot) \to \widetilde{\mathbb{C}}^{\mathrm{ex}}_0(z_0)(x,\cdot) \quad \text{in } \mathrm{L}^1(Y;\mathbb{M}_{\mathbb{B}}(\alpha,\beta)). \tag{3.9}$$

The case $x \in \Omega$:

Since Ω is assumed to be open for fixed $x \in \Omega$ due to (2.3) there exists $\varepsilon_0 > 0$ such that $x \in \Omega^-_\varepsilon$ for all $\varepsilon \in (0,\varepsilon_0)$. Note, that for $(x,y) \in \Omega^-_\varepsilon\times Y$ we have $z_\varepsilon(x) = z_\varepsilon(\mathcal{N}_\varepsilon(x))$, $\mathcal{N}_\varepsilon(\mathcal{N}_\varepsilon(x) + \varepsilon y) = \mathcal{N}_\varepsilon(x)$ and $\{\frac{\mathcal{N}_\varepsilon(x)+\varepsilon y}{\varepsilon}\}_Y = y$. Keeping this observation in mind when applying $\mathcal{T}_\varepsilon$ to the tensor $\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)$ given by (3.3) results in

$$\mathcal{T}_\varepsilon\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)(x,y) = \widetilde{\mathbb{C}}(z_\varepsilon(x))(y) \quad \text{for all } (x,y) \in \Omega\times Y. \tag{3.10}$$

According to $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and $z_\varepsilon \to z_0$ in $\mathrm{L}^1(\Omega)^m$, there exists a subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$ such that

$$z_{\varepsilon'}(x) \to z_0(x) \quad \text{for almost every } x \in \Omega. \tag{3.11}$$

Exploiting the continuity of $\widetilde{\mathbb{C}}$ combining (3.10) and (3.11), for almost every $x \in \Omega$ results in

$$\mathcal{T}_{\varepsilon'}\widetilde{\mathbb{C}}_{\varepsilon'}(z_{\varepsilon'})(x,\cdot) \to \widetilde{\mathbb{C}}(z_0(x))(\cdot) \overset{(3.8)}{=} \widetilde{\mathbb{C}}_0(z_0)(x,\cdot) \quad \text{in } \mathrm{L}^1(Y;\mathbb{M}_{\mathbb{B}}(\alpha,\beta)). \tag{3.12}$$

Here, we have applied the Theorem of dominated convergence and the fact that for $0 < \varepsilon < \varepsilon_0$ the coefficients $\mathcal{T}_\epsilon\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)$ and $\widetilde{\mathbb{C}}_0(z_0)$ are uniformly bounded on $\Omega_{\varepsilon_0}\times Y$ by a constant depending on β, see (1.4).

Combining (3.9) and (3.12) and exploiting $\mu_d(\partial\Omega) = 0$ (see (2.3)) we finally showed for almost every $x \in \mathbb{R}^d$

$$\mathcal{T}_{\varepsilon'}\widetilde{\mathbb{C}}_{\varepsilon'}(z_{\varepsilon'})(x,\cdot) \to \widetilde{\mathbb{C}}_0^{\mathrm{ex}}(z_0)(x,\cdot) \quad \text{in } \mathrm{L}^1(Y;\mathbb{M}_{\mathbb{B}}(\alpha,\beta)). \tag{3.13}$$

Applying once more the Theorem of dominated convergence, for $N = n^2{+}d^2$ we finally arrive at

$$\|\mathcal{T}_{\varepsilon'}\widetilde{\mathbb{C}}_{\varepsilon'}(z_{\varepsilon'}) - \widetilde{\mathbb{C}}_0^{\mathrm{ex}}(z_0)\|_{\mathrm{L}^1(\mathbb{R}^d\times Y)^N} = \int_{\mathbb{R}^d} \|\mathcal{T}_{\varepsilon'}\widetilde{\mathbb{C}}_{\varepsilon'}(z_{\varepsilon'})(x,\cdot) - \widetilde{\mathbb{C}}_0^{\mathrm{ex}}(z_0)(x,\cdot)\|_{\mathrm{L}^1(Y)^N}\,\mathrm{d}x \to 0$$

for $\varepsilon' \to 0$, which is nothing else but

$$\widetilde{\mathbb{C}}_{\varepsilon'}(z_{\varepsilon'}) \overset{s}{\to} \widetilde{\mathbb{C}}_0(z_0) \quad \text{in } \mathrm{L}^1(\Omega{\times}Y;\mathbb{M}_{\mathbb{B}}(\alpha,\beta)). \tag{3.14}$$

Via a standard contradiction argument we finally conclude the strong two-scale convergence of the whole sequence $(\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon))_{\varepsilon>0}$ to $\widetilde{\mathbb{C}}_0(z_0)$ with respect to the L^1-topology.

3. For $(v,V) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ arbitrary but fixed choose $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ as in Proposition 2.7, such that $\nabla v_\varepsilon \overset{s}{\to} \nabla_x Ev + \nabla_y V$ in $\mathrm{L}^2(\Omega{\times}Y)^{n\times d}$.

4. Choose a further subsequence $(\varepsilon'')_{\varepsilon''>0}$ of $(\varepsilon')_{\varepsilon'>0}$ such that $(\mathcal{T}_{\varepsilon''}\widetilde{\mathbb{C}}_{\varepsilon''}(z_{\varepsilon''})))_{\varepsilon''>0}$ converges almost everywhere in $\mathbb{R}^d{\times}Y$ (available due to (3.14)). Then, according to Corollary 2.5 combining the results of step 2 and 3 leads to

$$\widetilde{\mathbb{C}}_{\varepsilon''}(z_{\varepsilon''})\nabla v_{\varepsilon''} \overset{s}{\to} \widetilde{\mathbb{C}}_0(z_0)(\nabla_x Ev + \nabla_y V) \qquad \text{in } \mathrm{L}^2(\Omega{\times}Y)^{n\times d}. \tag{3.15}$$

5. Considering the left hand side of the weak formulation of (3.4) gives us

$$\langle\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)\nabla u_\varepsilon, \nabla v_\varepsilon\rangle_{\mathrm{L}^2(\Omega)^{n\times d}} = \langle\nabla u_\varepsilon, \widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)\nabla v_\varepsilon\rangle_{\mathrm{L}^2(\Omega)^{n\times d}},$$

where we already plugged in the particular test function $v_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ chosen in step 3. According to Proposition 2.4(a), the convergence result of step 1 and (3.15) we have

$$\lim_{\varepsilon''\to 0}\langle\widetilde{\mathbb{C}}_{\varepsilon''}(z_{\varepsilon''})\nabla u_{\varepsilon''}, \nabla v_{\varepsilon''}\rangle_{\mathrm{L}^2(\Omega)^{n\times d}} = \langle\widetilde{\mathbb{C}}_0(z_0)(\nabla_x Eu_0 + \nabla_y U_1), \nabla_x Ev + \nabla_y V\rangle_{\mathrm{L}^2(\Omega\times Y)^{n\times d}}.$$

Since $v_\varepsilon \rightharpoonup v$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ (see Proposition 2.7), the right hand side of the weak formulation of (3.4) converges to $\langle\ell, v\rangle$. Hence, for all $(v,V) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ the function $(u_0,U_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ is the unique solution of

$$\langle\widetilde{\mathbb{C}}_0(z_0)(\nabla_x Eu_0 + \nabla_y U_1), \nabla_x Ev + \nabla_y V\rangle_{\mathrm{L}^2(\Omega\times Y)^{n\times d}} = \langle\ell, v\rangle,$$

which is the two-scale equation stated in (3.7). Due to the uniqueness of the solution $(u_0,U_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ a contradiction argument yields the convergence of the whole sequence $(u_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$. □

Finally, we show that the two-scale equation (3.7) can be identified with a one-scale problem. For given functions $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ and $z_0 \in \mathrm{L}^p(\Omega)^m$ we now consider the unique solution $U_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ of the corrector equation

$$\langle \widetilde{\mathbb{C}}_0(z_0)(\nabla_x E u_0 + \nabla_y U_1), \nabla_y V\rangle_{\mathrm{L}^2(\Omega\times Y)^{n\times d}} = 0 \quad \forall\, V \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n. \tag{3.16}$$

The next proposition yields a crucial property of $U_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ enabling us to prove the equivalence of the limit systems given by (3.5) and (3.7). For this purpose, we introduce one more operator. For $z\in\mathbb{R}^m$, $\xi\in\mathbb{R}^{n\times d}$ and $v\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$ let

$$I_z(\xi,v) = \tfrac{1}{2}\int_Y \langle \widetilde{\mathbb{C}}(z)(y)(\xi+\nabla_y v(y)), \xi+\nabla_y v(y)\rangle_{n\times d}\,\mathrm{d}y.$$

The operator $\mathcal{L}_z : \mathbb{R}^{n\times d}\to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$ is defined as $\mathcal{L}_z(\xi) = \mathrm{Argmin}\{\, I_z(\xi,v)\,;\, v\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})\,\}$.

Proposition 3.4. *For every $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$, $z_0\in\mathrm{L}^1(\Omega)^m$ and $U_1\in\mathrm{L}^2(\Omega,\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ the following statements are equivalent:*

(i) *U_1 is the unique solution of* (3.16).

(ii) *$U_1 = \mathcal{L}_{z_0(\cdot)}(\nabla_x u_0(\cdot))$.*

(iii) *For all $v\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ and $V\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$, U_1 satisfies*

$$\langle \widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0)\nabla u_0, \nabla v\rangle_{\mathrm{L}^2(\Omega)^n} = \langle \widetilde{\mathbb{C}}_0(z_0)(\nabla_x E u_0+\nabla_y U_1), \nabla_x E v + \nabla_y V\rangle_{\mathrm{L}^2(\Omega\times Y)^{n\times d}}.$$

Proof. (ii)⇒(i): Observe first that by basic density properties for Bochner spaces the linear span of $\{(f_1v_1,\dots,f_nv_n)^T \mid f_i\in\mathrm{L}^2(\Omega), v_i\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})\}$ is dense in $\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$. Hence, it is sufficient to prove that $U_1 := \mathcal{L}_{z_0(\cdot)}(\nabla_x u_0(\cdot))$ satisfies (3.16) for every $V = fv$ with $f\in L^2(\Omega)$ and $v\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^n$. By definition, for almost every $x\in\Omega$ and all $v\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^n$ the function $v_x^* := \mathcal{L}_{z_0(x)}(\nabla_x u_0(x)) = U_1(x,\cdot)\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$ fulfills the Euler-Lagrange equation

$$0 = \mathrm{D}_v I_{z_0(x)}(\nabla_x u_0(x), v_x^*)[v] = \langle \widetilde{\mathbb{C}}(z_0(x))(\nabla_x E u_0(x) + \nabla_y v_x^*), \nabla_y v\rangle_{\mathrm{L}^2(Y)^{n\times d}}.$$

After multiplication with $f\in\mathrm{L}^2(\Omega)$ and integrating with respect to Ω we obtain (3.16).

(i)⇒(ii): For given $(u_0,z_0)\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n\times\mathrm{L}^p(\Omega)^m$ let $U_1\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ be the unique solution of the two-scale equation (3.16). As already proven in the first step $U_1^*(x,y) := \mathcal{L}_{z_0(x)}(\nabla_x u_0(x))(y)$ is also a solution of equation (3.16). According to the uniqueness of solutions this results in $U_1 = U_1^*$.

(i),(ii) ⇔ (iii): The prove of the equivalence with statement (iii) relies on the following identity for the derivative of the mapping $\xi\to\langle\mathbb{C}_{\mathrm{eff}}(z)\xi,\xi\rangle_{\mathbb{R}^{n\times d}}$: For all $\xi,\eta\in\mathbb{R}^{n\times d}$ and $z\in\mathbb{R}^m$ it holds

$$\langle \mathbb{C}_{\mathrm{eff}}(z)\xi,\eta\rangle_{n\times d} = \langle \widetilde{\mathbb{C}}(z)(\xi+\nabla_y\mathcal{L}_z(\xi)),\eta\rangle_{\mathrm{L}^2(Y)^{n\times d}}. \tag{3.17}$$

Assume that $U_1 = \mathcal{L}_{z_0(\cdot)}(\nabla u_0(\cdot)) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$ and hence satisfies (i). Then for all $v\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ and $V\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$, U_1 satisfies

$$\begin{aligned}&\langle \widetilde{\mathbb{C}}(z_0)(\nabla_x E u_0+\nabla_y U_1), \nabla_x E v+\nabla_y V\rangle_{\mathrm{L}^2(\Omega\times Y)^{n\times d}}\\ &\quad\overset{(3.16)}{=} \langle \widetilde{\mathbb{C}}(z_0)(\nabla_x E u_0+\nabla_y U_1), \nabla_x E v\rangle_{\mathrm{L}^2(\Omega\times Y)^{n\times d}} \overset{(3.17)}{=} \langle \mathbb{C}_{\mathrm{eff}}(z_0)\nabla u_0,\nabla v\rangle_{\mathrm{L}^2(\Omega)^{n\times d}}.\end{aligned}$$

On the other hand assume that $U_1 \in \mathrm{L}^2(\Omega, \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ satisfies (iii). Then, again by (3.17), for all $v \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ and $V \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ it holds

$$\begin{aligned}
&\langle \widetilde{\mathbb{C}}(z_0)(\nabla_x E u_0 + \nabla_y U_1), \nabla_x E v + \nabla_y V\rangle_{\mathrm{L}^2(\Omega\times Y)^{n\times d}} \\
&\qquad = \langle \mathbb{C}_{\mathrm{eff}}(z_0)\nabla u_0, \nabla v\rangle_{\mathrm{L}^2(\Omega)^{n\times d}} \overset{(3.17)}{=} \langle \widetilde{\mathbb{C}}(z_0)(\nabla_x E u_0 + \nabla_y U_1), \nabla_x E v\rangle_{\mathrm{L}^2(\Omega\times Y)^{n\times d}},
\end{aligned}$$

which implies (3.16) and (i). □

We are now in the position to prove Theorem 3.1.

Proof of Theorem 3.1. According to Proposition 3.3, there exist functions $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ and $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ written as a 2-tuple being the unique solution of (3.7) satisfying $u_\varepsilon \rightharpoonup u_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$. Choosing $v = 0$ in (3.7) shows that U_1 satisfies (3.16). Exploiting Proposition 3.4(iii), this finally implies that u_0 satisfies (3.5). □

Example 3.5. Here we are going to consider a linear elastic model, where for fixed $\varepsilon > 0$ the piecewise constant function $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ describes the distribution of two different types of material. This example is related to the time-dependent damage model investigated in the forthcoming paper [10], where for every $t \in [0, T]$ the function $z_\varepsilon(t) \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ will be given by a flow rule modeling the evolution of damage. There, by the decrease of $z_\varepsilon : [0, T]\times\Omega \to \mathbb{R}^m$ with respect to time the decrease of the amount of undamaged material is modeled.

Let $m = 1$, $n = d$ and let $\mathbb{B}(\xi) := \frac{1}{2}(\xi+\xi^T)$ for $\xi \in \mathbb{R}^{d\times d}$. Moreover, for $\mathbb{C}_1, \mathbb{C}_2 \in \mathbb{M}_{\mathbb{B}}(\alpha, \beta)$ we define

$$\widetilde{\mathbb{C}} : \mathbb{R} \to \mathcal{M}(Y; \alpha, \beta), \quad \widetilde{\mathbb{C}}(z)(y) = \begin{cases} \mathbb{1}_{Y\setminus B(r(z))}(y)\mathbb{C}_1 + \mathbb{1}_{B(r(z))}(y)\mathbb{C}_2 & \text{if } z \in [0,1] \\ \mathbb{1}_{Y\setminus B(r(0))}(y)\mathbb{C}_1 + \mathbb{1}_{B(r(0))}(y)\mathbb{C}_2 & \text{if } z < 0 \\ \mathbb{1}_{Y\setminus B(r(1))}(y)\mathbb{C}_1 + \mathbb{1}_{B(r(1))}(y)\mathbb{C}_2 & \text{if } z > 1 \end{cases}.$$

Here, $\mathbb{1}_A$ denotes the indicator function of the set A and $B(r(z))$ is the closed ball with radius $r(z) := R_Y(1-z)$ and the same center as Y (see Figure 1). The maximal radius $R_Y > 0$ is chosen such that $B(R_Y) \subset Y$.

To apply the convergence theory of this section to this example we have to verify the measurability condition (3.1) and the continuity condition (3.2). For $\widetilde{\mathbb{C}} : \mathbb{R} \to \mathcal{M}(Y; \alpha, \beta)$ defined as above the assumption (3.2) is fulfilled trivially. But note, that for fixed $y \in Y$ the mapping $z \mapsto \widetilde{\mathbb{C}}(z)(y)$ is not continuous and hence does not satisfy the Carathéodory condition, which would imply measurability of the composed function $\widetilde{\mathbb{C}}(z(\cdot))(\cdot)$.

To verify (3.1) let $z : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary measurable function. According to the definition of $\widetilde{\mathbb{C}} : \mathbb{R} \to \mathcal{M}(Y; \alpha, \beta)$, the mapping $\widetilde{\mathbb{C}}(z(\cdot))(\cdot) : \mathbb{R}^d\times Y \to \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}; \mathbb{R}^{d\times d})$ is constant on $M := \bigcup_{x\in\mathbb{R}^d} \Big(\{x\}\times B(r(z(x)))\Big)$ and on $(\mathbb{R}^d\times Y) \setminus M$. Hence, (3.1) is proven by showing that M is a measurable subset of $\mathbb{R}^d\times Y$.

To show the measurability of M we start by choosing a countable sequence $(z_\delta)_{(\delta>0)}$ of simple functions approximating z from below, i.e. $z_\delta(x) = \sum_{k=1}^{N_\delta} \mathbb{1}_{A_k^\delta}(x) z_k^\delta$ with $z_k^\delta = const$, $A_k^\delta \subset \mathbb{R}^d$ measurable and $\dot{\bigcup}_{k=1}^{N_\delta} A_k = \mathbb{R}^d$ with $z_\delta(x) \nearrow z(x)$ for all $x \in \mathbb{R}^d$. Let $M_\delta :=$

$\bigcup_{x\in\mathbb{R}^d}\Big(\{x\}\times B(r(z_\delta(x)))\Big)$. Note that M_δ is measurable for all $\delta>0$, since due to

$$M_\delta = \dot{\bigcup}_{k=1}^{N_\delta}\Big(\bigcup_{x\in A_k^\delta}\{x\}\times B(r(z_\delta(x)))\Big) = \dot{\bigcup}_{k=1}^{N_\delta}\Big(A_k^\delta\times B(r(z_k^\delta))\Big)\,,$$

it is the disjoint union of finitely many measurable sets and ergo measurable. By definition, $M\subset M_\delta$ for every $\delta>0$. The opposite relation $\bigcap_{\delta>0}M_\delta\subset M$ is shown by the following contradiction argument:

Let $(x^*,y^*)\in\bigcap_{\delta>0}M_\delta$ but $(x^*,y^*)\notin M$. Then for all $\delta>0$

$$y^*\in B(r(z_\delta(x^*))) \tag{3.18}$$

but $\operatorname{dist}(y^*,B(r(z(x^*))))=:2\Delta>0$ since $B(r(z(x^*)))$ was assumed to be closed. Hence,

$$y^*\notin B(r(z(x^*))+\Delta) \tag{3.19}$$

Since $z_\delta(x^*)\to z(x^*)$ by assumption, there exists $\delta_0>0$ such that for all $\delta\in(0,\delta_0)$ we have $|r(z_\delta(x^*))-r(z(x^*))|\le\Delta$. Hence, by (3.18),

$$y^*\in B(r(z_\delta(x^*)))\subset B(r(z(x^*))+\Delta),$$

which is a contradiction to (3.19). Altogether we proved $M=\bigcap_{\delta>0}M_\delta$. Since the countable intersection of measurable sets is measurable again, this shows the measurability of M.

Given a sequence $(z_\varepsilon)_{\varepsilon>0}$ with $z_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega)$, $z_\varepsilon\in[0,1]$ a.e. in Ω and $z_\varepsilon\to z_0$ strongly in $\mathrm{L}^1(\Omega)$, according to Theorem 3.1 the effective tensor is given as follows: for all $\xi\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and almost all $x\in\Omega$ we have

$$\begin{aligned}&\langle\widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0(x))\xi,\xi\rangle_{d\times d}\\&=\min_{v\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d}\int_Y\Big\langle\Big(\mathbb{1}_{Y\setminus B(r(z_0(x)))}\mathbb{C}_1+\mathbb{1}_{B(r(z_0(x)))}\mathbb{C}_2\Big)(\xi+\nabla_y v(y)),(\xi+\nabla_y v(y))\Big\rangle_{d\times d}\mathrm{d}y.\end{aligned}$$

For every $x\in\Omega$, this corresponds to the cell formula for periodic homogenization with respect to the geometry defined by $z_0(x)$, see also Figure 2. This example is a first step to give some mathematical background to the two-scale damage models investigated in [21].

4 Discrete gradients of piecewise constant functions

As already mentioned in the introduction, the second part of this paper is devoted to the Γ-convergence of a sequence of functionals $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ being related to the homogenization result of Section 3. Additionally to u_ε, in these functionals z_ε is considered as an additional unknown and we are interested in the Γ-convergence of $\mathcal{E}_\varepsilon(\cdot,\cdot)$ with respect to the weak topology induced by the functional, see Section 5. A major assumption of Theorem 3.1 is the strong convergence in $\mathrm{L}^1(\Omega)$ of the sequence $(z_\varepsilon)_{\varepsilon>0}$ to some limit function z_0. One could enforce this strong convergence by assuming that the sequence $(z_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in some Sobolev space $\mathrm{W}^{1,p}(\Omega)$ and add corresponding gradient terms to the energy functionals $\mathcal{E}_\varepsilon$. However, in view of Example 3.5 with piecewise constant z_ε, this assumption is not suitable for the application we have in mind.

Hence, this section is about the definition and the properties of a discrete gradient for piecewise constant functions z_ε and related weak compactness results. Note, that this section is independent of the homogenization results of the previous one. That means that this calculus first of all stands on its own concerning the notation and, probably more important, it is not restricted to the application presented in Section 5.

The aim of this section is the definition of a discrete gradient for piecewise constant functions on a lattice in such a way that only an overall constant function has gradient zero. Furthermore an in some sense bounded sequence of those piecewise constant functions, where the spacing of the lattice tends to zero, should lead to a limit belonging to a Sobolev space $\mathrm{W}^{1,p}$. Roughly spoken we want to introduce a penalty term, extracting those sequences of BV-functions that converge strongly in L^p to a Sobolev function, so that the discrete gradient of these sequences converge weakly in L^p to the gradient of this Sobolev function.

For technical reasons we now assume that the periodic lattice Λ defined by (2.1) is based on the orthonormal basis $\{e_1, e_2, \dots, e_d\}$ of $\mathbb{R}^d$. Moreover, let the associated unit cell be given by $Y = [0,1)^d$. According to this choice of the periodic lattice Λ we have $\varepsilon\Lambda \subset \frac{\varepsilon}{2}\Lambda$ and due to the choice of the associated unit cell Y for every $\lambda \in \Lambda$ there exist exactly 2^d elements $\lambda_1, \lambda_2, \dots, \lambda_{2^d} \in \frac{1}{2}\Lambda$ so that

$$\varepsilon(\lambda+Y) = \bigcup_{j=1}^{2^d} \frac{\varepsilon}{2}(\lambda_j+Y). \tag{4.1}$$

Note, that this property (which would not be valid with $Y = [-\frac{1}{2}, \frac{1}{2})^d$ for instance) makes the definition of our discrete gradient less technical. Moreover, we introduce the extension operator $V_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)$ extending a piecewise constant function $v \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)$ for every $\lambda \in \Lambda_\varepsilon^+ \backslash \Lambda_\varepsilon^-$ on $\varepsilon(\lambda+Y)\backslash\Omega$ constantly by the (constant) value of v on $\varepsilon(\lambda+Y) \cap \Omega$.

With all this, $\mathrm{K}_{\varepsilon\Lambda}(\Omega) \subset \mathrm{BV}(\Omega)$, and we introduce the discrete gradient in the following way:

$$R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}; \quad v \mapsto \sum_{i=1}^{d} \widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}(V_\varepsilon v), \tag{4.2}$$

where $\widetilde{R}^{(i)}_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ is defined via

$$\widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}(\widetilde{v})(x) := \begin{cases} \frac{1}{\varepsilon}\big(\widetilde{v}(x+\frac{\varepsilon}{2}e_i) - \widetilde{v}(x-\frac{\varepsilon}{2}e_i)\big) \otimes e_i & \text{if } x+\frac{\varepsilon}{2}e_i \in \Omega_\varepsilon^+ \text{ and } x-\frac{\varepsilon}{2}e_i \in \Omega_\varepsilon^+, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

This construction of the discrete Gradient is inspired by the so called lifting operator introduced by A. Buffa and C. Ortner in [3] defined via

$$\begin{gathered} \mathrm{R}^{\mathrm{BO}}_{\varepsilon,\eta} : \mathrm{W}^{1,p}_{\varepsilon\Lambda}(\Omega)^m \to S^\eta_{\varepsilon\Lambda}(\Omega)^{m\times d} \\ \int_\Omega \mathrm{R}^{\mathrm{BO}}_{\varepsilon,\eta}(w)(x) : \phi(x)\mathrm{d}x = -\int_{\Gamma^\varepsilon_{\mathrm{int}}} [\![w(s)]\!] : \{\{\phi(s)\}\}\mathrm{d}s \qquad \forall\, \phi \in S^\eta_{\varepsilon\Lambda}(\Omega)^{m\times d}, \end{gathered} \tag{4.4}$$

with $[\![w(s)]\!] = w^+(s) \otimes n^+ + w^-(s) \otimes n^-$ and $\{\{\phi(s)\}\} = \frac{1}{2}(\phi^+(s) + \phi^-(s))$, where $w^\pm$ and $\phi^\pm$ are the traces of w and ϕ with respect to the outward normals $n^\pm$ for $s \in \Gamma^\varepsilon_{\mathrm{int}} := \Omega \cap \bigcup_{\lambda\in\Lambda} \varepsilon(\lambda+\partial Y)$. Here, $\mathrm{W}^{1,p}_{\varepsilon\Lambda}(\Omega) := \{w \in \mathrm{L}^1(\Omega) : w|_{\varepsilon(\lambda+Y)\cap\Omega} \in \mathrm{W}^{1,p}(\varepsilon(\lambda+Y)\cap\Omega) \quad \forall\, \lambda \in \Lambda\}$ is the so called broken Sobolev space and $S^\eta_{\varepsilon\Lambda}(\Omega)$ denotes the set of all piecewise polynomial functions (in the same sense as in the piecewise constant case) with a degree $\eta \in \mathbb{N}_0$. Observing $\mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m \subset \mathrm{W}^{1,p}_{\varepsilon\Lambda}(\mathbb{R}^d)^m$ one very important difference between our definition (4.2) and the definition (1.5) from [3] is that for $\eta = 0$ in (4.4) the definition in [3] leads to the following discrete gradient for piecewise constant functions:

$$\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} : \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m \to \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^{m\times d} \tag{4.5}$$
$$\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}(v)(x) := \sum_{i=1}^{d} \tfrac{1}{2\varepsilon}\big(v(x+\varepsilon e_i) - v(x-\varepsilon e_i)\big) \otimes e_i$$

Here, we replaced Ω by $\mathbb{R}^d$ such that we do not have to care about what is happening in cells $\varepsilon(\lambda+Y)$ intersecting the boundary $\partial\Omega$. Observe that for $v \in \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m$ the function $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}(v)$ is piecewise constant with respect to the lattice $\varepsilon\Lambda$, while $R_{\frac{\varepsilon}{2}}(v)$ is piecewise constant on the finer lattice $\frac{\varepsilon}{2}\Lambda$. According to (4.5), the value of the discrete gradient $(\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}(v)(x))_{k,l}$, $k \in \{1,\dots,m\}$, $l \in \{1,\dots,d\}$, is defined by the values of the function v in the "next" $(v(x+\varepsilon e_i))$ and in the "previous" $(v(x-\varepsilon e_i))$ cell, but is independent of the value of the "actual" cell $(v(x))$. This leads to the following problems:

1. Considering a periodic piecewise constant function satisfying $v(x+\varepsilon e_i) = v(x-\varepsilon e_i)$ and $v(x) \neq v(x+\varepsilon e_i)$ for every $i \in \{1,\dots,d\}$ we obtain $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}(v) \equiv 0$ for $v \not\equiv const.$

2. For $d = m = 1$ the sequence $(v_\varepsilon)_{(\varepsilon>0)} \subset \mathrm{K}_{\varepsilon^p\Lambda}(\mathbb{R})$ of piecewise constant functions ($k \in \mathbb{Z}$) with

$$v_\varepsilon(x) = \begin{cases} 2 & \text{if } x \in \varepsilon^p[2k, 2k+1) \\ -2 & \text{if } x \in -\varepsilon^p[(2|k|+1), 2|k|) \\ 0 & \text{if } x \in \varepsilon^p[(2|k|+1), 2|k|) \end{cases} \tag{4.6}$$

converges weakly in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R})$ due to its periodicity to the Heaviside function $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ otherwise. But H does not belong to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R})$. According to the definition of the lifting operator we have $|\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}(v_\varepsilon)(x)| = \frac{1}{\varepsilon}$ for $x \in [0, \varepsilon^p)$ and $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}(v_\varepsilon) \equiv 0$ otherwise. This gives $\|\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}(v_\varepsilon)\|_{\mathrm{L}^p(\mathbb{R})} = 1$ which shows that this lifting operator is not the right penalty term in the sense mentioned in the beginning of this section. There is another comment on that in Remark 4.2.

As opposed to this, the discrete gradient defined in (4.2) evaluated for v_ε from (4.6) gives us $|R_{\frac{\varepsilon}{2}}(v_\varepsilon)(x)| = \frac{4}{\varepsilon}$ for $x < \frac{\varepsilon^p}{2}$ and $|R_{\frac{\varepsilon}{2}}(v_\varepsilon)(x)| = \frac{2}{\varepsilon}$ otherwise, which leads to $\|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|^p_{\mathrm{L}^p(\Omega)} \geq \mathrm{vol}(\Omega)(\frac{2}{\varepsilon})^p$ for any bounded subset Ω of $\mathbb{R}$. This shows that this term along $(v_\varepsilon)_{\varepsilon>0}$ is unbounded, which correlates with the fact that this sequence does not have a limit belonging to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R})$. This indicates that the L^p-norm of the discrete gradient defined in (4.2) is suitable as a penalty term filtering out sequences of piecewise constant functions converging to elements of $\mathrm{W}^{1,p}(\Omega)^m$ as it is stated in the following theorem:

Theorem 4.1 (Compactness result). *For $p \in (1,\infty)$ and every sequence $(v_\varepsilon)_{\varepsilon>0}$ of functions belonging to $\mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and satisfying*

$$\sup_{\varepsilon>0} \big(\|v_\varepsilon\|_{\mathrm{L}^p(\Omega)^m} + \|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}\big) \leq C < \infty \tag{4.7}$$

there exist a function $v_0 \in \mathrm{W}^{1,p}(\Omega)^m$ and a sub-sequence $(v_{\varepsilon'})_{\varepsilon'>0}$ of $(v_\varepsilon)_{\varepsilon>0}$ with

$$v_{\varepsilon'} \to v_0 \text{ in } \mathrm{L}^q(\Omega)^m \quad \text{and} \quad R_{\frac{\varepsilon}{2}}(v_{\varepsilon'}) \rightharpoonup \nabla v_0 \text{ in } \mathrm{L}^p(\Omega)^{m\times d},$$

where $1 \leq q < p^*$, and p^* denotes the Sobolev conjugate of p.

Remark 4.2. Our Theorem 4.1 is a modification of Theorem 5.2 from [3]. There, condition (4.7) is formulated with the penalty term $\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p} |[\![v_\varepsilon(s)]\!]|^p_{m\times d} \mathrm{d}s$ instead of our regularization term $\|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}$. The authors of [3] end up with a similar convergence result with respect to their discrete gradient $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}$. But due to this procedure a regularized (ε-dependent) model based on functionals depending on $\mathrm{K}_{\varepsilon\Lambda}$-functions has to contain two ingredients to arrive at a limit model described by functionals depending solely on Sobolev functions. First, the penalty term $\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p} |[\![v_\varepsilon(s)]\!]|^p_{m\times d} \mathrm{d}s$ forcing the sequence $(v_\varepsilon)_{\varepsilon>0}$ of $\mathrm{K}_{\varepsilon\Lambda}$-functions to converge to a Sobolev function, and second, the lifted function $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}(v_\varepsilon)$ to find a gradient in the limit. Thereby a further issue arises, namely, the identification and interpretation of the penalty term after passing to the limit. Clearly, due to our replacement this problem is solved. Since the proof of our Theorem 4.1 is based on [3, Theorem 5.2], we need the estimate of Lemma 4.3 below to adapt the proof from [3].

Lemma 4.3. *Let $p \in [1,\infty)$. Then there exist constants $\widehat{C} > 0$ and $C > 0$, such that for every $\varepsilon > 0$ and for all $v \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ it holds*

$$|Dv|(\Omega) \leq \widehat{C}\Big(\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p} |[\![v(s)]\!]|^p_{m\times d} \mathrm{d}s\Big)^{\frac{1}{p}} \leq \widehat{C}C \|R_{\frac{\varepsilon}{2}}(v)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}},$$

where Dv is the measure representing the distributional derivative of v and $|Dv|(\Omega)$ its total variation. Moreover, $\Gamma^\varepsilon_{\text{int}} := \Omega \cap \bigcup_{\lambda\in\Lambda} \varepsilon(\lambda+\partial Y)$.

Proof. The proof of the first inequality is a straight forward generalization of Theorem 3.26 from [13] to the case of $p \neq 2$ and can be found in [3] (Lemma 2) as a brief sketch, for example.

The second inequality results from the special structure of the discrete gradient. For a better understanding the calculations are split up so that the left hand side of every numbered equations is the same (starting point) and the only changes are on the right hand side. First of all (4.8) is valid since every face of the cell $\varepsilon(\lambda+Y)$ is taken twice when summing up on the right hand side:

$$\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p} |[\![v(s)]\!]|^p_{m\times d} \mathrm{d}s = \tfrac{1}{2} \sum_{\lambda\in\Lambda} \int_{\varepsilon(\lambda+\partial Y)} \varepsilon^{1-p} |[\![v(s)]\!]|^p_{m\times d} \mathbb{1}_\Omega(s) \mathrm{d}s \tag{4.8}$$

Since the integrand contains the characteristic function $\mathbb{1}_\Omega$, the function $v \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ can be replaced by any extension $\widetilde{v} \in \mathrm{L}^1(\Omega^+_\varepsilon)$ satisfying $\widetilde{v}|_\Omega = v$. We choose $\widetilde{v} := (V_\varepsilon(v)) \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ and exploit that due to decomposition (4.1) for every cell $\varepsilon(\lambda+Y) \subset \Omega^+_\varepsilon$ we have $[\![\widetilde{v}(s)]\!] = 0$ for $s \in \frac{\varepsilon}{2}(\lambda_j+\partial Y)\backslash\varepsilon(\lambda+\partial Y)$, since $\widetilde{v} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ is constant on $\varepsilon(\lambda+Y)$. Hence, the following equality is valid, since there only zeros are added:

$$\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p} |[\![v(s)]\!]|^p_{m\times d} \mathrm{d}s = \tfrac{1}{2} \sum_{\lambda\in\Lambda} \sum_{j=1}^{2^d} \varepsilon^{1-p} \int_{\frac{\varepsilon}{2}(\lambda_j+\partial Y)} |[\![\widetilde{v}(s)]\!]|^p_{m\times d} \mathbb{1}_\Omega(s) \mathrm{d}s. \tag{4.9}$$

Now we first of all increase the domain of integration in (4.9) by replacing $\mathbb{1}_\Omega$ by $\mathbb{1}_{\overline{\Omega^+_\varepsilon}}$ and then calculate the integral by splitting $\frac{\varepsilon}{2}(\lambda_j+\partial Y)$ into its $2d$ faces of $\frac{\varepsilon}{2}(\lambda_j+Y)$. For $s \in \partial\Omega^+_\varepsilon$ the jump term $[\![\widetilde{v}(s)]\!]$ is not well-defined since $\mathrm{supp}(\widetilde{v}) \subset \overline{\Omega^+_\varepsilon}$. Therefore, we set $[\![\widetilde{v}(s)]\!] := 0$ for $s \in \partial\Omega^+_\varepsilon$. Since the integrand is constant on every face, the integral gives the constant multiplied with $\left(\frac{\varepsilon}{2}\right)^{d-1}$, which is just the volume of one face. Moreover, the jump term of $\widetilde{v}$ is replaced by its definition, where $v^+ = \widetilde{v}(\frac{\varepsilon}{2}\lambda_j)$, $v^- = \widetilde{v}(\frac{\varepsilon}{2}(\lambda_j+e_i))$ and $n^+ = -n^- = e_i$ is used for one face of $\frac{\varepsilon}{2}(\lambda_j+Y)$ and $v^+ = \widetilde{v}(\frac{\varepsilon}{2}\lambda_j)$, $v^- = \widetilde{v}(\frac{\varepsilon}{2}(\lambda_j-e_i))$ and $n^+ = -n^- = -e_i$ for the opposite one. Altogether, we obtain:

$$\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p} |[\![v(s)]\!]|^p_{m\times d} \mathrm{d}s \le \tfrac{1}{2} \sum_{\lambda\in\Lambda^+_\varepsilon} \sum_{j=1}^{2^d} \varepsilon^{1-p} \sum_{i=1}^{d} \left(\tfrac{\varepsilon}{2}\right)^{d-1} \left| \Big(\widetilde{v}(\tfrac{\varepsilon}{2}\lambda_j) - \widetilde{v}(\tfrac{\varepsilon}{2}(\lambda_j+e_i)) \Big) \otimes e_i \right|^p_{m\times d} \delta^{(\lambda)}_{i,j}$$

$$+ \left(\tfrac{\varepsilon}{2}\right)^{d-1} \left| \Big(\widetilde{v}(\tfrac{\varepsilon}{2}(\lambda_j-e_i)) - \widetilde{v}(\tfrac{\varepsilon}{2}\lambda_j) \Big) \otimes e_i \right|^p_{m\times d} \widetilde{\delta}^{(\lambda)}_{i,j}, \tag{4.10}$$

where

$$\delta^{(\lambda)}_{i,j} := \begin{cases} 0 & \text{if } \frac{\varepsilon}{2}(\lambda_j+e_i) \notin \Omega^+_\varepsilon \\ 1 & \text{otherwise} \end{cases} \qquad \widetilde{\delta}^{(\lambda)}_{i,j} := \begin{cases} 0 & \text{if } \frac{\varepsilon}{2}(\lambda_j-e_i) \notin \Omega^+_\varepsilon \\ 1 & \text{otherwise} \end{cases}.$$

As already, mentioned a lot of zeros are added in (4.9) and this results in the following: Observe that for the λ_j as in (4.1) we have $\frac{\varepsilon}{2}\lambda_j \in \varepsilon(\lambda+Y)$. Moreover, either we have $\frac{\varepsilon}{2}(\lambda_j+e_i) \in \varepsilon(\lambda+Y)$ or $\frac{\varepsilon}{2}(\lambda_j-e_i) \in \varepsilon(\lambda+Y)$, which gives us either $\widetilde{v}(\frac{\varepsilon}{2}(\lambda_j+e_i)) = \widetilde{v}(\frac{\varepsilon}{2}\lambda_j)$ or $\widetilde{v}(\frac{\varepsilon}{2}(\lambda_j-e_i)) = \widetilde{v}(\frac{\varepsilon}{2}\lambda_j)$ for fixed $i \in \{1,\dots,d\}$ and $j \in \{1,\dots,2^d\}$. With this, always one of the terms of the right hand side of (4.10) is zero and the other can be replaced in the following way:

$$\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p} |[\![v(s)]\!]|^p_{m\times d} \mathrm{d}s \le \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d} \frac{\varepsilon^d}{2^d} \sum_{i=1}^{d} \varepsilon^{-p} \left| \Big(\widetilde{v}(\tfrac{\varepsilon}{2}(\lambda_j-e_i)) - \widetilde{v}(\tfrac{\varepsilon}{2}(\lambda_j+e_i)) \Big) \otimes e_i \right|^p_{m\times d} \delta^{(\lambda)}_{i,j} \widetilde{\delta}^{(\lambda)}_{i,j}. \tag{4.11}$$

The next step is interchanging the sum $\sum_{i=1}^d$ with the matrix norm $|\cdot|_{m\times d}$ on the right hand side of (4.11). Therefore, we set $f^{\lambda_j}_\varepsilon(e_i) := \frac{1}{\varepsilon}\big(\widetilde{v}(\frac{\varepsilon}{2}(\lambda_j-e_i)) - \widetilde{v}(\frac{\varepsilon}{2}(\lambda_j+e_i))\big)$ to shorten notation and observe that for all $i,k = 1,\dots,d$ we have $(f^{\lambda_j}_\varepsilon(e_i)\otimes e_i)e_k = f^{\lambda_j}_\varepsilon(e_i)\delta_{ik}$. With this the interchange is based on the following trivial calculation:

$$\begin{aligned} \sum_{i,k=1}^{d} \left| (f^{\lambda_j}_\varepsilon(e_i) \otimes e_i) e_k \right|^p_m &= \sum_{i,k=1}^{d} \left| f^{\lambda_j}_\varepsilon(e_i)\delta_{ik} \right|^p_m = \sum_{k=1}^{d} \left| f^{\lambda_j}_\varepsilon(e_k) \right|^p_m \\ &= \sum_{k=1}^{d} \Big| \sum_{i=1}^{d} (f^{\lambda_j}_\varepsilon(e_i)\delta_{ik}) \Big|^p_m = \sum_{k=1}^{d} \Big| \Big(\sum_{i=1}^{d} (f^{\lambda_j}_\varepsilon(e_i) \otimes e_i) \Big) e_k \Big|^p_m. \end{aligned} \tag{4.12}$$

For $A \in \mathbb{R}^{m\times d}$ and the orthonormal basis $\{e_1,\dots,e_d\}$ of $\mathbb{R}^d$ let $|\cdot|_{\{e_1,\dots,e_d\}}$ denote the matrix norm defined by $|A|^p_{\{e_1,\dots,e_d\}} := \sum_{k=1}^d |Ae_k|^p_m$. Then the following calculation yields the desired interchange:

$$\begin{aligned} \sum_{i=1}^{d} \left| f^{\lambda_j}_\varepsilon(e_i) \otimes e_i \right|^p_{m\times d} &\le C_1 \sum_{i=1}^{d} \left| f^{\lambda_j}_\varepsilon(e_i) \otimes e_i \right|^p_{\{e_1,\dots,e_d\}} = C_1 \sum_{i,k=1}^{d} \left| (f^{\lambda_j}_\varepsilon(e_i) \otimes e_i) e_k \right|^p_m \\ &\overset{(4.12)}{=} C_1 \sum_{k=1}^{d} \Big| \Big(\sum_{i=1}^{d} f^{\lambda_j}_\varepsilon(e_i) \otimes e_i \Big) e_k \Big|^p_m = C_1 \Big| \sum_{i=1}^{d} f^{\lambda_j}_\varepsilon(e_i) \otimes e_i \Big|^p_{\{e_1,\dots,e_d\}} \\ &\le C_1 C_2 \Big| \sum_{i=1}^{d} f^{\lambda_j}_\varepsilon(e_i) \otimes e_i \Big|^p_{m\times d}, \end{aligned}$$

where the norm equivalence in dimension md was exploited two times. For $C^p := C_1C_2$ this estimate turns the right hand side of (4.11) into

$$\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p}|[\![v(s)]\!]|^p_{m\times d}\mathrm{d}s \le C^p \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d} \tfrac{\varepsilon^d}{2^d}\Big|\sum_{i=1}^d \tfrac{1}{\varepsilon}\Big(\widetilde{v}\big(\tfrac{\varepsilon}{2}(\lambda_j-e_i)\big) - \widetilde{v}\big(\tfrac{\varepsilon}{2}(\lambda_j+e_i)\big)\Big)\otimes e_i\Big|^p_{m\times d} \delta^{(\lambda)}_{i,j}\widetilde{\delta}^{(\lambda)}_{i,j}.$$

Replacing $\frac{\varepsilon^d}{2^d}$ by the integral over $\frac{\varepsilon}{2}(\lambda_j+Y)$ we finally end up with

$$\begin{aligned}
&\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p}|[\![v(s)]\!]|^p_{m\times d}\mathrm{d}s\\
&\qquad\le C^p \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d} \int_{\frac{\varepsilon}{2}(\lambda_j+Y)} \Big|\sum_{i=1}^d \tfrac{1}{\varepsilon}\Big(\widetilde{v}\big(\tfrac{\varepsilon}{2}(\lambda_j-e_i)\big) - \widetilde{v}\big(\tfrac{\varepsilon}{2}(\lambda_j+e_i)\big)\Big)\otimes e_i\Big|^p_{m\times d} \delta^{(\lambda)}_{i,j}\widetilde{\delta}^{(\lambda)}_{i,j}\mathrm{d}x\\
&\qquad= C^p \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d} \int_{\frac{\varepsilon}{2}(\lambda_j+Y)} \Big|\sum_{i=1}^d \delta^{(\lambda)}_{i,j}\widetilde{\delta}^{(\lambda)}_{i,j}\tfrac{1}{\varepsilon}\Big(\widetilde{v}\big(x-\tfrac{\varepsilon}{2}e_i\big) - \widetilde{v}\big(x+\tfrac{\varepsilon}{2}e_i\big)\Big)\otimes e_i\Big|^p_{m\times d}\mathrm{d}x\\
&\qquad= C^p\Big\|\sum_{i=1}^d \widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}(\widetilde{v})\Big\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}},
\end{aligned}$$

where we used $\widetilde{v}(x\pm\frac{\varepsilon}{2}e_i) \equiv \widetilde{v}(\frac{\varepsilon}{2}\lambda_j\pm\frac{\varepsilon}{2}e_i)$ for $x\in\frac{\varepsilon}{2}(\lambda_j+Y)\subset\Omega^+_\varepsilon$, which is valid for all functions belonging to $\mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ due to their special structure. Replacing $\widetilde{v}$ by $V_\varepsilon v$ concludes the proof. ☐

Since for $v\in\mathrm{K}_{\varepsilon\Lambda}(\Omega)^m\subset\mathrm{W}^{1,p}_{\varepsilon\Lambda}(\Omega)$ the proof of compactness Theorem 5.2 in [3] relies on the definition of $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}(v)\in\mathrm{K}_{\varepsilon\Lambda}(\Omega)^{m\times d}$ by the identity (4.4), in the next lemma we state that the discrete gradient $R_{\frac{\varepsilon}{2}}(v)\in\mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega^+_\varepsilon)^{m\times d}$ of v fulfills a similar relation.

Lemma 4.4. *For $\varepsilon>0$ and for all $v\in\mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and every $\varphi\in\mathrm{K}_{\varepsilon\Lambda}(\Omega^-_\varepsilon)^{m\times d}$ it holds*

$$\int_\Omega R_{\frac{\varepsilon}{2}}(v)(x):\varphi^{\mathrm{ex}}(x)\mathrm{d}x = -\int_{\Gamma^\varepsilon_{\mathrm{int}}}[\![v(s)]\!]:\{\{\varphi^{\mathrm{ex}}(s)\}\}\mathrm{d}s, \tag{4.13}$$

where $\varphi^{\mathrm{ex}}\in\mathrm{L}^1(\mathbb{R}^d)$ is the extension with 0 to $\mathbb{R}^d$ of the function $\varphi\in\mathrm{K}_{\varepsilon\Lambda}(\Omega^-_\varepsilon)^{m\times d}$.

Proof. We start with rearranging the right hand side of (4.13). Since we are only testing with functions $\varphi\in\mathrm{K}_{\varepsilon\Lambda}(\Omega^-_\varepsilon)^{m\times d}$, analogously to the proof of Lemma 4.3 the function $v\in\mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ can be replaced by the extension $\widetilde{v}:=(V_\varepsilon(v))\in\mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$.

Let $\lambda\in\Lambda$ and $s\in\varepsilon(\lambda+\partial Y)$. Then $\{\{\varphi^{\mathrm{ex}}(s)\}\}\neq0$ implies $s\in\Gamma^\varepsilon_{\mathrm{int}}$, which is why the domain of integration can be increased to $\cup_{\lambda\in\Lambda}\varepsilon(\lambda+\partial Y)$. Therefore, $\widetilde{v}\in\mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ needs to be replaced by its extension $\widetilde{v}^{\mathrm{ex}}\in\mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m$ extending it with 0 to $\mathbb{R}^d$. Note, that according to $\{\{\varphi^{\mathrm{ex}}(s)\}\}\equiv0$ for $s\in\partial\Omega^+_\varepsilon$ the additional jump $[\![\widetilde{v}^{\mathrm{ex}}(s)]\!]\neq0$ does not play any role in the following calculations. On the right hand side of (4.14) below, every face of a cell $\varepsilon(\lambda+Y)$ is taken twice when summing up which is why this is an equality:

$$\int_{\Gamma^\varepsilon_{\mathrm{int}}}[\![v(s)]\!]:\{\{\varphi^{\mathrm{ex}}(s)\}\}\mathrm{d}s = \tfrac12\sum_{\lambda\in\Lambda}\int_{\varepsilon(\lambda+\partial Y)}[\![\widetilde{v}^{\mathrm{ex}}(s)]\!]:\{\{\varphi^{\mathrm{ex}}(s)\}\}\mathrm{d}s. \tag{4.14}$$

Analogously to the proof of Lemma 4.3 we calculate the integral which gives the factor ε^{d-1}. Furthermore, the jump term of $\widetilde{v}^{\mathrm{ex}}$ and the mean value term of φ^{ex} are replaced by $\big(\widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda) - \widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big) \otimes e_i$ and $\frac{1}{2}\big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big)$ for one face of $\varepsilon(\lambda{+}Y)$ and by $\big(\widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda) - \widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda{-}e_i))\big) \otimes (-e_i)$ and $\frac{1}{2}\big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{-}e_i))\big)$ for the opposite one:

$$\int_{\Gamma^{\varepsilon}_{\mathrm{int}}} [\![v(s)]\!] : \{\{\varphi^{\mathrm{ex}}(s)\}\}\mathrm{d}s$$

$$= \tfrac{1}{2}\sum_{\lambda\in\Lambda} \varepsilon^{d-1} \sum_{i=1}^{d} \Big(\big(\widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda) - \widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big)\otimes e_i : \tfrac{1}{2}\big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big) \tag{4.15a}$$

$$+ \ \big(\widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda{-}e_i)) - \widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda)\big)\otimes e_i : \tfrac{1}{2}\big(\varphi^{\mathrm{ex}}(\varepsilon(\lambda{-}e_i)) + \varphi^{\mathrm{ex}}(\varepsilon\lambda)\big)\Big). \tag{4.15b}$$

Now, the sums are interchanged and the translation $\lambda^* = \lambda{-}e_i$ is applied to line (4.15b) for every $i = 1, \dots, d$, such that we end up with

$$\int_{\Gamma^{\varepsilon}_{\mathrm{int}}} [\![v(s)]\!] : \{\{\varphi^{\mathrm{ex}}(s)\}\}\mathrm{d}s$$

$$= \tfrac{\varepsilon^{d-1}}{2} \sum_{i=1}^{d} \sum_{\lambda\in\Lambda} \big(\widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda) - \widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big) \otimes e_i : \big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{+}e_i))\big). \tag{4.16}$$

For rearranging the left hand side of (4.13) we introduce $Y_{e_i} = \{y \in Y : y{-}\frac{1}{2}e_i \in Y\}$ ($Y = [0,1)^d \Rightarrow Y_{e_1} = [\frac{1}{2},1){\times}[0,1)^{d-1}$) and $f^{(i)}_{\varepsilon}(x) := \frac{1}{\varepsilon}\big(\widetilde{v}(x{+}\frac{\varepsilon}{2}e_i) - \widetilde{v}(x{-}\frac{\varepsilon}{2}e_i)\big) \otimes e_i$ to shorten notation. Since $\operatorname{supp}(\varphi) \subset \overline{\Omega^-_{\varepsilon}}$, again v can be replaced by $\widetilde{v} := V_{\varepsilon}v$ on the left hand side of (4.13), which leads to

$$\int_{\Omega} R_{\frac{\varepsilon}{2}}(v)(x) : \varphi^{\mathrm{ex}}(x)\mathrm{d}x = \sum_{\lambda\in\Lambda^-_{\varepsilon}} \int_{\varepsilon(\lambda+Y)} \sum_{i=1}^{d} f^{(i)}_{\varepsilon}(x) : \varphi(\varepsilon\lambda)\mathrm{d}x, \tag{4.17}$$

where we already used $\varphi(x) \equiv \varphi(\varepsilon\lambda)$ for $x \in \varepsilon(\lambda{+}Y)$ and $\lambda \in \Lambda^-_{\varepsilon}$. Observing that

$$f^{(i)}_{\varepsilon}(x) = \begin{cases} \frac{1}{\varepsilon}\big(\widetilde{v}(\varepsilon(\lambda{+}e_i)) - \widetilde{v}(\varepsilon\lambda)\big) \otimes e_i & \text{if } x \in \varepsilon(\lambda{+}Y_{e_i}), \\ \frac{1}{\varepsilon}\big(\widetilde{v}(\varepsilon\lambda) - \widetilde{v}(\varepsilon(\lambda{-}e_i))\big) \otimes e_i & \text{if } x \in \varepsilon(\lambda{+}Y\backslash Y_{e_i}) \end{cases}$$

we are able to reformulate the right hand side of (4.17) by interchanging integration and summation in the following way:

$$\int_{\Omega} R_{\frac{\varepsilon}{2}}(v)(x) : \varphi^{\mathrm{ex}}(x)\mathrm{d}x$$

$$= \sum_{\lambda\in\Lambda^-_{\varepsilon}} \sum_{i=1}^{d} \Big(\int_{\varepsilon(\lambda+Y_{e_i})} f^{(i)}_{\varepsilon}(x) : \varphi(\varepsilon\lambda)\mathrm{d}x + \int_{\varepsilon(\lambda+Y\backslash Y_{e_i})} f^{(i)}_{\varepsilon}(x) : \varphi(\varepsilon\lambda)\mathrm{d}x \Big)$$

$$= \sum_{\lambda\in\Lambda^-_{\varepsilon}} \sum_{i=1}^{d} \tfrac{1}{2}\varepsilon^d \tfrac{1}{\varepsilon}\big(\widetilde{v}(\varepsilon(\lambda{+}e_i)) - \widetilde{v}(\varepsilon\lambda)\big) \otimes e_i : \varphi(\varepsilon\lambda) \tag{4.18a}$$

$$+ \tfrac{1}{2}\varepsilon^d \tfrac{1}{\varepsilon}\big(\widetilde{v}(\varepsilon\lambda) - \widetilde{v}(\varepsilon(\lambda{-}e_i))\big) \otimes e_i : \varphi(\varepsilon\lambda). \tag{4.18b}$$

Here, we already used, that $f^{(i)}_{\varepsilon}$ is constant on the domain of integration. Since $\varphi^{\mathrm{ex}}(\varepsilon\lambda) = 0$ for all $\lambda \in \Lambda\backslash\Lambda^-_{\varepsilon}$, the first sum in (4.18) can be replaced by the sum of $\lambda \in \Lambda$. Afterwards, again the sums are interchanged and the translation $\lambda^* = \lambda - e_i$ is applied to line (4.18b) for every $i = 1, \dots, d$, such that we end up with

$$\int_\Omega R_{\frac{\varepsilon}{2}}(v)(x) : \varphi^{\mathrm{ex}}(x)\mathrm{d}x$$
$$= \tfrac{\varepsilon^{d-1}}{2} \sum_{i=1}^{d} \sum_{\lambda\in\Lambda} \big(\widetilde{v}^{\mathrm{ex}}(\varepsilon(\lambda+e_i)) - \widetilde{v}^{\mathrm{ex}}(\varepsilon\lambda)\big) \otimes e_i : \big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda+e_i))\big). \qquad (4.19)$$

Comparing (4.19) and (4.16) we find that (4.13) is valid. □

Now we are in the position to prove Theorem 4.1.

Proof. Here, we mainly follow the steps of the proof of Theorem 5.2 of [3] and explain the main differences.

As already mentioned in [3], the distributional derivative Du of a broken Sobolev function $u \in \mathrm{W}^{1,p}_{\varepsilon\Lambda}(\Omega)^m$ is given by

$$\langle Du, \psi\rangle = \int_\Omega \nabla u : \psi \mathrm{d}x - \int_{\Gamma^\varepsilon_{\mathrm{int}}} [\![u]\!] : \psi \mathrm{d}s \qquad \forall\, \psi \in \mathrm{C}^\infty_c(\Omega)^{m\times d}. \qquad (4.20)$$

This can be seen by using integration by parts on each cell $\varepsilon(\lambda+Y)$.

Now, let $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ satisfy condition (4.7) of Theorem 4.1. Since L^p is reflexive ($p \in (1,\infty)$), there exists a subsequence and limit elements $v_0 \in \mathrm{L}^p(\Omega)^m$, $V_0 \in \mathrm{L}^p(\Omega)^{m\times d}$ such that $v_{\varepsilon'} \rightharpoonup v_0$ in $\mathrm{L}^p(\Omega)^m$ and $R_{\frac{\varepsilon'}{2}} v_{\varepsilon'} \rightharpoonup V_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$. The goal is to show that $v_0 \in \mathrm{W}^{1,p}(\Omega)^m$ with $Dv_0 = V_0$. Using (4.20) for $v_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ we find with $\psi \in \mathrm{C}^\infty_c(\Omega)^{m\times d}$ arbitrary but fixed

$$\langle Dv_\varepsilon, \psi\rangle = -\int_{\Gamma^\varepsilon_{\mathrm{int}}} [\![v_\varepsilon]\!] : \psi \mathrm{d}s. \qquad (4.21)$$

Choosing $\varepsilon_0 > 0$ so small such that $\mathrm{supp}(\psi) \subset \overline{\Omega^-_{\varepsilon_0}}$ we are able to find a sequence $(\varphi_\varepsilon)_{(0<\varepsilon<\varepsilon_0)}$ with $\varphi_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^-_\varepsilon)^{m\times d}$ such that $\|\psi - \varphi^{\mathrm{ex}}_\varepsilon\|_{\mathrm{L}^\infty(\Omega)^{m\times d}} \to 0$ for $\varepsilon \to 0$. By adding and subtracting $\varphi^{\mathrm{ex}}_\varepsilon$ we find with (4.21)

$$\begin{aligned}
\langle Dv_\varepsilon, \psi\rangle &= -\int_{\Gamma^\varepsilon_{\mathrm{int}}} [\![v_\varepsilon]\!]:\{\{\psi - \varphi^{\mathrm{ex}}_\varepsilon\}\}\mathrm{d}s - \int_{\Gamma^\varepsilon_{\mathrm{int}}} [\![v_\varepsilon]\!]:\{\{\varphi^{\mathrm{ex}}_\varepsilon\}\}\mathrm{d}s \\
&\overset{(4.13)}{=} -\int_{\Gamma^\varepsilon_{\mathrm{int}}} [\![v_\varepsilon]\!]:\{\{\psi - \varphi^{\mathrm{ex}}_\varepsilon\}\}\mathrm{d}s + \int_\Omega R_{\frac{\varepsilon}{2}}(v_\varepsilon):\varphi^{\mathrm{ex}}_\varepsilon \mathrm{d}x \\
&= -\int_{\Gamma^\varepsilon_{\mathrm{int}}} [\![v_\varepsilon]\!]:\{\{\psi - \varphi^{\mathrm{ex}}_\varepsilon\}\}\mathrm{d}s + \int_\Omega R_{\frac{\varepsilon}{2}}(v_\varepsilon):(\varphi^{\mathrm{ex}}_\varepsilon - \psi)\mathrm{d}x + \int_\Omega R_{\frac{\varepsilon}{2}}(v_\varepsilon):\psi \mathrm{d}x \qquad (4.22)
\end{aligned}$$

As we will see below, the first two terms of (4.22) are bounded by $C\|\psi - \varphi^{\mathrm{ex}}_\varepsilon\|_{\mathrm{L}^\infty(\Omega)^{m\times d}}$ and hence tend to 0 as $\varepsilon \to 0$. Therefore, since $R_{\frac{\varepsilon'}{2}} v_{\varepsilon'} \rightharpoonup V_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$, we end up with

$$\lim_{\varepsilon'\to 0} \langle Dv_{\varepsilon'}, \psi\rangle = \int_\Omega V_0 : \psi \mathrm{d}s \qquad \forall\, \psi \in \mathrm{C}^\infty_c(\Omega)^{m\times d}. \qquad (4.23)$$

To show the boundedness of the first two terms of (4.22) we use Hölder's inequality to conclude with Lemma 4.3

$$
\begin{aligned}
\Big| -\int_{\Gamma^\varepsilon_{\mathrm{int}}} [\![v_\varepsilon]\!] : \{\{\psi - \varphi^{\mathrm{ex}}_\varepsilon\}\} \mathrm{d}s \Big| &\le \|[\![v_\varepsilon]\!]\|_{\mathrm{L}^p(\Gamma^\varepsilon_{\mathrm{int}})^{m\times d}} \|\{\{\psi - \varphi^{\mathrm{ex}}_\varepsilon\}\}\|_{\mathrm{L}^{p'}(\Gamma^\varepsilon_{\mathrm{int}})^{m\times d}} \\
&\le \varepsilon^{\frac{p-1}{p}} \|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} \|\psi - \varphi^{\mathrm{ex}}_\varepsilon\|_{\mathrm{L}^\infty(\Omega)^{m\times d}} \mathrm{area}(\Gamma^\varepsilon_{\mathrm{int}})^{\frac{1}{p'}} \\
&\le \|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} \|\psi - \varphi^{\mathrm{ex}}_\varepsilon\|_{\mathrm{L}^\infty(\Omega)^{m\times d}} (d\mathrm{vol}(\Omega^+_\varepsilon))^{\frac{1}{p'}}
\end{aligned}
$$

and

$$
\begin{aligned}
\Big| \int_\Omega R_{\frac{\varepsilon}{2}}(v_\varepsilon) : (\varphi^{\mathrm{ex}}_\varepsilon - \psi) \mathrm{d}x \Big| &\le \|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} \|\varphi^{\mathrm{ex}}_\varepsilon - \psi\|_{\mathrm{L}^{p'}(\Omega)^{m\times d}} \\
&\le \|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} \|\varphi^{\mathrm{ex}}_\varepsilon - \psi\|_{\mathrm{L}^\infty(\Omega)^{m\times d}} \mathrm{vol}(\Omega)^{\frac{1}{p'}}.
\end{aligned}
$$

Here, we already used $\mathrm{area}(\Gamma^\varepsilon_{\mathrm{int}}) \le d\,\mathrm{vol}(\Omega)\varepsilon^{-1}$, which is valid since $\mathrm{area}(\Gamma^\varepsilon_{\mathrm{int}})$ is bounded by the product of the number of cells contained in Ω^+_ε, which is $\mathrm{vol}(\Omega^+_\varepsilon)\varepsilon^{-d}$, and the volume of the part of $\Gamma^\varepsilon_{\mathrm{int}}$ contained in one cell, which is $d\varepsilon^{d-1}$. With this, the assumed uniform bound of the term $\|R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}$ yields the result.

On the other hand using the definition of the distributional derivative of $v_{\varepsilon'} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and $v_{\varepsilon'} \rightharpoonup v_0$ in $\mathrm{L}^p(\Omega)^m$, we have

$$
\lim_{\varepsilon'\to 0} \langle Dv_{\varepsilon'}, \psi\rangle = \lim_{\varepsilon'\to 0} -\int_\Omega v_{\varepsilon'} \cdot \mathrm{div}\psi \mathrm{d}x = -\int_\Omega v_0 \cdot \mathrm{div}\psi \mathrm{d}x \qquad \forall\, \psi \in \mathrm{C}^\infty_c(\Omega)^{m\times d}. \tag{4.24}
$$

Now, combining (4.23) and (4.24) we obtain

$$
\int_\Omega V_0 : \psi \mathrm{d}x = -\int_\Omega v_0 \cdot \mathrm{div}\psi \mathrm{d}x \qquad \forall\, \psi \in \mathrm{C}^\infty_c(\Omega)^{m\times d},
$$

which gives us $v_0 \in \mathrm{W}^{1,p}(\Omega)^m$ and $Dv_0 = V_0$.

Finally, we use the fact that $v_{\varepsilon'} \overset{*}{\rightharpoonup} v_0$ in $\mathrm{BV}(\Omega)^m$ implies $v_{\varepsilon'} \to v_0$ in $\mathrm{L}^1(\Omega)^m$ in order to conclude $v_{\varepsilon'} \to v_0$ in $\mathrm{L}^q(\Omega)^m$ for every $q \in [1, p^*)$. Thereby we use the following interpolation inequality obtained by Hölder's inequality for every $\theta \in (0,1)$:

$$
\|v_\varepsilon - v_0\|_{\mathrm{L}^q(\Omega)^m} \le \|v_\varepsilon - v_0\|^{1-\theta}_{\mathrm{L}^{p^*}(\Omega)^m} \|v_\varepsilon - v_0\|^\theta_{\mathrm{L}^1(\Omega)^m},
$$

and the term $\|v_\varepsilon - v_0\|_{\mathrm{L}^{p^*}(\Omega)^m}$ is bounded due to the following Sobolev–Poincare inequality proved in Theorem 4.1 of [3] and Lemma 4.3:

$$
\|v_\varepsilon\|_{\mathrm{L}^{p^*}(\Omega)^m} \le C_S \left(\|v_\varepsilon\|_{\mathrm{L}^1(\Omega)^m} + \left(\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p} |[\![v_\varepsilon(s)]\!]|^p \mathrm{d}s \right)^{\frac{1}{p}} \right).
$$

This finishes the proof. □

Definition 4.5 (Projector to piecewise constant functions)**.** Let $\varepsilon > 0$ and $p \in [1, \infty)$. The projector $P_\varepsilon : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)$ to piecewise constant functions is defined via

$$
P_\varepsilon w(x) := \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} w(\xi)\mathrm{d}\xi,
$$

where $\fint_A g(a)\mathrm{d}a := \frac{1}{\mathrm{vol}(A)} \int_A g(a)\mathrm{d}a$ is the average of the function g over A and $\mathcal{N}_\varepsilon : \mathbb{R}^d \to \varepsilon\Lambda$ maps every point $x \in \varepsilon(\lambda + Y) \subset \mathbb{R}^d$ to the lattice point $\varepsilon\lambda \in \varepsilon\Lambda$.

Remark 4.6. Note, that the mapping $\mathcal{N}_\varepsilon : \mathbb{R}^d \to \varepsilon\Lambda$ is well-defined for arbitrary choices of Y, as long as $\bigcup_{\lambda\in\Lambda}(\lambda + Y) = \mathbb{R}^d$ and $(\lambda_1 + Y) \cap (\lambda_2 + Y) = \emptyset$ for all $\lambda_1 \neq \lambda_2$ are fulfilled. In this way $\mathcal{N}_\varepsilon : \mathbb{R}^d \to \varepsilon\Lambda$ does not depend on the choice of Y, so we do not need to worry about it in the following sections.

Moreover note, that $V_\varepsilon((P_\varepsilon w^{\mathrm{ex}})|_\Omega) \equiv (P_\varepsilon w^{\mathrm{ex}})|_{\Omega_\varepsilon^+}$ for $w \in \mathrm{L}^p(\Omega)$.

Theorem 4.7 (Approximation result)**.** *For every function $v_0 \in \mathrm{W}^{1,p}(\Omega)^m$ there exists a sequence $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ so that*

$$\lim_{\varepsilon\to 0}\left(\|v_0 - v_\varepsilon\|_{\mathrm{L}^p(\Omega)^m} + \|(\nabla v_0)^{\mathrm{ex}} - R_{\frac{\varepsilon}{2}}(v_\varepsilon)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\right) = 0. \tag{4.25}$$

Proof. Choose $\varepsilon_0 > 0$ and $\delta > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have $\Omega_\varepsilon^+ \subset B_\delta(\Omega)$. Here, $B_\delta(\Omega)$ denotes a δ-neighborhood of Ω. Let $v_0 \in \mathrm{C}^\infty(\Omega)^m \cap \mathrm{W}^{1,p}(\Omega)^m$ and $\widetilde{v}_0 \in \mathrm{W}_0^{1,p}(B_\delta(\Omega))^m$ with $\widetilde{v}_0|_\Omega = v_0$ which exists according to Theorem A 6.12 in [2]. For $\varepsilon \in (0, \varepsilon_0)$ we define $v_\varepsilon := (P_\varepsilon \widetilde{v}_0^{\mathrm{ex}})|_\Omega$ and prove that the sequence $(v_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ satisfies (4.25).

1. Proving $v_\varepsilon \to v_0$ in $\mathrm{L}^p(\Omega)^m$ we start by decomposing Ω into Ω_ε^- and $\Omega\backslash\Omega_\varepsilon^-$, which allows us to exploit $(P_\varepsilon \widetilde{v}_0^{\mathrm{ex}})|_{\Omega_\varepsilon^-} \equiv (P_\varepsilon v_0^{\mathrm{ex}})|_{\Omega_\varepsilon^-}$, since $\widetilde{v}_0|_{\Omega_\varepsilon^-} \equiv v_0|_{\Omega_\varepsilon^-}$ by definition. Afterwards we increase the domain of integration and apply the triangle inequality. Then again the domain of integration is increased and at last $\|P_\varepsilon w\|_{\mathrm{L}^p(\Omega_\varepsilon^\pm)} \leq \|w\|_{\mathrm{L}^p(\Omega_\varepsilon^\pm)}$ is used for $w \in \mathrm{L}^p(\mathbb{R}^d)$:

$$\begin{aligned}
\|v_0 - P_\varepsilon \widetilde{v}_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega)^m} &= \|v_0 - P_\varepsilon v_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega_\varepsilon^-)^m} + \|v_0 - P_\varepsilon \widetilde{v}_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega\backslash\Omega_\varepsilon^-)^m} \\
&\leq \|v_0 - P_\varepsilon v_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega)^m} + \|v_0\|^p_{\mathrm{L}^p(\Omega\backslash\Omega_\varepsilon^-)^m} + \|P_\varepsilon \widetilde{v}_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)^m} \\
&\leq \|v_0 - P_\varepsilon v_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega)^m} + \|v_0\|^p_{\mathrm{L}^p(\Omega\backslash\Omega_\varepsilon^-)^m} + \|\widetilde{v}_0\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)^m}.
\end{aligned}$$

Since $P_\varepsilon w^{\mathrm{ex}} \to w$ in $\mathrm{L}^p(\Omega)$ for every $w \in \mathrm{L}^p(\Omega)$ and since $0 \leq \mathrm{vol}(\Omega\backslash\Omega_\varepsilon^-) \leq \mathrm{vol}(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-) \to 0$ according to (2.3), this inequality proves $v_\varepsilon \to v_0$ in $\mathrm{L}^p(\Omega)^m$.

2. For $R_{\frac{\varepsilon}{2}}(v_\varepsilon)|_\Omega \to \nabla v_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$ we prove $\lim_{\varepsilon\to0} \|(\nabla v_0)^{\mathrm{ex}} e_i - (R_{\frac{\varepsilon}{2}}(v_\varepsilon))e_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} = 0$ for every $i \in \{1, \ldots, d\}$. Thereto, let $i \in \{1, \ldots, d\}$ be fixed. In the following calculations we start by adding and subtracting $(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i$ to apply the triangle inequality.

$$\begin{aligned}
&\|(\nabla v_0)^{\mathrm{ex}} e_i - (R_{\frac{\varepsilon}{2}}(v_\varepsilon))e_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\quad\leq \|(\nabla v_0)^{\mathrm{ex}} e_i - (P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} + \|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - (R_{\frac{\varepsilon}{2}}(v_\varepsilon))e_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m}
\end{aligned}$$

Then analogously to step 1 the first term tends to zero when $\varepsilon \to 0$. Moreover, $(R_{\frac{\varepsilon}{2}}(v_\varepsilon))e_i = (\widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}(V_\varepsilon v_\varepsilon))e_i$ on Ω_ε^+ see (4.3) and the identity $V_\varepsilon v_\varepsilon = (P_\varepsilon \widetilde{v}_0^{\mathrm{ex}})|_{\Omega_\varepsilon^+}$ can be used to transform the second term in the following way.

$$\begin{aligned}
&\|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - (R_{\frac{\varepsilon}{2}}(v_\varepsilon))e_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\quad= \|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - (\widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}((P_\varepsilon \widetilde{v}_0^{\mathrm{ex}})|_{\Omega_\varepsilon^+}))e_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\quad\leq \|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - \tfrac{1}{\varepsilon}\big(P_\varepsilon \widetilde{v}_0^{\mathrm{ex}}(\cdot + \tfrac{\varepsilon}{2}e_i) - P_\varepsilon \widetilde{v}_0^{\mathrm{ex}}(\cdot - \tfrac{\varepsilon}{2}e_i)\big)\|_{\mathrm{L}^p(A_\varepsilon)^m} &(4.26)\\
&\qquad + \|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i\|_{\mathrm{L}^p(B_\varepsilon)^m}, &(4.27)
\end{aligned}$$

where, $A_\varepsilon := \{x \in \Omega_\varepsilon^+ \,|\, (x + \frac{\varepsilon}{2}e_i) \in \Omega_\varepsilon^+ \text{ and } (x - \frac{\varepsilon}{2}e_i) \in \Omega_\varepsilon^+\}$ and $B_\varepsilon := \Omega_\varepsilon^+\backslash A_\varepsilon$ for fixed $i \in \{1, \ldots, d\}$. Since $B_\varepsilon \subset \Omega_\varepsilon^+\backslash\Omega_\varepsilon^-$, the term in line (4.27) is bounded. Moreover,

$$\|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)^m} \leq \|(\nabla\widetilde{v}_0)e_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)^m} \overset{\varepsilon\to0}{\longrightarrow} 0,$$

where again $\|P_\varepsilon w\|_{\mathrm{L}^p(\Omega_\varepsilon^\pm)} \le \|w\|_{\mathrm{L}^p(\Omega_\varepsilon^\pm)}$ for $w \in \mathrm{L}^p(\mathbb{R}^d)$ and $\mathrm{vol}(\Omega_\varepsilon^+ \backslash \Omega_\varepsilon^-) \to 0$ for $\varepsilon \to 0$ is used. The term in line (4.26) can be estimated by increasing the domain of integration, exploiting $\|P_\varepsilon w\|_{\mathrm{L}^p(\Omega_\varepsilon^\pm)} \le \|w\|_{\mathrm{L}^p(\Omega_\varepsilon^\pm)}$ for $w \in \mathrm{L}^p(\mathbb{R}^d)$ and replacing $\frac{1}{\varepsilon}[\widetilde{v}_0(x+\frac{\varepsilon}{2}e_i) - \widetilde{v}_0(x-\frac{\varepsilon}{2}e_i)]$ by $\frac{1}{2}\int_{-1}^1 \nabla\widetilde{v}_0(x+\frac{\varepsilon}{2}e_i t)e_i \mathrm{d}t$ in the following way

$$\begin{aligned}
&\|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - \tfrac{1}{\varepsilon}\big(P_\varepsilon\widetilde{v}_0^{\mathrm{ex}}(\cdot + \tfrac{\varepsilon}{2}e_i) - P_\varepsilon\widetilde{v}_0^{\mathrm{ex}}(\cdot - \tfrac{\varepsilon}{2}e_i)\big)\|_{\mathrm{L}^p(A_\varepsilon)^m} \\
&\qquad \le \|(P_\varepsilon(\nabla\widetilde{v}_0)^{\mathrm{ex}})e_i - \tfrac{1}{\varepsilon}\big(P_\varepsilon\widetilde{v}_0^{\mathrm{ex}}(\cdot + \tfrac{\varepsilon}{2}e_i) - P_\varepsilon\widetilde{v}_0^{\mathrm{ex}}(\cdot - \tfrac{\varepsilon}{2}e_i)\big)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\qquad \le \|(\nabla\widetilde{v}_0)e_i - \tfrac{1}{\varepsilon}\big(\widetilde{v}_0(\cdot + \tfrac{\varepsilon}{2}e_i) - \widetilde{v}_0(\cdot - \tfrac{\varepsilon}{2}e_i)\big)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\qquad = \|(\nabla\widetilde{v}_0)e_i - \tfrac{1}{2}\textstyle\int_{-1}^1 \big(\nabla\widetilde{v}_0(\cdot + \tfrac{\varepsilon}{2}e_i t)\big)e_i \mathrm{d}t\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m},
\end{aligned}$$

which is valid for $\varepsilon \in (0, \varepsilon_0)$ small enough such that from $x \in \Omega_\varepsilon^+$ it follows $x+\frac{\varepsilon}{2}e_i \in B_\delta(\Omega)$ and $x-\frac{\varepsilon}{2}e_i \in B_\delta(\Omega)$.

With this estimate it is easy to prove for $v_0 \in \mathrm{C}^\infty(\Omega)^m \cap \mathrm{W}^{1,p}(\Omega)^m$ that the term in line (4.26) converges to zero, too. Then, by density, the claim of Theorem 4.7 holds for arbitrary $v_0 \in \mathrm{W}^{1,p}(\Omega)^m$, too. ☐

5 Γ-convergence

We will now study the Γ-convergence of a functional related to the homogenization problem formulated in Section 3, which depends on both, the variable u and the phase variable z. It is defined as follows:

$$\begin{aligned}
&\mathcal{E}_\varepsilon : \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n \times \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathbb{R} \\
&\mathcal{E}_\varepsilon(u_\varepsilon, z_\varepsilon) := \tfrac{1}{2}\langle \widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)\nabla u_\varepsilon, \nabla u_\varepsilon\rangle_{\mathrm{L}^2(\Omega)^{n\times d}} + \|R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \langle \ell, u_\varepsilon\rangle.
\end{aligned} \tag{5.1}$$

As in Section 3, for a function $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ the tensor $\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon) \in \mathcal{M}(\Omega; \alpha, \beta)$ is based on the given tensor $\widetilde{\mathbb{C}} : \mathbb{R}^m \to \mathcal{M}(Y; \alpha, \beta)$ and is defined by

$$\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)(x) := \widetilde{\mathbb{C}}(z_\varepsilon(x))(\{\tfrac{x}{\varepsilon}\}_Y) \quad \text{for almost every } x \in \Omega.$$

In the application we have in mind, namely a damage model, the time dependent microscopic damage state is encoded in a time dependent variable $z_\varepsilon : [0, T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$. In the forthcoming paper [10], we are interested in deriving effective evolution models as $\varepsilon \to 0$. This necessitates the investigation of the limit behavior of the sequence of functionals $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ with respect to both variables u_ε and z_ε. Moreover, the following Γ-convergence result motivates the somehow artificial assumptions on the sequence $(z_\varepsilon)_{\varepsilon>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ in Section 3.

The following theorem states that $\mathcal{E}_0 : \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n \times \mathrm{W}^{1,p}(\Omega)^m \to \mathbb{R}$ defined via

$$\mathcal{E}_0(u_0, z_0) := \tfrac{1}{2}\langle \widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0)\nabla u_0, \nabla u_0\rangle_{\mathrm{L}^2(\Omega)^{n\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle \ell, u_0\rangle, \tag{5.2}$$

where the tensor $\widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0) \in \mathcal{M}(\Omega, \alpha, \beta)$ for almost every $x \in \Omega$ is given by

$$\langle \widetilde{\mathbb{C}}_{\mathrm{eff}}(z_0)(x)\xi, \xi\rangle_{n\times d} = \min_{v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^n} \int_Y \langle \widetilde{\mathbb{C}}(z_0(x))(y)(\xi + \nabla_y v(y)), \xi + \nabla_y v(y)\rangle_{n\times d} \mathrm{d}y.$$

is the Γ-limit of $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ with respect to the topology induced by assuming uniform a priori estimates of ε-dependent functionals.

Theorem 5.1 (Γ-convergence of $\mathcal{E}_\varepsilon$). *Let $p>1$, let $\widetilde{\mathbb{C}}:\mathbb{R}^m\to\mathcal{M}(Y;\alpha,\beta)$ satisfy the conditions* (3.1) *and* (3.2) *and let the functionals $\mathcal{E}_\varepsilon:\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n\times\mathrm{K}_{\varepsilon\Lambda}(\Omega)^m\to\mathbb{R}$ and $\mathcal{E}_0:\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n\times\mathrm{W}^{1,p}(\Omega)^m\to\mathbb{R}$ be defined by* (5.1) *and* (5.2)*, respectively. Moreover, let $(u_\varepsilon,z_\varepsilon)_{\varepsilon>0}$ be a sequence such that $(u_\varepsilon,z_\varepsilon)\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n\times\mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and*

$$u_\varepsilon\rightharpoonup u_0\quad in\ \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n,\qquad z_\varepsilon\to z_0\quad in\ \mathrm{L}^p(\Omega)^m,\qquad R_{\frac{\varepsilon}{2}}(z_\varepsilon)|_\Omega\rightharpoonup\nabla z_0\quad in\ \mathrm{L}^p(\Omega)^{m\times d}.$$

Then it holds $\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon(u_\varepsilon,z_\varepsilon)\geq\mathcal{E}_0(u_0,z_0)$.

Moreover, for every $(\widetilde{u}_0,\widetilde{z}_0)\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n\times\mathrm{W}^{1,p}(\Omega)^m$ there exists a sequence $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)_{\varepsilon>0}$ such that $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n\times\mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$,

$$\widetilde{u}_\varepsilon\to\widetilde{u}_0\quad in\ \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n,\qquad \widetilde{z}_\varepsilon\to\widetilde{z}_0\quad in\ \mathrm{L}^p(\Omega)^m,\qquad R_{\frac{\varepsilon}{2}}(\widetilde{z}_\varepsilon)|_\Omega\to\nabla\widetilde{z}_0\quad in\ \mathrm{L}^p(\Omega)^{m\times d}$$

and $\lim_{\varepsilon\to0}\mathcal{E}_\varepsilon(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)=\mathcal{E}_0(\widetilde{u}_0,\widetilde{z}_0)$.

Proof. $\liminf$**-inequality:** According to the assumption we already have $\lim_{\varepsilon\to0}\langle\ell(t),u_\varepsilon\rangle=\langle\ell(t),u_0\rangle$ and $\liminf_{\varepsilon\to0}\|R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|_{\mathrm{L}^p(\Omega)^{m\times d}}\geq\|\nabla z_0\|_{\mathrm{L}^p(\Omega)^{m\times d}}$. According to Proposition 2.6 there exists a function $U_1\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ such that $\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\rightharpoonup\nabla_x u_0^{\mathrm{ex}}+\nabla_y U_1^{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R}^d\times Y)^{n\times d}$. Moreover, due to $z_\varepsilon\to z_0$ in $\mathrm{L}^p(\Omega)^m$, we have $\mathcal{T}_\varepsilon\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)\to\widetilde{\mathbb{C}}_0^{\mathrm{ex}}(z_0)$ with respect to the strong L^1-topology analogously to (3.14). Now, we are in the position to apply Theorem 3.23 of [6] (see also [11, 12]) yielding the following $\liminf$-inequality ($\mathbf{L}^2:=\mathrm{L}^2(\mathbb{R}^d\times Y)^{n\times d}$):

$$\liminf_{\varepsilon\to0}\langle\mathcal{T}_\varepsilon\widetilde{\mathbb{C}}_\varepsilon(z_\varepsilon)\mathcal{T}_\varepsilon\nabla u_\varepsilon,\mathcal{T}_\varepsilon\nabla u_\varepsilon\rangle_{\mathbf{L}^2}\geq\langle\widetilde{\mathbb{C}}_0^{\mathrm{ex}}(z_0)(\nabla_x Eu_0^{\mathrm{ex}}+\nabla_y U_1^{\mathrm{ex}}),\nabla_x Eu_0^{\mathrm{ex}}+\nabla_y U_1^{\mathrm{ex}}\rangle_{\mathbf{L}^2}$$

Altogether we proved

$$\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon(u_\varepsilon,z_\varepsilon)\geq\mathbf{E}(u_0,U_1,z_0)\geq\min_{U\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n}\mathbf{E}(u_0,U,z_0)=\mathcal{E}_0(u_0,z_0)$$

by taking into account the integral identity (2.4), Proposition 3.4(iii) and $\mathrm{supp}(\widetilde{\mathbb{C}}_0^{\mathrm{ex}}(z_0))\subset\overline{\Omega\times Y}$.

$\lim(\sup)$**-(in)equality:** For a given function $(\widetilde{u}_0,\widetilde{z}_0)\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n\times\mathrm{W}^{1,p}(\Omega)^m$ choose $\widetilde{U}_1\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ as the unique solution of (3.16). With this, let the first component $\widetilde{u}_\varepsilon\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n$ of the recovery sequence be constructed as follows. Adopting the notation of Proposition 2.7 let $w_\varepsilon\in\mathrm{H}^1_0(\Omega)^n$ be the solution of the elliptic problem stated there with $w_0=0\in\mathrm{H}^1_0(\Omega)^n$ and $W_1=\widetilde{U}_1\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$. Then according to Proposition 2.7 we have $w_\varepsilon\rightharpoonup0$ in $\mathrm{H}^1_0(\Omega)^n$, $w_\varepsilon\xrightarrow{s}0$ in $\mathrm{L}^2(\Omega\times Y)^n$ and $\nabla w_\varepsilon\xrightarrow{s}\nabla_y\widetilde{U}_1$ in $\mathrm{L}^2(\Omega\times Y)^{n\times d}$. Now, the first component of the recovery sequence is defined via $\widetilde{u}_\varepsilon:=\widetilde{u}_0+w_\varepsilon$. Using property (b) of Proposition 2.4 and the convergence results for $(w_\varepsilon)_{\varepsilon>0}$ we find

$$\begin{aligned}\widetilde{u}_\varepsilon&\rightharpoonup\widetilde{u}_0&&\text{in }\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^n,\\ \widetilde{u}_\varepsilon&\xrightarrow{s}E\widetilde{u}_0&&\text{in }\mathrm{L}^2(\Omega\times Y)^n,\\ \nabla\widetilde{u}_\varepsilon&\xrightarrow{s}\nabla_x E\widetilde{u}_0+\nabla_y\widetilde{U}_1&&\text{in }\mathrm{L}^2(\Omega\times Y)^{n\times d}.\end{aligned}$$

According to Theorem 4.7 for $\widetilde{z}_0\in\mathrm{W}^{1,p}(\Omega)^m$ there exists a sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ satisfying $\widetilde{z}_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and

$$\begin{aligned}\widetilde{z}_\varepsilon&\to\widetilde{z}_0&&\text{in }\mathrm{L}^p(\Omega)^m,\\ R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|_\Omega&\to\nabla\widetilde{z}_0&&\text{in }\mathrm{L}^p(\Omega)^{m\times d}.\end{aligned}\tag{5.3}$$

Analogously to (3.14) this gives us $\widetilde{\mathbb{C}}_\varepsilon(\widetilde{z}_\varepsilon) \xrightarrow{s} \widetilde{\mathbb{C}}_0(\widetilde{z}_0)$ in $\mathrm{L}^1(\Omega\times Y;\mathbb{M}_{\mathbb{B}}(\alpha,\beta))$. By applying Corollary 2.5 for $m_\varepsilon := \widetilde{\mathbb{C}}_\varepsilon(\widetilde{z}_\varepsilon)$, $M_0 := \widetilde{\mathbb{C}}_0(\widetilde{z}_0)$ and $v_\varepsilon := \nabla\widetilde{u}_\varepsilon$, $V_0 := \nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1$, the combination of these convergence results for the first and the second component of the recovery sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ gives

$$w_\varepsilon := \widetilde{\mathbb{C}}_\varepsilon(\widetilde{z}_\varepsilon)\nabla\widetilde{u}_\varepsilon \xrightarrow{s} \widetilde{\mathbb{C}}_0(\widetilde{z}_0)(\nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1) =: W_0 \quad \text{in } \mathrm{L}^2(\Omega\times Y)^{n\times d}.$$

Finally, this gives

$$\lim_{\varepsilon\to 0}\langle \widetilde{\mathbb{C}}_\varepsilon(\widetilde{z}_\varepsilon)\nabla\widetilde{u}_\varepsilon, \nabla\widetilde{u}_\varepsilon\rangle_{\mathrm{L}^2(\Omega)^{n\times d}} = \langle \widetilde{\mathbb{C}}_0(\widetilde{z}_0)(\nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1), \nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1\rangle_{\mathrm{L}^2(\Omega\times Y)^{n\times d}}, \tag{5.4}$$

by exploiting Proposition 2.4(a). Combining (5.3), (5.4) and $\lim_{\varepsilon\to 0}\langle \ell(t), \widetilde{u}_\varepsilon\rangle = \langle \ell(t), \widetilde{u}_0\rangle$ together wit Proposition 3.4(iii) proves

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) = \mathbf{E}(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) = \min_{\widetilde{U}\in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n} \mathbf{E}(\widetilde{u}_0, \widetilde{U}, \widetilde{z}_0) = \mathcal{E}_0(\widetilde{u}_0, \widetilde{z}_0).$$

□

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