On Difference and Differential Equations for Modifications of Classical Orthogonal Polynomials

Habilitation Thesis

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Contents

Chapter 1

Introduction

A sequence of polynomials (P_n), where $P_n(x)$ is of exact degree n in x, is said to be orthogonal with respect to a Lebesgue-Stieltjes measure $d\mu(x)$ defined on the interval $[a,\, b]$ (which may be infinite) if

$$\int_a^b P_m(x)\, P_n(x)\, d\mu(x) = 0,\ m \neq n. \tag{1.1}$$

The previous relation, called *orthogonality relation,* assumes implicitly that the moments

$$\mu_n = \int_a^b x^n\, d\mu(x) = 0,\ n = 0,\, 1,\, \ldots, \tag{1.2}$$

are finite.

If the nondecreasing, real-valued, bounded function $\mu(x)$ happens to be absolutely continuous with $d\mu(x) = w(x)\, dx,\ w(x) \geq 0$, then (1.1) reduces to

$$\int_a^b P_m(x)\, P_n(x)\, w(x)\, dx = 0,\ m \neq n, \tag{1.3}$$

and the sequence (P_n) is said to be orthogonal with respect the *weight* function $w(x)$ defined on the interval $[a,\, b]$. However, if $\mu(x)$ is a step-function with jumps w_j at the point $x = x_j,\ j = 0,\, 1,\, 2,\, \ldots$, then (1.1) takes the form

$$\sum_{j=0}^{\infty} P_m(x_j)\, P_n(x_j)\, w_j = 0,\ m \neq n. \tag{1.4}$$

In this case, the variable $x = x_j$ is discrete instead of being continuous like in (1.3); we therefore refer to the sequence (P_n) as orthogonal polynomials of a discrete variable.

Orthogonal polynomials (P_n) (of a continuous or a discrete variable) satisfy a three-term recurrence relation given (in the monic form) by

$$\begin{aligned} P_{n+1}(x) &= (x - \beta_n)\, P_n(x) - \gamma_n\, P_{n-1}(x),\ n \geq 1, \\ P_{-1}(x) &= 0,\ P_0(x) = 1, \end{aligned} \tag{1.5}$$

where β_n and γ_n are real numbers with $\gamma_n > 0,\ n \geq 1$.

Conversely, if a sequence (P_n) satisfies (1.5) where γ_n and β_n are complex numbers with $\gamma_n \neq 0,\ n \geq 0$, then there exists a function of bounded variation ψ on $(-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} P_n(x)\, P_m(x)\, d\psi(x) = \gamma_0\, \gamma_1\, \ldots \gamma_n\, \delta_{n,m},\ n, m \geq 0, \tag{1.6}$$

where $\delta_{n,m}$ is the Kronecker symbol defined by

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Since (P_n) satisfies (1.5), each P_n is of exact degree n, thus (P_n) thanks to (1.6) is orthogonal with respect to the Lebesgue-Stieltjes measure $d\psi(x)$. The function ψ can be chosen to be real valued if and only if (β_n) and (γ_n) are real sequences. ψ can also be chosen to be nondecreasing with an infinite number of points of increase if β_n and γ_n are real and $\gamma_n \neq 0,\ n \geq 0$. This result is known as Favard's theorem [17] (see also [52, 62, 58, 21]).

Favard's theorem can be used to generate new families of orthogonal polynomials. For example, if the sequence (P_n) is orthogonal and satisfies equation (1.5) where β_n and γ_n are complex numbers with $\gamma_n \neq 0,\ n \geq 0$, then, for a given nonnegative integer r, we can define a polynomial sequence $(P_n^{(r)})$ from the relation

$$\begin{aligned} P_{n+1}^{(r)}(x) &= (x - \beta_{n+r})\, P_n^{(r)}(x) - \gamma_{n+r}\, P_{n-1}^{(r)}(x),\ n \geq 1, \\ P_{-1}^{(r)}(x) &= 0,\ P_0^{(r)}(x) = 1. \end{aligned} \tag{1.7}$$

Since $(P_n^{(r)})$ satisfies a three-term recurrence relation with $\gamma_{n+r} \neq 0,\ \forall n$, then by Favard's theorem, $(P_n^{(r)})$ is orthogonal.

The sequence $(P_n^{(r)})$ is called the r-th associated of (P_n). It should be mentioned that the order of association, r, can also be a real or a complex number; provided that β_{n+r} and γ_{n+r} (which should be seen as the expressions β_n and γ_n in which we replace n by $n + r$) are well defined for all n and $\gamma_{n+r} \neq 0$. This is, in general, the case for the classical orthogonal polynomials since for these classes, β_n and γ_n are rational functions of n or q^n (see the next chapter for the definition of classical orthogonal polynomials).

By imposing restrictive conditions on the weight function $w(x)$, one gets interesting new systems of orthogonal polynomials. Among these systems are the classical orthogonal polynomials. These orthogonal polynomials, which are very important because of their applications in many areas of science (Numerical analysis, Physics, Statistics ...) are known to satisfy (depending on whether the variable is continuous or discrete) a second-order linear homogenous differential or difference equation with polynomial coefficients.

Classical orthogonal polynomials of a continuous variable satisfy an equation of the type

$$\sigma(x)\, \frac{d^2}{dx^2}\, y(x) + \tau(x)\, \frac{d}{dx}\, y(x) + \lambda_n\, y(x) = 0. \tag{1.8}$$

On the other hand, classical orthogonal polynomials of a discrete variable satisfy (depending on the type of the discrete variable) three types of difference equations:

- a second-order difference equation

$$\sigma(x)\,\Delta\,\nabla y(x) + \tau(x)\,\nabla\,y(x) + \lambda_n\,y(x) = 0, \tag{1.9}$$

when the variable is of the form (discrete linear)

$$x_j = j,\ j = 0,\,1,\,\ldots;$$

- a second-order q-difference equation

$$\sigma(x)\,D_q\,D_{\frac{1}{q}}y(x) + \tau(x)\,D_q\,y(x) + \lambda_n\,y(x) = 0 \tag{1.10}$$

for the variable of the form (q-linear)

$$x_j = q^j,\ j = 0,\,1,\,\ldots \text{ or } j = \ldots -2,\,-1,\,0,\,1,\,2,\,\ldots;$$

- or a second-order divided difference equation

$$\sigma(x(s))\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla}{\nabla x(s)}y(x(s)) + \frac{\tau(x(s))}{2}\left(\frac{\Delta}{\Delta x(s)} + \frac{\nabla}{\nabla x(s)}\right)y(x(s)) + \lambda_n y(x(s)) = 0 \tag{1.11}$$

for the discrete variable of quadratic or q-quadratic form:

$$\begin{cases} x_s = x(s) = a_2\,s^2 + a_1\,s + a_0, & \text{I} \\ x_s = x(s) = b_2\,q^s + b_1\,q^{-s} + b_0. & \text{II} \end{cases} \tag{1.12}$$

The functions σ and τ appearing in equations (1.8)-(1.11) are polynomials. σ is of degree maximum 2 and τ is of degree exactly 1. λ_n is a constant depending on the coefficients of σ and τ, and Δ, ∇ and D_q are respectively the forward difference operator, the backward difference operator and the Hahn operator [35] defined by:

$$\begin{aligned} \Delta f(s) &= f(s+1) - f(s), \\ \nabla f(s) &= f(s) - f(s-1), \\ D_q f(s) &= \frac{f(q\,s) - f(s)}{(q-1)\,s},\quad s \neq 0,\ D_q f(0) = f'(0), \end{aligned} \tag{1.13}$$

assuming that $q \neq 1$ and f is differentiable at $s = 0$.

Orthogonal polynomial solutions of (1.8) are called *classical orthogonal polynomials of a continuous variable*. They are mainly the Jacobi, the Laguerre, the Hermite and the Bessel polynomials.

Those solutions of (1.9), called *classical orthogonal polynomials of a discrete variable on a linear lattice*, are mainly the Hahn, the Meixner, the Charlier and the Krawtchouk polynomials.

Also, orthogonal polynomial solutions of (1.10) are called *classical orthogonal polynomials of a discrete variable on a q-linear lattice (in short q-classical orthogonal polynomials)*. They are the q-analogues of those solutions of (1.8) and (1.9). Explicitly, they are [39, 41]: The

Big q-Jacobi, Big q-Laguerre, Little q-Jacobi, Little q-Laguerre (Wall), q-Laguerre, Alternative q-Charlier, Al-Salam-Carlitz I, Al-Salam-Carlitz II, Stieltjes-Wigert, Discrete q-Hermite, Discrete q^{-1}-Hermite II, q-Hahn, q-Meixner, Quantum q-Krawtchouk, q-Krawtchouk, Affine q-Krawtchouk, the q-Charlier and the q-Charlier II polynomials.

From (1.11), one can recover (1.8)-(1.10) by choosing appropriate values for the parameters appearing in (1.12). Therefore, the three classes of orthogonal polynomials (called in short; *the very classical orthogonal polynomials*), namely, the classical orthogonal polynomials of a continuous variable, the classical orthogonal polynomials of a discrete variable on a linear lattice and q-classical orthogonal polynomials, are the special case or limiting case of orthogonal polynomials solution of (1.11).

The polynomials solutions of (1.11) are called classical orthogonal polynomials of a discrete variable on a quadratic or a q-quadratic lattice (depending on whether the lattice in (1.12) is of form I or II). They are: The Askey-Wilson, the q-Racah, the continuous dual q-Hahn, the continuous q-Hahn, the dual q-Hahn, the Al-Salam Chihara, the q-Meixner-Pollaczek, the continuous q-Jacobi, the the continuous dual q-Krawtchouk, the continuous big q-Hermite, the continuous q-Laguerre, the continuous q-Hermite, Wilson, the Racah, the dual Hahn, the continuous dual Hahn, the continuous Hahn and the Meixner-Pollaczek polynomials.

The first 12 families and the last 6 families are those of the classical orthogonal polynomials of a discrete variable on a q-quadratic or a quadratic lattice respectively.

Notice that by classical orthogonal polynomials we refer to the very classical orthogonal polynomials or to the classical orthogonal polynomials of a discrete variable on a quadratic or a q-quadratic lattice.

Let (P_n) be a system of very classical orthogonal polynomials. As mentioned above, (P_n) satisfies a second-order linear homogenous differential or difference equation with polynomial coefficients. Even though the r-th associated $(P_n^{(r)})$ is obtained by an *innocent-looking alteration* of the three-term recurrence relation satisfied by (P_n), $(P_n^{(r)})$ is in general not classical and *does not* satisfy any second-order homogenous linear differential or difference equation with polynomial coefficients [48, 47]. It satisfies, instead, a fourth-order linear homogenous differential or difference equation with polynomial coefficients [14]. As illustration, let us mention that the classical orthogonal polynomials of a continuous variable (P_n) satisfies a second-order differential equation of the form

$$\sigma(x)\, P_n''(x) + \tau(x)\, P_n'(x) + \lambda_n\, P_n(x) = 0,$$

where σ is a polynomial of degree at most 2, τ a first-degree polynomial and

$$\lambda_n = -\frac{n}{2}((n-1)\,\sigma'' + 2\,\tau';$$

and that the r-associated $(P_n^{(r)})$ of (P_n) satisfies [14, 30] the following fourth-order differential equation

$$\mathbb{F}_n^{(r)}\,(P_n) = 0,$$

where

$$\mathbb{F}_n^{(r)} = \sigma^2\,\frac{d^4}{dx^4} + 5\sigma\,\sigma'\,\frac{d^3}{dx^3} + (6\sigma\,\sigma'' - 2\tau'\,\sigma + 2\tau\,\sigma' + 2\lambda_{n+r}\,\sigma + 2\lambda_{r-1}\,\sigma - \tau^2 + 3\sigma'^2)\,\frac{d^2}{dx^2}$$

$$+ \quad 3(\lambda_{r-1}\,\sigma' + \lambda_{n+r}\,\sigma' - \tau\,\tau' + \tau\,\sigma'' + \sigma'\,\sigma'')\,\frac{d}{dx}$$
$$+ \quad [(\lambda_{n+r} - \lambda_{r-1})^2 + (\lambda_{n+r} + \lambda_{r-1})\sigma'' + \tau'\,\sigma'' - \tau'^2].$$

The main goal of this thesis is to present the *derivation, the factorization and the solutions* of the fourth-order differential or difference equation satisfied by orthogonal polynomials obtained by modification of classical orthogonal polynomials. These results are therefore valid for:

- the 4 families of the classical orthogonal polynomials of a continuous variable;
- the 4 families of the classical orthogonal polynomials of discrete variable on a linear lattice;
- the 18 families of the q-classical orthogonal polynomials;
- the 12 families of the classical orthogonal polynomials of a discrete variable on a q-quadratic variable;
- the 6 families of the classical orthogonal polynomials of a discrete variable on a quadratic variable.

Here, by modification of classical orthogonal polynomials we refer to the associated classical orthogonal polynomials and also to orthogonal polynomials obtained by making finite modifications to the three-term recurrence relation of the classical orthogonal polynomial systems or its associated.

As motivation of our work, we would like to mention that classical orthogonal polynomials are very useful in many areas of science because of their properties and particularly the second-order differential or difference equation of hypergeometric type they satisfy.

Modifications of classical orthogonal polynomial systems lead to new systems of orthogonal polynomials which in general are not classical and do not satisfy any second-order differential or difference equation with polynomial coefficients [48, 47]. They satisfy, instead, a fourth-order difference or differential equation with polynomial coefficients. However, these new systems of orthogonal polynomials are also important since they are involved in the solutions of many problems. As example, we mention that, given (P_n) a system of polynomials orthogonal with respect to the Stieltjes-Lebesgue measure $d\mu(x)$, satisfying the three-term recurrence relation (1.5), we have:

- The first associated $(P^{(1)}_{n-1})$ and (P_n) are two linearly independent solutions of (1.5);
- $(P^{(1)}_{n-1})$ is the nth partial numerator of the Jacobi continued fraction [7] corresponding to (P_n), and is involved in the Stieltjes inversion formula [17];
- The associated $(P^{(r)}_n)$ is involved in the representation of the transitional probabilities of the birth-death process with killing (see [36, 18, 37] and references therein);
- Orthogonal polynomials obtained by finite modification of the recurrence coefficients of (P_n) are involved in the quantum mechanical study of the many-body systems. For more information we refer to the next chapter.

Chapter 2 is devoted to the fundament and definitions. In Chapter 3, which is the main part of this thesis, we derive, factorize and solve the fourth-order divided-difference equation for orthogonal polynomials obtained by modifying classical orthogonal polynomials of a discrete variable on a quadratic or a q-quadratic lattice. Chapter 4 contains specializations and some applications. As special cases, we point out the three papers [29]-[31] containing the results on the derivation, the factorization and the solutions of the fourth-order differential or difference equation satisfied by the modifications of the very classical orthogonal polynomials. The thesis ends with the conclusion and the perspectives for further works.

Here we would like to mention that as far as we know, the results obtained in Chapter 3 are new and generalize several works based on the derivation, the factorization and the solutions of the fourth-order differential or difference equation satisfied by the modifications of the very classical orthogonal polynomials (see [24]-[31], [11, 42, 54, 55] and references therein).

Chapter 2

Basic theory

2.1 Notations

By $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, we refer respectively to the set of positive integers, the set of real numbers and the set of complex numbers. Also, we define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and if $\mathbb{S}$ is $\mathbb{R}$ or $\mathbb{C}$, then $\mathbb{S}[X]$ refers to the set of polynomials with coefficients in $\mathbb{S}$ while $\mathbb{S}(X)$ refers to the set of rational functions with coefficients in $\mathbb{S}$.

2.2 Hypergeometric and basic hypergeometric series

In the next subsections we recall the definitions of hypergeometric and basic hypergeometric series since classical orthogonal polynomials are represented in terms of such functions.

2.2.1 Hypergeometric series

The generalized hypergeometric series ${}_rF_s$ is defined by

$$ {}_rF_s\left(\left.\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}\right| x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\,\ldots\,(a_r)_k}{(b_1)_k\,\ldots\,(b_s)_k}\frac{x^k}{k!}, $$

with the Pochhammer symbol $(a)_k$ defined by

$$ (a)_n = a\,(a+1)\,(a+2)\ldots(a+n-1),\; (a)_0 = 1. \tag{2.1} $$

The parameters must be such that the denominator factors in the terms of the series are never zero. When one of the numerator parameters a_i equals $-n$ where n is a nonnegative integer, this hypergeometric series is a polynomial in x. Otherwise the radius of convergence ρ of the hypergeometric series is given by

$$ \rho = \begin{cases} \infty & \text{if} \quad r < s+1 \\ 1 & \text{if} \quad r = s+1 \\ 0 & \text{if} \quad r > s+1. \end{cases} $$

2.2.2 Basic hypergeometric functions

The basic hypergeometric series (or q-hypergeometric series) ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\middle| q;x\right) = \sum_{k=0}^{\infty} \frac{(a_1;\dots;a_r;q)_k}{(b_1;\dots\ b_s;q)_k} (-1)^{(1+s-r)\,k}\, q^{(1+s-r)\binom{k}{2}} \frac{x^k}{(q;q)_k},$$

with

$$(a_1;\dots;a_r;q)_k = (a_1;q)_k \dots (a_r;q)_k,\ (a_1;\dots;a_r;q)_\infty = (a_1;q)_\infty \dots (a_r;q)_\infty,$$

where the q-Pochhammer symbols $(a;q)_k$ and $(a;q)_\infty$ are defined by

$$\begin{aligned} (a;q)_k &= (1-a)(1-a\,q)(1-a\,q^2)\dots(1-a\,q^{k-1}),\ k\ge 1,\ (a;q)_0 = 1; \\ (a;q)_\infty &= (1-a)(1-a\,q)(1-a\,q^2)\dots = \prod_{k=0}^{\infty}(1-a\,q^k). \end{aligned}$$

Again, we assume that the parameters are such that the denominator factors in the terms of the series are never zero. If one of the numerator parameters a_i equals q^{-n} where n is a nonnegative integer, this basic hypergeometric series is a polynomial in x. Otherwise the radius of convergence ρ of the basic hypergeometric series is given by

$$\rho = \begin{cases} \infty & \text{if} \quad r < s+1 \\ 1 & \text{if} \quad r = s+1 \\ 0 & \text{if} \quad r > s+1. \end{cases}$$

2.3 Three-term recurrence relation

Let $\mu(x)$ denote a nondecreasing real-valued, bounded function with an infinite number of points of increase in the interval $[a,\, b]$. The latter interval may be infinite. We assume that moments of all orders exist, that is,

$$\int_a^b x^n\, d\mu(x)$$

exists for all $n \in \mathbb{N}$.

Definition 2.1 *The polynomial sequence (P_n) is said to be orthogonal with respect to the Lebesgue-Stieltjes measure $d\mu(x)$ if*

$$\begin{cases} degree(P_n) = n,\ n \ge 0; \\ \\ \int_a^b P_n(x)\, P_m(x)\, d\mu(x) = h_n\, \delta_{n,m},\ n,\ m \ge 0, \end{cases} \tag{2.2}$$

where $h_n \neq 0,\ n \ge 0$ and $\delta_{n,m}$ is the Kronecker symbol defined by

$$\delta_{n,m} = \begin{cases} 1 & \text{if}\, n = m \\ 0 & \text{if}\, n \neq m. \end{cases}$$

In fact, given a Lebesgue-Stieltjes measure $d\mu(x)$, one can construct a corresponding system of orthogonal polynomials satisfying (2.2) from the linearly independent set of monomials (x^n), using the Gram-Schmidt orthogonalization method for the inner product

$$(p,q) \longrightarrow \int_a^b p(x)\, q(x)\, d\mu(x),$$

defined in $\mathbb{R}[X]$.

Remark 2.1 *1. Equations (2.2) are equivalent to*

$$\begin{cases} degree(P_n) = n,\ n \geq 0; \\ \int_a^b x^m\, P_n(x)\, d\mu(x) = h_n\, \delta_{n,m},\ n \geq 0,\ m = 0,\ 1,\ \ldots,\ n,\ h_n \neq 0. \end{cases} \tag{2.3}$$

2. *Given a Lebesgue-Stieltjes measure $d\mu(x)$, (P_n) is uniquely determined up to an arbitrary non-zero constant multiplier factor, that is, if (Q_n) is also orthogonal with respect to $d\mu(x)$, then there are non-zero constants c_n such that $P_n(x) = c_n\, Q_n(x)$, $n \geq 0$.*

As immediate consequence of the previous definition, we have a relation linking three consecutive terms of an orthogonal polynomial system.

Theorem 2.1 *Let (P_n) be a system of polynomials orthogonal with respect to a Lebesgue-Stieltjes measure $d\mu(x)$. (P_n) satisfies*

$$P_{n+1}(x) = (A_n\, x + B_n)\, P_n(x) - C_n\, P_{n-1}(x),\ n \geq 0, \tag{2.4}$$

where we set $P_{-1}(x) = 0$. Here, $A_n,\ B_n$ and C_n are real constants for $n \geq 0$, and $A_{n-1}\, A_n\, C_n > 0,\ n \geq 1$.

Moreover, if k_n is the highest coefficient of P_n and h_n defined by (2.2), then we have

$$A_n = \frac{k_{n+1}}{k_n},\ C_{n+1} = \frac{A_{n+1}}{A_n}\,\frac{h_{n+1}}{h_n}. \tag{2.5}$$

The relation (2.4) is called a *three-term recurrence relation*. It is very useful because it allows for example to express P_{n+1} in terms of P_n and P_{n-1}. Therefore, using this relation, one can express any term P_n in terms of the coefficients $A_i,\ B_i,\ C_i,\ 0 \leq i \leq n-1$ and the initial values P_{-1} and P_0. The proof of the previous theorem as well as the proof of those following in this section can be found in many books on orthogonal polynomials (see for example [4, 17, 51, 60]).

Remark 2.2 *1. When the system (P_n) is monic, that is*

$$k_n = 1,\ n \geq 0 \text{ or } P_n(x) = x^n + \text{lower terms}\,,\ n \geq 0,$$

(2.4) reads

$$\begin{array}{rcl} P_{n+1}(x) & = & (x - \beta_n)\, P_n(x) - \gamma_n\, P_{n-1}(x),\ n \geq 1, \\ P_{-1}(x) & = & 0,\ P_0(x) = 1, \end{array} \tag{2.6}$$

where

$$\beta_n = -B_n,\ \ \gamma_n = C_n,\ \ h_n = \gamma_0\, \gamma_1\, \ldots\, \gamma_n,\ n \geq 0, \quad \text{with } \gamma_0 \equiv \int_a^b d\mu(x).$$

2. *On the other hand, if (P_n) is orthonormal, that is*

$$h_n = 1,\ n \geq 0 \text{ or } \int_a^b P_n(x)\, P_n(x)\, d\mu(x) = 1,\ n \geq 0,$$

(2.4) reads

$$\begin{aligned} P_{n+1}(x) &= (a_n\, x + b_n)\, P_n(x) - c_n\, P_{n-1}(x),\ n \geq 1, \\ P_{-1}(x) &= 0,\ P_0(x) = \left(\int_a^b d\mu(x)\right)^{\frac{-1}{2}}, \end{aligned} \tag{2.7}$$

where

$$a_n = \frac{k_{n+1}}{k_n},\ b_n = -a_n \int_a^b x\, P_n(x)\, P_n(x)\, d\mu(x),\ c_n = \frac{a_n}{a_{n-1}},\ c_0 = 0.$$

An important consequence of the three-term recurrence relation is the following result, called the Christoffel-Darboux formula. We state this result for monic orthogonal polynomial systems.

Theorem 2.2 *Let (P_n) be a system of monic polynomials satisfying (2.6). Then*

$$\sum_{k=0}^{n} \frac{P_k(x)\, P_k(y)}{h_k} = \frac{1}{h_n} \frac{P_{n+1}(x)\, P_n(y) - P_n(x)\, P_{n+1}(y)}{x - y}, \tag{2.8}$$

where h_n is defined by

$$h_n = \gamma_0\, \gamma_1\, \ldots\, \gamma_n,\ n \geq 0, \quad \text{with } \gamma_0 \equiv \int_a^b d\mu(x).$$

The following theorem gives the confluent form of the Christoffel-Darboux formula, that is (2.8) when $y = x$.

Theorem 2.3 *If (P_n) is defined as in the Theorem 2.2, then*

$$\sum_{k=0}^{n} \frac{P_k^2(x)}{h_k} = \frac{1}{h_n} (P'_{n+1}(x)\, P_n(x) - P'_n(x)\, P_{n+1}(x)). \tag{2.9}$$

Corollary 2.1

$$P'_{n+1}(x)\, P_n(x) - P'_n(x)\, P_{n+1}(x) > 0, \quad \text{for all } x.$$

The previous corollary is very important for the study of the properties of the zeros of P_n.

Theorem 2.4 *Let (P_n) be a system of polynomials orthogonal with respect to a Lebesgue-Stieltjes measure $d\mu(x)$ on the interval $[a,\, b]$. (P_n) has the following properties:*

1. *P_n has n simple zeros in $(a,\, b)$.*

2. *The zeros of P_n and P_{n+1} separate each other, that is, between two consecutive zeros of P_n, there is a zero of P_{n+1} and conversely. More precisely, if we denote by $x_{k,n}$, $1 \leq k \leq n$ the zeros of P_n ordered by increasing order, then we have*

$$x_{k,n+1} < x_{k,n} < x_{k+1,n+1},\ 1 \leq k \leq n.$$

The zeros of P_n are of decisive importance in the Gauss quadrature formula. We refer the reader to the books on orthogonal polynomials already mentioned above.

To conclude this section, we would like to mention the following: To a Stieltjes-Lebesgue measure $d\mu(x)$ corresponds a unique set of monic orthogonal polynomials. This set satisfies a three-term recurrence relation given by (1.5). In general, it is very difficult to get the coefficients β_n and γ_n when the measure is $d\mu(x)$ is given. However, when the measure is absolutely continuous, that is $d\mu(x) = w(x)\,dx$, and the weight function $w(x)$ satisfies a Pearson-type equation, then the recurrence coefficients β_n and γ_n are the solution of two nonlinear recurrence equations called the Laguerre-Freud equations [15, 23, 13]. For some special cases such as the very classical orthogonal polynomials, the β_n and γ_n can be expressed explicitly in terms of n. We will return to the computation of the recurrence coefficients later.

2.4 Modifications of the recurrence coefficients

Some Reasons to investigate the modifications of the recurrence coefficients

In quantum mechanical study of many-body systems, one often writes the Hamiltonian H in the form of a tridiagonal matrix. This may be done either by using a Lanczoz-like algorithm or by means of the tight-binding approximation. In both cases, one is led to the so-called chain model of the system.

Since the energies of the quantum levels of the physical system are represented by the eigenvalues of the associated Jacobi matrix, these energies can also be represented by the zeros of orthogonal polynomials corresponding to the associated Jacobi matrix. This is possible because the characteristic polynomials of the principal submatrices of a tridiagonal matrix form a system of orthogonal polynomials.

Then one can transform the eigenvalue problem associated to H into a Jacobi matrix eigenvalue problem or, equivalently, into a problem of determining orthogonal polynomials from its associated three-term recurrence relation. This is a very important link between orthogonal polynomial systems and the Hamiltonian of many-body systems.

So, within this framework, a perturbation of the physical system may be visualized as a perturbation of an element of the associated fictitious chain, which is equivalent to a modification of one or both links of that element with its two nearest neighbors.

On the other hand, because of their physical relevance, experiments are made, with lasers which are able to produce an extremely localized perturbation anywhere in the surface of the target manybody system.

To what extent, the global properties (e.g., the spectroscopical one such as the distribution of its quantum level energies) of the system get modified by the induced local perturbation.

Since to consider local perturbation of an arbitrary element of the chain and search for its effects on the chain properties is a very important physical question, taking into account the relation between the so-called chain model of the system and orthogonal polynomials, we deduce that, to consider a given system of orthogonal polynomials satisfying a three-term recurrence relation and study the properties of the new system of orthogonal polynomials obtained by making a perturbation at any arbitrary level of the recurrence coefficients of the initial system is a very important mathematical question.

This problem was considered for the first time in the theory of orthogonal polynomials by Chihara [17] and later on, many authors have investigated analysis of the orthogonal polynomials which fulfil a three-term recurrence relation with perturbated initial conditions. For more information and references, we refer to [46]. Here, we would like to mention works by Marcellàn, Dehesa and Ronveaux [46] who have studied this problem in a more general approach.

It is also important to notice that the main part of the above introduction is taken from [46]; therefore, we refer to it for more information and references. The perturbations considered in the above mentioned works are one of the main subjects of the following subsections.

2.4.1 The associated orthogonal polynomials

One of the most important perturbation of the recurrence coefficients is the association procedure. This modification of the recurrence coefficients is not finite since it acts on all the recurrence coefficients.

Given $r \in \mathbb{N}_0$, (P_n) a sequence of orthogonal polynomial satisfying

$$\begin{aligned} P_{n+1}(x) &= (x-\beta_n)\,P_n(x) - \gamma_n\,P_{n-1}(x),\ n \geq 1, \\ P_{-1}(x) &= 0,\ P_0(x) = 1, \end{aligned} \tag{2.10}$$

where β_n and γ_n are complex numbers with $\gamma_n \neq 0,\ n \geq 1$, the rth associated of (P_n) is the unique family of monic orthogonal polynomial denoted $(P_n^{(r)})$, defined by the recurrence relation

$$\begin{aligned} P_{n+1}^{(r)}(x) &= (x-\beta_{n+r})\,P_n^{(r)}(x) - \gamma_{n+r}\,P_{n-1}^{(r)}(x),\ n \geq 1, \\ P_{-1}^{(r)}(x) &= 0,\ P_0^{(r)}(x) = 1. \end{aligned} \tag{2.11}$$

As already mentioned in the previous chapter, $(P_n^{(1)})$ provides a second linearly independent solution of (2.10):

Theorem 2.5 *Let (P_n) be a sequence of orthogonal polynomial satisfying (2.10). Then we have the following:*

1. *$(P_n^{(r)})$ satisfies*

$$P_n^{(r)}P_n^{(r+1)} - P_{n+1}^{(r)}P_{n-1}^{(r+1)} = \prod_{k=1}^{n}\gamma_{r+k},\ n \geq 1. \tag{2.12}$$

2. *$(P_{n-1}^{(1)})$ and (P_n) are two linearly independent solutions of (2.10).*

3. *$P_n^{(r)}$ is related to $P_n^{(1)}$ and P_n by the relation*

$$P_n^{(r)}(x) = \frac{P_{r-1}(x)}{\Gamma_{r-1}}\, P_{n+r-1}^{(1)}(x) - \frac{P_{r-2}^{(1)}(x)}{\Gamma_{r-1}}\, P_{n+r}(x), \quad n \geq 0, \quad r \geq 1, \tag{2.13}$$

where the sequence $(\Gamma_n)_n$ is defined by

$$\Gamma_n = \prod_{i=1}^{n} \gamma_i, \; n \geq 1, \quad \Gamma_0 \equiv 1. \tag{2.14}$$

Proof: 1.) We write (2.10) for $P_n^{(r)}$ and $P_{n-1}^{(r+1)}$

$$P_{n+1}^{(r)}(x) = (x - \beta_{n+r})P_n^{(r)}(x) - \gamma_{n+r}P_{n-1}^{(r)}(x), \tag{2.15}$$

$$P_n^{(r+1)}(x) = (x - \beta_{n+r})P_{n-1}^{(r+1)}(x) - \gamma_{n+r}P_{n-2}^{(r+1)}(x). \tag{2.16}$$

In the second step we subtract the two equations obtained after multiplying (2.15) and (2.16) by $P_{n-1}^{(r+1)}$ and $P_n^{(r)}$, respectively,

$$P_n^{(r)}P_n^{(r+1)} - P_{n+1}^{(r)}P_{n-1}^{(r+1)} = \gamma_{n+r}(P_{n-1}^{(r)}P_{n-1}^{(r+1)} - P_n^{(r)}P_{n-2}^{(r+1)}).$$

Then relation (2.12) results by iterating the latter.

2.) $P_n^{(1)}$ and P_n are solutions of (2.10) by definition. They are linearly independent because they satisfy the relation (2.12) for $r = 0$.

3.) From the point 2 of the previous theorem, we deduce that $(P_{n+r-1}^{(1)})$ and P_{n+r} are two linearly independent solutions of the equation

$$X_{n+1}(x) = (x - \beta_{n+r})\, X_n(x) - \gamma_{n+r}\, X_{n-1}(x), \; n \geq 1. \tag{2.17}$$

Therefore, there are coefficients $A(x)$ and $B(x)$ such that

$$P_n^{(r)}(x) = A(x)\; P_{n+r-1}^{(1)}(x) + B(x)\; P_{n+r}(x), \quad n \geq 0.$$

Finally, (2.13) is obtained by computing the coefficients $A(x)$ and $B(x)$ from the initial values $P_{-1}^{(r)} = 0, \; P_0^{(r)} = 1$ and using (2.12) to simplify the result obtained.

□

As an illustration of the importance of the association procedure, we mention that to the recurrence coefficients of (2.10) corresponds the continued fraction

$$S(x) = \frac{\gamma_0}{x - \beta_0 - \frac{\gamma_1}{x-\beta_1-\frac{\gamma_2}{x-\beta_2\cdots}}}. \tag{2.18}$$

It happens that for $x \notin [a,\, b]$, $S(x)$ is related to the polynomials (P_n) and the measure $d\mu(x)$ with respect to which (P_n) is orthogonal by [4, 17]

$$S(x) = \lim_{n\to\infty} \gamma_0\, \frac{P_{n-1}^{(1)}(x)}{P_n(x)} = \int_a^b \frac{d\mu(t)}{x-t} = \sum_{n=0}^{\infty} \frac{\mu_n}{x^{n+1}}, \tag{2.19}$$

where $\mu_n = \int_a^b x^n \, d\mu(x)$ is the moment of order n of the measure $d\mu(x)$, and $P_{n-1}^{(1)}(x)$ is the first associate of (P_n).

If $S(x)$ is known, then the distribution function μ can be recovered from $S(x)$ by means of the *Stieltjes inversion formula* [57] (see also [1, 61]):

$$\mu(t) - \mu(s) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_s^t \mathrm{Im}(S(x + i\, y)\, dx.$$

2.4.2 Finite modifications

The aim of this section is to define the different modifications we will deal with and find relations between the new systems of orthogonal polynomials obtained after modifications, and the initial ones.

Global approach

We start with (P_n), a system of orthogonal polynomials satisfying equation (2.10). Then given k a nonnegative integer, ξ_k and ζ_k two complex numbers with $\zeta_k \neq 0$, the general modification consists by replacing the terms β_k and γ_k of the sequence (β_n) and (γ_n) by $\beta_k + \xi_k$ and $\zeta_k \, \gamma_k$ respectively.

By doing so, we get the polynomial system $(\tilde{P}_n)$ defined by

$$\begin{aligned} \tilde{P}_{n+1}(x) &= (x - \beta_n(k))\, \tilde{P}_n(x) - \gamma_n(k)\, \tilde{P}_{n-1}(x), \; n \geq 1, \\ \tilde{P}_{-1}(x) &= 0, \; \tilde{P}_0(x) = 1, \end{aligned} \tag{2.20}$$

with

$$\begin{cases} \beta_n(k) = \beta_n, \; \gamma_n(k) = \gamma_n & \text{if} \quad n \neq k, \\ \beta_k(k) = \beta_k + \xi_k, \; \gamma_k(k) = \zeta_k \, \gamma_k. \end{cases} \tag{2.21}$$

$(\tilde{P}_n)$ is orthogonal by Favard's theorem. In order to express the new orthogonal system in terms of the initial one, one has to remark that (P_n), $(P_{n-1}^{(1)})$ and $(\tilde{P}_n)$ satisfy the same recurrence equation

$$X_{n+1}(x) = (x - \beta_n)\, X_n(x) - \gamma_n \, X_{n-1}(x), \; n \geq k+1.$$

Therefore, there exist constants (with respect to n) $A_k(x)$ and $B_k(x)$ such that

$$\tilde{P}_n(x) = A_k(x)\, P_n(x) + B_k(x)\, P_{n-1}^{(1)}(x), \; n \geq k+1. \tag{2.22}$$

The coefficients A_k and B_k are obtained from the previous equation using the initial values $\tilde{P}_k$ and $\tilde{P}_{k+1}$ deduced from (2.20) and (2.21)

$$\tilde{P}_k(x) = P_k(x), \; \tilde{P}_{k+1}(x) = P_{k+1}(x) - \xi_k \, P_k(x) + (1 - \zeta_k)\, \gamma_k \, P_{k-1}(x).$$

After the description of the general approach for the modifications, we list several specific types of theses modifications with some relations and references.

1- **The co-recursive $(P_n^{[\mu]})_n$ and the generalized co-recursive orthogonal polynomials $(P_n^{[k,\mu]})_n$**

The orthogonal polynomial sequence $(P_n^{[\mu]})_n$ was introduced for the first time by Chihara [16] as the family of polynomials generated by the recursion formula (2.10) in which β_0 is replaced by $\beta_0 + \mu$:

$$P_{n+1}^{[\mu]}(x) = (x - \beta_n)\, P_n^{[\mu]}(x) - \gamma_n\, P_{n-1}^{[\mu]}(x), \quad n \geq 1, \tag{2.23}$$

with the initial conditions

$$P_0^{[\mu]}(x) = 1, \quad P_1^{[\mu]}(x) = x - \beta_0 - \mu, \tag{2.24}$$

where μ denotes a real number.

This notion was extended to the generalized co-recursive orthogonal polynomials in [19, 20, 56] by modifying the sequence $(\beta_n)_n$ at the level k. This yields an orthogonal polynomial sequence denoted by $(P_n^{[k,\mu]})_n$ generated by the recursion formula

$$P_{n+1}^{[k,\mu]}(x) = (x - \beta_n^*)\, P_n^{[k,\mu]}(x) - \gamma_n\, P_{n-1}^{[k,\mu]}(x), \quad n \geq 1, \tag{2.25}$$

with the initial conditions

$$P_0^{[k,\mu]}(x) = 1, \quad P_1^{[k,\mu]}(x) = x - \beta_0^*, \tag{2.26}$$

where $\beta_n^* = \beta_n$ for $n \neq k$ and $\beta_k^* = \beta_k + \mu$.

The orthogonal polynomial sequence $(P_n^{[k,\mu]})_n$ is related to $(P_n)_n$ and its associated by [46]

$$\begin{aligned} P_n^{[k,\mu]}(x) &= P_n(x) - \mu\, P_k(x)\, P_{n-(k+1)}^{(k+1)}(x),\ n \geq k+1, \\ P_n^{[k,\mu]}(x) &= P_n(x),\ n \leq k. \end{aligned} \tag{2.27}$$

Use of (2.13) transforms the previous equations in

$$\begin{aligned} P_n^{[k,\mu]}(x) &= -\frac{\mu\, P_k^2(x)}{\Gamma_k}\, P_{n-1}^{(1)}(x) + \left(1 + \frac{\mu\, P_k(x) P_{k-1}^{(1)}}{\Gamma_k}\right) P_n(x),\ n \geq k+1, \\ P_n^{[k,\mu]}(x) &= P_n(x),\ n \leq k. \end{aligned} \tag{2.28}$$

Obviously we have the relations $P_n^{[0,\mu]} = P_n^{[\mu]}$, and $P_n^{[0]} = P_n$.

2- **The co-recursive associated $(P_n^{\{r,\mu\}})_n$ and the generalized co-recursive associated orthogonal polynomials $(P_n^{\{r,k,\mu\}})_n$**

The co-recursive associated as well as the generalized co-recursive associated of the orthogonal polynomial sequence $(P_n)_n$, denoted by $(P_n^{\{r,\mu\}})_n$ and $(P_n^{\{r,k,\mu\}})_n$ respectively, are, the co-recursive and the generalized co-recursive (with modification on β_k) of the associated $(P_n^{(r)})_n$ of $(P_n)_n$, respectively. Thanks to (2.27), they are related with $(P_n)_n$ and its associated by

$$P_n^{\{r,0,\mu\}} = P_n^{\{r,\mu\}},$$

and

$$\begin{aligned} P_n^{\{r,k,\mu\}}(x) &= P_n^{(r)}(x) - \mu\, P_k^{(r)}(x)\, P_{n-(k+1)}^{(r+k+1)}(x),\ n \geq k+1, \\ P_n^{\{r,k,\mu\}}(x) &= P_n^{(r)}(x),\ n \leq k. \end{aligned} \tag{2.29}$$

The generalized co-recursive associated orthogonal polynomials can also be expressed using (2.13) and (2.29) by

$$\begin{aligned} P_n^{\{r,k,\mu\}}(x) &= \left(\frac{P_{r-1}(x)}{\Gamma_{r-1}} - \frac{\mu\, P_{k+r}(x)\, P_k^{(r)}(x)}{\Gamma_{r+k}} \right) P_{n+r-1}^{(1)}(x) \\ &- \left(\frac{P_{r-2}^{(1)}(x)}{\Gamma_{r-1}} - \frac{\mu\, P_{k+r-1}^{(1)}(x)\, P_k^{(r)}(x)}{\Gamma_{r+k}} \right) P_{n+r}(x), \quad n \geq k+1, \\ P_n^{\{r,k,\mu\}}(x) &= P_n^{(r)}(x), \quad n \leq k. \end{aligned} \tag{2.30}$$

3- **The co-dilated $(P_n^{|\lambda|})_n$ and the generalized co-dilated orthogonal polynomials $(P_n^{|k,\lambda|})_n$**

The co-dilated of the orthogonal polynomial sequence $(P_n)_n$, denoted by $(P_n^{|\lambda|})_n$, was introduced by Dini [19], as the family of polynomials generated by the recursion formula (2.10) in which γ_1, is replaced by $\lambda\, \gamma_1$ i.e.,

$$P_{n+1}^{|\lambda|}(x) = (x - \beta_n)\, P_n^{|\lambda|}(x) - \gamma_n\, P_{n-1}^{|\lambda|}(x), \quad n \geq 2 \tag{2.31}$$

with the initial conditions

$$P_0^{|\lambda|}(x) = 1, \quad P_1^{|\lambda|}(x) = x - \beta_0,\ P_2^{|\lambda|}(x) = (x-\beta_0)(x-\beta_1) - \lambda\, \gamma_1, \tag{2.32}$$

where λ is a non-zero real number.

This notion was extended to the generalized co-dilated orthogonal polynomials in [20, 56] by modifying the sequence $(\gamma_n)_n$ at the level k. This yields an orthogonal polynomial sequence denoted by $(P_n^{|k,\lambda|})_n$ and generated by the recurrence equation

$$P_{n+1}^{|k,\lambda|}(x) = (x - \beta_n)\, P_n^{|k,\lambda|}(x) - \gamma_n^*\, P_{n-1}^{|k,\lambda|}(x), \quad n \geq 1, \tag{2.33}$$

with the initial conditions

$$P_0^{|k,\lambda|}(x) = 1, \quad P_1^{|k,\lambda|}(x) = x - \beta_0, \tag{2.34}$$

where $\gamma_n^* = \gamma_n$ for $n \neq k$ and $\gamma_k^* = \lambda\, \gamma_k$.

The orthogonal polynomial sequence $(P_n^{|k,\lambda|})_n$ is related to $(P_n)_n$ and its associated by [46]

$$\begin{aligned} P_n^{|k,\lambda|}(x) &= P_n(x) + (1-\lambda)\, \gamma_k\, P_{k-1}(x)\, P_{n-(k+1)}^{(k+1)}(x),\ n \geq k+1, \\ P_n^{|k,\lambda|}(x) &= P_n(x),\ n \leq k. \end{aligned} \tag{2.35}$$

Use of (2.13) transforms the previous equation in

$$\begin{aligned} P_n^{|k,\lambda|}(x) &= \left(1 - \frac{(1-\lambda)\ P_{k-1}(x)\ P_{k-1}^{(1)}(x)}{\Gamma_{k-1}}\right) P_n(x) + \frac{(1-\lambda)\ P_{k-1}(x)\ P_k(x)}{\Gamma_{k-1}} P_{n-1}^{(1)}(x), \quad n \geq k+1, \\ P_n^{|k,\lambda|}(x) &= P_n(x), \quad n \leq k. \end{aligned} \tag{2.36}$$

For $k = 1$ or $\lambda = 1$, we have

$$P_n^{|1,\lambda|} = P_n^{|\lambda|}, \quad P_n^{|k,1|} = P_n.$$

4- **The generalized co-modified orthogonal polynomials** $(P_n^{[k,\mu,\lambda]})_n$

New families of orthogonal polynomials can also be generated by modifying at the same time the sequences $(\beta_n)_n$ and $(\gamma_n)_n$ at the levels k and k' respectively. When $k = k'$, the new family obtained [46], denoted by $(P_n^{[k,\mu,\lambda]})_n$ is generated by the three-term recurrence relation

$$P_{n+1}^{[k,\mu,\lambda]}(x) = (x - \beta_n^*)\ P_n^{[k,\mu,\lambda]}(x) - \gamma_n^*\ P_{n-1}^{[k,\mu,\lambda]}(x), \quad n \geq 1, \tag{2.37}$$

with the initial conditions

$$P_0^{[k,\mu,\lambda]}(x) = 1, \quad P_1^{[k,\mu,\lambda]}(x) = x - \beta_0^*, \tag{2.38}$$

where $\beta_n^* = \beta_n, \quad \gamma_n^* = \gamma_n$ for $n \neq k$ and $\beta_k^* = \beta_k + \mu, \quad \gamma_k^* = \lambda\ \gamma_k$. This family is represented in terms of the starting polynomials and their associated by [46]

$$\begin{aligned} P_n^{[k,\mu,\lambda]}(x) &= P_n(x) + ((1-\lambda)\ \gamma_k\ P_{k-1}(x) - \mu\ P_k(x))\ P_{n-(k+1)}^{(k+1)}(x), \ n \geq k+1, \\ P_n^{|k,\lambda|}(x) &= P_n(x), \ n \leq k. \end{aligned} \tag{2.39}$$

The latter relation can also be written as

$$\begin{aligned} P_n^{[k,\mu,\lambda]}(x) &= \left(1 - \frac{(1-\lambda)\, P_{k-1}(x)\, P_{k-1}^{(1)}(x)}{\Gamma_{k-1}} + \frac{\mu\, P_k(x)\, P_{k-1}^{(1)}(x)}{\Gamma_k}\right) P_n(x) \\ &+ \left(\frac{(1-\lambda) P_{k-1}(x)\ P_k(x)}{\Gamma_{k-1}} - \frac{\mu\ P_k^2(x)}{\Gamma_k}\right) P_{n-1}^{(1)}(x), \quad n \geq k+1, \\ P_n^{[k,\lambda]}(x) &= P_n(x), \quad n \leq k. \end{aligned} \tag{2.40}$$

Remark 2.3 *1. By iteration of the modification described by (2.20)-(2.21), one is led to the following modification*

$$\begin{cases} \beta_n(k) = \beta_n, \ \gamma_n(k) = \gamma_n, & n \geq k+1 \\ \beta_j(k) = \beta_j + \xi_j, \ \gamma_j(k) = \zeta_j\, \gamma_j, & 0 \leq j \leq k, \end{cases}$$

where ξ_j and ζ_j are given complex numbers with $\zeta_j \neq 0, \ 0 \leq j \leq k$.

Orthogonal polynomials corresponding to the recurrence coefficients $(\beta_n(k))$ and $(\gamma_n(k))$ can also be represented as in (2.22). For more information, we refer to [48].

2. *Starting by the system (P_n) satisfying (2.10), by applying the association procedure (2.11) or/and the finite modification (2.20) on (P_n), one is led to a new orthogonal family $(\tilde{P}_n)$, which is related to the initial one by a relation of the form*

$$\tilde{P}_n(x) = I_{n,r,k}(x)\, P_{n+r}(x) + J_{n,r,k}(x)\, P^{(1)}_{n+r-1}(x), \tag{2.41}$$

where r and k are nonnegative integers and $I_{n,r,k}$ and $J_{n,r,k}$ are polynomials in the variable x with the specific property: **They don't depend on** n **for** $n \geq k$*, that is*

$$I_{n,r,k}(x) := I_{r,k}(x), \quad J_{n,r,k}(x) := J_{r,k}(x), \quad \text{for} \quad n \geq k; \tag{2.42}$$

and in addition

$$J_{r,k}(x) \neq 0.$$

To conclude this subsection, we would like to mention that Equation (2.41) giving a link between the initial orthogonal system and the modified one is valid for all modifications mentioned in this chapter.

2.5 Classical orthogonal polynomials

In this section we define what we mean by classical orthogonal polynomials and use this definition and the orthogonality relation to establish the second-order differential or difference equation (1.8)-(1.11) satisfied by these polynomials. Also, we compute explicitly the recurrence coefficients β_n and γ_n in terms of the coefficients of the polynomials σ and τ appearing in the Pearson equation (2.44).

Second-order differential or difference equations (1.8)-(1.11) are the key for the derivation of the fourth-order differential or difference equations for the modified classical orthogonal polynomials.

Several properties of classical orthogonal polynomials are derived from these second-order differential or difference equations. We mention for example the orthogonality of the derivatives, the second-order differential or difference equations for the derivatives, the Rodrigues formula and several differential-difference relations. Here, we will avoid going into these details. Readers are referred to the books [4, 17, 51, 60] and also to the papers [10, 59].

2.5.1 Classical orthogonal polynomials of a continuous variable

Definition 2.2 *A polynomial sequence (P_n) is said to be classical if there exists a weight function $w(x)$ defined on the interval $]a, b[$ (with $w(x) \geq 0$) such that:*

1. *(P_n) is orthogonal with respect to $w(x)$, i.e.,*

$$\begin{cases} degree(P_n) = n,\ n \geq 0; \\ \int_a^b P_n(x)\, P_m(x)\, w(x)\, dx = h_n\, \delta_{n,m},\ n,\ m \geq 0,\ h_n \neq 0,\ n \geq 0; \end{cases} \tag{2.43}$$

2. *The weight function $w(x)$ satisfies a first-order differential equation (called* Pearson equation*) of the form*

$$\frac{d}{dx}(\sigma(x)\,w(x)) = \tau(x)\,w(x), \tag{2.44}$$

with the border conditions

$$\lim_{x\to a} x^n\,\sigma(x)\,w(x) = \lim_{x\to b} x^n\,\sigma(x)\,w(x) = 0,\ \forall n \in \mathbb{N}_0, \tag{2.45}$$

where σ is a polynomial of degree at most two and τ is a first degree polynomial.

Remark 2.4 *If we denote by I the orthogonality interval, then the border condition (2.45) is equivalent to*

$$\begin{array}{ll} \sigma(a)\,w(a) = \sigma(b)\,w(b) = 0, & \text{for } I = (a,b),\ \ a,b \in \mathbb{R}; \\ \sigma(a)\,w(a) = 0,\ \ \lim\limits_{x\to\infty} x^n\,\sigma(x)\,w(x) = 0,\ \forall n \in \mathbb{N}_0, & \text{for } I = (a,\infty),\ \ a \in \mathbb{R}; \\ \lim\limits_{x\to\pm\infty} x^n\,\sigma(x)\,w(x) = 0,\ \forall n \in \mathbb{N}_0, & \text{for } I = (-\infty,\infty). \end{array}$$

Solving the Pearson equation (2.44) and taking into account the border conditions (2.45), one gets depending on the degree of σ and after a linear change of variables three main classes of classical orthogonal polynomials [39]:

1. The Jacobi polynomials $(P_n^{(\alpha,\beta)})$ for which

$$\begin{array}{rcl} \sigma(x) &=& 1-x^2,\ \tau(x) = -(\beta+\alpha+2)\,x+\beta-\alpha,\ \alpha > -1,\ \beta > -1; \\ w(x) &=& (1-x)^\alpha\,(1+x)^\beta,\ x \in\ (a,\,b)\ =\ (-1,\,1); \end{array}$$

2. the Laguerre polynomials (L_n^α) for which

$$\sigma(x) = x,\ \tau(x) = -x+\alpha+1,\ \alpha > -1,\ w(x) = x^\alpha\,e^{-x},\ x \in\ (a,\,b)\ =\ (0,\,\infty);$$

3. the Hermite polynomials (H_n) for which

$$\sigma(x) = 1,\ \tau(x) = -2\,x,\ w(x) = e^{-x^2},\ x \in\ (a,\,b)\ =\ (-\infty,\,\infty).$$

Remark 2.5 *When the polynomial σ is of degree two with double zero, one obtains the Bessel polynomials, whose data are [17, 41]*

$$\sigma(x) = (x-a)^2,\ \tau(x) = (\alpha+2)\,(x-a)+\delta,\ w(x) = (x-a)^\alpha\,e^{-\frac{\delta}{x-a}},\ \delta > 0,\ a,\ \alpha \in \mathbb{R}.$$

In general, the Bessel polynomials are orthogonal not on the real line but on the circle $|z-a| = 1$,

$$\int_{|z-a|=1} \tilde{w}(z)\,P_n(z)\,P_m(z)\,dz = h_n\,\delta_{n,m},\ h_n \neq 0,\ n,m \geq 0.$$

However, the orthogonality on the real line

$$\int_a^\infty (x-a)^\alpha\,e^{-\frac{\delta}{x-a}}\,P_n(x)\,P_m(x)\,dx = h_n\,\delta_{n,m},\ h_n \neq 0,$$

is still possible, but only for finite number of P_n since for the previous integral to exist, it is necessary that

$$n + m + \alpha < -1.$$

The Bessel polynomials, which can be expressed in terms of the Laguerre ones [51], can also be generated by the Rodrigues formula [41]

$$B_n(a, \alpha, \delta, x) = \frac{(x-a)^{-\alpha}\, e^{\frac{\delta}{x-a}}}{(n+\alpha+1)_n} \frac{d^n}{dx^n} \left[(x-a)^{2\,n+\alpha}\, e^{-\frac{\delta}{x-a}}\right],\ n = 0,\ 1,\ \ldots N,$$

where N is a positive integer and $(a)_n$ is the Pochhammer symbol defined by (2.1).

A complete characterization of the classical orthogonal polynomials can be found in the recent book by Lesky [41] (see also Al-Salam [2] and references therein).

Classical orthogonal polynomials of a continuous variable satisfy a second-order differential equation of the hypergeometric type:

Theorem 2.6 *If (P_n) is a classical orthogonal polynomial system, orthogonal with respect to the weight function $w(x)$ satisfying (2.44) and (2.45), then each P_n satisfies*

$$\sigma(x)\, P_n''(x) + \tau(x)\, P_n'(x) + \lambda_n\, P_n(x) = 0, \tag{2.46}$$

with

$$\lambda_n = -\frac{n}{2}(\,(n-1)\,\sigma'' + 2\,\tau').$$

Proof: We assume that the system (P_n) is monic. For all nonnegative integers n, we set

$$Q_n(x) = \sigma(x)\, P_n''(x) + \tau(x)\, P_n'(x),\ n \geq 1,\ Q_0(x) = 1.$$

Then by integrating by parts twice and using (2.43)-(2.45), one gets

$$\int_a^b w(x)\, x^m\, Q_n(x)\, dx = m\,[\tau' + (m-1)\,\frac{\sigma''}{2}] \int_a^b w(x)\, x^m\, P_n(x)\, dx,$$

where n and m are given integers with $m \leq n$. Therefore,

$$\begin{aligned} \int_a^b w(x)\, x^m\, Q_n(x)\, dx &= 0,\ 0 \leq m \leq n-1,\ n \geq 1; \\ \int_a^b w(x)\, x^n\, Q_n(x)\, dx &= n\,[\tau' + (n-1)\,\frac{\sigma''}{2}]\, h_n,\ n \geq 1; \\ \int_a^b w(x)\ Q_0(x)\, dx &= h_0, \end{aligned} \tag{2.47}$$

where h_n ($\neq 0$) is defined by (2.43). It remains to prove that

$$[\tau' + (n-1)\,\frac{\sigma''}{2}] \neq 0,\ \forall n \geq 1. \tag{2.48}$$

For this, we multiply (2.44) by x^n and integrate using (2.45) and obtain

$$(\tau' + n\,\frac{\sigma''}{2})\,\mu_{n+1} = -(n\sigma_1 + \tau_0)\,\mu_n - n\,\sigma_0\,\mu_{n-1}, \tag{2.49}$$

where μ_n is the moment of order n

$$\mu_n = \int_a^b w(x)\,x^n\,dx = 0,\ n = 0,\,1,\,\ldots,$$

and $\sigma(x) = \sigma_2\,x^2 + \sigma_1\,x + \sigma_0,\ \tau(x) = \tau_1\,x + \tau_0$.

In order to be able to compute uniquely all the moments μ_n from the previous ones using (2.49), is is necessary that $(\tau' + n\,\frac{\sigma''}{2}) \neq 0,\ n \geq 0$. Therefore, (2.48) is proved. Since for $n \geq 1$, $Q_n(x) = n\,(\tau' + (n-1)\,\frac{\sigma''}{2})\,x^n + \ldots$., the degree of each Q_n is exactly n. This together with (2.47) allows us to conclude thanks to the Remark 2.1 that (Q_n) is orthogonal with respect to the weight function $w(x)$. From the uniqueness of the system of orthogonal polynomials corresponding to a given weight, we deduce that there exists a sequence (λ_n) such that

$$Q_n(x) = -\lambda_n\,P_n(x),\ n \geq 0.$$

By comparing the leading coefficients of P_n and Q_n from the previous equation, one gets

$$\lambda_n = -\frac{n}{2}\,((n-1)\,\sigma'' + 2\,\tau').$$

□

The converse of the previous theorem is also true in the following sense:

Theorem 2.7 *If (P_n) is a polynomial system such that*

$$\begin{aligned} &\text{degree}(P_n) = n,\ n \geq 0, \\ &\sigma(x)\,P_n''(x) + \tau(x)\,P_n'(x) + \lambda_n\,P_n(x) = 0,\ n \geq 0, \end{aligned} \tag{2.50}$$

where σ is a polynomial of degree at most two and τ is a first degree polynomial with

$$\lambda_n = -\frac{n}{2}\,((n-1)\,\sigma'' + 2\,\tau'),$$

then the family (P_n) is classical.

Proof: We have to prove that (P_n) is orthogonal with respect to a weight function which satisfies a Pearson-type equation and the border condition (see equations (2.44) and (2.45)).

First, we solve the Pearson equation

$$(\sigma(x)\,w(x))' = \tau(x)\,w(x), \tag{2.51}$$

where σ and τ are those of (2.50), and get

$$w(x) = \frac{1}{\sigma(x)}\,e^{\int \frac{\tau(x)}{\sigma(x)}\,dx},$$

assuming that σ is positive on the orthogonality interval $(a,\, b)$.

The possible forms for the weight function $w(x)$ corresponding to the possible degrees of $\sigma(x)$ can, after a linear change of variable, be reduced (up to a multiplier factor) to the four following canonical forms:

$$w(x) = \begin{cases} e^{-x^2}, & \text{for } \sigma(x) = 1 \text{ (Hermite)};\\ x^\alpha\, e^{-x}, & \text{for } \sigma(x) = x, \text{ (Laguerre)};\\ x^\alpha\, e^{-\frac{\delta}{x}}, & \text{for } \sigma(x) = x^2, \text{ (Bessel)};\\ (1-x)^\alpha\,(1+x)^\beta, & \text{for } \sigma(x) = 1-x^2; \text{ (Jacobi)}. \end{cases}$$

Here, $\alpha,\ \beta$ and δ are constants with some restrictions in order to ensure the border conditions.

In the second step, we multiply the differential equation in (2.50) by $w(x)$ and use (2.51) to get the self-adjoint form of the second-order differential equation

$$(\sigma(x)\, w(x)\, P_n'(x))' + \lambda_n\, w(x)\, P_n(x) = 0,\ n \geq 0.$$

Next, use of the previous equation for P_n and P_m gives

$$\begin{aligned}(\lambda_m - \lambda_n)\, w(x)\, P_n(x)\, P_m(x) &= P_m(x)[\sigma(x)\, w(x)\, P_n'(x)]' - P_n(x)[\sigma(x)\, w(x)\, P_m'(x)]'\\ &= \left[\sigma(x)\, w(x)\, (P_n(x)P_m'(x) - P_n'(x)P_m(x))\right]'.\end{aligned}$$

Hence,

$$(\lambda_m - \lambda_n)\int_a^b w(x)\, P_n(x)\, P_m(x)\, dx = \left[\sigma(x)\, w(x)\, (P_n(x)P_m'(x) - P_n'(x)P_m(x))\right]_a^b.$$

The right hand-side of the previous equation vanishes thanks to the border condition (see (2.45)). Since $w(x) > 0$ over $(a,\, b)$ and $\lambda_n \neq \lambda_m$ for $n \neq m$, we deduce that

$$\int_a^b w(x)\, P_n(x)\, P_m(x)\, dx = 0,\ \text{for } n \neq m.$$

Therefore, (P_n) is orthogonal with respect to the weight $w(x)$.

Finally, (P_n) is classical since the orthogonality weight $w(x)$ satisfies the Pearson equation (2.51) with degree$(\sigma) \leq 2$ and degree$(\tau) = 1$. □

We will now use the second-order differential equation for classical orthogonal polynomials established above to compute the coefficients β_n and γ_n of the three-terms recurrence relation, this following works by Lesky [41] (see also [40]).

Proposition 2.1 *Let (P_n) be a system of monic classical orthogonal polynomials satisfying the second-order differential equation (2.46). Then the recurrence coefficients β_n and γ_n of the three-terms recurrence relation (2.10) satisfied by (P_n) are given by*

$$\beta_n = -\frac{(-2\,\tau_0 - 2\,\sigma_1\, n + 2\,\sigma_1\, n^2)\,\sigma_2 + \tau_1\,(\tau_0 + 2\,\sigma_1\, n)}{((2\,n-2)\,\sigma_2 + \tau_1)\,(2\,\sigma_2\, n + \tau_1)},$$

$$\begin{aligned}\gamma_n &= \frac{-n\,(\sigma_2\, n + \tau_1 - 2\,\sigma_2)\,\sigma_0}{(\tau_1 - 3\,\sigma_2 + 2\,\sigma_2\, n)\,(\tau_1 - \sigma_2 + 2\,\sigma_2\, n)}\\ &+ \frac{n\,(\sigma_2\, n + \tau_1 - 2\,\sigma_2)\,(\tau_0 + \sigma_1\, n - \sigma_1)\,(\sigma_1\, n\,\sigma_2 - \sigma_1\,\sigma_2 + \sigma_1\,\tau_1 - \tau_0\,\sigma_2)}{(\tau_1 - 3\,\sigma_2 + 2\,\sigma_2\, n)\,(2\,\sigma_2\, n + \tau_1 - 2\,\sigma_2)^2\,(\tau_1 - \sigma_2 + 2\,\sigma_2\, n)},\end{aligned}$$

where $\sigma(x) = \sigma_2\, x^2 + \sigma_1\, x + \sigma_0,\ \tau(x) = \tau_1\, x + \tau_0$.

Proof: In the first step we write

$$P_n(x) = x^n + T_{n,1}\, x^{n-1} + T_{n,2}\, x^{n-2} + U_n(x),$$

where $(T_{n,k})$ are real numbers and $U_n(x)$ is a polynomial of degree at most $n-3$. Then we substitute the previous relation into the second-order differential equation (2.46) and get

$$\begin{aligned} & [n\,(n-1)\,\sigma_2 + n\,\tau_1 + \lambda_n]\, x^n \\ + \;& [n\,(n-1)\,\sigma_1 + (n-1)\,(n-2)\sigma_2\, T_{n,1} + \tau_0\, n + (n-1)\,\tau_1\, T_{n,1} + \lambda_n\, T_{n,1}]\, x^{n-1} \\ + \;& [n\,(n-1)\,\sigma_0 + (n-1)\,(n-2)\sigma_1\, T_{n,1} + (n-2)\,(n-3)\sigma_2\, T_{n,2} \\ + \;& (n-1)\,\tau_0\, T_{n,1} + (n-2)\,\tau_1\, T_{n,2} + \lambda_n\, T_{n,2}]\, x^{n-2} + V_n(x) = 0, \end{aligned}$$

where V_n is a polynomial of degree at most $n-3$.

Next, we solve in terms of the unknowns λ_n, $T_{n,1}$ and $T_{n,2}$ the system of three linear equations obtained from the previous equation

$$\begin{aligned} & n\,(n-1)\,\sigma_2 + n\,\tau_1 + \lambda_n = 0; \\ & n\,(n-1)\,\sigma_1 + (n-1)\,(n-2)\sigma_2\, T_{n,1} + \tau_0\, n + (n-1)\,\tau_1\, T_{n,1} + \lambda_n\, T_{n,1} = 0; \\ & n\,(n-1)\,\sigma_0 + (n-1)\,(n-2)\sigma_1\, T_{n,1} + (n-2)\,(n-3)\sigma_2\, T_{n,2} \\ & \qquad + (n-1)\,\tau_0\, T_{n,1} + (n-2)\,\tau_1\, T_{n,2} + \lambda_n\, T_{n,2} = 0 \end{aligned}$$

and get

$$\begin{aligned} \lambda_n &= -n\,(n-1)\,\sigma_2 - n\,\tau_1; \quad T_{n,1} = \frac{n\,(\tau_0 + \sigma_1\, n - \sigma_1)}{2\,\sigma_2\, n + \tau_1 - 2\,\sigma_2}; \\ T_{n,2} &= \frac{n\,(n-1)\,({\sigma_1}^2\, n^2 + 2\,\sigma_1\,\tau_0\, n - 3\,{\sigma_1}^2\, n + 2\,\sigma_2\,\sigma_0\, n + \tau_1\,\sigma_0 + {\tau_0}^2 - 3\,\tau_0\,\sigma_1 - 2\,\sigma_2\,\sigma_0 + 2\,{\sigma_1}^2)}{2\,(2\,\sigma_2\, n + \tau_1 - 2\,\sigma_2)\,(\tau_1 - 3\,\sigma_2 + 2\,\sigma_2\, n)}. \end{aligned}$$

Finally, we use the previous results and the following relations between the coefficients $T_{n,j}$ [17, 23]

$$T_{n+1,1} = T_{n,1} - \beta_n,\; T_{n+1,2} = T_{n,2} - \beta_n\, T_{n,1} - \gamma_n, \tag{2.52}$$

to compute the coefficients β_n and γ_n. □

The following table contains data for the classical orthogonal polynomials.

Table 2.1

Family	Hermite	Laguerre	Bessel	Jacobi
$(a,\, b)$	$(-\infty,\, \infty)$	$(0,\, \infty)$	$(a,\, \infty)$ or the circle: $\lvert z-a\rvert = 1$	$(-1,\, 1)$
$w(x)$	e^{-x^2}	$x^\alpha\, e^{-x}$	$(x-a)^\alpha\, e^{-\frac{\delta}{x-a}}$	$(1-x)^\alpha\,(1+x)^\beta$
$\sigma(x)$	1	x	$(x-a)^2$	$1-x^2$
$\tau(x)$	$-2\,x$	$\alpha+1-x$	$(\alpha+2)\,(x-a)+\delta$	$\beta-\alpha-(\alpha+\beta+2)\,x$
λ_n	$2\,n$	n	$n\,(n+\alpha+1)$	$n\,(n+\alpha+\beta+1)$
β_n	0	$2\,n+\alpha+1$	$\frac{4\,a\,n\,(n+\alpha+1)+\alpha(a\,(\alpha+2)-\delta)}{(2\,n+\alpha)\,(2\,n+\alpha+2)}$	$\frac{\beta^2-\alpha^2}{(2\,n+\alpha+\beta)\,(2\,n+\alpha+\beta+2)}$
γ_n	$\frac{n}{2}$	$n\,(n+\alpha)$	$\frac{-n\,(n+\alpha)\,\delta^2}{(2\,n+\alpha-1)\,(2\,n+\alpha)^2\,(2\,n+\alpha+1)}$	$\frac{4\,n(n+\alpha)\,(n+\beta)\,(n+\alpha+\beta)}{(2\,n+\alpha+\beta-1)\,(2\,n+\alpha+\beta)^2\,(2\,n+\alpha+\beta+1)}$

To conclude this subsection, we mention in the following table the hypergeometric representation of the classical orthogonal polynomials of a continuous variable in the monic case. The notations we use differ slightly (by the tilde) from those of [39] because here we deal with monic orthogonal polynomials.

Table 2.2

Family	Hypergeometric representation	Conditions
Hermite	$\tilde{H}_n(x) = x^n\, {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix}\middle\vert -\frac{1}{x^2}\right)$	-
Laguerre	$\tilde{L}_n^{\alpha}(x) = (-1)^n\,(\alpha+1)_n\,{}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}\middle\vert x\right)$	$\alpha > -1$
Bessel	$\tilde{B}_n(a,\alpha,\delta,x) = \frac{\delta^n}{(n+\alpha+1)_n}\,{}_2F_0\left(\begin{matrix} -n, n+\alpha+1 \\ - \end{matrix}\middle\vert -\frac{x-a}{\delta}\right)$	$\delta > 0,\ \alpha \leq -3,$ $n = 0,\, 1,\, \ldots,\, N$
Jacobi	$\tilde{P}_n^{(\alpha,\beta)}(x) = \frac{2^n\,(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\,{}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}\middle\vert \frac{1-x}{2}\right)$	$\alpha > -1,\ \beta > -1.$

2.5.2 Classical orthogonal polynomials of a discrete variable on a linear lattice

In this subsection, we define classical orthogonal polynomials of a discrete variable on linear lattice and state the results and properties similar to those given for the classical orthogonal polynomials of a continuous variable. The proofs are similar to those given in the previous subsection. For more details, the reader is refereed to the next section and also to [51].

Definition 2.3 *A polynomial sequence (P_n) is said to be a classical orthogonal polynomials of a discrete variable on a linear lattice if there exists a weight function $w(x)$ defined on the interval $]a,b[$ (with $w(x) \geq 0$) such that:*

1. *(P_n) is orthogonal with respect to $w(x)$, i.e.,*

$$\begin{cases} degree(P_n) = n,\ n \geq 0; \\ \sum_{x=a}^{b} P_n(x)\,P_m(x)\,w(x) = h_n\,\delta_{n,m},\ n,\,m \geq 0,\ h_n \neq 0,\, n \geq 0; \end{cases}$$

2. *The weight function $w(x)$ satisfies a first-order difference equation (called* discrete Pearson equation*) of the form*

$$\Delta(\sigma(x)\,w(x)) = \tau(x)\,w(x), \tag{2.53}$$

with the border conditions

$$\lim_{x\to a} x^n\,\sigma(x)\,w(x) = \lim_{x\to b} x^n\,\sigma(x)\,w(x) = 0,\ \forall n \in \mathbb{N}_0, \tag{2.54}$$

where σ is a polynomial of degree at most two and τ is a first degree polynomial. The forward operator Δ as well as the backward operator ∇ are defined by

$$\Delta f(x) = f(x+1) - f(x),\ \nabla f(x) = f(x) - f(x-1). \tag{2.55}$$

Solving the discrete Pearson equation and taking into account the border conditions, one gets depending on the degree of σ and after a linear change of variables four main classes of classical orthogonal polynomials of a discrete variable on a linear lattice [39]:

1. The Hahn polynomials $(Q_n^{(\alpha,\beta)})$ for which $a = 0,\ b = N \in \mathbb{N}$ and

$$\begin{aligned} \sigma(x) &= x\,(N+\alpha-x),\ \tau(x) = -(\beta+\alpha+2)\,x + (\beta+1)\,(N-1), \\ w(x) &= \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x},\ x = 0,\,1,\,\ldots,\,N; \end{aligned}$$

   with the conditions

$$\alpha > -1,\ \beta > -1 \text{ or } \alpha < -\mathrm{N},\ \beta < -\mathrm{N};$$

2. the Meixner polynomials $(M_n(x;\beta,c))$ for which $a = 0,\ b = \infty$ and

$$\sigma(x) = x,\ \tau(x) = (c-1)\,x + \beta\,c,\ 0 < c < 1,\ \beta > 0,\ w(x) = \frac{(\beta)_x}{x!}\,c^x,\ x = 0,\,1,\,\ldots,\,\infty;$$

3. the Krawtchouk polynomials $(K_n(x;p,N))$ for which $a = 0,\ b = N \in \mathbb{N}$ and

$$\sigma(x) = x,\ \tau(x) = \frac{N\,P - x}{1-p},\ 0 < p < 1,\ w(x) = \binom{N}{x}\,p^x\,(1-p)^{N-x},\ x = 0,\,1,\,\ldots,\,N;$$

4. the Charlier polynomials $(C_n(x;a))$ for which $a = 0,\ b = \infty$ and

$$\sigma(x) = x,\ \tau(x) = a - x,\ a > 0,\ w(x) = \frac{a^x}{x!},\ x = 0,\,1,\,\ldots,\,\infty.$$

Classical orthogonal polynomials of a discrete variable on a linear lattice satisfy a second-order difference equation:

Theorem 2.8 *If (P_n) is a classical orthogonal polynomials of a discrete variable on a linear lattice, orthogonal with respect to the weight function $w(x)$ satisfying (2.53) and (2.54), then each P_n satisfies*

$$\sigma(x)\,\Delta\nabla P_n(x) + \tau(x)\,\Delta P_n(x) + \lambda_n\,P_n(x) = 0, \tag{2.56}$$

with

$$\lambda_n = -\frac{n}{2}(\,(n-1)\,\sigma'' + 2\,\tau'),$$

and the operators Δ and ∇ defined by (2.55).

The converse of the previous theorem is also true in the following sense:

Theorem 2.9 *If (P_n) is a polynomial system such that*

$$\begin{aligned} &\mathrm{degree}(P_n) = n,\ n \geq 0, \\ &\sigma(x)\,\Delta\nabla P_n(x) + \tau(x)\,\Delta P_n(x) + \lambda_n\,P_n(x) = 0,\ n \geq 0, \end{aligned} \tag{2.57}$$

where σ is a polynomial of degree at most two and τ is a first degree polynomial with

$$\lambda_n = -\frac{n}{2}\,((n-1)\,\sigma'' + 2\,\tau'),$$

then the family (P_n) is a classical orthogonal polynomial of a discrete variable on a linear lattice.

We will again use the second-order difference equation for classical orthogonal polynomials of a discrete variable on a linear lattice established above to compute the coefficients β_n and γ_n of the three-terms recurrence relation, this following works by Lesky [41] (see also [40]).

Proposition 2.2 *Let (P_n) be a system of monic classical orthogonal polynomials of a discrete variable on a linear lattice satisfying the second-order difference equation (2.56). Then the recurrence coefficients β_n and γ_n of the three-terms recurrence relation (2.10) satisfied by (P_n) are given by*

$$\beta_n = -\frac{(-2\,\tau_0 - \tau_1 n - 2\,\sigma_1 n + 2\,\sigma_1 n^2 + \tau_1 n^2)\,\sigma_2 + \tau_1\,(\tau_1 n + \tau_0 + 2\,\sigma_1 n)}{(2\,\sigma_2 n + \tau_1)\,((-2+2\,n)\,\sigma_2 + \tau_1)}, \tag{2.58}$$

$$\begin{aligned} \gamma_n \;&=\; -\frac{n\,(\sigma_2 n + \tau_1 - 2\,\sigma_2)\,\sigma_0}{(\tau_1 - \sigma_2 + 2\,\sigma_2 n)\;(\tau_1 - 3\,\sigma_2 + 2\,\sigma_2 n)} \\ &- \;\left[(2\,\sigma_2 n + \tau_1 - 2\,\sigma_2)^2\,(\tau_1 - 3\,\sigma_2 + 2\,\sigma_2 n)\,(\tau_1 - \sigma_2 + 2\,\sigma_2 n)\right]^{-1} \\ &\times\; n\;(\sigma_2 n + \tau_1 - 2\,\sigma_2)\left(\tau_0 + \sigma_1 n - \sigma_1 + \tau_1 n - \tau_1 + \sigma_2 + \sigma_2 n^2 - 2\,\sigma_2 n\right) \\ &\times\; \left(-2\,{\sigma_2}^2 n + {\sigma_2}^2 n^2 + {\sigma_2}^2 + \sigma_2\tau_1 n - \sigma_1 n\sigma_2 - \sigma_2\tau_1 + \tau_0\sigma_2 + \sigma_1\sigma_2 - \sigma_1\tau_1\right), \end{aligned} \tag{2.59}$$

where $\sigma(x) = \sigma_2\,x^2 + \sigma_1\,x + \sigma_0,\ \tau(x) = \tau_1\,x + \tau_0$.

The following table contains data for the classical orthogonal polynomials of a discrete variable on a linear lattice.

Table 2.3

Family	Krawtchouk	Charlier	Meixner	Hahn
$[a,\,b]$	$\{0,\,1,\,\ldots,\,N\}$	$\{0,\,1,\,\ldots,\,\infty\}$	$\{0,\,1,\,\ldots,\,\infty\}$	$\{0,\,1,\,\ldots,\,N\}$
$w(x)$	$\binom{N}{x}\,p^x\,(1-p)^{N-x}$	$\frac{a^x}{x!}$	$\frac{(\beta)_x}{x!}\,c^x$	$\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}$
$\sigma(x)$	x	x	x	$x\,(N+\alpha-x)$
$\tau(x)$	$\frac{N\,p-x}{1-p}$	$a-x$	$c\,\beta + (c-1)\,x$	$-(\alpha+\beta+2)\,x$ $+(\beta+1)\,(N-1)$
λ_n	$\frac{n}{1-p}$	n	$(1-c)\,n$	$n\,(n+\alpha+\beta+1)$
β_n	$p\,(N-n)+n\,(1-p)$	$n+a$	$\frac{n+(n+\beta)\,c}{1-c}$	$A_n + C_n$
γ_n	$n\,p\,(1-p)\,(N+1-n)$	$n\,a$	$\frac{n\,(n+\beta-1)\,c}{(1-c)^2}$	$A_{n-1}\,C_n,$

with

$$A_n = \frac{(N-n)\,(n+\alpha+\beta+1)\,(n+\alpha+1)}{(2\,n+\alpha+\beta+1)\,(2\,n+\alpha+\beta+2)},\ C_n = \frac{n\,(n+\alpha+\beta+N+1)\,(n+\beta)}{(2\,n+\alpha+\beta)\,(2\,n+\alpha+\beta+1)}.$$

Here $\binom{a}{b}$ refers to the binomial symbol defined by

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\,\Gamma(a-b+1)}.$$

To conclude this subsection, we mention in the following table the hypergeometric representation of the classical orthogonal polynomials of a discrete variable on a linear lattice in the monic case. The notations we use differ slightly (by the tilde) from those of [39] because here we deal with monic orthogonal polynomials.

Table 2.4

Family	Hypergeometric representation	Conditions
Krawtchouk	$\tilde{K}_n(x) = (-N)_n\, p^n\, {}_2F_1\left(\begin{matrix} -x, -n \\ -N \end{matrix} \,\middle\vert\, \frac{1}{p}\right)$	$0 < p < 1$ $n = 0, 1, \ldots, N,$
Charlier	$\tilde{C}_n(x, a) = (-a)^n\, {}_2F_0\left(\begin{matrix} -x, -n \\ - \end{matrix} \,\middle\vert\, -\frac{1}{a}\right)$	$a > 0,$
Meixner	$\tilde{M}_n(x, \beta, c) = (\beta)_n\, (\frac{c}{c-1})^n\, {}_2F_1\left(\begin{matrix} -x, -n \\ \beta \end{matrix} \,\middle\vert\, 1 - \frac{1}{c}\right)$	$\beta > 0,\ 0 < c < 1,$
Hahn	$\tilde{Q}_n(x; \alpha, \beta, N) = \frac{(\alpha+1)_n\,(-N)_n}{(n+\alpha+\beta+1)_n}\, {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix} \,\middle\vert\, 1\right)$	$\alpha > -1,\ \beta > -1$ or $\alpha < -N,\ \beta < -N$.

2.5.3 q-Classical orthogonal polynomials

In this subsection, we define the notion of q-classical orthogonal polynomials and state the results and properties similar to those given for the classical orthogonal polynomials of a continuous variable. For the proofs of these results and properties, the reader is referred to the reference [49] and also to the references therein.

Let $\mathbb{P}$ be the linear space of polynomial functions in $\mathbb{C}$ with complex coefficients and $\mathbb{P}^*$ its algebraic dual space, i.e., $\mathbb{P}^*$ is the linear space of all linear applications $\mathcal{L} \;:\; P \longrightarrow \mathbb{C}$. We call the elements of $\mathbb{P}^*$ as functionals and we denote the action of the functional $\mathcal{L}$ on the polynomial p by $\langle \mathcal{L}, p \rangle$.

Definition 2.4 *A polynomial sequence* (P_n) *is said to be a monic orthogonal polynomial sequence with respect to a functional* $\mathcal{L}$ *if*

$$P_n(x) = x^n + \text{lower terms} \;\text{ and }\; \langle \mathcal{L}, P_n\, P_m \rangle = k_n\, \delta_{n,m},\; k_n \neq 0,\, n \geq 0. \tag{2.60}$$

Notice that to a given functional $\mathcal{L}$, there corresponds at most one family of monic orthogonal polynomials (i.e. orthogonal with respect to $\mathcal{L}$).

Definition 2.5 *Let $\mathcal{L}$ be a functional and (P_n) a monic orthogonal polynomial sequence with respect to $\mathcal{L}$. The functional $\mathcal{L}$ is said to be a q-classical functional and (P_n) a q-classical monic orthogonal polynomial sequence if and only if there exists a pair of polynomials ϕ and ψ with* $\operatorname{degree}\phi \leq 2$ *and* $\operatorname{degree}\psi = 1$ *such that*

$$D_q(\phi\,\mathcal{L}) = \psi\mathcal{L}. \tag{2.61}$$

Here, the product $f\,\mathcal{L}$ of the functional $\mathcal{L}$ by the polynomial f and the q-derivative $D_q\mathcal{L}$ of the functional $\mathcal{L}$ are functionals defined by

$$\langle f\,\mathcal{L}, p\rangle = \langle \mathcal{L}, f\,p\rangle, \;\; \langle\, D_q\mathcal{L}, p\rangle = \langle \mathcal{L}, D_q\,p\rangle, \; \forall\, p \in \mathbb{P},$$

where the operator D_q is the Hahn operator [35]

$$D_q f(s) = \frac{f(q\,s) - f(s)}{(q-1)\,s}, \;\; s \neq 0, \; D_q f(0) = f'(0).$$

Using the previous definition, Medem, Álvarez-Nodarse and Marcellán [49] had identify seventeen q-classical orthogonal polynomial sequences among all the families in the so-called q-Askey scheme [39]. They are: The Big q-Jacobi, Big q-Laguerre, Little q-Jacobi, Little q-Laguerre (Wall), q-Laguerre, Alternative q-Charlier, Al-Salam-Carlitz I, Al-Salam-Carlitz II, Stieltjes-Wigert, Discrete q-Hermite, Discrete q^{-1}-Hermite II, q-Hahn, q-Meixner, Quantum q-Krawtchouk, q-Krawtchouk, Affine q-Krawtchouk, and q-Charlier polynomials.

It should be mentioned that the complete characterization of classical orthogonal polynomials has been given by Lesky in a recent book [41]. In fact, looking for all orthogonal polynomial sequence satisfying a q-difference equation of the form

$$\phi(x)\, D_q^2\, P_n(x) + \tau(x)\, D_q\, P_n(x) - \lambda_n\, P_n(q\,x) = 0,$$

where ϕ and ψ are polynomials with $\operatorname{degree}\phi \leq 2$ and $\operatorname{degree}\psi = 1$, Lesky has found eighteen orthogonal polynomial sequences: The seventeen families mentioned above plus the new one which he has called q-Charlier II. Of course all the eighteen families are not completely independent since some of them are special cases of others. For seek of completeness, we have included at the end of this section data relative to these eighteen families.

Remark 2.6 *For all q-classical orthogonal polynomials, the corresponding q-classical functionals (therefore the orthogonality relation (2.60)) are defined via the weight function $w(x)$ in one of the following ways:*

$$\langle \mathcal{L}, p\rangle = \sum_{k=0}^{\infty} w(k)\, p(q^k)\, q^k; \tag{2.62}$$

$$\langle \mathcal{L}, p\rangle = \sum_{k=0}^{\infty} w(k)\, p(q^{-k})\, q^{-k}; \tag{2.63}$$

$$\langle \mathcal{L}, p\rangle = \sum_{k=-\infty}^{\infty} w(k)\, p(c\,q^k)\, q^k; \tag{2.64}$$

$$\langle \mathcal{L}, p\rangle = \int_{\alpha}^{\beta} w(x)\, p(x)\, d_q x; \tag{2.65}$$

$$\langle \mathcal{L}, p\rangle = \int_{0}^{\infty} w(x)\, p(x)\, dx, \tag{2.66}$$

where c, α, β are constants and the integral in (2.65) refers to the q-integration defined by (see the book by Gasper and Rahman [32] for more details)

$$\int_0^z f(t)\, d_q t = (1-q) \sum_{n=0}^{\infty} q^n\, f(q^n\, z).$$

The following theorem and proposition give the equivalences between the functional equation (2.61), the q-Pearson equation for the weight and the second-order q-difference equation satisfied by the q-classical families [49, 35] (see also [6]).

Theorem 2.10 *Let $\mathcal{L}$ be a functional and (P_n) a monic orthogonal polynomial sequence with respect to $\mathcal{L}$. The following statements are equivalent:*

1. *The functional $\mathcal{L}$ is q-classical and verifies the functional equation (2.61), where ϕ and ψ are polynomials with* $\operatorname{degree}\phi \leq 2$ *and* $\operatorname{degree}\psi = 1$.

2. *The weight function $w(x)$, defined on the interval (a, b), associated to the q-classical functional $\mathcal{L}$ (as stated in the Remark 2.6) satisfies the q-Pearson equation*

$$D_q(\sigma(x)\, w(x)) = \tau(x)\, w(x), \tag{2.67}$$

with the conditions

$$\begin{aligned} w(x_i) > 0,\ a \leq x_i \leq b-1,\ (x_{i+1} = q\, x_i), \\ \sigma(x)\, w(x)\, x^k|_{x=a,b} = 0,\ k \geq 0. \end{aligned}$$

The polynomials σ and τ are related to ϕ and ψ in the following way

$$\sigma(x) = \phi(x) + (q-1)\, x\, \psi(x),\ \tau(x) = -\psi(x). \tag{2.68}$$

3. *Each P_n satisfies the second-order q-difference equation*

$$\sigma(x)\, D_q\, D_{\frac{1}{q}}\, P_n(x) + \tau(x)\, D_q\, P_n(x) + \lambda_n\, P_n(x) = 0, \tag{2.69}$$

where

$$\lambda_n = -[n]_q\, (\tau' + [n-1]_{\frac{1}{q}}\, \frac{\sigma''}{2\, q}),$$

with

$$[k]_a = \frac{a^k - 1}{a-1},\ \tau' \equiv \tau'(x) = \frac{d\tau(x)}{dx},\ \sigma''(x) \equiv \sigma'' = \frac{d^2\sigma(x)}{dx^2}.$$

We will now use the second-order q-difference equation for the q-classical orthogonal polynomials established above to compute the coefficients β_n and γ_n of the three-terms recurrence relations, this following works by Lesky [41] (see also [40]).

Proposition 2.3 *Let (P_n) be a system of monic q-classical orthogonal polynomials satisfying the second-order q-difference equation (2.69). Then the recurrence coefficients β_n and γ_n of the three-term recurrence relation (2.10) satisfied by (P_n) are given by*

$$\begin{aligned}\beta_n \;=\;& [(-1+\rho)\,(1+q)\,(q\tau_1\rho - q\sigma_2 - \tau_1\rho + \sigma_2\rho)\,\rho\,\sigma_1 \\ & - \tau_0\,(-1+q)\left(\rho\,\sigma_2 q^2 - q\tau_1\rho^2 - q\sigma_2 + \rho\,q\sigma_2 + \tau_1\rho^2 - \sigma_2\rho^2\right)\rho\big]\,/ \\ & \left(q^2\sigma_2 - q\tau_1\rho^2 + \tau_1\rho^2 - \sigma_2\rho^2\right)\left(-\sigma_2 + q\tau_1\rho^2 - \tau_1\rho^2 + \sigma_2\rho^2\right);\end{aligned}$$

$$\begin{aligned}\gamma_n \;=\;& \Big[(-1+\rho)\left(-q\tau_1\rho - \sigma_2\rho + \tau_1\rho + q^2\sigma_2\right) q\rho\,\left(q^2\sigma_2 - q\tau_1\rho^2 + \tau_1\rho^2 - \sigma_2\rho^2\right)^2\sigma_0 \\ - \;& (-1+\rho)\left(-q\tau_1\rho - \sigma_2\rho + \tau_1\rho + q^2\sigma_2 q^2\rho^2\,(-\rho\,\sigma_1 + \rho\,\tau_0 - q\rho\,\tau_0 + q\sigma_1)\right) \\ \times \;& \left(\tau_0 q^2\sigma_2 - q\sigma_1\tau_1\rho + \sigma_1 q\sigma_2 - \tau_0 q\sigma_2 + \rho\,\sigma_1\tau_1 - \rho\,\sigma_1\sigma_2\right)\Big]\,/ \\ & \left(q^2\sigma_2 - q\tau_1\rho^2 + \tau_1\rho^2 - \sigma_2\rho^2\right)^2\left(q^3\sigma_2 - q\tau_1\rho^2 - \sigma_2\rho^2 + \tau_1\rho^2\right)\left(q\sigma_2 - q\tau_1\rho^2 - \sigma_2\rho^2 + \tau_1\rho^2\right),\end{aligned}$$

where $\sigma(x) = \sigma_2\,x^2\sigma_1\,x + \sigma_0,\ \tau(x) = \tau_1\,x + \tau_0$ and $\rho = q^n$.

To conclude this subsection, we give data relative to the weight function $w(x)$, the polynomials σ and τ of the q-Pearson equation (2.67) satisfied by the weight function, as well as the recurrence coefficients β_n and γ_n and the q-hypergeometric representation (in the monic case) for each of the eighteen q-classical families. The orthogonality relations (also included) as well as the q-hypergeometric representation are taken from [39]; therefore, the assumption: $0 < q < 1$. It is worth pointing out that in the following data:

- The polynomials σ and τ and the constant λ_n are the coefficients of the second-order q-difference equation satisfied by each of the eighteen q-classical families

$$\sigma(x)\,D_q\,D_{\frac{1}{q}}\,P_n(x) + \tau(x)\,D_q\,P_n(x) + \lambda_n\,P_n(x) = 0;$$

- The coefficients β_n and γ_n are those of the recurrence relation of the monic form of the q-classical orthogonal polynomials

$$P_{n+1}(x) = (x - \beta_n)\,P_n(x) - \gamma_n\,P_{n-1}(x),\ n \geq 1,\ P_{-1} = 0,\ P_0(x) = 1.$$

  They are computed for each case via the proposition 2.3 using the coefficients of the polynomials σ and τ (of course they coincide with those given in [39]).

- The polynomials σ and τ are obtained by solving the q-Pearson equation

$$D_q(\sigma(x)\,w(x)) = \tau(x)\,w(x),$$

  for each of the eighteen weights (supposed known) with σ and τ as unknowns;

- The notations we use for the hypergeometric representation of q-classical families differ slightly (by the tilde) from those of [39] because here we deal with monic orthogonal polynomials.

1. **Big-q-Jacobi polynomials**

$$\tilde{P}_n(x;a,b,c;q) = \frac{(aq, c\,q;q)_n}{(abq^{n+1};q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, ab\,q^{n+1}, x \\ aq, cq \end{matrix} \middle| q;q \right), \quad w(x) = \frac{(\frac{x}{a}, \frac{x}{c};q)_\infty}{(x, \frac{bx}{c};q)_\infty},$$

$$\sigma(x) = (x-cq)\,(x-aq)\,, \tau(x) = \frac{xabq^2 - x + qc - abq^2 + aq - acq^2}{q-1},$$

$$\lambda_n = -\frac{(q^n abq - 1)\,q\,(q^n-1)}{q^n\,(q-1)^2},$$

$$\beta_n = \frac{\big(ab\,(a+c+ac+ab)\,q^{2n+1} - a\,(b+ab+cb+c)\,(q+1)\,q^n + a + c + ac + ab\big)\,q^{n+1}}{(abq^{2n+2}-1)\,(abq^{2n}-1)},$$

$$\gamma_n = \frac{a\,(q^n-1)\,(q^n ab-1)\,(q^n b-1)\,(q^n c-1)\,(q^n a-1)\,(q^n ab-c)\,q^{n+2}}{(abq^{2n+1}-1)\,(q-abq^{2n})\,(abq^{2n}-1)^2}.$$

The orthogonality relation is :

$$\int_{cq}^{aq} w(x)\,\tilde{P}_n(x;a,b,c;q)\tilde{P}_m(x;a,b,c;q)\,d_qx =$$
$$aq(1-q)\frac{(q, abq^2, a^{-1}c, ac^{-1}q;q)_\infty}{(aq, bq, cq, abc^{-1}q;q)_\infty}\left[\frac{(aq, cq;q)_n}{(abq^{n+1};q)_n}\right]^2$$
$$\times \quad \frac{(1-abq)}{(1-abq^{2n+1})}\,\frac{(q, bq, abc^{-1}q;q)_n}{(aq, abq, cq;q)_n}\,(-acq^2)^n\,q^{\binom{n}{2}}\,\delta_{n,m},$$

under the conditions $0 < a < q^{-1},\ 0 < b < q^{-1}$ and $c < 0$.

2. **q-Hahn polynomials**

$$\tilde{Q}_n(q^{-k};\alpha,\beta,N|q) = \frac{(\alpha\,q, q^{-N};q)_n}{(\alpha\beta\,q^{n+1};q)_n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha\beta\,q^{n+1}, q^{-k} \\ \alpha\,q\,q^{-N} \end{matrix} \middle| q;q \right),$$

$$w(q^{-k}) = \frac{\left(\alpha\,q, q^{-N};q\right)_k}{\left(q, q^{-N}\beta^{-1};q\right)_k}(\alpha\beta)^{-k},$$

$$\sigma(x) = (x-\alpha\,q)\,(x-q^{-N}),\ \tau(x) = \frac{x(\alpha\,\beta\,q^2-1) - \alpha\,\beta\,q^2 + \alpha\,q - \alpha\,qq^{-N} + q^{-N}}{q-1},$$

$$\lambda_n = -\frac{(q^n\alpha\,\beta\,q-1)\,q\,(q^n-1)}{q^n\,(q-1)^2},$$

$$\beta_n = \frac{\alpha\,\beta\,q^{3n+1}\left((\beta+1)\alpha\,q^{N+1} + \alpha + 1\right)}{(\alpha\,\beta\,q^{2n+2}-1)\,(-1+q^{2n}\alpha\,\beta)\,q^N}$$
$$+\frac{q^n\left(-\alpha\,(q+1)\left((\alpha+1)\beta\,q^{N+1} + 1 + \beta\right)q^n + (\beta+1)\alpha\,q^{N+1} + \alpha + 1\right)}{(\alpha\,\beta\,q^{2n+2}-1)\,(-1+q^{2n}\alpha\,\beta)\,q^N},$$

$$\gamma_n = \frac{\alpha\,q^{n-2N}\,(q^n-1)\,(q^n\alpha\,\beta-1)\,(-1+q^n\alpha)\,(q^n\beta-1)\left(-q^n+q^{N+1}\right)\left(q^{n+1}\alpha\,\beta\,q^N-1\right)}{(q^{2n}\alpha\,\beta\,q-1)\,(-q+q^{2n}\alpha\,\beta)\,(-1+q^{2n}\alpha\,\beta)^2}.$$

The orthogonality relation is :

$$\sum_{k=0}^{N} w(q^{-k})\,\tilde{Q}_n(q^{-k};\alpha,\beta,N|q)\,\tilde{Q}_m(q^{-k};\alpha,\beta,N|q)\,q^{-k} =$$

$$\frac{(\alpha\beta\,q^2;q)_N}{(\beta\,q;q)_N\,(\alpha\,q)^N}\,\frac{(q,\alpha\beta\,q^{N+2},\beta\,q;q)_n}{(\alpha\,q,\alpha\beta\,q,q^{-N};q)_N}\,\frac{(1-\alpha\beta\,q)\,(-\alpha\,q)^n}{(1-\alpha\beta\,q^{2n+1})}\,\left[\frac{(\alpha\,q,q^{-N};q)_n}{(\alpha\beta\,q^{n+1};q)_n}\right]^2,$$

under the conditions $0<\alpha<q^{-1}$ and $0<\beta<q^{-1}$ or $\alpha>q^{-N}$ and $\beta>q^{-N}$.

3. **Big q-Laguerre polynomials**

$$\tilde{P}_n(x;a,b;q) = (aq,bq;q)_n\,{}_3\phi_2\left(\begin{matrix} q^{-n},0,x \\ aq,bq \end{matrix}\middle|\, q;q\right),\ w(x) = \frac{(\frac{x}{a},\frac{x}{b};q)_\infty}{(x;q)_\infty},$$

$$\sigma(x) = (x-aq)\,(x-bq), \tau(x) = -\frac{x+abq^2-qb-aq}{q-1}, \lambda_n = \frac{q\,(q^n-1)}{q^n\,(q-1)^2},$$

$$\beta_n = -ab(q+1)q^{2n+1} + (ab+b+a)q^{n+1},$$

$$\gamma_n = baq^{n+1}(q^n b-1)(q^n a-1)(q^n-1).$$

The orthogonality relation is

$$\int_{bq}^{aq} w(x)\,\tilde{P}_n(x;a,b;q)\tilde{P}_m(x;a,b;q)\,d_qx =$$

$$aq(1-q)\,(-abq^2)^n\,(aq,bq;q)_n\,\frac{(q,a^{-1}b,ab^{-1}q;q)_\infty}{(aq,bq;q)_\infty}\,(q;q)_n\,q^{\binom{n}{2}}\,\delta_{n,m},$$

under the conditions $0<a<q^{-1}$ and $b<0$.

4. **Little q-Jacobi polynomials**

$$\tilde{p}_n(x;a,a|q) = \frac{(-1)^n\,q^{\binom{n}{2}}(aq,q)_n}{(abq^{n+1};q)_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n},-abq^{n+1} \\ aq \end{matrix}\middle|\, q;qx\right),\ w(q^k) = \frac{(bq;q)_k}{(q;q)_k}\,a^k,$$

$$\sigma(x) = x^2-x,\ \tau(x) = \frac{(abq^2-1)x}{q-1} + \frac{1-qa}{q-1},\ \lambda_n = -\frac{\left(abq^{n+1}-1\right)q\,(q^n-1)}{q^n\,(q-1)^2},$$

$$\beta_n = \frac{q^n\left(abq^{2n+1}\,(1+a) - a\,(1+b)\,(q+1)\,q^n + 1 + a\right)}{(abq^{2n+2}-1)\,(abq^{2n}-1)},$$

$$\gamma_n = \frac{aq^{2n}\,(q^n-1)\,(abq^n-1)\,(bq^n-1)\,(aq^n-1)}{(abq^{2n+1}-1)\,(abq^{2n}-q)\,(abq^{2n}-1)^2}.$$

The orthogonality relation is

$$\sum_{k=0}^{\infty} w(q^k)\,\tilde{p}_n(x;a,a|q)\,\tilde{p}_m(x;a,a|q)\,q^k =$$

$$\frac{(abq^2;q)_\infty}{(aq;q)_\infty}\,\frac{(1-abq)\,(aq)^n}{(1-abq^{2n+1})}\,\frac{(q,bq;q)_n}{(aq,abq;q)_n}\,\left[\frac{(aq;q)_n\,q^{\binom{n}{2}}}{(abq^{n+1};q)_n}\right]^2\,\delta_{n,m},$$

under the conditions $0<a<q^{-1}$ and $b<q^{-1}$.

5. **q-Meixner polynomials**

$$\tilde{M}_n(q^{-k};b,c;q) = (-1)^n\,(bq;q)_n\,c^n\,q^{-n^2}\,{}_2\phi_1\!\left(\begin{matrix} q^{-n},q^{-k} \\ bq \end{matrix}\,\middle|\, q;-\frac{q^{n+1}}{c}\right),$$

$$w(q^{-k}) = \frac{(bq;q)_k}{(q,-bcq;q)_k}\,c^k q^{\binom{k+1}{2}},$$

$$\sigma(x) = c(x-qb), \tau(x) = \frac{qx-c+bcq-q}{q-1},\ \lambda_n = -\frac{q\,(q^n-1)}{(q-1)^2},$$

$$\beta_n = (1-c-bc)q^{-n} + c(q+1)\,q^{-1-2n},\ \gamma_n = c\,(q^n b-1)\,(q^n+c)\,(q^n-1)\ q^{1-4n}.$$

The orthogonality relation is

$$\sum_{k=0}^{\infty} w(q^{-k})\,\tilde{M}_n(q^{-k};b,c;q)\,\tilde{M}_m(q^{-k};b,c;q)\,q^{-k} =$$
$$\frac{(-c;q)_\infty}{(-bcq;q)_\infty}\,(q,-c^{-1}\,q;q)_n\,(bq;q)_n\,q^{-n-2n^2}\,c^{2n}\,\delta_{n,m},$$

under the conditions $0<b<q^{-1}$ and $c>0$.

6. **Quantun q-Krawtchouk polynomials**

$$\tilde{K}_n^{qtm}(q^{-k};p,N;q) = (q^{-N};q)_n\,p^{-n}\,q^{-n^2}\,{}_2\phi_1\!\left(\begin{matrix} q^{-n},q^{-k} \\ q^{-N} \end{matrix}\,\middle|\, q;pq^{n+1}\right),$$

$$w(q^{-k}) = \frac{(pq;q)_{N-k}}{(q;q)_k\,(q;q)_{N-k}}\,(-1)^{N-k}\,q^{\binom{k+1}{2}},$$

$$\sigma(x) = -x+q^{-N},\ \tau(x) = \frac{pqx-q^{-N}-pq+1}{q-1},\ \lambda_n = -\frac{pq\,(q^n-1)}{(q-1)^2},$$

$$\beta_n = \frac{(p+1)\,q^{N+1}+1}{p\,q^{N+n+1}} - \frac{q+1}{p\,q^{2n+1}},\ \gamma_n = \frac{(pq^n-1)\,(q^{N+1}-q^n)\,(q^n-1)}{p^2\,q^{4n+N}}.$$

The orthogonality relation is

$$\sum_{k=0}^{N} w(q^{-k})\,\tilde{K}_n^{qtm}(q^{-k};p,N;q)\,\tilde{K}_m^{qtm}(q^{-k};p,N;q)\,q^{-k} =$$
$$\frac{(-1)^n\,p^N\,(q;q)_{N-n}\,(q,pq;q)_n}{(q,q;q)_N}\,q^{\binom{N+1}{2}-\binom{n+1}{2}+Nn}\left[(q^{-N};q)_n\,p^{-n}\,q^{-n^2}\right]^2\,\delta_{n,m},$$

under the condition $p>q^{-N}$.

7. **q-Krawtchouk polynomials**

$$\tilde{K}_n(q^{-k};p,N;q) = \frac{(q^{-N};q)_n}{(-pq^n;q)_n}\,{}_3\phi_2\!\left(\begin{matrix} q^{-n},q^{-k},-pq^n \\ q^{-N},0 \end{matrix}\,\middle|\, q;q\right),\ w(q^{-k}) = \frac{(q^{-N};q)_k}{(q;q)_k}\,(-p)^{-k}\,q^k,$$

$$\sigma(x) = px\left(-x+q^{-N}\right),\ \tau(x) = \frac{(p+q)x-q-pq^{-N}}{q-1},\ \lambda_n = -\frac{q\left(q^n+p\right)\left(q^n-1\right)}{q^n\left(q-1\right)^2},$$
$$\beta_n = \frac{q^{n-N}\left(\left(-p+q^N\right)q^{2n+1}+p\left(q+1\right)\left(q^{N+1}+1\right)q^n-pq\left(-p+q^N\right)\right)}{\left(q^{2n+1}+p\right)\left(pq+q^{2n}\right)},$$
$$\gamma_n = \frac{pq^{2n+1}\left(q^n-1\right)\left(pq+q^n\right)\left(-q^n+q^{N+1}\right)\left(q^{n+N}+p\right)}{\left(p+q^{2n}\right)\left(pq^2+q^{2n}\right)\left(pq+q^{2n}\right)^2q^{2N}}.$$

The orthogonality relation is

$$\sum_{k=0}^{N} w(q^{-k})\,\tilde{K}_n(q^{-k};p,N;q)\,\tilde{K}_m(q^{-k};p,N;q)\,q^{-k} =$$
$$\frac{(q,-pq^{N+1};q)_n}{(-p,q^{-N};q)_n}\,\frac{(1+p)}{(1+pq^{2n})}\left[\frac{(q^{-N};q)_n}{(-pq^n;q)_n}\right]^2(-pq;q)_N\,p^{-N}\,q^{-\binom{N+1}{2}}\,(-pq^{-N})^n\,q^{n^2}\,\delta_{n,m},$$

under the condition $p > 0$.

8. **Affine q-Krawtchouk polynomials**

$$\tilde{K}_n^{Aff}(q^{-k};p,N;q) = (pqq^{-N};q)_n\,{}_3\phi_2\left(\begin{matrix} q^{-n},0,-q^k \\ pq,q^{-N}\end{matrix}\,\middle|\,q;q\right),\ w(q^{-k}) = \frac{(pq;q)_k}{(q;q)_k\,(q;q)_{N-k}}\,p^{-k},$$
$$\sigma\left(x\right) = \left(x-pq\right)\left(x-q^{-N}\right),\ \tau\left(x\right) = \frac{-x+q^{-N}+pq-pq^{1-N}}{q-1},\ \lambda_n = \frac{\left(q^n-1\right)}{\left(q-1\right)^2}q^{1-n},$$
$$\beta_n = -p\left(q+1\right)q^{2n-N}+\left(pqq^N+1+p\right)q^{n-N},\ \gamma_n = \left(pq^n-1\right)\left(q^n-qq^N\right)\left(q^n-1\right)pq^{n-2N-1}.$$

The orthogonality relation is

$$\sum_{k=0}^{N} w(q^{-k})\,\tilde{K}_n^{Aff}(q^{-k};p,N;q)\,\tilde{K}_m^{Aff}(q^{-k};p,N;q)\,q^{-k} =$$
$$(pq)^{n-N}\,\frac{(q;q)_n\,(q;q)_{N-n}}{(pq;q)_n\,(q;q)_N}\,\left[(pq,q^{-N};q)_n\right]^2\,\delta_{n,m},\quad 0<p<q^{-1}.$$

9. **Little q-Laguerre / Wall polynomials**

$$\tilde{p}_n(x;a|q) = (-1)^n\,q^{\binom{n}{2}}\,(aq;q)_n\,{}_2\phi_1\left(\begin{matrix} q^{-n},0 \\ aq\end{matrix}\,\middle|\,q;qx\right),\ w(q^k) = \frac{a^k}{(q;q)_k},$$
$$\sigma\left(x\right) = x\left(x-1\right),\tau\left(x\right) = \frac{-x+1-aq}{q-1},\ \lambda_n = \frac{q\left(q^n-1\right)}{q^n\left(q-1\right)^2},$$
$$\beta_n = -a\left(q+1\right)q^{2n}+\left(1+a\right)q^n,\ \gamma_n = a\,q^{2n-1}\left(q^na-1\right)\left(q^n-1\right).$$

The orthogonality relation is

$$\sum_{k=0}^{\infty} w(q^k)\, \tilde{p}_n(q^k; a|q)\, \tilde{p}_m(q^k; a|q)\, q^k =$$
$$\frac{(q;q)_n}{(aq;q)_\infty}\, (aq;q)_n\, (aq)^n\, q^{n(n-1)}\, \delta_{n,m}, \quad 0 < a < q^{-1}.$$

10. q**-Laguerre polynomials**

$$\tilde{L}_n^{(\alpha)}(x;q) = (-1)^n\, q^{-n\,(n+\alpha)}\, {}_2\phi_1\!\left(\begin{matrix} q^{-n}, -x \\ 0 \end{matrix} \middle| q; q^{n+\alpha+1} \right),\ w(x) = \frac{x^\alpha}{(-x;q)_\infty},$$
$$\sigma(x) = x,\ \tau(x) = \frac{q^{\alpha+1}\, x + q^{\alpha+1} - 1}{q-1},\ \lambda_n = -\frac{q^\alpha q\,(q^n-1)}{(q-1)^2},$$
$$\beta_n = -(q^\alpha+1)\, q^{-n-\alpha} + (q+1)\, q^{-2n-\alpha-1},\ \gamma_n = \left(q^{n+\alpha}-1\right)(q^n-1)\, q^{1-4n-2\alpha}.$$

The orthogonality relations are

$$\int_0^\infty w(x)\, \tilde{L}_n^{(\alpha)}(x;q)\, \tilde{L}_m^{(\alpha)}(x;q)\, dx =$$
$$\frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\, (q^{\alpha+1};q)_n\, (q;q)_n\, q^{-n-2n(n+\alpha)}\, \Gamma(-\alpha)\, \Gamma(\alpha+1)\, \delta_{n,m},$$

under the conditions $\alpha > -1$, or

$$\sum_{k=-\infty}^{\infty} w(cq^k)\, \tilde{L}_n^{(\alpha)}(cq^k;q)\, \tilde{L}_m^{(\alpha)}(cq^k;q)\, q^k =$$
$$\frac{(q, -cq^{\alpha+1}, -c^{-1}\, q^{-\alpha}; q)_\infty}{(q^{\alpha+1}, -c, -c^{-1}q; q)_\infty}\, (q^{\alpha+1};q)_n\, (q;q)_n\, q^{-n-2n(n+\alpha)}\, c^{-\alpha}\, \delta_{n,m},$$

under the conditions $\alpha > -1$ and $c > 0$.

11. **Alternative** q**-Charlier polynomials**

$$\tilde{K}_n(x;a;q) = \frac{(-1)^n\, q^{\binom{n}{2}}}{(-a\, q^n;q)_n}\, {}_2\phi_1\!\left(\begin{matrix} q^{-n}, -a\, q^n \\ 0 \end{matrix} \middle| q; qx \right),\ w(q^k) = \frac{a^k\, q^{\binom{k}{2}}}{(q;q)_k},$$
$$\sigma\,(x) = x\,(x-1)\,,\ \tau\,(x) = \frac{-(qa+1)x+1}{q-1},\ \lambda_n = \frac{(aq^n+1)\,(q^n-1)}{q^{n-1}\,(q-1)^2},$$
$$\beta_n = -\frac{q^n\left(aq^{2n+1} - a\,(q+1)\,q^n - q\right)}{(aq^{2n+1}+1)\,(aq^{2n}+q)},\ \gamma_n = \frac{aq^{3n+1}\,(1-q^n)\,(aq^n+q)}{(1+aq^{2n})\,(a\,q^{2n}+q^2)\,(aq^{2n}+q)^2}.$$

The orthogonality relation is

$$\sum_{k=0}^{\infty} w(q^k)\, \tilde{K}_n(q^k;a;q)\, \tilde{K}_m(q^k;a;q)\, q^k =$$
$$(q;q)_n\,(-aq^n;q)_\infty\, \frac{a^n\, q^{\binom{n+1}{2}}}{(1+aq^{2n})} \left[\frac{q^{\binom{n}{2}}}{(-aq^n;q)_n}\right]^2 \delta_{n,m},$$

with $a > 0$.

12. **q-Charlier polynomials**

$$\tilde{C}_n(q^{-k};a;q) = a^n\,(-1)^n\, q^{-n^2}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-k} \\ 0 \end{matrix}\,\middle|\, q; -\frac{q^{n+1}}{a}\right),\ w(q^{-k}) = \frac{a^k\, q^{\binom{k+1}{2}}}{(q;q)_k},$$
$$\sigma(x) = a\,x,\ \tau(x) = \frac{qx-a-q}{q-1}, \lambda_n = \frac{(1-q^n)\,q}{(q-1)^2},$$
$$\beta_n = (1-a)q^{-n} + a(q+1)q^{-1-2n},\ \gamma_n = a(a+q^n)(1-q^n)q^{1-4n}.$$

The orthogonality relation is

$$\sum_{k=0}^{\infty} w(q^{-k})\, \tilde{C}_n(q^{-k};a;q)\, \tilde{C}_m(q^{-k};a;q)\, q^{-k} =$$
$$q^{-n-2n^2}\, a^{2n}\,(-a;q)_\infty\,(-a^{-1}q,q;q)_n\, \delta_{n,m},\quad a > 0.$$

13. **Al-Salam-Carlitz I polynomials**

$$\tilde{U}n^{(a)}(x;q) = (-a)^n\, q^{\binom{n}{2}}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, \frac{1}{x} \\ 0 \end{matrix}\,\middle|\, q; \frac{qx}{a}\right),\ w(x) = \left(qx, \frac{qx}{a}; q\right)_\infty,$$
$$\sigma(x) = (x-1)\,(x-a)\,,\ \tau(x) = \frac{-x+a+1}{q-1},$$
$$\lambda_n = \frac{(q^n-1)}{q^{n-1}\,(q-1)^2},\ \beta_n = (1+a)\,q^n,\ \gamma_n = aq^{n-1}\,(q^n-1)$$

The orthogonality relation is

$$\int_a^1 w(x)\, \tilde{U}n^{(a)}(x;q)\, \tilde{U}m^{(a)}(x;q)\, d_qx =$$
$$(-a)^n\,(1-q)\,(q;q)_n\,(q,a,q^{-1}q;q)_\infty\, q^{\binom{n}{2}}\, \delta_{n,m},\quad a < 0.$$

14. **Al-Salam-Carlitz II polynomials**

$$\tilde{V}n^{(a)}(x;q) = (-a)^n\, q^{-\binom{n}{2}}\, {}_2\phi_0\left(\begin{matrix} q^{-n}, x \\ - \end{matrix}\,\middle|\, q; \frac{q^n}{a}\right),\ w(q^{-k}) = \frac{a^k\, q^{k^2+k}}{(q;q)_k\,(aq;q)_k},$$

$$\sigma(x) = a, \tau(x) = \frac{x-1-a}{q-1},$$

$$\lambda_n = \frac{1-q^n}{(q-1)^2},\ \beta_n = (1+a)\,q^{-n},\ \gamma_n = a\,(1-q^n)\,q^{1-2n}.$$

The orthogonality relation is

$$\sum_{k=0}^{\infty} w(q^{-k})\,\tilde{V}n^{(a)}(q^{-k};q)\,\tilde{V}m^{(a)}(q^{-k};q)\,q^{-k} = \frac{(q;q)_n\,a^n}{(aq;q)_\infty\,q^{n^2}}\,\delta_{n,m},\quad a>0.$$

15. **Stieltjes - wigert polynomials**

$$\tilde{S}_n(x;q) = (-1)^n\, q^{-n^2}\, {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}\,\middle|\, q; -q^{n+1}\,x\right),\ w(x) = \frac{1}{\left(-x, -\frac{q}{x}; q\right)_\infty},$$

$$\sigma(x) = x,\ \tau(x) = \frac{qx-1}{q-1},$$

$$\lambda_n = \frac{q\,(1-q^n)}{(q-1)^2},\ \beta_n = -q^{-n} + (q+1)\,q^{-2n-1},\ \gamma_n = (1-q^n)\,q^{1-4n}.$$

The orthogonality relation is

$$\int_0^\infty w(x)\,\tilde{S}_n(x;q)\,\tilde{S}_m(x;q)\,dx = -\ln q\,q^{-n-2n^2}\,(q;q)_\infty\,(q;q)_n\,\delta_{n,m}.$$

16. **Discrete q-Hermite I polynomials**

$$\tilde{h}_n(x;q) = \tilde{U}_n^{(-1)}(x;q) = q^{\binom{n}{2}}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, \frac{1}{x} \\ 0 \end{matrix}\,\middle|\, q; -qx\right),\ w(x) = (qx, -qx; q)_\infty\,,$$

$$\sigma(x) = (x-1)\,(x+1)\,,\ \tau(x) = \frac{-x}{q-1},$$

$$\lambda_n = \frac{(q^n-1)}{q^{n-1}\,(q-1)^2},\ \beta_n = 0,\ \gamma_n = q^{n-1}\,(1-q^n)\,.$$

The orthogonality relation is

$$\int_0^\infty w(x)\,\tilde{h}_n(x;q)\,\tilde{h}_m(x;q)\,d_qx = (1-q)\,(q;q)_n\,(q,-1,-q;q)_\infty\,q^{\binom{n}{2}}\,\delta_{n,m}.$$

17. **Discrete q-Hermite II polynomials**

$$\tilde{\tilde{h}}_n(x;q) = i^{-n}\,\tilde{V}n^{(-1)}(ix;q) = i^{-n}\,q^{-\binom{n}{2}}\,{}_2\phi_0\left(\begin{matrix} q^{-n}, ix \\ - \end{matrix}\,\middle|\, q; -q^n\right) = x^n\,{}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-n+1} \\ 0 \end{matrix}\,\middle|\, q^2; -\frac{q^2}{x^2}\right)$$

$$w(q^k;q) = \frac{1}{(-q^{2k};q^2)_\infty},$$

$$\sigma(x) = 1, \tau(x) = \frac{x}{q-1}, \lambda_n = \frac{1-q^n}{(q-1)^2},\ \beta_n = 0,\ \gamma_n = (1-q^n)\,q^{1-2n}.$$

The orthogonality relation is

$$\sum_{k=-\infty}^{\infty} \left[\tilde{\tilde{h}}_n(cq^k;q)\,\tilde{\tilde{h}}_m(cq^k;q) + \tilde{\tilde{h}}_n(-cq^k;q)\,\tilde{\tilde{h}}_m(-cq^k;q)\right]\,w(cq^k;q)\,q^k =$$
$$2\frac{(q^2,-c^2q,-c^{-2}\,q;q^2)_\infty}{(q,-c^2,-c^{-2}\,q^2;q^2)_\infty}\,\frac{(q;q)_n}{q^{n^2}}\,\delta_{n,m},\quad c>0.$$

It should be mentioned that the relation $\tilde{\tilde{h}}_n(-x;q) = (-1)^n\,\tilde{\tilde{h}}_n(x;q)$ allows to transform the previous orthogonality relation into

$$\sum_{k=-\infty}^{\infty} \tilde{\tilde{h}}_n(cq^k;q)\,\tilde{\tilde{h}}_m(cq^k;q)\,w(cq^k;q)\,q^k = \frac{(q^2,-c^2q,-c^{-2}\,q;q^2)_\infty}{(q,-c^2,-c^{-2}\,q^2;q^2)_\infty}\,\frac{(q;q)_n}{q^{n^2}}\,\delta_{n,m},\quad c>0.$$

18. **q-Charlier II polynomials [41]**

$$\tilde{C}_n(q^{-k};u;q) = \frac{(-1)^n\,q^{-\binom{n+1}{2}}\,(q;q)_n}{(q^{-n},uq^n;q)_n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-k}, u\,q^n \\ 0, 0 \end{matrix}\,\middle|\, q;q\right),\ w(q^{-k}) = \frac{1}{(q;q)_k}\,(\frac{q}{u})^k,$$

$$\sigma(x) = x^2, \tau(x) = \frac{(qu-1)x - qu}{q-1},\ \lambda_n = \frac{(q^nu-1)\,(1-q^n)}{q^{n-1}\,(q-1)^2},$$

$$\beta_n = \frac{u\,q^{n+1}\left((q+1)\,q^n - 1 - u\,q^{2n}\right)}{(u\,q^{2n+1}-1)\,(q-u\,q^{2n})},\ \gamma_n = \frac{u^2\,q^{3n+2}\,(q-q^nu)\,(1-q^n)}{(u\,q^{2n}-1)\,(q^2-u\,q^{2n})\,(q-u\,q^{2n})^2}.$$

The orthogonality relation is

$$\sum_{k=0}^{\infty} w(q^{-k})\,\tilde{C}_n(q^{-k};u;q)\,\tilde{C}_m(q^{-k};u;q)\,q^{-k} = (-1)^n\,u^{2n}\,\frac{(q,u;q)_n}{(u;uq;q)_{2n}}\,q^{\frac{n\,(3n+1)}{2}}\,\delta_{n,m}.$$

2.5.4 Classical orthogonal polynomials of a discrete variable on a quadratic and a q-quadratic lattice

Introduction

The very classical orthogonal polynomials satisfy a second-order differential, difference or q-difference equation (see equations (1.8)-(1.10)). These equations have a special property: The derivative of any solution of the equation (1.8) satisfies an equation of the same type. The same applies for equations (1.9) and (1.10) with the usual derivative replaced by the difference or the q-difference derivative. Second-order differential, difference or q-difference equations with such properties are said to be of hypergeometric type. This property is crucial for the derivation of some specific properties of the very classical orthogonal polynomials such as the orthogonality of the derivatives and the Rodrigues formula.

Some families of orthogonal polynomials such as those of q-Racah [8] and Askey-Wilson [9] defined respectively by

$$R_n(\mu(s);a,b,c,N;q) = {}_4\phi_3\left(\begin{array}{c} q^{-n}, abq^{n+1}, q^{-s}, cq^{s-N} \\ aq, bcq, q^{-N} \end{array}\middle| q;q\right), \quad \mu(s) = q^{-s} + cq^{s-N}, \tag{2.70}$$

$$\frac{p_n(x;a,b,c,d|q)}{(ab,ac,ad;q)_n\, a^{-n}} = {}_4\phi_3\left(\begin{array}{c} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{array}\middle| q;q\right), \quad x = \cos\theta, \tag{2.71}$$

do not fulfil any of the three equations satisfied by the very classical orthogonal polynomials (see equations (1.8)-(1.10)). However, Askey and Wilson [9] obtained a difference equation as well as a Rodrigues-type formula for the Askey-Wilson polynomials (of course different from those of the very classical orthogonal polynomials) this using special properties of the ${}_4\phi_3$ polynomials in (2.71).

Andrews and Askey [3] (see also [10]) observed that all classical orthogonal polynomials with absolutely continuous measures can be obtained as special or limiting cases of the Askey-Wilson polynomials, and all the classical orthogonal polynomials with discrete positive measures that include Hahn, dual Hahn, Krawtchouk, Meixner, Charlier polynomials and their various q-analogues can be obtained as special or limiting cases of the q-Racah polynomials. From this remark, they suggested the following definition of the classical orthogonal polynomials:

Definition 2.6 *An orthogonal polynomial sequence is classical if it is a special or limiting cases of the ${}_4\phi_3$ polynomials given by the q-Racah polynomials in (2.70) or the Askey-Wilson polynomials in (2.71).*

This definition was reformulated in [10] like those of the very classical orthogonal polynomials, in terms of second-order difference equations. The aim of this subsection is mainly to review the characterization theorem of classical orthogonal polynomials, and use it later to improve the definition of classical orthogonal polynomials stated in [10].

Characterization theorem

Since the q-Racah as well as the Askey-Wilson polynomials are both special cases of the hypergeometric or the basic hypergeometric functions, it is natural to look for an equation which extends the differential, the difference or the q-difference equations (1.8)-(1.10), and whose solutions have the hypergeometric property and contain the q-Racah and the Askey-Wilson polynomials as special cases.

For this extension, Nikiforov, Uvarov and Suslov in their various works (see [10, 51] and references therein) have replaced the differential equation (1.8) by the difference equation of the form

$$\frac{\tilde\sigma(x(s))}{x(s+\frac{h}{2})-x(s-\frac{h}{2})}\left[\frac{y(s+h)-y(s)}{x(s+h)-x(s)}-\frac{y(s)-y(s-h)}{x(s)-x(s-h)}\right] \qquad (2.72)$$
$$+\frac{\tilde\tau(x(s))}{2}\left[\frac{y(s+h)-y(s)}{x(s+h)-x(s)}+\frac{y(s)-y(s-h)}{x(s)-x(s-h)}\right]+\lambda\, y(s)=0,$$

where $x(s)$ which is a continuously differentiable function of s on some domain of the complex plane, defines a lattice with variable mesh-size $\Delta x = x(s+h)-x(s)$. $\tilde\sigma(x(s))$ and $\tilde\tau(x(s))$ are polynomials in $x(s)$ of degree at most 2 and 1 respectively. It can be shown that for small h, the previous equation approximates (1.8) to second-order in h. The latter equation, by rescaling the variable s, is equivalent to

$$\tilde\sigma(x(s))\,\frac{\Delta}{\Delta\, x(s-\frac{1}{2})}\,\frac{\nabla\, y(s)}{\nabla\, x(s)}+\frac{\tilde\tau(x(s))}{2}\left[\frac{\Delta\, y(s)}{\Delta\, x(s)}+\frac{\nabla\, y(s)}{\nabla\, x(s)}\right]+\lambda\, y(s)=0 \qquad (2.73)$$

where the operators Δ and ∇ are defined in (1.13).

Use of the notations

$$x_\mu(x)=x(s+\frac{\mu}{2}),\ \mu\in\mathbb{C},\ \ v_0(s)\equiv\ y(s),\ \ v_k(s)=\frac{\Delta\, v_{k-1}(s)}{\Delta\, x_{k-1}(s)},\ k=1,\,2,\,\ldots, \qquad (2.74)$$

as well as the relations

$$\begin{cases} \frac{1}{2}\left[\frac{\Delta\, y(s)}{\Delta\, x(s)}+\frac{\nabla\, y(s)}{\nabla\, x(s)}\right]=\frac{\Delta\, y(s)}{\Delta\, x(s)}-\frac{1}{2}\,\Delta\,\left[\frac{\nabla\, y(s)}{\nabla\, x(s)}\right] \\ \\ \frac{1}{2}\left[\frac{\Delta\, y(s)}{\Delta\, x(s)}+\frac{\nabla\, y(s)}{\nabla\, x(s)}\right]=\frac{\nabla\, y(s)}{\nabla\, x(s)}+\frac{1}{2}\,\Delta\,\left[\frac{\nabla\, y(s)}{\nabla\, x(s)}\right] \end{cases} \qquad (2.75)$$

transform equation (2.73) in a more suggestive equivalent form:

$$\sigma(s)\,\frac{\nabla\, v_1(s)}{\nabla\, x_1(s)}+\tau(s)\, v_1(s)+\lambda\, v_0(s)=0, \qquad (2.76)$$

where

$$\tau(s)\equiv\tau(x(s))=\tilde\tau(x(s)),\ \sigma(s)=\tilde\sigma(x(s))\,-\frac{1}{2}\,\tilde\tau(x(s))\,\nabla\, x_1(s). \qquad (2.77)$$

If we apply the difference-derivative operator $\frac{\Delta}{\Delta\, x(s)}$ on (2.76) and use the relation

$$\Delta\,\left[f(s)\, g(s)\right]=f(s+1)\,\Delta\, g(s)+f(s)\,\Delta\, g(s), \qquad (2.78)$$

we obtain

$$\sigma_1(s)\,\frac{\nabla\, v_2(s)}{\nabla\, x_2(s)}+\tau_1(s)\, v_2(s)+\mu_1\, v_1(s)=0, \qquad (2.79)$$

where

$$\begin{aligned} \sigma_1(s) &= \sigma(s), \\ \tau_1(s) &= \frac{\sigma(s+1)-\sigma(s)+\tau(s+1)\,\nabla\, x_1(s+1)}{\nabla\, x_2(s)}, \\ \mu_1 &= \lambda+\frac{\Delta\,\tau(s)}{\Delta\, x(s)}. \end{aligned} \qquad (2.80)$$

Successive applications of the difference operators $\frac{\Delta}{\Delta x_j(s)}$, $j = 0,\ 1\ \dots,\ k$ to (2.76) lead to

$$\sigma_k(s)\, \frac{\nabla v_{k+1}(s)}{\nabla x_{k+1}(s)} + \tau_k(s)\, v_{k+1}(s) + \mu_k\, v_k(s) = 0, \tag{2.81}$$

where

$$\begin{aligned} \sigma_k(s) &= \sigma(s), \\ \tau_k(s) &= \frac{\sigma(s+k) - \sigma(s) + \tau(s+k)\, \nabla x_1(s+k)}{\nabla x_{k+1}(s)}, \\ \mu_k &= \lambda + \sum_{j=0}^{k-1} \frac{\Delta\, \tau_j(s)}{\Delta\, x_j(s)},\ \ \tau_0 \equiv \tau,\ \ \mu_0 \equiv \lambda. \end{aligned} \tag{2.82}$$

For equation (2.76) to be called *the difference equation of hypergeometric type*, it should have a property, analogous to the one of the differential equation of hypergeometric type, of retaining its form after differentiation. For this extent, Equations (2.76) and (2.81) should be of the same type, that is, the coefficient $\tau_k(s)$ of (2.81) should be polynomial (in the variable $x_k(s)$) of degree at most 1, and μ_k a constant. This requirement, which is in fact equivalent to impose to the lattice $x(s)$ to satisfy two difference equations (see [10], eq. 2.1 and 2.2) is the key to the following theorem characterizing the classical orthogonal polynomials of a discrete variable on nonuniform lattices.

Theorem 2.11 *Let λ be independent of s. Let $\sigma(s)$ be defined by (2.77) where $\tilde\sigma(s)$ and $\tau(s) = \tilde\tau(s)$ are polynomials (in the variable $x(s)$) of degree at most 2 and 1 respectively. Then (2.76) has polynomial solutions, and is of hypergeometric type, that is, its successive difference-derivatives $v_k(s)$ satisfy equations of the same kind, namely, (2.81) with constant μ_k, if and only if $x(s)$ is a linear, q-linear, quadratic, or q-quadratic lattice of the form*

$$x(s) = \begin{cases} C_1\, q^{-s} + C_2\, q^s & \text{if} \quad q \neq 1, \\ C_3\, s^2 + C_4\, s & \text{if} \quad q = 1, \end{cases} \tag{2.83}$$

where $q \in \mathbb{C}$ and $C_1,\ C_2,\ C_3,\ C_4$ are arbitrary constants such that $(C_1, C_2) \neq (0,0)$ and $(C_3, C_4) \neq (0,0)$.

The proof of this theorem, which can be found in [10] is based on the following lemma:

Lemma 2.1 *Let $\tilde\sigma(s)$ and $\tau(s) = \tilde\tau(x(s))$ be polynomials (in the variable $x(s)$) of degree at most 2 and 1, respectively, such that $\sigma(s)$ and $\tau(s)$ defined by (2.77) are not both constants. Then, $\tau_k(s)$ in (2.82) is linear in $x_k(s)$ if and only if constants α and β exist such that the lattice $x(s)$ satisfies the conditions*

$$x(s+k) - x(s) = \gamma_k\, \nabla x_{k+1}(s), \tag{2.84}$$

$$\frac{x(s+k) + x(s)}{2} = \alpha_k\, x_k(s) + \beta_k, \tag{2.85}$$

for $k = 0,\ 1,\ \dots,$ with

$$\alpha_0 = 1,\ \alpha_1 = \alpha,\ \beta_0 = 0,\ \beta_1 = \beta, \gamma_0 = 0,\ \gamma_1 = 1, \tag{2.86}$$

and the sequences (α_k), (β_k), (γ_k) *satisfy the following relations*

$$\begin{aligned} \alpha_{k+1} - 2\,\alpha\,\alpha_k + \alpha_{k-1} &= 0, \\ \beta_{k+1} - 2\,\beta_k + \beta_{k-1} &= 2\,\beta\,\alpha_k, \\ \gamma_{k+1} - \gamma_{k-1} &= 2\,\alpha_k, \end{aligned} \tag{2.87}$$

for $k = 0,\, 1,\, \ldots$.

The following remark gives the general solutions for the lattices $x(s)$ as well as the expressions of α_k, β_k and γ_k in terms of the constants α and β.

Remark 2.7.

1. *If we replace* s *successively by* $s - \frac{k+1}{2}$ *and* $s + \frac{k-1}{2}$ *in (2.85) and sum the two equations obtained for* $k = 1$*, we get*

$$x(s+1) + 2\,x(s) + x(s-1) = 2\,\alpha\,(x(s+\frac{1}{2}) + x(s-\frac{1}{2}) + 4\,\beta.$$

Use of (2.85) for $k = 1$ *and* s *replaced by* $s - \frac{1}{2}$ *together with the previous relation, yields the second-order difference equation for the lattice* $x(s)$

$$x(s+1) - 2\,(2\,\alpha^2 - 1)\,x(s) + x(s-1) = 4\,\beta\,(\alpha+1),$$

whose general solution is given by

$$x(s) = \begin{cases} C_5\,q^{-s} + C_6\,q^{s} + \frac{\beta}{1-\alpha} & \text{if} \quad \alpha \neq 1, \\ 4\,\beta\,s^2 + C_7\,s + C_8 & \text{if} \quad \alpha = 1, \end{cases} \tag{2.88}$$

where α, β *and* q *are related through (2.85)-(2.86), and* C_5, C_6, C_7, C_8 *are arbitrary constants. The constants* $\frac{\beta}{1-\alpha}$ *and* C_8 *play no essential role in the solution of* $x(s)$*, therefore they can be omitted without loss of generality.*

2. *If we assume that* $q \neq 0$ *and* $q \neq 1$*, the coefficients* α_k, β_k *and* γ_k *that satisfy the second-order difference equations (2.87) with constant coefficients can be expressed as*

$$\alpha_k = \begin{cases} 1 \\ \frac{1}{2}(q^{\frac{k}{2}} + q^{-\frac{k}{2}}) \end{cases}, \quad \beta_k = \begin{cases} \beta\,k^2 \\ \beta\left(\frac{q^{\frac{k}{4}} - q^{-\frac{k}{4}}}{q^{\frac{1}{4}} - q^{-\frac{1}{4}}}\right)^2 \end{cases}, \quad \gamma_k = \begin{cases} k \\ \frac{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \end{cases},$$

with the first line of the array valid for $\alpha = 1$ *and the second one valid for*

$$\alpha = \frac{1}{2}(q^{\frac{1}{2}} + q^{-\frac{1}{2}}).$$

After having stated and proved the characterization theorem, Atakishiyev, Rahman and Suslov [10] proposed the following definition:

Definition 2.7 *An orthogonal polynomial sequence* $(P_n(x(s))$ *on the interval* $(x(a), x(b))$ *is classical if and only if :*

1. *$P_n(x(s))$ satisfies a difference equation of the form (2.73) with $x(s)$ given by (2.83);*

2. *A positive weight function $\rho(s)$ satisfying the Pearson-type difference equation*

$$\frac{\nabla(\sigma(s+1)\,\rho(s+1))}{\nabla\, x_1(s)} = \tau(s)\,\rho(s); \tag{2.89}$$

   exists.

3. *The boundary conditions*

$$\sigma(s)\,\rho(s)\,x^k(s-\frac{1}{2})\bigg|_{s=a,b} = 0 \tag{2.90}$$

   hold for $k = 0,\, 1,\, 2,\, \ldots$ (in the discrete case).

This definition covers the q-Racah polynomials as well as the Askey-Wilson polynomials and their limiting and special cases. In short, it covers the classical orthogonal polynomials which consist mainly of 44 families of orthogonal polynomials among which are covered the 26 families of very classical orthogonal polynomials already treated in this chapter. For more details, the reader is referred to the next chapter.

On top of the definition of the classical orthogonal polynomials, we define the classical orthogonal polynomials on quadratic lattices as the families of classical orthogonal polynomials for which the lattice is quadratic (i.e. excluding the very classical orthogonal polynomials).

In connection with the classical orthogonal polynomials (P_n) defined above, there is an important function called function of second kind denoted Q_n and connected to P_n in the following way [59]

$$Q_n(x(z)) = \frac{1}{\rho(z)} \sum_{s=a}^{b-1} \frac{P_n(x(s))\,\rho(s)\,\nabla x_1(s)}{x(s)-x(z)}, \quad z \neq a,\, a+1,\, \ldots,\, b-1. \tag{2.91}$$

Also, the first associated $P_n^{(1)}$ of P_n, defined by the recurrence relation (2.11) for $r = 1$ is related to P_n by

$$P_n^{(1)}(x(z)) = \sum_{s=a}^{b-1} \frac{[P_{n+1}(x(s)) - P_{n+1}(x(z))]\,\rho(s)\,\nabla x_1(s)}{x(s)-x(z)}, \quad z \neq a,\, a+1,\, \ldots,\, b-1,$$

for the discrete orthogonality.

The following properties [59] will be used in the next chapter.

Theorem 2.12 *The function Q_n obeys:*

1. *$\forall n \in \mathbb{N}_0$, P_n and Q_n are two linearly independent solutions of (2.73).*

2. *P_n and Q_n are the two linearly independent solutions of the recurrence relation*

$$X_{n+1}(x(s)) = (x(s)-\beta_n)\,X_n(x(s)) - \gamma_n\,X_{n-1}(x(s)),\ n \geq 1.$$

3. *P_n and Q_n and the first associated $P_n^{(1)}$ are related by*

$$Q_n(x(s)) = P_n(x(s))\, Q_0(x(s)) + \frac{\gamma_0}{\rho(s)}\, P_{n-1}^{(1)}(x(s)),\ n \geq 1,\ s \neq a,\, a+1,\, \ldots,\, b-1. \quad (2.92)$$

4. *Q_n fulfils the following asymptotic property*

$$Q_n(x(s)) = -\frac{\prod_{i=0}^{n} \gamma_i}{\rho(s)\, x^{n+1}(s)} \left[1 + O\left(\frac{1}{x(s)}\right)\right],\ \ x(s) \to \infty, \quad (2.93)$$

with

$$\gamma_0 = \sum_{s=0}^{N} \rho(s)\, \nabla x_1(s) \ \text{ or } \ \gamma_0 = \int_C \rho(s)\, \nabla x_1(s)\, ds,$$

for the discrete and the continuous orthogonality respectively (see definition 3.1).

Chapter 3

Fourth-order difference equations

In this chapter, we simplify the definition of the classical orthogonal polynomials given in the previous chapter and provide data for classical orthogonal polynomials on quadratic and q-quadratic lattices [39], excluding the case with complex derivative. More precisely, we identify the lattice and compute the polynomials $\tilde{\sigma}$ and $\tilde{\tau}$ and λ_n of (2.73). Also, we compute the recurrence coefficients β_n and γ_n in terms of $\tilde{\sigma}$, $\tilde{\tau}$, α, β and δ; state and prove some intermediate results which will be used later to establish the fourth-order difference equations for modifications of the classical orthogonal polynomials.

3.1 Intermediate results

Let $x(s)$ be a lattice satisfying equations

$$\begin{aligned} x(s+k)-x(s) &= \gamma_k \nabla x_{k+1}(s), && (3.1)\\ \frac{x(s+k)+x(s)}{2} &= \alpha_k\, x_k(s)+\beta_k, && (3.2)\end{aligned}$$

for $k=0,\,1,\,\ldots,$ with

$$\alpha_0=1,\ \alpha_1=\alpha,\ \beta_0=0,\ \beta_1=\beta, \gamma_0=0,\ \gamma_1=1, \tag{3.3}$$

and the sequences $(\alpha_k),\ (\beta_k),\ (\gamma_k)$ satisfy the following relations

$$\begin{aligned} \alpha_{k+1}-2\,\alpha\,\alpha_k+\alpha_{k-1} &= 0,\\ \beta_{k+1}-2\,\beta_k+\beta_{k-1} &= 2\,\beta\,\alpha_k,\\ \gamma_{k+1}-\gamma_{k-1} &= 2\,\alpha_k,\end{aligned} \tag{3.4}$$

for $k=0,\,1,\,\ldots.$ Such a lattice has the property

$$x(s+1)^2+x(s)^2=2\,A_2(x_1(s))=a_2\,x_1(s)^2+a_1\,x_1(s)+a_0, \tag{3.5}$$

where the a_j are constants (to be found later in (3.17)), with $a_2\neq 0$.

3.1.1 The operators $\mathbb{D}_x$ and $\mathbb{S}_x$

We define the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ by

$$\mathbb{D}_x\, f(x(s)) = \frac{\Delta\, f(x(s))}{\Delta\, x(s)}, \quad \mathbb{S}_x\, f(x(s)) = \frac{f(x(s+1)) + f(x(s))}{2}, \tag{3.6}$$

and outline some of their properties.

Proposition 3.1 *If $P_n(x(s))$ is a polynomial of degree $n \geq 1$ in $x(s)$, then*

$$\mathbb{D}_x(P_n(x(s))) = q_{n-1}(x_1(s)), \quad \mathbb{S}_x(P_n(x(s))) = r_n(x_1(s)), \tag{3.7}$$

where q_{n-1} and r_n are polynomials of degree $n-1$ and n respectively.

More generally, we have

$$\mathbb{D}_{x_\mu}(P_n(x_\mu(s))) = q_{n-1}(x_{\mu+1}(s)), \quad \mathbb{S}_{x_\mu}(P_n(x_\mu(s))) = r_n(x_{\mu+1}(s)). \tag{3.8}$$

Proof: First, for fixed $n \geq 1$, we write the relation

$$\Delta(f(s)\, g(s)) = \frac{f(s+1) + f(s)}{2} \Delta\, g(s) + \frac{g(s+1) + g(s)}{2} \Delta\, f(s),$$

for $f(s) = x^{n-1}(s)$ and $g(s) = x(s)$ and obtain using relations (3.2) for $k = 1$

$$\mathbb{D}_x\, x^n(s) = (\alpha\, x_1(s) + \beta)\, \mathbb{D}_x\, x^{n-1}(s) + \mathbb{S}_x\, x^{n-1}(s).$$

In the same way using in addition (3.5), we obtain by taking $f(s) = x^{n-1}(s)\, e^{i\,\pi\, s}$ and $g(s) = x(s)$

$$\mathbb{S}_x\, x^n(s) = (\alpha\, x_1(s) + \beta)\, \mathbb{S}_x\, x^{n-1}(s) + \left[A_2(x_1(s)) - (\alpha\, x_1(s) + \beta)^2\right]\, \mathbb{D}_x\, x^{n-1}(s).$$

From the two previous relations, it is easy to show by induction that $\mathbb{D}_x\, x^n(s)$ and $\mathbb{S}_x\, x^n(s)$ are polynomials of degree at most $n-1$ and n respectively in the variable $x_1(s)$. Next, we write

$$\mathbb{D}_x\, x^n(s) = \sum_{k=0}^{n-1} D_{n,k}\, x_1^k(s), \; \mathbb{S}_x\, x^n(s) = \sum_{k=0}^{n} S_{n,k}\, x_1^k(s), \tag{3.9}$$

and get from the previous equation a system of two recurrence relations in $D_{n,n-1}$ and $S_{n,n}$. Solving this system with the initial conditions $D_{1,0} = 1, \; D_{2,1} = 2\,\alpha, \; S_{0,0} = 1, \; S_{1,1} = \alpha$, one obtains that

$$D_{n,n-1} \neq 0, \; n \geq 1 \text{ and } S_{n,n} \neq 0, \; n \geq 0.$$

This proves the first assertion of the proposition. Equation (3.8) is obtained by replacing s by $s + \frac{\mu}{2}$ in (3.7). The coefficients $D_{n,k-1}$ and $S_{n,k}$ for $k = n,\, n-1$ and $n-2$ will be given explicitly in this section. □

In order to express the polynomial A_2 in (3.5) in terms of α, β and δ and also to compute the coefficients $D_{n,k-1}$, we state the derivative and the quotient rules for the companion operators $\mathbb{D}_x$ and $\mathbb{S}_x$.

Theorem 3.1 *The following statements hold.*

1. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ obey the following product rules:*

$$\mathbb{D}_x\left(f(x(s))g(x(s))\right) = \mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s)), \tag{3.10}$$

$$\mathbb{S}_x\left(f(x(s))g(x(s))\right) = Q_2(x_1(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s)), \tag{3.11}$$

where Q_2 is a polynomial of degree 2

$$Q_2(x_1(s)) = (\alpha^2 - 1)\,x_1^2(s) + 2\,\beta\,(\alpha+1)\,x_1(s) + \delta, \tag{3.12}$$

and δ is a constant depending on $\alpha,\ \beta$ and the initial values of $x(s)$.

2. *They also satisfy the quotient rule*

$$\mathbb{D}_x\left(\frac{f(x(s))}{g(x(s))}\right) = \frac{\mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s)) - \mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s))}{Q_2(x_1(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2}; \tag{3.13}$$

$$\mathbb{S}_x\left(\frac{f(x(s))}{g(x(s))}\right) = \frac{Q_2(x_1(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) - \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s))}{Q_2(x_1(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2}, \tag{3.14}$$

provided that $g(x(s)) \neq 0,\ s \in (a,b)$.

3. *More generally, relations (3.10)-(3.14) remain valid if we replace x and x_1 by x_μ and $x_{\mu+1}$ respectively, $\mu \in \mathbb{C}$.*

Proof: First, we solve the equations

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+1)) - f(x(s))}{x(s+1) - x(s)},\ \mathbb{S}_x f(x(s)) = \frac{f(x(s+1)) + f(x(s))}{2},$$

in terms of $f(x(s+1))$ and $f(x(s))$. Then, we substitute this result for f and g in the equations

$$\mathbb{D}_x\left(f(x(s))\,g(x(s))\right) = \frac{f(x(s+1))g(x(s+1)) - f(x(s))g(x(s))}{x(s+1) - x(s)},$$

$$\mathbb{S}_x\left(f(x(s))\,g(x(s))\right) = \frac{f(x(s+1))g(x(s+1)) + f(x(s))g(x(s))}{2}$$

and obtain respectively (3.10) and

$$\mathbb{S}_x\left(f(x(s))g(x(s))\right) = \frac{(x(s+1) - x(s))^2}{4}\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s)).$$

By taking $f(x(s)) = g(x(s)) = x(s)$ in the previous equation, we get

$$Q_2(s) \equiv \frac{(x(s+1) - x(s))^2}{4} = \mathbb{S}_x x^2(s) - \left(\mathbb{S}_x x(s)\right)^2.$$

By means of Proposition 3.1, $Q_2(x(s))$ is a polynomial of degree at most 2 in the variable $x_1(s)$. Hence,

$$(x(s+1) - x(s))^2 = \delta_2\,x_1^2(s) + \delta_1\,x_1(s) + \delta_0,$$

where δ_j are constants. Application of the operator $\mathbb{D}_{x_1}$ on both hand-sides of the previous equation and use of Equations (3.1) and (3.2) for $k = 2$ produce

$$\begin{aligned}\delta_2\,(x_1(s+1)+x_1(s))+\delta_1 &= \frac{(x(s+2)-2\,x(s+1)+x(s))\,(x(s+2)-x(s))}{x_1(s+1)-x_1(s)}\\ &= 2\,\gamma_2\,[(\alpha_2-1)\,x(s+1)+\beta_2].\end{aligned}$$

The previous equation gives by means of (3.2) for $k = 1$ and s replaced by $s+\frac{1}{2}$

$$2\delta_2\,(\alpha\,x(s+1)+\beta)+\delta_1 = 2\,\gamma_2\,[(\alpha_2-1)\,x(s+1)+\beta_2].$$

Therefore,

$$\delta_2 = \frac{\gamma_2\,(\,\alpha_2-1)}{\alpha} = 4\,(\alpha^2-1),\;\; \delta_1 = 8\beta\,(\alpha+1)$$

and

$$Q_2(x_1(s)) = \frac{(x(s+1)-x(s))^2}{4} = (\alpha^2-1)\,x_1^2(s)+2\,\beta\,(\alpha+1)\,x_1(s)+\delta. \tag{3.15}$$

2. To prove the second statement, we take $f(x(s)) = \frac{1}{g(x(s))}$ in (3.10) and (3.11) to get

$$\begin{aligned}\mathbb{D}_x g(x(s))\,\mathbb{S}_x\frac{1}{g(x(s))}+\mathbb{S}_x g(x(s))\,\mathbb{D}_x\frac{1}{g(x(s))} &= 0,\\ \mathbb{S}_x g(x(s))\,\mathbb{S}_x\frac{1}{g(x(s))}+Q_2(x_1(s))\,\mathbb{D}_x g(x(s))\,\mathbb{D}_x\frac{1}{g(x(s))} &= 1.\end{aligned}$$

The determinant of the previous system with respect to the unknowns

$$\mathbb{S}_x\frac{1}{g(x(s))} \;\text{ and }\; \mathbb{D}_x\frac{1}{g(x(s))}$$

is

$$Q_2(x_1(s))\,[\mathbb{D}_x g(x(s))]^2-[\mathbb{S}_x g(x(s))]^2 = 4\,g(x(s))\,g(x(s+1)) \neq 0. \tag{3.16}$$

Hence,

$$\begin{aligned}\mathbb{S}_x\frac{1}{g(x(s))} &= \frac{-\mathbb{S}_x g(x(s))}{Q_2(x_1(s))\,[\mathbb{D}_x g(x(s))]^2-[\mathbb{S}_x g(x(s))]^2},\\ \mathbb{D}_x\frac{1}{g(x(s))} &= \frac{\mathbb{D}_x g(x(s))}{Q_2(x_1(s))\,[\mathbb{D}_x g(x(s))]^2-[\mathbb{S}_x g(x(s))]^2}.\end{aligned}$$

Application of the product rules (3.10)-(3.11) to the product $f(x(s)) \times \frac{1}{g(x(s))}$ produces (3.13) and (3.14). The third statement of the theorem is proved by replacing s by $s+\frac{\mu}{2}$ in (3.10)-(3.14). □

Remark 3.1 *The parameters α, β and δ are the ingredients for the classical orthogonal polynomials. As will be shown later, the recurrence coefficients of classical orthogonal polynomials are expressed explicitly in terms of these three parameters and the coefficients of the polynomials $\tilde{\sigma}$ and $\tilde{\tau}$ involved in the Pearson-type equation satisfied by the orthogonality weight function (see (2.89)).*

Remark 3.2 *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ appeared already in the works of Magnus [44, 45]. In [44] Magnus gave also the product and quotient rules for a more general divided-difference operator. Theorem 3.1 is more specific to quadratic lattices and gives in details the coefficients appearing in the product and quotient rules, in terms of the parameters α, β and δ of the lattice.*

Corollary 3.1 *As direct consequence of the previous theorem we have:*

1. *From the quotient rules (3.13) and (3.14), one observes that the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ transform a rational function of the variable $x(s)$ into a rational function of the variable $x_1(s)$.*
2. *If we square both members of (3.2) for $k = 1$ and combine with the previous equation, then we obtain*

$$\begin{aligned} x^2(s+1) + x^2(s) &= 2\,(2\,\alpha^2 - 1)\, x_1^2(s) + 4\,\beta\,(2\,\alpha + 1)\, x_1(s) + 2\,(\beta^2 + \delta), && (3.17) \\ x(s)\, x(s+1) &= x_1^2(s) - 2\,\beta\, x_1(s) + \beta^2 - \delta. && (3.18) \end{aligned}$$

3. *From Equation (3.16), on remarks that if $g(x(s))$ is a polynomial of degree n in $x(s)$, then $g(x(s))\, g(x(s+1))$ is a polynomial of degree $2\,n$ in the variable $x_1(s)$.*

The product rules for $\mathbb{D}_x$ and $\mathbb{S}_x$ provide the recurrence relations for the coefficients $D_{n,k}$ and $S_{n,k}$.

Proposition 3.2 *The coefficients $D_{n,k}$ and $S_{n,k}$ of the expansions (3.9) satisfy*

$$\begin{aligned} S_{n,k} &= -\alpha\, D_{n,k-1} - \beta\, D_{n,k} + D_{n+1,k}, \quad 0 \le k \le n, && (3.19) \\ S_{n+1,k} &= (\alpha^2 - 1)\, D_{n,k-2} + 2\,(\alpha + 1)\,\beta\, D_{n,k-1} + \delta\, D_{n,k} \\ &\quad + \alpha\, S_{n,k-1} + \beta\, S_{n,k}, \quad 0 \le k \le n+1, && (3.20) \end{aligned}$$

with the convention

$$D_{n,n} = D_{n,n+1} = S_{n,n+1} = D_{n,-1} = D_{n,-2} = S_{n,-1} = 0, \; n \ge 0. \tag{3.21}$$

Proof: Equations (3.19) and (3.20) are obtained from the expansion formulaes (3.9) and the product rules (3.10) and (3.11) for $f(x(s)) = x^n(s)$ and $g(x(s)) = x(s)$. □

Substitution of (3.19) in (3.20) reads

$$D_{n+2,k} - 2\,\alpha\, D_{n+1,k-1} + D_{n,k-2} = 2\,\beta\, D_{n+1,k} + (\delta - \beta^2)\, D_{n,k} + 2\,\beta\, D_{n,k-1}. \tag{3.22}$$

The previous equation for $k = n + 1$ gives a second-order homogenous linear difference equation with constant coefficients

$$D_{n+2,n+1} - 2\,\alpha\, D_{n+1,n} + D_{n,n-1} = 0,$$

whose solution with the initial conditions $D_{0,-1} = 0,\ D_{1,0} = 1$ is

$$D_{n,n-1} = \begin{cases} n & \text{if} \quad \alpha = 1, \\ \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}})} & \text{if} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}. \end{cases} \tag{3.23}$$

$S_{n,n}$ is therefore deduced using (3.19) for $k = n$

$$S_{n,n} = \begin{cases} 1 & \text{if} \quad \alpha = 1, \\ \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{2} & \text{if} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}. \end{cases} \tag{3.24}$$

The coefficients $D_{n,n-2}$ and $S_{n,n-1}$ are deduced by solving (3.22) for $k = n - 1$ with the initial conditions $D_{0,-2} = D_{1,-1} = 0$ and using (3.23) and (3.24). Using the computer algebra software Maple 9 [50] we get with $p = q^2$

$$D_{n,n-2} = \begin{cases} \frac{1}{3}\beta\, n\,(n-1)\,(2\,n-1) & \text{if} \quad \alpha = 1, \\ \frac{2\,p\,\beta\,n\,(p^{1-n}+p^n)}{(p+1)\,(p-1)^2} - \frac{2\,p^2\,\beta\,(p^n - p^{-n})}{(p+1)\,(p-1)^3} & \text{if} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}; \end{cases} \tag{3.25}$$

$$S_{n,n-1} = \begin{cases} \beta\, n\,(2\,n-1) & \text{if} \quad \alpha = 1, \\ -\frac{\beta\, n\,(p^{1-n} - p^n)}{p-1} & \text{if} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}. \end{cases} \tag{3.26}$$

The coefficients $D_{n,n-3}$ and $S_{n,n-2}$ are obtained likewise using Equations (3.19)-(3.26) with the initial conditions $D_{1,-2} = D_{2,-1} = 0$

$$D_{n,n-3} = \frac{1}{30} n\,(n-1)\,(n-2)\,(4\,\beta^2\,n^2 - 8\,\beta^2\,n + 5\,\delta + 3\,\beta^2), \text{ for } \alpha = 1; \tag{3.27}$$

$$\begin{aligned} D_{n,n-3} \;=\; & \left(\frac{p^{2-n} + p^n}{(p^2-1)^2} p\, n - \frac{p^n - p^{-n}}{(p^2-1)^3} 2\,p^3\right)\delta \\ & + \left(\frac{p^n - p^{2-n}}{(p-1)^3\,(p+1)} 2\,p\,n^2 + \frac{p^{3-n} - 7\,p^{2-n} + 7\,p^{n+1} + p^n}{(p-1)^4\,(p+1)}\,p\,n \right. \\ & \left. + \frac{p^n - p^{-n}}{(p-1)^5\,(p+1)}\,6\,p^3\right)\beta^2; \end{aligned} \tag{3.28}$$

for $\alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}$ and

$$S_{n,n-2} = \frac{1}{6}\,n\,(n-1)\,(4\,\beta^2\,n^2 - 8\,\beta^2\,n + 3\,\delta + 3\,\beta^2), \text{ for } \alpha = 1; \tag{3.29}$$

$$S_{n,n-2} \;=\; \frac{p^n - p^{2-n}}{2\,(p^2-1)}\delta\, n + \left(\frac{p^n + p^{2-n}}{(p-1)^2}\,n^2 + \frac{p^n - 3\,p^{n+1} + 3\,p^{2-n} - p^{3-n}}{2\,(p-1)^3}\,n\right)\beta^2, \tag{3.30}$$

for $\alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}$. The remaining coefficients $D_{n,k-1}$ and $S_{n,k},\ k = 1 \ldots n-3$ can be computed by following the same procedure.

3.1.2 The operators $\mathbb{F}_x$ and $\mathbb{M}_x$

For the manipulation of the difference equations for orthogonal polynomials, it is sometimes more convenient to work with the operators $\mathbb{F}_x$ and $\mathbb{M}_x$ defined by

$$\mathbb{F}_x f(s) = \frac{\Delta}{\Delta x(s-\frac{1}{2})} \frac{\nabla f(s)}{\nabla x(s)}, \quad \mathbb{M}_x f(s) = \frac{1}{2}\left(\frac{\Delta f(s)}{\Delta x(s)} + \frac{\nabla f(s)}{\nabla x(s)}\right). \tag{3.31}$$

These operators satisfy properties similar to those of $\mathbb{D}_x$ and $\mathbb{S}_x$.

Theorem 3.2 *The following properties hold:*

1. *If $P_n(x(s))$ is a polynomial of degree n in the variable $x(s)$, then $\mathbb{F}_x(P_n(x(s)))$ and $\mathbb{M}_x(P_n(x(s)))$ are polynomials of degree $n-2$ and $n-1$ respectively in $x(s)$.*

2. *$\mathbb{F}_x$ and $\mathbb{M}_x$ obey the product rules*

$$\begin{aligned}\mathbb{F}_x\mathbb{M}_x f(s) &= (2\,\alpha^2-1)\,\mathbb{M}_x\mathbb{F}_x f(s) + 2\,\alpha\, U_1(s)\,\mathbb{F}_x\mathbb{F}_x f(s); \quad (3.32)\\ \mathbb{M}_x\mathbb{M}_x f(s) &= \alpha\,\mathbb{F}_x f(s) + 2\,\alpha\, U_1(s)\,\mathbb{M}_x\mathbb{F}_x f(s) + (2\,\alpha^2-1)\,U_2(s)\,\mathbb{F}_x\mathbb{F}_x f(s), \quad (3.33)\end{aligned}$$

where the expression $\mathbb{F}_x\mathbb{M}_x f(s)$ refers to $\mathbb{F}_x\left(\mathbb{M}_x f(s)\right)$ and

$$U_1(s) = (\alpha+1)\,[(\alpha-1)\,x(s)+\beta],\; U_2(s) = (\alpha^2-1)\,x^2(s)+2\,\beta\,(\alpha+1)\,x(s)+\delta. \quad (3.34)$$

Proof: From the definition of $\mathbb{D}_x$, $\mathbb{S}_x$, $\mathbb{F}_x$ and $\mathbb{M}_x$ (see (3.6) and (3.31)) we have

$$\begin{aligned}\mathbb{F}_x(f(x(s))) &= \frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla f(x(s))}{\nabla x(s)} = \mathbb{D}_{x_{-1}}\mathbb{D}_{x_{-2}} f(x_{-2}(s)), \quad (3.35)\\ \mathbb{M}_x(f(x(s))) &= \frac{1}{2}\left(\frac{\Delta f(x(s))}{\Delta x(s)} + \frac{\nabla f(x(s))}{\nabla x(s)}\right) = \mathbb{S}_{x_{-1}}\mathbb{D}_{x_{-2}} f(x_{-2}(s)). \quad (3.36)\end{aligned}$$

Let $P_n(x(s))$ be a polynomial of degree n in $x(s)$. From (3.35) and Proposition 3.1, we deduce that $\mathbb{D}_{x_{-2}}$ transforms $P_n(x_{-2}(s))$ into a polynomial of degree $n-1$ in $x_{-1}(s)$ and $\mathbb{D}_{x_{-1}}$ transforms $\mathbb{D}_{x_{-2}} P_n(x_{-2}(s))$ into a polynomial of degree $n-2$ in $x_0(s) = x(s)$. Therefore, $\mathbb{F}_x P_n(x(s))$ is a polynomial of degree $n-2$ in $x(s)$. Similarly, we deduce that $\mathbb{M}_x P_n(x(s))$ is a polynomial of degree $n-1$ in $x(s)$. For the proof of the product rules, we proceed as follows:

1. First, we consider Equation (3.1) for $k=2,\; s=s-1$ and $k=2,\, s=s-\frac{3}{2}$; and Equation (3.2) for $k=1$; $k=1,\, s=s-\frac{1}{2}$ and $k=2,\; s=s-1$. Then we solve the system of five linear equations obtained for the unknowns

$$x_{-3}(s),\, x_{-1}(s),\, x_{-2}(s),\, x_1(s),\, x_3(s)$$

and obtain the expressions

$$\begin{aligned}2\alpha\, x_{-3}(s) &= -(2\alpha-1)(2\alpha+1)\,x_2(s) + (4\alpha^2+2\alpha-1)(4\alpha^2-2\alpha-1)\,x(s)\\ &\quad +2\beta\,(2\alpha+1)(4\alpha^2+2\alpha-1),\\ 2\alpha\, x_{-1}(s) &= -x_2(s) + (2\alpha-1)(2\alpha+1)\,x(s) + 2\beta\,(2\alpha+1), \quad (3.37)\\ x_{-2}(s) &= -x_2(s) + (4\alpha^2-2)\,x(s) + 4\beta\,(\alpha+1),\\ 2\alpha\, x_1(s) &= x_2(s) + x(s) - 2\beta,\\ 2\alpha\, x_3(s) &= (2\alpha-1)(2\alpha+1)\,x_2(s) - x(s) + 2\beta\,(2\alpha+1);\end{aligned}$$

from which we deduce

$$\begin{aligned}x_{-4}(s) &= -(4\alpha^2-2)\,x_2(s) + (4\alpha^2-1)\,(4\alpha^2-3)\,x(s) + 4\beta(\alpha+1)(4\alpha^2-1),\\ x_4(s) &= (4\alpha^2-2)\,x_2(s) - x(s) + 4\beta(\alpha+1). \quad (3.38)\end{aligned}$$

2. In the second step, we remark that the right-hand side of the equations

$$
\begin{aligned}
\mathbb{F}_x f(s) &= \frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla f(s)}{\nabla x(s)},\\
\mathbb{M}_x f(s) &= \frac{1}{2}\left(\frac{\Delta f(s)}{\Delta x(s)}+\frac{\nabla f(s)}{\nabla x(s)}\right),\\
\mathbb{M}_x\mathbb{F}_x f(s) &= \frac{1}{2}\left(\frac{\Delta}{\Delta x(s)}+\frac{\nabla}{\nabla x(s)}\right)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla f(s)}{\nabla x(s)},\\
\mathbb{F}_x\mathbb{F}_x f(s) &= \frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla}{\nabla x(s)}\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla f(s)}{\nabla x(s)},
\end{aligned}
\tag{3.39}
$$

are linear combinations of $f(s+j)$, $j=-2,\,-1,\,1,\,2$. Therefore, this system has a unique solution with respect to the unknowns $f(s+j)$, $j=-2,\,-1,\,1,\,2$ since its determinant is

$$
\frac{1}{\nabla x_{-2}(s)\,\nabla x_{-1}(s)\,[\nabla x(s)\,\Delta x(s)\,\nabla x_1(s)]^2\,\Delta x_1(s)\,\Delta x_2(s)} \neq 0.
$$

Computations with Maple 9 [50] produce

$$
\begin{aligned}
f(s-2) &= f(s)+\Big(x_{-2}(s)\,x_{-1}(s)-\frac{1}{2}x_{-4}(s)\,x_{-1}(s)-\frac{1}{2}x_{-1}(s)\,x(s)-x_{-3}(s)\,x_{-2}(s)\\
&+ x_{-4}(s)\,x_{-3}(s)-\frac{1}{2}x_{-4}(s)\,x_{-1}(s)+\frac{1}{2}x_1(s)\,x(s)\Big)\,\mathbb{F}_x f(s)\\
&- (\nabla x_{-2}(s)+\nabla x(s))\,\mathbb{M}_x f(s)-\nabla x_{-2}(s)\,\nabla x_{-1}(s)\,\nabla x(s)\,\mathbb{M}_x\mathbb{F}_x f(s)\\
&+ \frac{1}{2}\nabla x_{-2}(s)\,\nabla x_{-1}(s)\,\nabla x(s)\,\nabla x_1(s)\mathbb{F}_x\mathbb{F}_x f(s),\\
f(s-1) &= f(s)-\nabla x(s)\mathbb{M}_x f(s)+\frac{1}{2}\nabla x(s)\,\nabla x_1(s)\,\mathbb{F}_x f(s),\\
f(s+1) &= f(s)+\Delta x(s)\mathbb{M}_x f(s)+\frac{1}{2}\nabla x_1(s)\,\Delta x(s)\,\mathbb{F}_x f(s),\\
f(s+2) &= f(s)+\Big(\frac{1}{2}x_{-1}(s)\,x(s)-\frac{1}{2}x(s)\,x_1(s)+x_1(s)\,x_2(s)-\frac{1}{2}x_1(s)\,x_4(s)\\
&- x_2(s)\,x_3(s)-\frac{1}{2}x_{-1}(s)\,x_4(s)+x_4(s)\,x_3(s)\Big)\,\mathbb{F}_x f(s)\\
&+ (\Delta x_2(s)+\Delta x(s))\,\mathbb{M}_x f(s)+\Delta x(s)\,\Delta x_1(s)\,\Delta x_2(s)\,\mathbb{M}_x\mathbb{F}_x f(s)\\
&- \frac{1}{2}\nabla x_1(s)\,\Delta x(s)\,\Delta x_1(s)\,\Delta x_2(s)\mathbb{F}_x\mathbb{F}_x f(s).
\end{aligned}
\tag{3.40}
$$

3. In the third step, we substitute the systems of Equations (3.37), (3.38) and (3.40) into the right hand-side of the equations

$$
\begin{aligned}
\mathbb{F}_x\mathbb{M}_x f(s) &= \frac{1}{2}\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla}{\nabla x(s)}\left(\frac{\Delta f(s)}{\Delta x(s)}+\frac{\nabla f(s)}{\nabla x(s)}\right),\\
\mathbb{M}_x\mathbb{M}_x f(s) &= \frac{1}{4}\left(\frac{\Delta}{\Delta x(s)}+\frac{\nabla}{\nabla x(s)}\right)\left(\frac{\Delta f(s)}{\Delta x(s)}+\frac{\nabla f(s)}{\nabla x(s)}\right)
\end{aligned}
$$

and get after some computations with Maple 9 [50]

$$\begin{aligned}\mathbb{F}_x\mathbb{M}_x f(s) &= (2\,\alpha^2-1)\,\mathbb{M}_x\mathbb{F}_x f(s) + 2\,\alpha\,U_1(s)\,\mathbb{F}_x\mathbb{F}_x f(s); && (3.41)\\ \mathbb{M}_x\mathbb{M}_x f(s) &= \alpha\,\mathbb{F}_x f(s) + 2\,\alpha\,U_1(s)\,\mathbb{M}_x\mathbb{F}_x f(s) + (2\,\alpha^2-1)\,\tilde{U}_2(s)\,\mathbb{F}_x\mathbb{F}_x f(s), && (3.42)\end{aligned}$$

with

$$U_1(s) = (\alpha+1)[(\alpha-1)x(s)+\beta],\ \tilde{U}_2(s) = \frac{[x(s+1)+(1-2\alpha^2)x(s)-2\beta(\alpha+1)]^2}{4\alpha^2}.$$

It remains to prove that $\tilde{U}_2(s) = U_2(s)$ given by (3.34). To do this, we first remark by writing (3.42) for $f(s) = x^4(s)$ that $\tilde{U}_2(s)$ is a polynomial of degree at most 2 in $x(s)$. Then we apply the operator $\mathbb{F}_x$ to both members of the equation

$$\frac{[x(s+1)+(1-2\alpha^2)x(s)-2\beta(\alpha+1)]^2}{4\alpha^2} = a_2\,x^2(s) + a_1\,x(s) + a_0,$$

use (3.37)-(3.38) and get $a_2 = \alpha^2 - 1$. Next we apply $\mathbb{M}_x$ to the equation

$$\frac{[x(s+1)+(1-2\alpha^2)x(s)-2\beta(\alpha+1)]^2}{4\alpha^2} - (\alpha^2-1)x^2(s) = a_1\,x(s) + a_0,$$

use again (3.37)-(3.38) to get $a_1 = 2\beta(\alpha+1)$. Finally, we obtain the constant term a_0 by taking $s = 0$ in the previous equation

$$a_0 = \frac{x^2(1)+x^2(0)}{4\alpha^2} - \frac{(2\alpha^2-1)\,x(0)\,x(1)}{2\alpha^2} - \frac{\beta\,(\alpha+1)\,(x(1)+x(0))}{\alpha^2} + \frac{\beta^2\,(1+\alpha)^2}{\alpha^2}.$$

The term a_0, by means of the expression of $x_1(s)$ given in (3.37), is equal to the constant term δ obtained from (3.15) by taking $s = 0$. Hence

$$\begin{aligned}\tilde{U}_2(s) = U_2(s) &= \frac{[x(s+1)+(1-2\alpha^2)x(s)-2\beta(\alpha+1)]^2}{4\alpha^2} && (3.43)\\ &= (\alpha^2-1)\,x^2(s) + 2\,\beta\,(\alpha+1)\,x(s) + \delta.\end{aligned}$$

□

Theorem 3.3 *The following product and quotient rules hold:*

$$\begin{aligned}\mathbb{F}_x\,(fg) &= \mathbb{F}_x(f)\,g + f\,\mathbb{F}_x(g) + 2\,\alpha\,\mathbb{M}_x(f)\,\mathbb{M}_x(g) && (3.44)\\ &+ 2\,U_1\,[\mathbb{F}_x(f)\,\mathbb{M}_x(g) + \mathbb{M}_x(f)\,\mathbb{F}_x(g)] + 2\,\alpha\,U_2\,\mathbb{F}_x(f)\,\mathbb{F}_x(g);\end{aligned}$$

$$\begin{aligned}\mathbb{M}_x\,(fg) &= \mathbb{M}_x(f)\,g + f\,\mathbb{M}_x(g) + 2\,U_1\,\mathbb{M}_x(f)\,\mathbb{M}_x(g) && (3.45)\\ &+ 2\,\alpha\,U_2\,[\mathbb{F}_x(f)\,\mathbb{M}_x(g) + \mathbb{M}_x(f)\,\mathbb{F}_x(g)] + 2\,U_1\,U_2\,\mathbb{F}_x(f)\,\mathbb{F}_x(g);\end{aligned}$$

$$\begin{aligned}
\mathbb{F}_x\left(\frac{f}{g}\right) &= \left\{2\,\alpha\, f\ [\mathbb{M}_x(g)]^2 - 2\,\alpha\,\mathbb{M}_x(f)\,\mathbb{M}_x(g)\,g + \mathbb{F}_x(f)\,g^2 - f\,g\,\mathbb{F}_x(g)\right. \\
&+ 2\,\alpha\,U_2\,\mathbb{F}_x(f)\,\mathbb{F}_x(g)\,g - 2\,\alpha\,U_2\,f\ [\mathbb{F}_x(g)]^2 \\
&+ \left. 2\,U_1\,\mathbb{F}_x(f)\,\mathbb{M}_x(g)\,g - 2\,U_1\,\mathbb{M}_x(f)\,\mathbb{F}_x(g)\,g\ \right\} / \\
&\quad \left\{U_2\ [U_1\,\mathbb{F}_x(g) + \alpha\,\mathbb{M}_x(g)]^2 - [g + 2\,U_1\,\mathbb{M}_x(g) + 2\,\alpha\,U_2\,\mathbb{F}_x(g)]^2\right\}\ g;
\end{aligned} \tag{3.46}$$

$$\begin{aligned}
\mathbb{M}_x\left(\frac{f}{g}\right) &= \left\{2\,U_1\,U_2\,f\ [\mathbb{F}_x(g)]^2 - 2\,U_1\,U_2\,\mathbb{F}_x(f)\,\mathbb{F}_x(g)\,g + \mathbb{M}_x(f)\,g^2 - f\,g\,\mathbb{M}_x(g)\right. \\
&+ 2\,\alpha\,U_2\,\mathbb{M}_x(f)\,\mathbb{F}_x(g)\,g - 2\,\alpha\,U_2\,\mathbb{F}_x(f)\,\mathbb{M}_x(g)\,g \\
&+ \left. 2\,U_1\,\mathbb{M}_x(f)\,\mathbb{M}_x(g)\,g - 2\,U_1\,f\ [\mathbb{M}_x(g)]^2\right\} / \\
&\quad \left\{U_2\ [U_1\,\mathbb{F}_x(g) + \alpha\,\mathbb{M}_x(g)]^2 - [g + 2\,U_1\,\mathbb{M}_x(g) + 2\,\alpha\,U_2\,\mathbb{F}_x(g)]^2\right\}\ g,
\end{aligned} \tag{3.47}$$

with

$$f \equiv f(s), U_j \equiv U_j(s) \text{ and } g \equiv g(s) \neq 0,\ \forall s \in (a,b).$$

Proof: The derivation of Equations (3.44) and (3.45) are similar to those of (3.32) and (3.33). More precisely, we replace the expressions of $f(s+j)$, $g(s+j)$, $j = -2, -1, 1, 2$ obtained from (3.40), in the right hand-side of the equations

$$\begin{aligned}
\mathbb{F}_x(f(s)\,g(s)) &= \frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla[f(s)\,g(s)]}{\nabla x(s)}, \\
\mathbb{M}_x(f(s)\,g(s)) &= \frac{1}{2}\left(\frac{\Delta[f(s)\,g(s)]}{\Delta x(s)} + \frac{\nabla[f(s)\,g(s)]}{\nabla x(s)}\right),
\end{aligned}$$

and obtain (3.44) and (3.45) after some calculations taking into account (3.37).

The quotient rules (3.46) and (3.47) can be derived by applying the product rules (3.44) and (3.45) on $f(s) \times \frac{1}{g(s)}$. Therefore we have to express $\mathbb{F}_x\left(\frac{1}{g(s)}\right)$ and $\mathbb{M}_x\left(\frac{1}{g(s)}\right)$ for $g(s) \neq 0$, $\forall s \in (a,b)$, in terms of $g(s)$, $\mathbb{F}_x g(s)$ and $\mathbb{M}_x g(s)$. For this purpose, we take $f(s) = \frac{1}{g(s)}$ in (3.44) and (3.45) and get the linear system in $\mathbb{F}_x\left(\frac{1}{g}\right)$ and $\mathbb{M}_x\left(\frac{1}{g}\right)$

$$\begin{aligned}
&[g + 2\,U_1\,\mathbb{M}_x(g) + 2\,\alpha\,U_2\,\mathbb{F}_x(g)]\,\mathbb{F}_x\left(\frac{1}{g}\right) + 2\,[U_1\,\mathbb{F}_x(g) + \alpha\,\mathbb{M}_x(g)]\,\mathbb{M}_x\left(\frac{1}{g}\right) = -\frac{1}{g}\mathbb{F}_x(g) \\
&2U_2\ [U_1\,\mathbb{F}_x(g) + \alpha\,\mathbb{M}_x(g)]\,\mathbb{F}_x\left(\frac{1}{g}\right) + [g + 2\,U_1\,\mathbb{M}_x(g) + 2\,\alpha\,U_2\,\mathbb{F}_x(g)]\,\mathbb{M}_x\left(\frac{1}{g}\right) = -\frac{1}{g}\mathbb{M}_x(g)
\end{aligned}$$

whose determinant is

$$\begin{aligned}
[g + 2\,U_1\,\mathbb{M}_x(g) + 2\,\alpha\,U_2\,\mathbb{F}_x(g)]^2 - 4U_2\ [U_1\,\mathbb{F}_x(g) + \alpha\,\mathbb{M}_x(g)]^2 &= g(s-1)\,g(s+1) \\
&\neq 0,\ \forall s \in (a,b).
\end{aligned}$$

Therefore, $\mathbb{F}_x\left(\frac{1}{g}\right)$ and $\mathbb{M}_x\left(\frac{1}{g}\right)$ are uniquely determined from the previous linear system, and the quotient rules $\mathbb{F}_x\left(\frac{f}{g}\right)$ and $\mathbb{M}_x\left(\frac{f}{g}\right)$ are deduced by application of the product rules (3.44) and (3.45) on $f(s)\times\frac{1}{g(s)}$. □

The product rules provide the recurrence relation for the coefficients $F_{n,k}$ and $M_{n,k}$ of the expansion

$$\mathbb{F}_x x^n(s) = \sum_{k=0}^{n-2} F_{n,k}\, x^k(s),\quad \mathbb{M}_x x^n(s) = \sum_{k=0}^{n-1} M_{n,k}\, x^k(s). \tag{3.48}$$

Proposition 3.3 *The coefficients $F_{n,k}$ and $M_{n,k}$ satisfy*

$$F_{n+1,k} - 2\,\alpha\, M_{n,k} - 2\,\beta\,(\alpha+1)\,F_{n,k} + (1-2\,\alpha^2)\,F_{n,k-1} = 0,\ 0\le k\le n-1, \tag{3.49}$$

$$\begin{aligned} &M_{n+1,k} - 2\,\beta\,(\alpha+1)\,M_{n,k} - 2\,\alpha\,(\alpha^2-1)\,F_{n,k-2} - 2\,\alpha\,\delta\,F_{n,k} \\ &\quad -4\,\alpha\,\beta\,(\alpha+1)\,F_{n,k-1} + (1-2\,\alpha^2)\,M_{n,k-1} - \delta_{n,k} = 0,\ 0\le k\le n, \end{aligned} \tag{3.50}$$

where $\delta_{n,k}$ is the Kronecker symbol; with the convention

$$F_{n+1,n} = F_{n,n+1} = M_{n,n+1} = F_{n,n} = M_{n,n} = F_{n,-1} = F_{n,-2} = M_{n,-1} = 0,\ n\ge 0.$$

Proof: The proof is obtained using (3.48) and the product rules (3.44) and (3.45) for $f(s) = x^n(s),\ g(s) = x(s)$. □

The previous proposition allows to compute recurrently the coefficients $F_{n,k}$ and $M_{n,k}$. Furthermore, these coefficients can also be computed via the following relations with the coefficients $D_{n,k}$ and $S_{n,k}$.

Proposition 3.4 *The coefficients $D_{n,k}$ and $S_{n,k}$ of the expansions (3.9) are related to the coefficients $F_{n,k}$ and $M_{n,k}$ of the expansions (3.48) by*

$$F_{n,k} = \sum_{j=k+1}^{n-1} D_{n,j}\, D_{j,k},\ 0\le k\le n-2, \tag{3.51}$$

$$M_{n,k} = \sum_{j=k}^{n-1} D_{n,j}\, S_{j,k},\ 0\le k\le n-1. \tag{3.52}$$

Proof: The proof follows from Equations (3.9), (3.35), (3.36) and (3.48). □

As a second consequence of the product rules, we shall prove another important result concerning the linear divided-difference equation of higher order satisfied by products of functions each of them satisfying a linear divided-difference equation. For this purpose, we start with the following preliminaries

Lemma 3.1 *Let $f(s)$ be a function of the variable $x(s)$ satisfying a second-order divided-difference equation*

$$\mathbb{F}_x f(s) + a_1(s)\,\mathbb{M}_x f(s) + a_0(s)\,f(s) = 0, \tag{3.53}$$

where a_0 and a_1 are given functions of $x(s)$.

Then the expressions $\mathbb{F}_x\mathbb{F}_x f(s)$, $\mathbb{M}_x\mathbb{F}_x f(s)$, $\mathbb{F}_x\mathbb{M}_x f(s)$, $\mathbb{M}_x\mathbb{M}_x f(s)$ and $\mathbb{F}_x f(s)$ can be written in the form

$$c_1(s)\,f(s) + c_2(s)\,\mathbb{M}_x f(s),$$

where $c_1(s)$ and $c_2(s)$ are functions of $a_0(s)$ and $a_1(s)$.

Proof: Assuming that the lattice $x(s)$ is not constant ($x(s) \neq x(t)$ for $s \neq t$), Equation (3.53) is equivalent to

$$(\nabla x_1(s)\, a_1(s) + 2)\, f(s+1) + C_0(s)\, f(s) + (\nabla x_1(s)\, a_1(s) - 2)\, f(s-1) = 0, \qquad (3.54)$$

where $C_0(s)$ is a function of $a_1(s),\ a_0(s)$ and $x(s)$.

1. If $a_1(s) = \pm\frac{2}{\nabla x_1(s)}$, then the previous equation is equivalent to $\frac{f(s+1)}{f(s)} = C_1(s)$; therefore, $f(s+j),\ j = 1\ldots$ is proportional to $f(s)$.

2. If $a_1(s) \neq \pm\frac{2}{\nabla x_1(s)}$, then from (3.54), we get

$$\begin{aligned} f(s-2) &= \frac{([-4 + 2\, a_1(s)\, \nabla x_1(s) - 2\, a_1(s-1)\, \nabla x_1(s-1) + a_1(s-1)\, \nabla x_1(s-1)\, a_1(s)\, \nabla x_1(s)}{(-2 + a_1(s)\, \nabla x_1(s))\, (-2 + a_1(s-1)\, \nabla x_1(s-1))} \\ &\quad + \frac{(C_0(s-1)\, C_0(s))\, f(s)}{(-2 + a_1(s)\, \nabla x_1(s))\, (-2 + a_1(s-1)\, \nabla x_1(s-1))} \\ &\quad + \frac{C_0(s-1)\, (2 + a_1(s)\, \nabla x_1(s))\, f(s+1)}{(-2 + a_1(s)\, \nabla x_1(s))\, (-2 + a_1(s-1)\, \nabla x_1(s-1))}; \\ f(s-1) &= \frac{(\nabla x_1(s)\, a_1(s) + 2)\, f(s+1) + C_0(s)\, f(s)}{\nabla x_1(s)\, a_1(s) - 2}; \\ f(s+2) &= \frac{(\Delta x_1(s)\, a_1(s+1) - 2)\, f(s) - C_0(s+1)\, f(s+1)}{\Delta x_1(s)\, a_1(s+1) + 2}. \end{aligned}$$

Next, we use the previous equations to write the expressions $\mathbb{F}_x\mathbb{F}_x f(s)$, $\mathbb{M}_x\mathbb{F}_x f(s)$, $\mathbb{F}_x\mathbb{M}_x f(s)$ and $\mathbb{M}_x\mathbb{M}_x f(s)$ as linear combination of $f(s)$ and $f(s+1)$. Finally we use (3.53) and the following equation taken from (3.40)

$$f(s+1) = f(s) + \Delta x(s)\mathbb{M}_x f(s) + \frac{1}{2}\nabla x_1(s)\, \Delta x(s)\, \mathbb{F}_x f(s)$$

to convert the expressions $\mathbb{F}_x\mathbb{F}_x f(s)$, $\mathbb{M}_x\mathbb{F}_x f(s)$, $\mathbb{F}_x\mathbb{M}_x f(s)$ and $\mathbb{M}_x\mathbb{M}_x f(s)$ from the linear combination of $f(s)$ and $f(s+1)$ to the linear combination of $f(s)$ and $\mathbb{M}_x f(s)$.

□

Theorem 3.4 *Let $f(s)$ and $g(s)$ be two functions of the variable $x(s)$ satisfying respectively*

$$\mathbb{F}_x f(s) + a_1(s)\, \mathbb{M}_x f(s) + a_0(s)\, f(s) = 0,\ \mathbb{F}_x g(s) + b_1(s)\, \mathbb{M}_x g(s) + b_0(s)\, g(s) = 0, \quad (3.57)$$

where a_j and b_j are given functions of $x(s)$.

Then, the product $f(s)\, g(s)$ is a solution of a fourth-order divided-difference equation of the form

$$I_4(s)\, \mathbb{F}_x\mathbb{F}_x y(s) + I_3(s)\, \mathbb{M}_x\mathbb{F}_x y(s) + I_2(s)\, \mathbb{F}_x y(s) + I_1(s)\, \mathbb{M}_x y(s) + I_0(s)\, y(s) = 0$$

where I_j are functions of a_j and b_j. If the $a_j(s),\ j = 0,\, 1$ and the $b_j(s),\ j = 0,\, 1$ are polynomials in the variable $x(s)$, then the coefficients $I_j(s),\ j = 0\ \ldots\ 4$ can be chosen to be polynomials in the variable $x(s)$.

Proof: We apply the identity operator as well as the operators $\mathbb{F}_x\mathbb{F}_x$, $\mathbb{M}_x\mathbb{F}_x$, $\mathbb{F}_x$ and $\mathbb{M}_x$ to the equation

$$y(s) = f(s)\, g(s),$$

and use the product rules (3.44) and (3.45) to get five equations whose right-hand sides are linear combinations of expressions of the form $p_1(s)\, p_2(s)$ with

$$p_j(s) \in \{\mathbb{F}_x\mathbb{F}_x h_j(s),\, \mathbb{M}_x\mathbb{F}_x h_j(s),\, \mathbb{F}_x\mathbb{M}_x h_j(s),\, \mathbb{M}_x\mathbb{M}_x h_j(s),\, \mathbb{F}_x h_j(s)\}\,,\ j = 1,\, 2,$$

with $h_1 = f,\ h_2 = g$. These right-hand sides are transformed by means of the previous lemma into linear combinations of

$$f(s)\, g(s),\ f(s)\,\mathbb{M}_x g(s),\ [\mathbb{M}_x f(s)]\, g(s) \text{ and } [\mathbb{M}_x f(s)]\,\mathbb{M}_x g(s).$$

Thus, these five equations can be written as

$$\begin{aligned}
&X_{0,0} = y(s),\\
&c_{2,1}(s)\, X_{0,0}(s) + c_{2,2}(s)\, X_{0,1}(s) + c_{2,3}(s)\, X_{1,0}(s) + c_{2,4}(s)\, X_{1,1}(s) = \mathbb{M}_x y(s),\\
&c_{3,1}(s)\, X_{0,0}(s) + c_{3,2}(s)\, X_{0,1}(s) + c_{3,3}(s)\, X_{1,0}(s) + c_{3,4}(s)\, X_{1,1}(s) = \mathbb{F}_x y(s),\\
&c_{4,1}(s)\, X_{0,0}(s) + c_{4,2}(s)\, X_{0,1}(s) + c_{4,3}(s)\, X_{1,0}(s) + c_{4,4}(s)\, X_{1,1}(s) = \mathbb{M}_x\mathbb{F}_x y(s),\\
&c_{5,1}(s)\, X_{0,0}(s) + c_{5,2}(s)\, X_{0,1}(s) + c_{5,3}(s)\, X_{1,0}(s) + c_{5,4}(s)\, X_{1,1}(s) = \mathbb{F}_x\mathbb{F}_x y(s),
\end{aligned} \tag{3.58}$$

with the notations

$$X_{0,0} = f(s)\, g(s),\ X_{0,1} = f(s)\,\mathbb{M}_x g(s),\ X_{1,0} = [\mathbb{M}_x f(s)]\ g(s),\ X_{1,1} = [\mathbb{M}_x f(s)]\ \mathbb{M}_x g(s)$$

where $c_{j,k}$ are functions of a_j and b_j. The system (3.58) contains 5 linear equations for 4 unknowns, namely $X_{j,k}(s),\ j,k = 0,1$. For the solutions of this system to exist, it is necessary for $y(s)$ to satisfy the equation

$$\begin{vmatrix}
1 & 0 & 0 & 0 & y(s)\\
c_{2,1}(s) & c_{2,2}(s) & c_{2,3}(s) & c_{2,4}(s) & \mathbb{M}_x y(s)\\
c_{3,1}(s) & c_{3,2}(s) & c_{3,3}(s) & c_{3,4}(s) & \mathbb{F}_x y(s)\\
c_{4,1}(s) & c_{4,2}(s) & c_{4,3}(s) & c_{4,4}(s) & \mathbb{M}_x\mathbb{F}_x y(s)\\
c_{5,1}(s) & c_{5,2}(s) & c_{5,3}(s) & c_{5,4}(s) & \mathbb{F}_x\mathbb{F}_x y(s)
\end{vmatrix} = 0, \tag{3.59}$$

which is the fourth-order divided-difference equation desired. □

As consequence of this theorem, we claim the following:

Corollary 3.2 *If $f_j,\ j = 1, \dots n$ are functions of the variable $x(s)$ such that any f_j satisfies a linear divided-difference equation of order r_j involving only the operators $\mathbb{F}_x$ and $\mathbb{M}_x$, then the product $f = \prod_{j=1}^{n} f_j$ satisfies a divided-difference equation of order $r = \prod_{j=1}^{n} r_j$ involving only (at most) the operators*

$$\mathbb{M}_x^j\, \mathbb{F}_x^k,\ j = 0,\, 1,\ \text{and } 0 \le j + 2\,k \le r = \prod_{j=0}^{n} r_j.$$

3.2 Recurrence coefficients for classical orthogonal polynomials

3.2.1 From orthogonality to second-order difference equation

In the last chapter, we have seen that the very classical orthogonal polynomials are defined as the ones orthogonal with respect to a positive weight function satisfying the Pearson-type differential, difference or q-difference equation with some border conditions. This definition implies that the very classical orthogonal polynomials satisfy a second-order differential, difference or q-difference equation with polynomial coefficients (see Theorems 2.6, 2.8 and 2.9).

Also, any family of polynomials (P_n) with $\mathrm{degree}(P_n) = n$ satisfying equations (1.8), (1.9) or (1.10) is very classical.

The definition of classical orthogonal polynomials given in [10] is not similar to those of the very classical orthogonal polynomials, because, according to this definition, for a family of polynomials to be classical, it should be orthogonal with respect to a weight function satisfying a Pearson-type equation; and, should in addition, satisfy a second-order difference equation. The requirement for P_n to satisfy a second-order difference equation, which in the case of the very classical orthogonal polynomials is a consequence of the orthogonality, is redundant. This condition can be omitted. Therefore, we propose the following definition which in our opinion is a natural extension of those of the very classical orthogonal polynomials (see definitions 2.1, 2.3 and 2.5).

Definition 3.1 *An orthogonal polynomial sequence (P_n) on the real interval $(x(a), x(b))$ is classical if and only if:*

1. *(P_n) is orthogonal with respect to the weight function $\rho(s)$ i.e.*

$$\begin{cases} \mathrm{degree}(P_n) = n, \; n \geq 0, \\ \sum_{i=0}^{N} P_n(x(s_i))\, P_m(x(s_i))\, \rho(s_i)\, \nabla x_1(s_i) = k_n\, \delta_{n,m}, \quad k_n \neq 0, \, n, \, m \geq 0, \end{cases} \tag{3.60}$$

with

$$s_0 = a, s_{i+1} = s_i + 1, \, s_{N+1} = b, \, N \in \mathbb{N}_0 \cup \{\infty\} \tag{3.61}$$

for the discrete orthogonality or

$$\begin{cases} \mathrm{degree}(P_n) = n, \; n \geq 0, \\ \int_C P_n(x(s))\, P_m(x(s))\, \rho(s)\, \nabla x_1(s)\, ds = k_n\, \delta_{n,m}, \quad k_n \neq 0, \, n, \, m \geq 0, \end{cases} \tag{3.62}$$

where C is a contour in the complex s-plane, for the continuous orthogonality;

2. *The weight ρ satisfies the Pearson-type difference equation*

$$\frac{\Delta}{\nabla x_1(s)}(\sigma(s)\, \rho(s)) = \tau(x(s))\, \rho(s), \tag{3.63}$$

where $\tau(s)$ is a polynomial of degree 1 in $x(s)$ and

$$\tilde{\sigma}(s) = \sigma(s) + \frac{1}{2}\tau(s)\,\nabla x_1(s) \tag{3.64}$$

is a non-zero polynomial of degree at most 2 in the variable $x(s)$, with the border conditions

$$\begin{cases} \sigma(s)\,\rho(s)\,x^k(s-\frac{1}{2})\big|_{s=a,\,b} = 0,\ k = 0,\,1,\,2,\ldots, \\ \int_C \Delta\left[\sigma(s)\,\rho(s)\,x^k(s-\frac{1}{2})\right] ds = 0,\ k = 0,\,1,\,2,\ldots. \end{cases} \tag{3.65}$$

for the discrete orthogonality and the continuous orthogonality respectively.

Remark 3.3 *It should be noticed that in case of the continuous orthogonality, for the family $(P_n(x(s)))$ to be a classical orthogonal polynomials in the real variable $x(s)$, it should be possible [10] to choose a contour C in such a way that the second relation of (3.62) can be expressed as a real orthogonality relation*

$$\int_a^b P_n(x)\,P_n(x)\,\rho(x)\,dx = k_n\,\delta_{n,m},\ k_n \neq 0,\ n,\ m \geq 0,$$

with $\rho(x) > 0,\ x \in (a,b)$.

Let us remind that Atakishiyev, Rahman and Suslov [10], using the second-order difference equation (2.73) satisfied by an orthogonal family (P_n), the Pearson-type equation (2.89) satisfied by the orthogonality weight and the border conditions (2.90), obtained the orthogonality relation (3.60). Here, we prove the converse: The orthogonality relation provided above plus the Pearson-type equation and the border conditions lead to the second-order divided-difference equation of form (2.73).

Theorem 3.5 *Let (P_n) be a sequence of classical orthogonal polynomials satisfying the orthogonality relation (3.60). Then, each P_n satisfies*

$$\tilde{\sigma}(x(s))\,\mathbb{F}_x P_n(x(s)) + \tilde{\tau}(x(s))\,\mathbb{M}_x P_n(x(s)) + \lambda_n\,P_n(x(s)) = 0, \tag{3.66}$$

where λ_n is a constant term given by

$$\lambda_n = -D_{n,n-1}\,(\tilde{\sigma}_2\,D_{n-1,n-2} + \tau_1\,S_{n-1,n-1}), \tag{3.67}$$

with $\tilde{\sigma}(s) = \tilde{\sigma}_2\,x^2(s) + \tilde{\sigma}_1\,x(s) + \tilde{\sigma}_0, \tau(s) = \tau_1\,x(s) + \tau_0$.

In order to simplify the proof, we state and prove the following lemma

Lemma 3.2 *Under the hypothesis of the previous theorem, the following identities hold:*

$$\begin{aligned}&\Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_n(x(s))}{\nabla x(s))}\right] P_m(x(s)) - \Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_m(x(s))}{\nabla x(s))}\right] P_n(x(s)) \\ &= \Delta\left\{\sigma(s)\,\rho(s)\,W(P_n(x(s)),P_m(x(s)))\right\};\end{aligned} \tag{3.68}$$

$$\begin{aligned}&\Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_m(x(s))}{\nabla x(s))}\right] \\ &= \left[\tilde{\sigma}(s)\,\mathbb{F}_x P_m(x(s)) + \tau(s)\,\mathbb{M}_x P_m(x(s)\right]\,\rho(s)\,\nabla x_1(s),\end{aligned} \tag{3.69}$$

where

$$W(P_n(x(s)),P_m(x(s))) = P_n(x(s))\frac{\nabla P_m(x(s))}{\nabla x(s))} - P_m(x(s))\frac{\nabla P_n(x(s))}{\nabla x(s))} \tag{3.70}$$

is the discrete analog of the Wronskian.

The wronskian $W(P_n(x(s)),P_m(x(s)))$ is a polynomial of degree at most $n+m-1$ in the variable $x_{-1}(s)=x(s-\frac{1}{2})$.

Proof: First, we observe that the Pearson-type equation (3.63) is equivalent to

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s)+\tau(s)\,\nabla x_1(s)}{\sigma(s+1)}. \tag{3.71}$$

Using the relation

$$\Delta(f(s)\,g(s)) = f(s)\,\Delta g(s) + g(s+1)\,\Delta f(s) = f(s+1)\,\Delta g(s) + g(s)\,\Delta f(s), \tag{3.72}$$

we obtain for given $n,\ m\in\mathbb{N}$,

$$\begin{aligned}&\Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_n(x(s))}{\nabla x(s))}\right]P_m(x(s)) - \Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_m(x(s))}{\nabla x(s))}\right]P_n(x(s))\\ &= \Delta\left[\sigma(s)\,\rho(s)\,W(P_n(x(s)),P_m(x(s)))\right],\end{aligned}$$

where

$$W(P_n(x(s)),P_m(x(s))) = P_n(x(s))\frac{\nabla P_m(x(s))}{\nabla x(s))} - P_m(x(s))\frac{\nabla P_n(x(s))}{\nabla x(s))}. \tag{3.73}$$

The Wronskian $W(P_n(x(s)),P_m(x(s)))$ thanks to (3.72) can also be written as

$$\begin{aligned}&W(P_n(x(s)),P_m(x(s))) = \\ &\quad \mathbb{S}_{x_{-2}}P_n(x_{-2}(s))\,\mathbb{D}_{x_{-2}}P_m(x_{-2}(s)) - \mathbb{S}_{x_{-2}}P_m(x_{-2}(s))\,\mathbb{D}_{x_{-2}}P_n(x_{-2}(s)).\end{aligned} \tag{3.74}$$

Therefore, from (3.8), we remark that $W(P_n(x(s)),P_m(x(s)))$ is a polynomial of degree at most $n+m-1$ in the variable $x_{-1}(s)=x(s-\frac{1}{2})$.

For the second identity, we use again (3.72), then (3.63) together with (3.71) and finally (2.75) to obtain

$$\begin{aligned}&\Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_m(x(s))}{\nabla x(s))}\right]\\ &= \Delta\left[\sigma(s)\,\rho(s)\right]\,\frac{\nabla P_m(x(s))}{\nabla x(s))} + \sigma(s+1)\,\rho(s+1)\Delta\frac{\nabla P_m(x(s))}{\Delta x(s))}\\ &= \tau(s)\,\nabla x_1(s)\,\rho(s)\frac{\nabla P_m(x(s))}{\nabla x(s))} + (\sigma(s)+\tau(s)\,\nabla x_1(s))\rho(s)\Delta\frac{\nabla P_m(x(s))}{\nabla x(s))}\\ &= \tau(s)\,\nabla x_1(s)\,\rho(s)\left[\mathbb{M}_x P_m(x(s)) - \frac{1}{2}\Delta\frac{\nabla P_m(x(s))}{\nabla x(s))}\right]\\ &\quad +(\sigma(s)+\tau(s)\,\nabla x_1(s))\rho(s)\Delta\frac{\nabla P_m(x(s))}{\nabla x(s))}\\ &= \left[\tilde{\sigma}(s)\,\mathbb{F}_x P_m(x(s)) + \tau(s)\,\mathbb{M}_x P_m(x(s)\right]\,\rho(s)\,\nabla x_1(s),\end{aligned}$$

with $\tilde{\sigma}$ given by (3.64). $\square$

Next we give the proof of Theorem 3.5.

Proof: We set $n,\, m \in \mathbb{N}$ and write

$$V_n(x(s)) = \tilde{\sigma}(s)\, \mathbb{F}_x P_n(x(s)) + \tau(s)\, \mathbb{M}_x P_n(x(s),\, n \geq 1,\, V_0(x(s)) = 1.$$

We shall prove that the family (V_n) is also orthogonal with respect to the weight $\rho(s)$. First we prove that $\mathrm{degree}(V_n) = n,\, n \geq 1$. To do this, we assume that (P_n) is monic and use the expansions (3.51) and (3.52) of $\mathbb{F}_x x^n(s)$ and $\mathbb{M}_x x^n(s)$ to obtain that the leading coefficient of V_n, which we denote by h_n, is

$$\begin{aligned} h_n &= \tilde{\sigma}_2\, F_{n,n-2} + \tau_1\, M_{n,n-1} \\ &= D_{n,n-1}\, (\tilde{\sigma}_2\, D_{n-1,n-2} + \tau_1\, S_{n-1,n-1}),\; n \geq 1,\; h_0 = 1, \end{aligned} \tag{3.75}$$

with the latter identity obtained thanks to Proposition 3.4.

Following [10] page 204, if we define the moment of the weight function ρ by

$$M_n = \sum_{s=a}^{b-1} [x(s) - x(a+n-1)]^{(n)}\, \rho(s)\, \nabla x_1(s)$$

or

$$M_n = \frac{1}{2\pi i} \int_C [x(s) - x(a+n-1)]^{(n)}\; \rho(s)\, \nabla x_1(s)\, ds$$

for the discrete or the continuous orthogonality respectively, with $\sigma(a) = 0$ and the generalized power of the lattice $x(s)$ defined as

$$[x_r(z) - x_r(s)]^{(k)} = \prod_{j-0}^{k-1} [x_r(z) - x_r(s-j)],$$

it turns out that M_n satisfies (see [10], Eqn. (6.8))

$$(\tilde{\sigma}_2\, D_{n,n-1} + \tau_1\, S_{n,n}) M_{n+1} = -\tau_n(a)\, M_n,\; n \geq 0.$$

For all the moments $M_n,\; n \geq 0$ to exist, it is necessary to have

$$\tilde{\sigma}_2\, D_{n,n-1} + \tau_1\, S_{n,n} \neq 0,\;\; n \geq 0.$$

Since $D_{n,n-1} \neq 0,\; n \geq 1$, we deduce that $h_n \neq 0,\; n \geq 1$ and $\mathrm{degree}(V_n) = n,\; n \geq 1$.

Next, we assume without any loss of generality that $m \leq n$. Then, using (3.68) and (3.69), we get

$$\begin{aligned} &\sum_{i=0}^{N} V_n(x(s_i))\, P_m(x(s_i))\, \rho(s_i)\, \nabla x_1(s_i) \\ = \quad &\sum_{i=0}^{N} P_n(x(s_i))\, V_m(x(s_i))\, \rho(s_i)\, \nabla x_1(s_i) + \sigma(s)\, \rho(s)\, W(P_n(x(s_i)), P_m(x(s_i)))|_{i=0}^{N+1}\,. \end{aligned}$$

Finally, use of the previous relation, the orthogonality relation for (P_n) (3.60), the equations (3.69), (3.75), the border conditions (3.65) and the fact that $W(P_n(x(s)), P_m(x(s)))$ is a polynomial in the variable $x(s-\frac{1}{2})$, give

$$\sum_{i=0}^{N} V_n(x(s_i))\, P_m(x(s_i))\, \rho(s_i)\, \nabla x_1(s_i) = \sum_{i=0}^{N} P_n(x(s_i))\, V_m(x(s_i))\, \rho(s_i)\, \nabla x_1(s_i) = h_n\, k_n\, \delta_{n,m}.$$

The latter relation, combined with the fact that V_n is of degree n assures that the family (V_n) is orthogonal with respect to the weight function ρ. Hence, (P_n) and (V_n) are proportional since they are orthogonal with respect to the same weight; therefore there exists a constant term λ_n such that

$$V_n(x(s)) = -\lambda_n\, P_n(x(s)),\ n \geq 0.$$

Comparison of the leading terms in both members of the previous equation yields

$$\lambda_n = -D_{n,n-1}\, (\tilde{\sigma}_2\, D_{n-1,n-2} + \tau_1\, S_{n-1,n-1}).$$

The proof using the continuous orthogonality is obtained in the same way. □

3.2.2 Three-term recurrence coefficients

Before we compute the recurrence coefficients of the classical orthogonal polynomials, let us make some comments relative to the coefficients α, β and δ involved in the definition of the lattice $x(s)$ and also in the functions U_1 and U_2 appearing in the quadratic rules (3.32), (3.33) and (3.44)-(3.45).

Remark 3.4 *The monic polynomial P_n, solution of (3.66), depends only on α, β, δ and the coefficients of the polynomials $\tilde{\sigma}$ and $\tilde{\tau}$. This can be seen easily since the coefficients $F_{n,k}$ and $M_{n,k}$ of the expansion (3.9) depend only on α, β, δ (see (3.51)-(3.52)and (3.23)-(3.30)). The lattice $x(s)$, satisfying (3.1) and (3.2) is given by (2.88), therefore, the selection of coefficients α, β, δ leads to different classes of orthogonal polynomials:*

1. *If $\alpha = 1$, then $x(s) = 4\beta\, s^2 + C_7\, s + C_8$ and we have the following special cases:*

   1.1. *If $\beta = C_7 = 0$, then $x(s) = C_8$, and $\delta = 0$ thanks to (3.15). In this case, $\mathbb{D}_x \equiv \frac{d}{dx}$ and (3.66) is equivalent to*

   $$\tilde{\sigma}(x)\, \frac{d^2}{dx^2} P_n(x) + \tau(x)\, \frac{d}{dx} P_n(x) + \lambda_n\, P_n(x) = 0.$$

   Therefore, the special case $\alpha = 1$, $\beta = \delta = 0$ corresponds to the classical orthogonal polynomials of a continuous variable.

   1.2. *If $\beta = C_8 = 0$ and $C_7 = 1$, then $x(s) = s$ and $\delta = \frac{1}{4}$ by (3.15). Equation (3.66) reads*

   $$\phi(s)\, \Delta\nabla P_n(s) + \psi(s)\, \Delta P_n(s) + \lambda_n\, P_n(s) = 0,$$

   with $\phi(s) = \tilde{\sigma}(s) - \frac{1}{2}\, \tilde{\tau}(s)$, $\psi(s) = \tilde{\tau}(s)$. Thus the case $\alpha = 1, \beta = 0$ and $\delta = \frac{1}{4}$ corresponds to the classical orthogonal polynomials of a discrete variable on a linear lattice.

1.3. If $\beta \neq 0$, then the corresponding families are the classical orthogonal polynomials of a discrete variable on a quadratic lattice.

2. For $\alpha = \frac{q^{\frac{1}{2}}+q^{\frac{-1}{2}}}{2}$ with $q \neq 0,\ 1$, it follows from (2.88) that

$$x(s) = C_5\, q^{-s} + C_6\, q^{s} + \frac{\beta}{1-\alpha}.$$

The coefficient β which is only involved in the constant term of the lattice can be omitted without any loss of generality. Therefore, we take $\beta = 0$.

2.1. If $C_5 = 0$ and $C_6 = 1$, then $x(s) = q^s$ and $\delta = 0$. We have

$$\mathbb{F}_x = q^2\, D_q\, D_{\frac{1}{q}},\ \mathbb{M}_x = \frac{1}{2}(D_q + D_{\frac{1}{q}}),$$

and Equation (3.66) in this case is equivalent to

$$\phi(x(s))\, D_q\, D_{\frac{1}{q}} P_n(x(s)) + \psi(x(s))\, D_q P_n(x(s)) + \lambda_n\, P_n(x(s)) = 0,\ \ x(s) = q^s$$

with

$$\phi(x(s)) = q^2\, \tilde{\sigma}(s) - \frac{1}{2}(q-1)\, x(s)\, \tilde{\tau}(s),\ \psi(x(s)) = \tilde{\tau}(s).$$

Similarly, for $C_5 = 1$ and $C_6 = 0$, we get $x(s) = q^{-s}$ and $\delta = 0$. For these parameters, (3.66) is again equivalent to a second-order q-difference equation of the same type as the previous one.

Hence, the case $\alpha = \frac{q^{\frac{1}{2}}+q^{\frac{-1}{2}}}{2}$, $\beta = \delta = 0$ corresponds to the q-classical orthogonal polynomials.

2.2. If $C_5\, C_6 \neq 0$, then $\delta \neq 0$. Hence, the case $\alpha = \frac{q^{\frac{1}{2}}+q^{\frac{-1}{2}}}{2}$, $\beta = 0$ with $\delta \neq 0$ (depending on the two initial values of the lattice) corresponds to the classical orthogonal polynomials of a discrete variable on a q-quadratic lattice.

The method we will use here to compute the recurrence coefficients is the same used for the very classical orthogonal polynomials.

We assume that (P_n) is a system of monic classical orthogonal polynomials such that each P_n satisfies the equation (3.66) which we relabel as

$$\phi(x(s))\, \mathbb{F}_x P_n(x(s)) + \psi(x(s))\, \mathbb{M}_x P_n(x(s)) + \lambda_n\, P_n(x(s)) = 0, \tag{3.76}$$

where

$$\phi(s) = \tilde{\sigma}(s) = \sigma(s) + \frac{1}{2}\tau(s)\, \nabla x_1(s),\ \psi(s) = \tau(s), \tag{3.77}$$

and

$$\phi(x(s)) = \phi_2\, x^2(s) + \phi_1\, x(s) + \phi_0,\ \psi(x(s)) = \psi_1\, x(s) + \psi_0, \tag{3.78}$$

with $\phi(x(s)) \neq 0$ and $\psi_1 \neq 0$. Because of the orthogonality, (P_n) satisfies the three-term recurrence relation (2.6) which we recall here:

$$P_{n+1}(x) = (x - \beta_n)\, P_n(x) - \gamma_n\, P_{n-1}(x),\ n \geq 1,\ P_{-1} = 0,\ P_0(x) = 1.$$

First, we write

$$P_n(x(s)) = \sum_{j=0}^{n} T_{n,n-j}\, x^j(s), \quad T_{n,0} = 1, \tag{3.79}$$

and replace this expression into the previous recurrence relation, then use the expansions (3.48) to get

$$\begin{gathered}
\left[\phi_2\, x^2(s) + \phi_1\, x(s) + \phi_0\right] \sum_{k=0}^{n-2} \left(\sum_{j=k+2}^{n} T_{n,n-j}\, F_{j,k} \right) x^k(s) + \\
\left[\psi_1\, x(s) + \psi_0\right] \sum_{k=0}^{n-1} \left(\sum_{j=k+1}^{n-1} T_{n,n-j}\, M_{j,k} \right) x^k(s) + \lambda_n \sum_{k=0}^{n} T_{n,n-k}\, x^k(s) = 0.
\end{gathered}$$

Then we look for the coefficients of the monomials $x^n(s)$ on the previous equation and obtain

$$\lambda_n = -\phi_2\, F_{n,n-2} - \psi_1\, M_{n,n-1}. \tag{3.80}$$

Next, we collect the coefficients of the monomials $x^j(s),\ j = n-1,\ n-2$ to get the following linear system with respect to the unknowns $T_{n,1},\ T_{n,2}$

$$(\lambda_n + \phi_2 F_{n-1,n-3} + \psi_1 M_{n-1,n-2})\, T_{n,1} + \phi_2 F_{n,n-3} + \phi_1 F_{n,n-2} + \psi_1 M_{n,n-2} + \psi_0 M_{n,n-1} = 0,$$

$$\begin{aligned}
&(\psi_0 M_{n-1,n-2} + \psi_1 M_{n-1,n-3} + \phi_1 F_{n-1,n-3} + \phi_2 F_{n-1,n-4})\, T_{n,1} \\
&\quad + (\phi_2 F_{n-2,n-4} + \psi_1 M_{n-2,n-3} + \lambda_n)\, T_{n,2} \\
&\qquad + \psi_0 M_{n,n-2} + \psi_1 M_{n,n-3} + \phi_1 F_{n,n-3} + \phi_2 F_{n,n-4} + \phi_0 F_{n,n-2} = 0.
\end{aligned}$$

Solving this system produces the coefficients $T_{n,1}$ and $T_{n,2}$.

Computations taking into account the relations

$$\begin{aligned}
&M_{n,n-1} = D_{n,n-1} S_{n-1,n-1}, \\
&M_{n,n-2} = D_{n,n-1} S_{n-1,n-2} + D_{n,n-2} S_{n-2,n-2}, \\
&M_{n,n-3} = D_{n,n-1} S_{n-1,n-3} + D_{n,n-2} S_{n-2,n-3} + D_{n,n-3} S_{n-3,n-3}, \\
&F_{n,n-2} = D_{n,n-1} D_{n-1,n-2}, \\
&F_{n,n-3} = D_{n,n-1} D_{n-1,n-3} + D_{n,n-2} D_{n-2,n-3}, \\
&F_{n,n-4} = D_{n,n-1} D_{n-1,n-4} + D_{n,n-2} D_{n-2,n-4} + D_{n,n-3} D_{n-3,n-4},
\end{aligned}$$

obtained from (3.51) and (3.52), give the following expressions for the coefficients $F_{n,k-1}$ and $M_{n,k}$ for $k = n,\ n-1$ and $n-3$.

First case: $\alpha = 1$:

$$\begin{aligned}
F_{n,n-2} &= n\,(n-1)\,, \\
F_{n,n-3} &= \frac{4}{3}\,\beta\, n\,(n-2)\,(n-1)^2\,, \\
F_{n,n-4} &= \frac{1}{45}\, n\,(n-1)\,(n-2)\,(n-3)\,\left(32\,\beta^2 n^2 - 96\,\beta^2 n + 52\,\beta^2 + 15\,\delta\right),
\end{aligned}$$

$$\begin{aligned} M_{n,n-1} &= n, & (3.81)\\ M_{n,n-2} &= \frac{2}{3}\,\beta\, n\,(n-1)\,(4\,n-5)\,, & (3.82)\\ M_{n,n-3} &= \frac{2}{15}\, n\,(n-1)\,(n-2)\left(16\,\beta^2 n^2 - 52\,\beta^2 n + 32\,\beta^2 + 5\,\delta\right). \end{aligned}$$

Second case: $\alpha = \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\ \beta = 0$:

$$\begin{aligned} F_{n,n-2} &= \frac{(q^n-1)(q^n-q)q^{\frac{1}{2}}}{(q-1)^2\, q^n},\\ F_{n,n-3} &= 0,\\ F_{n,n-4} &= \left(\frac{(q+1)(q^{2n}-q^3)\,n}{(q-1)^3\, q^{n+\frac{1}{2}}} - \frac{(q^n-1)(3q^{n+1}+q^n-q^3-3q^2)q^{\frac{1}{2}}}{(q-1)^4\, q^n}\right)\delta,\\ M_{n,n-1} &= \frac{(q^n-1)(q^n+q)}{2\,(q-1)q^n}, & (3.83)\\ M_{n,n-2} &= 0,\\ M_{n,n-3} &= \frac{\delta}{2}\left(\frac{(q+1)(q^{2n}+q^3)\,n}{(q-1)^2\, q^{n+1}} - \frac{(q+1)(q^n-1)(q^n+q^2)}{(q-1)^3\, q^n}\right). \end{aligned}$$

The coefficients β_n and γ_n are deduced from $T_{n,1}$ and $T_{n,2}$ using (2.52) and the previous expressions of $F_{n,k}$ and $M_{n,k}$. We can now state the following explicit results obtained with the help of Maple [50]:

Theorem 3.6 *The coefficients β_n and γ_n of the recurrence coefficients (2.6) of the classical orthogonal polynomials satisfying (3.76) are given explicitly by:*
Case 1: $\alpha_x = 1$**:**

$$\begin{aligned} \beta_n &= -\frac{4n\left(2\,\psi_1 n - \psi_1 - 2\,\phi_2 n + 2\,\phi_2 n^2\right)(\psi_1 + \phi_2 n - \phi_2)}{(2\,\phi_2 n + \psi_1)\,(2\,\phi_2(n-1) + \psi_1)}\beta_x \qquad (3.84)\\ &\quad -\frac{-2\,\psi_0\phi_2 + \psi_1\psi_0 - 2\,\phi_1 n\phi_2 + 2\,\phi_1\psi_1 n + 2\,\phi_1 n^2\phi_2}{(2\,\phi_2 n + \psi_1)\,(2\,\phi_2(n-1) + \psi_1)}; \end{aligned}$$

$$\begin{aligned} \gamma_n &= -\frac{n\,(n-1)\,(\psi_1 + \phi_2 n - \phi_2)\,(\psi_1 + \phi_2 n - 2\,\phi_2)}{(2\,\phi_2 n - \phi_2 + \psi_1)\,(2\,\phi_2 n - 3\,\phi_2 + \psi_1)}\delta_x\\ &\quad + \frac{16\,(n-1)^3\,(\psi_1 + \phi_2 n - 2\,\phi_2)\,(\psi_1 + \phi_2 n - \phi_2)^3\, n}{(2\,\phi_2 n - 3\,\phi_2 + \psi_1)\,(2\,\phi_2 n - \phi_2 + \psi_1)\,(2\,\phi_2 n - 2\,\phi_2 + \psi_1)^2}\beta_x^2\\ &\quad + \left(2\,\phi_1 n^2\phi_2 - 4\,\phi_1 n\phi_2 + 2\,\phi_1\psi_1 n + 2\,\phi_2\phi_1 - 2\,\phi_1\psi_1 + \psi_1\psi_0\right) \times \qquad (3.85)\\ &\quad \frac{4\,(n-1)\,(\psi_1 + \phi_2 n - \phi_2)\,(\psi_1 + \phi_2 n - 2\,\phi_2)\, n}{(2\,\phi_2 n - 3\,\phi_2 + \psi_1)\,(2\,\phi_2 n - \phi_2 + \psi_1)\,(2\,\phi_2 n - 2\,\phi_2 + \psi_1)^2}\beta_x\\ &\quad - \frac{(\psi_1 + \phi_2 n - 2\,\phi_2)\, n}{(2\,\phi_2 n - 3\,\phi_2 + \psi_1)\,(2\,\phi_2 n - \phi_2 + \psi_1)\,(2\,\phi_2 n - 2\,\phi_2 + \psi_1)^2} \times\\ &\quad \left(4\,\phi_0{\phi_2}^2 n^2 - 8\,\phi_0{\phi_2}^2 n + 4\,\phi_0{\phi_2}^2 - {\phi_1}^2 n^2\phi_2 + 2\,{\phi_1}^2 n\phi_2 + 4\,\phi_0\phi_2 n\psi_1 + {\psi_0}^2\phi_2\right.\\ &\quad \left. -4\,\phi_0\phi_2\psi_1 - {\phi_1}^2\phi_2 - {\phi_1}^2 n\psi_1 + {\phi_1}^2\psi_1 + \phi_0{\psi_1}^2 - \psi_0\phi_1\psi_1\right), \end{aligned}$$

Case 2: *For* $\alpha_x = \frac{p+p^{-1}}{2}$ *and* $\beta_x = 0$*, with the notations*

$$p = q^{\frac{1}{2}},\ \ q \neq 0,\ 1,\ \text{and}\ \zeta = p^n,$$

$$\begin{aligned}
\beta_n \;=\; & \big[\left(-2\,\phi_1 p^3 + \psi_0 p^4 - 2\,\psi_0 p^2 + \psi_0 - 2\,p\phi_1\right)\left(\psi_1 p^2 + 2\,\phi_2 p - \psi_1\right)\zeta^6 \\
- & \left(p^2+1\right)\left(p^6\psi_1\psi_0 + 2\,p^5\phi_1\psi_1 - 2\,p^5\psi_0\phi_2 - 4\,p^4\phi_2\phi_1 - p^4\psi_1\psi_0\right. \\
& \left. +4\,p^3\psi_0\phi_2 - 4\,p^3\psi_1\phi_1 - 4\,p^2\phi_1\phi_2 - p^2\psi_1\psi_0 + 2\,p\phi_1\psi_1 - 2\,\psi_0\phi_2 p + \psi_1\psi_0\right)\zeta^4 \\
- & p^2\left(-2\,\phi_1 p^3 + \psi_0 p^4 - 2\,\psi_0 p^2 + \psi_0 - 2\,p\phi_1\right)\left(\psi_1 p^2 - 2\,\phi_2 p - \psi_1\right)\zeta^2\big] \,/ \\
& \big\{\left(\left(\psi_1 p^2 + 2\,\phi_2 p - \psi_1\right)\zeta^4 + \psi_1 p^2 - 2\,\phi_2 p - \psi_1\right)\times \\
& \left(\left(\psi_1 p^2 + 2\,\phi_2 p - \psi_1\right)\zeta^4 + p^4\left(\psi_1 p^2 - 2\,\phi_2 p - \psi_1\right)\right)\big\}\,,
\end{aligned} \tag{3.86}$$

$$\begin{aligned}
\gamma_n \;=\; & \big\{-2\,\left(-\psi_1 p^4 - \psi_1\zeta^2 + \psi_1 p^6 + \psi_1 p^2\zeta^2 + 2\,\phi_2 p\,\zeta^2 - 2\,\phi_2 p^5\right)\times \\
& \left(\zeta^2-1\right)p^3\zeta^2\left(p^2-1\right)^2\left(-\psi_1 p^4 - \psi_1\zeta^4 + \psi_1 p^6 + \psi_1\zeta^4 p^2 + 2\,\phi_2 p\,\zeta^4 - 2\,\phi_2 p^5\right)^2\phi_0 \\
& + \left(-\psi_1 p^4 - \psi_1\zeta^2 + \psi_1 p^6 + \psi_1 p^2\zeta^2 + 2\,\phi_2 p\,\zeta^2 - 2\,\phi_2 p^5\right)\left(\zeta^2-1\right)p^2\left(p^2-\zeta^2\right)\times \\
& \left(-2\,\phi_2 p^3 + 2\,\phi_2 p\,\zeta^2 - \psi_1 p^2 + \psi_1 p^4 - \psi_1\zeta^2 + \psi_1 p^2\zeta^2\right)\times \\
& \left(-\psi_1 p^4 - \psi_1\zeta^4 + \psi_1 p^6 + \psi_1\zeta^4 p^2 + 2\,\phi_2 p\,\zeta^4 - 2\,\phi_2 p^5\right)^2\delta \\
& + \left(-\psi_1 p^4 - \psi_1\zeta^2 + \psi_1 p^6 + \psi_1 p^2\zeta^2 + 2\,\phi_2 p\,\zeta^2 - 2\,\phi_2 p^5\right)\times \\
& \left(\zeta^2-1\right)p^4\zeta^4\left(p^2-1\right)^2\left(\psi_0 p^4 - 2\,\phi_1 p^3 - \psi_0 p^2 + \zeta^2\psi_0 p^2 + 2\,\phi_1\zeta^2 p - \zeta^2\psi_0\right)\times \\
& \left(p^6\psi_1\psi_0 - 2\,p^5\psi_0\phi_2 + 2\,p^5\phi_1\psi_1 - p^4\psi_1\zeta^2\psi_0 - 2\,p^4\psi_1\psi_0 - 4\,p^4\phi_2\phi_1 - 2\,p^3\zeta^2\phi_2\psi_0 + 2\,p^3\zeta^2\phi_1\psi_1\right. \\
& \quad +2\,p^3\psi_0\phi_2 - 2\,p^3\psi_1\phi_1 + 2\,p^2\psi_1\zeta^2\psi_0 \\
& \quad \left. +4\,p^2\zeta^2\phi_1\phi_2 + p^2\psi_1\psi_0 + 2\,p\,\zeta^2\phi_2\psi_0 - 2\,p\,\zeta^2\phi_1\psi_1 - \psi_1\zeta^2\psi_0\right)\big\} \,/ \\
& \big\{(p^2-1)^2\,\left(\left(\psi_1 p^2 + 2\,\phi_2 p - \psi_1\right)\zeta^4 + p^6\left(\psi_1 p^2 - 2\,\phi_2 p - \psi_1\right)\right)\times \\
& \quad \left(\left(\psi_1 p^2 + 2\,\phi_2 p - \psi_1\right)\zeta^4 + p^2\left(\psi_1 p^2 - 2\,\phi_2 p - \psi_1\right)\right)\times \\
& \quad \left(\left(\psi_1 p^2 + 2\,\phi_2 p - \psi_1\right)\zeta^4 + p^4\left(\psi_1 p^2 - 2\,\phi_2 p - \psi_1\right)\right)^2\big\}\,.
\end{aligned} \tag{3.87}$$

The coefficient λ_n of (3.76) is given by

$$\lambda_n = -n\left(\psi_1 + (n-1)\,\phi_2\right), \tag{3.88}$$

for $\alpha_x = 1$ and

$$\lambda_n = \frac{\left(-\psi_1 p^2 - 2\,\phi_2 p + \psi_1\right)\zeta^4 + \left(2\,\psi_1 p^2 + 2\,\phi_2 p - \psi_1 + 2\,\phi_2 p^3 - \psi_1 p^4\right)\zeta^2 + p^2\left(\psi_1 p^2 - 2\,\phi_2 p - \psi_1\right)}{2\,(p^2-1)^2\,\zeta^2} \tag{3.89}$$

for $\alpha_x = \frac{p+p^{-1}}{2}$, $p = q^{\frac{1}{2}}$ and $\zeta = p^n$.

Remark 3.5 *1. By selecting specific values of the parameters α_x, β_x and δ_x (as indicated in the Remark 3.4) in the previous theorem, we recover after appropriate changes in ϕ and ψ the coefficients β_n and γ_n of the very classical orthogonal polynomials.*

2. The relations (3.84)-(3.87) are important tools for computer algebra since they provide in 4 relations the recurrence coefficients of all classical orthogonal polynomials [39].

3.2.3 Classical orthogonal polynomials (not very classical)

In this subsection, we provide data for the classical (but not very classical) orthogonal polynomials. Here, we deal with 14 families out of 18 mentioned in [39]. The four remaining families, namely, the Wilson, the continuous dual Hahn, the continuous Hahn and the Meixner-Pollaczek polynomials deal with the complex difference-derivative which is not included in this thesis and will be treated later separately. However, these families can be reached by limiting procedures from the Askey-Wilson polynomials, therefore, the difference equations satisfied by modifications of such polynomials can be deduced from those of the Askey-Wilson polynomials, since the latter polynomials are among those considered in this work.

The information we provide for each family is

1. the hypergeometric representation of the polynomials;
2. the weight function;
3. the orthogonality relation;
4. the explicit form of the lattice $x(s)$ and the determination of the coefficients α, β and δ (**relabelled α_x, β_x and δ_x to avoid confusion with parameters α, β and δ involved in the definition of diverse classical families**) in connection with $x(s)$;
5. the coefficients ϕ and ψ for which (3.63) and (3.76) are satisfied, of course taking into account the relations (3.77);
6. the recurrence coefficients β_n, γ_n and the coefficients λ_n of (3.76).

Notice that all these informations have been taken from [39] except the polynomials ϕ and ψ which we have computed from the difference equations for P_n provided in [39]. We have also computed for each family the coefficients β_n and γ_n using Theorem 3.6 and the entries α_x, β_x, δ_x, ϕ and ψ obtained in this thesis. We have checked positively that they coincide with those given in [39]. In the following lines we illustrate the determination of the polynomials ϕ and ψ for the Askey-Wilson polynomials. Those of the remaining families are obtained likewise.

The notations we use for the classical orthogonal families differ slightly (by the tilde) from those of [39] because here we deal with monic orthogonal polynomials.

Part I: Cases of the q-quadratic lattices

1. **Askey-Wilson polynomials**

$$\frac{2^n a^n (abcd\, q^{n-1};q)_n}{(ab,ac,ad;q)_n}\,\tilde{p}_n(x;a,b,c,d|q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\,\middle|\, q;q\right),\ x=\cos\theta.$$

$$\rho(x) := \rho(x;a,b,c,d|q) = \frac{1}{\sqrt{1-x^2}}\left|\frac{(e^{2i\theta};q)_\infty}{(ae^{i\theta},be^{i\theta},ce^{i\theta},de^{i\theta};q)_\infty}\right|^2,\ x=\cos\theta.$$

If $a,\ b,\ c$ and d are real, or occur in complex conjugate pairs if complex, and if

$$\max(|a|,|b\,|c|,|d|)<1,$$

then we have the following orthogonality relation

$$\frac{1}{2\pi}\int_{-1}^{1}\rho(x)\,\tilde{p}_n(x;a,b,c,d|q)\,\tilde{p}_m(x;a,b,c,d|q)\,dx=h_n\,\delta_{n,m},$$

where

$$h_n=\frac{\left[2^n\ (abcdq^{n-1};q)_n\right]^{-2}\ (abcdq^{n-1};q)_n\ (abcdq^{2n};q)_\infty}{(q^{n+1},abq^n,acq^n,adq2^n,bcq^n,bdq^n,cdq^n;q)_\infty}.$$

For the lattice we write

$$x=\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{q^s+q^{-s}}{2}:=x(s),\ q^s:=e^{i\theta}.$$

It follows from (2.88) that $\alpha_x=\frac{q^{\frac12}+q^{-\frac12}}{2}$, $\beta_x=0$, $C_5=C_6=\frac12$ and $\delta_x=-\frac{(q-1)^2}{4q}$ by (3.15).

The Askey-Wilson operator

$$\mathcal{D}_q f(x):=\frac{\delta_q f(x)}{\delta_q x},\ x=\cos\theta,$$

with

$$\delta_q f(e^{i\theta})=f(q^{\frac12}e^{i\theta})-f(q^{-\frac12}e^{i\theta}),$$

can be written in terms of $\mathbb{D}_x$ as

$$\mathcal{D}_q f(x)=\mathbb{D}_{x_{-1}}f(x_{-1}(s)),\ x:=x(s)=\frac{q^s+q^{-s}}{2}. \tag{3.90}$$

In order to compute the polynomials ϕ and ψ, we write the Pearson-type equation (3.63) in its equivalent form (using (3.77))

$$\frac{\rho(s+1)}{\rho(s)}=\frac{\phi(s)+\frac12\psi(s)\nabla x_1(s)}{\phi(s+1)-\frac12\psi(s+1)\Delta x_1(s)}$$

and use the previous expression of the Askey-Wilson weight $\rho(x)\equiv\rho(x(s))\equiv\rho(s)$ to get

$$\frac{\rho(s+1)}{\rho(s)}=\frac{\left(q^{s+1}-1\right)\left(q^{s+1}+1\right)(-1+aq^s)\,q\,(-1+bq^s)\,(-1+cq^s)\,(-1+dq^s)}{(q^{s+1}-d)\,(-q^{s+1}+c)\,(q^{s+1}-b)\,(-q^{s+1}+a)\,(1+q^s)\,(-1+q^s)}.$$

Then we combine the last two equations and use the expansion (3.78) with $x(s)=\frac{q^s+q^{-s}}{2}$ to obtain a polynomial equation in q^s with coefficients depending linearly on those of the polynomials ϕ and ψ. Collection of different coefficients of the powers of q^s leads to a system of linear equations in the variables $\phi_2,\ \phi_1,\ \phi_0,\ \psi_1$ and ψ_0. Solving this system, we obtain the polynomials ϕ and ψ:

$$\begin{aligned}\phi\,(s)\ &=\ 2\,(dcba+1)\,x^2\,(s)-(a+b+c+d+abc+abd+acd+bcd)\,x\,(s)\\&\qquad+ab+ac+ad+bc+bd+cd-abcd-1,\\ \psi\,(s)\ &=\ \frac{4\,(abcd-1)\,q^{\frac12}\,x\,(s)}{q-1}+\frac{2\,(a+b+c+d-abc-abd-acd-bcd)\ q^{\frac12}}{q-1}.\end{aligned}$$

By using the relations (3.7), (3.10), (3.35), (3.36) and (3.90), we have checked that the difference equation (Equation (3.1.6) in [39])

$$(1-q)^2\,\mathcal{D}_q\left[\rho(x;aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}},|q)\,\mathcal{D}_q\tilde{p}_n(x)\right]+\lambda_n\,\rho(x;a,b,c,d|q)\,\tilde{p}_n(x)=0,\ x=\cos\theta,$$

with $\tilde{p}_n(x;a,b,c,d|q)\equiv\tilde{p}_n(x)$, is equivalent to

$$\phi(s)\,\mathbb{F}_x\tilde{p}_n(x(s))+\psi(s)\,\mathbb{M}_x\tilde{p}_n(x(s))+\lambda_n\,\tilde{p}_n(x)=0,$$

where the constant λ_n is given by (3.89) with the polynomials $\phi(s)$ and $\psi(s)$ given as above.

The recurrence coefficients are

$$\begin{aligned}\beta_n &= \frac{1}{2}(a+a^{-1}-A_n-C_n),\\ \gamma_n &= \frac{1}{4}A_{n-1}\,C_n,\end{aligned}$$

with

$$\begin{aligned}A_n &= \frac{(1-abq^n)\,(1-acq^n)\,(1-adq^n)\,(1-abcdq^{n-1})}{a\,(1-abcdq^{2\,n-1})\,(1-abcdq^{2\,n})},\\ C_n &= \frac{a\,(1-q^n)\,(1-bcq^{n-1})\,(1-bdq^{n-1})\,(1-cdq^{n-1})}{(1-abcdq^{2\,n-1})\,(1-abcdq^{2\,n-2})}.\end{aligned}$$

2. **q-Racah polynomials**

2.1. Hypergeometric representation:

$$\tilde{R}_n\left(x(s);\alpha,\beta,\gamma,\delta|q\right)=\frac{(\alpha q,\beta\delta q,\gamma q;q)_n}{(\alpha\beta q^{n+1};q)_n}\,{}_4\phi_3\left(\left.\begin{array}{c}q^{-n},\alpha\beta q^{n+1},q^{-s},\gamma\delta q^{s+1}\\ \alpha q,\beta\delta q,\gamma q\end{array}\right|q;q\right),\ n=0,\,1,\,2,\,\ldots,\,N,$$

where $x(s)=q^{-s}+\gamma\delta q^{s+1}$, and

$$\alpha q=q^{-N}\ \text{or}\ \beta\delta q=q^{-N}\ \text{or}\ \gamma q=q^{-N},\ \text{with } N \text{ a nonnegative integer.}$$

2.2. Weight function:

$$\rho\left(s;\alpha,\beta,\gamma,\delta|q\right)=\frac{(\alpha q,\beta\delta q,\gamma q,\gamma\delta q;q)_s}{(q,\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_s}\frac{(1-\gamma\delta q^{2s+1})}{(\alpha\beta q)^s\,(1-\gamma\delta q)}\frac{1}{\nabla x_1(s)}.$$

2.3. Orthogonality relation:

$$\sum_{s=0}^{N}\tilde{R}_n\left(x(s);\alpha,\beta,\gamma,\delta|q\right)\,\tilde{R}_m\left(x(s);\alpha,\beta,\gamma,\delta|q\right)\,\rho\left(s;\alpha,\beta,\gamma,\delta|q\right)\,\nabla x_1(s)=h_n\,\delta_{n,m},$$

where

$$\begin{aligned}h_n &= \frac{\left(\alpha^{-1}\beta^{-1}\gamma,\alpha^{-1}\delta,\beta^{-1},\gamma\delta q^2;q\right)_\infty}{(\alpha^{-1}\beta^{-1}q^{-1},\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_\infty}\times\\ &\quad\frac{(1-\alpha\beta q)(\gamma\delta q)^n}{(1-\alpha\beta q^{2n+1})}\frac{\left(q,\alpha\beta\gamma^{-1}q,\alpha\delta^{-1}q,\beta q;q\right)_n}{(\alpha q,\alpha\beta q,\beta\gamma q,\beta\delta q,\gamma q;q)_n}\left[\frac{(\alpha q,\beta\delta q,\gamma q;q)_n}{(\alpha\beta q^{n+1};q)_n}\right]^2.\end{aligned}$$

2.4. The lattice:

$$x(s) = q^{-s} + \gamma\delta q^{s+1},\ \alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -(q-1)^2\gamma\delta.$$

2.5. Polynomials ϕ and ψ:

$$\begin{aligned}
\phi(s) &= \left(q^2\alpha\,\beta+1\right)x^2(s) - \left(q\beta\delta\,\gamma + q\alpha\,\gamma + q\alpha\,\beta\delta + \alpha\,\beta\,q + \alpha + \beta\delta + \gamma\delta + \gamma\right) q\,x(s) \\
&\quad -2\left(q^2\alpha\,\gamma\delta\,\beta - q\beta\delta\,\gamma - q\alpha\,\beta\delta - q\gamma^2\delta - q\alpha\,\gamma\,\delta - q\alpha\,\gamma - q\gamma\,\delta^2\beta + \gamma\delta\right) q, \\
\psi(s) &= \frac{2\left(q^2\alpha\,\beta-1\right)q^{\frac{1}{2}}x(s)}{q-1} - \frac{2q\left(q\beta\delta\,\gamma + q\alpha\,\gamma + q\alpha\,\beta\delta + \alpha\,\beta\,q - \alpha - \beta\delta - \gamma\delta - \gamma\right)q^{\frac{1}{2}}}{q-1}.
\end{aligned}$$

2.6. Recurrence coefficients:

$$\begin{aligned}
\beta_n &= 1 + \gamma\delta q - A_n - C_n, \\
\gamma_n &= A_{n-1}\,C_n,
\end{aligned}$$

with

$$\begin{aligned}
A_n &= \frac{(1-\alpha q^{n+1})(1-\alpha\beta q^{n+1})(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})}, \\
C_n &= \frac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}.
\end{aligned}$$

3. **Continuous dual q-Hahn polynomials**

3.1. Hypergeometric representation:

$$\tilde{p}_n(x;a,b,c|q) = 2^{-n}a^{-n}\,(ab,ac;q)_n\,{}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac \end{matrix}\,\middle|\, q;q\right),\ x = \cos\theta.$$

3.2. Weight function:

$$\rho(x) := \rho(x;a,b,c,d|q) = \frac{1}{\sqrt{1-x^2}}\left|\frac{(e^{2i\theta};q)_\infty}{(ae^{i\theta},be^{i\theta},ce^{i\theta};q)_\infty}\right|^2,\ x = \cos\theta.$$

3.3. Orthogonality relation:
If $a,\ b$ and d are real, or one is real and the other two are complex conjugates, and if

$$\max(|a|,|b\,|c|) < 1,$$

then we have

$$\frac{1}{2\pi}\int_{-1}^{1}\rho(x)\,\tilde{p}_n(x;a,b,c|q)\,\tilde{p}_m(x;a,b,c|q)\,dx = h_n\,\delta_{n,m},$$

where

$$h_n = \frac{4^{-n}}{(q^{n+1},abq^n,acq^n,bcq^n;q)_\infty}.$$

3.4. The lattice:

$$x(s)=\frac{q^{s}+q^{-s}}{2},\ \alpha_x=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\ \beta_x=0,\ \delta_x=-\frac{(q-1)^2}{4q}$$

3.5. Polynomials ϕ and ψ:

$$\begin{aligned}\phi\,(s) &= 2\,x^2\,(s)-(a+b+c+abc)\,x\,(s)+\,ab+ac+bc-1,\\ \psi\,(s) &= -\frac{4\,q^{\frac{1}{2}}\,x\,(s)}{q-1}+\frac{2\,(a+b+c-abc)\;q^{\frac{1}{2}}}{q-1}.\end{aligned}$$

3.6. Recurrence coefficients:

$$\begin{aligned}\beta_n &= \frac{1}{2}(a+a^{-1}-A_n-C_n),\\ \gamma_n &= \frac{1}{4}A_{n-1}\,C_n,\end{aligned}$$

with

$$\begin{aligned}A_n &= a^{-1}\,(1-abq^n)\,(1-acq^n)\,,\\ C_n &= a\,(1-q^n)\left(1-bcq^{n-1}\right).\end{aligned}$$

4. **Continuous q-Hahn polynomials**

4.1. Hypergeometric representation:

$$\tilde{p}_n(x;a,b,c,d|q)=\frac{(abe^{2i\varphi},ac,ad;q)}{(2ae^{i\varphi})^n\;(abcdq^{n-1};q)_n}{}_4\phi_3\left(\begin{matrix}q^{-n},abcdq^{n-1},ae^{i(\theta+2\varphi)},ae^{-i\theta}\\ abe^{2i\varphi},ac,ad\end{matrix}\,\middle|\,q;q\right),\ x=\cos(\theta+\varphi).$$

4.2. Weight function:

$$\rho(x):=\rho(x;a,b,c,d|q)=\left|\frac{(e^{2i(\theta+\varphi)};q)_\infty}{(ae^{i(\theta+2\varphi)},be^{i(\theta+2\varphi)},ce^{i\theta},de^{i\theta};q)_\infty}\right|^2,\ x=\cos(\theta+\varphi).$$

4.3. Orthogonality relation:
If $c=a$ and $d=b$ then we have, if a and b are real with $\max(|a|,|b|)<1$ or if $b=\bar{a}$ and $|a|<1$ then we have

$$\frac{1}{2\pi}\int_{-1}^{1}\rho(x)\,\tilde{p}_n(x;a,b,c,d|q)\,\tilde{p}_m(x;a,b,c,d|q)\,dx=h_n\,\delta_{n,m},\ x=\cos(\theta+\varphi),$$

where

$$h_n=\frac{4^{-n}\;(abcdq^{n-1};q)_n^{-1}\;(abcdq^{2n};q)_\infty}{(q^{n+1},abq^ne^{2i\varphi},acq^n,adq2^n,bcq^n,bdq^n,cdq^ne^{-2i\varphi};q)_\infty}.$$

4.4. The lattice:

$$x(s)=\frac{e^{i\varphi}\,q^{s}+e^{-i\varphi}\,q^{-s}}{2},\ \alpha_x=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\ \beta_x=0,\ \delta_x=-\frac{(q-1)^2}{4q},\ e^{i\theta}:=q^s.$$

4.5. Polynomials ϕ and ψ:

$$\begin{aligned}\phi(s) &= 2(dcba+1)x^2(s) - \frac{\left(d+dcb+at^2+bt^2ad+cbat^2+c+cda+bt^2\right)x(s)}{t}\\ &+\frac{cat^2+bt^2d-t^2cbad+cbt^2+cd+t^2+bt^4a+t^2ad}{t^2},\\ \psi(s) &= \frac{4(-1+dcba)q^{\frac{1}{2}}x(s)}{q-1}-\frac{2q^{\frac{1}{2}}\left(-c-d+cda-bt^2-at^2+dcb+cbat^2+bt^2ad\right)}{(q-1)t},\end{aligned}$$

with the notation $t=e^{i\varphi}$.

4.6. Recurrence coefficients:

$$\begin{aligned}\beta_n &= \frac{1}{2}(ae^{i\varphi}+a^{-1}e^{-i\varphi}-A_n-C_n),\\ \gamma_n &= \frac{1}{4}A_{n-1}C_n,\end{aligned}$$

with

$$\begin{aligned}A_n &= \frac{(1-abq^ne^{2i\varphi})(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{ae^{i\varphi}(1-abcdq^{2n-1})(1-abcdq^{2n})},\\ C_n &= \frac{ae^{i\varphi}(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1}e^{-2i\varphi})}{(1-abcdq^{2n-1})(1-abcdq^{2n-2})}.\end{aligned}$$

5. **Dual q-Hahn polynomials**

5.1. Hypergeometric representation:

$$\tilde{R}_n(x(s);\gamma,\delta,N|q)=\left(\gamma q,q^{-N};q\right)_n\,{}_3\phi_2\left(\left.\begin{array}{c}q^{-n},q^{-s},q^{-s},\gamma\delta q^{s+1}\\ \gamma q,q^{-N}\end{array}\right|q;q\right),\ n=0,\,1,\,2,\,\ldots,\,N,$$

where $x(s)=q^{-s}+\gamma\delta q^{s+1}$ and N a nonnegative integer.

5.2. Weight function:

$$\rho(s;\alpha,\beta,\gamma,\delta|q)=\frac{(\alpha q,\beta\delta q,\gamma q,\gamma\delta q;q)_s}{(q,\gamma\delta q^{N+2},\delta q;q)_s}\frac{(1-\gamma\delta q^{2s+1})}{(-\gamma q)^s(1-\gamma\delta q)}\frac{q^{Ns-\binom{s}{2}}}{\nabla x_1(s)}.$$

5.3. Orthogonality relation:

For $0<\gamma<q^{-1}$ and $0<\delta<q^{-1}$ or $\gamma>q^{-1}$ and $\delta>q^{-1}$, the orthogonality relation is

$$\sum_{s=0}^{N}\tilde{R}_n(x(s);\alpha,\beta,\gamma,\delta|q)\ \tilde{R}_m(x(s);\alpha,\beta,\gamma,\delta|q)\ \rho(s;\alpha,\beta,\gamma,\delta|q)\ \nabla x_1(s)=h_n\,\delta_{n,m},$$

where

$$h_n=\frac{(\gamma\delta q^2;q)_N\,\left(q,\delta^{-1}q^{-N};q\right)\,(\gamma\delta q)^n}{(\gamma q)^N(\delta q;q)_N}.$$

5.4. The lattice:

$$x(s)=q^{-s}+\gamma\delta q^{s+1},\ \alpha_x=\frac{q^{\frac12}+q^{-\frac12}}{2},\ \beta_x=0,\ \delta_x=-(q-1)^2\gamma\delta.$$

5.5. Polynomials ϕ and ψ:

$$\begin{aligned}\phi(s) &= x^2(s)-\frac{\left(\gamma q+q^Nq\gamma\delta+\gamma q^Nq+1\right)x(s)}{2q^N}+\frac{\gamma\left(q^Nq\gamma\delta-\delta q^N+\delta+1\right)q}{2q^N},\\ \psi(s) &= -\frac{2q^{\frac12}x(s)}{q-1}+\frac{\left(q^Nq\gamma\delta+\gamma q^Nq-\gamma q+1\right)q^{\frac12}}{(q-1)q^N}.\end{aligned}$$

5.6. Recurrence coefficients:

$$\begin{aligned}\beta_n &= 1+\gamma\delta q-A_n-C_n,\\ \gamma_n &= A_{n-1}C_n,\end{aligned}$$

with

$$\begin{aligned}A_n &= \left(1-q^{n-N}\right)\left(1-\gamma q^{n+1}\right),\\ C_n &= \gamma q\left(1-q^n\right)\left(\delta-q^{n-N-1}\right).\end{aligned}$$

6. **Al-Salam-Chihara polynomials**

6.1. Hypergeometric representation:

$$\tilde Q_n(x;a,b|q)=2^{-n}a^{-n}(ab;q)_n\,{}_3\phi_2\left(\left.\begin{matrix}q^{-n},ae^{i\theta},ae^{-i\theta}\\ ab,0\end{matrix}\right|q;q\right),\ x=\cos\theta.$$

6.2. Weight function:

$$\rho(x):=\rho(x;a,b|q)=\frac{1}{\sqrt{1-x^2}}\left|\frac{(e^{2i\theta};q)_\infty}{(ae^{i\theta},be^{i\theta};q)_\infty}\right|^2,\ x=\cos\theta.$$

6.3. Orthogonality relation:

If a and b are real, or complex conjugates and $\max(|a|,|b|)<1$, then we have the following orthogonality relation

$$\frac{1}{2\pi}\int_{-1}^{1}\rho(x)\,\tilde Q_n(x;a,b|q)\,\tilde Q_m(x;a,b|q)\,dx=\frac{\delta_{n,m}}{4^n\,(q^{n+1},abq^n;q)}.$$

6.4. The lattice:

$$x(s)=\frac{q^s+q^{-s}}{2},\ \alpha_x=\frac{q^{\frac12}+q^{-\frac12}}{2},\ \beta_x=0,\ \delta_x=-\frac{(q-1)^2}{4q}.$$

6.5. Polynomials ϕ and ψ:

$$\phi(s) = 2\,x^2(s) - (a+b)\,x(s) + ab - 1, \quad \psi(s) = -\frac{4q^{\frac{1}{2}}\,x(s)}{q-1} + \frac{2\,(a+b)\,q^{\frac{1}{2}}}{q-1}.$$

6.6. Recurrence coefficients:

$$\beta_n = \frac{1}{2}(a+b)q^n, \quad \gamma_n = \frac{1}{4}\,(1-q^n)\left(1-abq^{n-1}\right).$$

7. **q-Meixner-Pollaczek polynomials**

7.1. Hypergeometric representation:

$$\tilde{P}_n(x;a|q) = \frac{(a^2;q)_n\,(q;q)_n}{(2\,a\,e^{i\varphi})^n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i(\theta+2\varphi)}, ae^{-i\theta} \\ a^2, 0 \end{matrix}\,\middle|\, q;q\right), \; x = \cos(\theta+\varphi).$$

7.2. Weight function:

$$\rho(x) := \rho(x;a,b,c,d|q) = \left|\frac{(e^{2i(\theta+\varphi)};q)_\infty}{(ae^{i(\theta+2\varphi)}, ae^{i\theta};q)_\infty}\right|^2, \; x = \cos(\theta+\varphi).$$

7.3. Orthogonality relation:

For $0 < a < 1$ we have

$$\frac{1}{2\pi}\int_{-1}^{1}\rho(x)\,\tilde{P}_n(x;a|q)\,\tilde{P}_m(x;a|q)\,dx = \frac{(q;q)_n}{4^n\,(q,a^2q^n;q)_\infty}\,\delta_{n,m}, \; x = \cos(\theta+\varphi).$$

7.4. The lattice:

$$x(s) = \frac{e^{i\varphi}\,q^s + e^{-i\varphi}\,q^{-s}}{2}, \; \alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}, \; \beta_x = 0, \; \delta_x = -\frac{(q-1)^2}{4q}, \; e^{i\theta} := q^s.$$

7.5. Polynomials ϕ and ψ:

$$\phi(s) = 2\,x^2(s) - 2\,a\,\cos\varphi\,x(s) + a^2 - 1, \; \psi(s) = -\frac{4\,q^{\frac{1}{2}}\,x(s)}{q-1} + \frac{4\,a\,q^{\frac{1}{2}}\,\cos\varphi}{q-1}.$$

7.6. Recurrence coefficients:

$$\beta_n = a\,q^n\,\cos\varphi, \quad \gamma_n = \frac{1}{4}(1-q^n)(1-a^2q^{n-1}).$$

8. **Continuous q-Jacobi polynomials**

8.1. Hypergeometric representation:

$$\tilde{P}_n^{(\alpha,\beta)}(x|q) = \frac{\left(q^{\alpha+1};q\right)_n\left(q, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)};q\right)_n}{2^n\,(q;q)_n\,q^{(\frac{1}{2}\alpha+\frac{1}{4})n}\,\left(q^{n+\alpha+\beta+1};q\right)_n} \times$$
$${}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{1}{2}\alpha+\frac{1}{4}}\,e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}\,e^{-i\theta} \\ q^{\alpha+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)} \end{matrix}\,\middle|\, q;q\right), \; x = \cos\theta.$$

8.2. Weight function:

$$\rho(x) := \rho(x;a,b,c,d|q) = \frac{1}{\sqrt{1-x^2}} \left| \frac{(e^{i\theta}, -e^{i\theta}; q^{\frac{1}{2}})_\infty}{\left(q^{\frac{1}{2}\alpha+\frac{1}{4}}\, e^{i\theta}, -q^{\frac{1}{2}\beta+\frac{1}{4}}\, e^{i\theta}; q^{\frac{1}{2}}\right)_\infty} \right|^2,\ x = \cos\theta.$$

8.3. Orthogonality relation:

For $\alpha \geq -\frac{1}{2}$ and $\beta \geq -\frac{1}{2}$ we have

$$\frac{1}{2\pi}\int_{-1}^{1} \rho(x)\, \tilde{P}_n^{(\alpha,\beta)}(x|q)\, \tilde{P}_m^{(\alpha,\beta)}(x|q)\, dx = h_n\, \delta_{n,m},$$

where

$$\begin{aligned} h_n \;=\; & \left[\frac{\left(q, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)}; q\right)_n}{2^n\, q^{(\frac{1}{2}\alpha+\frac{1}{4})n}\, (q^{n+\alpha+\beta+1};q)_n} \right]^2 \times \\ & \frac{\left(q^{\frac{1}{2}(\alpha+\beta+2)}, q^{\frac{1}{2}(\alpha+\beta+3)}; q\right)_\infty}{\left(q, q^{\alpha+1}, q^{\beta+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)}; q\right)_\infty} \times \\ & \frac{\left(1-q^{\alpha+\beta+1}\right)\left(q^{\alpha+1}, q^{\beta+1}, -q^{\frac{1}{2}(\alpha+\beta+3)}, ; q\right)_n}{(1-q^{2n+\alpha+\beta+1})\left(q, q^{\alpha+\beta+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}; q\right)_n}. \end{aligned}$$

8.4. The lattice:

$$x(s) = \frac{q^s+q^{-s}}{2},\ \alpha_x = \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -\frac{(q-1)^2}{4q}$$

8.5. Coefficients of the polynomials ϕ and ψ:

$$\begin{aligned} \phi_2 &= p^{2\alpha+2\beta+4}+1,\ \ \phi_1 = \frac{1}{2}\,(p+1)\,p^{\frac{1}{2}}\left(p^{2\alpha+\beta+2}-p^{\alpha}-p^{\alpha+2\beta+2}+p^{\beta}\right), \\ \phi_0 &= -\frac{1}{2}\,p^{2\alpha+2\beta+4}-\frac{1}{2}\,p^{\alpha+\beta+3}+\frac{1}{2}\,p^{2\alpha+2}-p^{\alpha+\beta+2}+\frac{1}{2}\,p^{2\beta+2}-\frac{1}{2}\,p^{1+\alpha+\beta}-\frac{1}{2}, \\ \psi_1 &= \frac{2\,p\,(p^{2\alpha+2\beta+4}-1)}{(p-1)\,(p+1)},\ \ \psi_0 = -\frac{p^{\frac{3}{2}}\left(-p^{2\alpha+\beta+2}-p^{\alpha}+p^{\alpha+2\beta+2}+p^{\beta}\right)}{p-1}, \end{aligned}$$

with $q = p^2$.

8.6. Recurrence coefficients:

$$\begin{aligned} \beta_n &= \frac{1}{2}(q^{\frac{1}{2}\alpha+\frac{1}{4}}+q^{-\frac{1}{2}\alpha-\frac{1}{4}}-A_n-C_n), \\ \gamma_n &= \frac{1}{4}A_{n-1}\,C_n, \end{aligned}$$

with

$$A_n = \frac{\left(1-q^{n+\alpha+1}\right)\left(1-q^{n+\alpha+\beta+1}\right)\left(1+q^{n+\frac{1}{2}(\alpha+\beta+1)}\right)\left(1+q^{n+\frac{1}{2}(\alpha+\beta+2)}\right)}{q^{\frac{1}{2}\alpha+\frac{1}{4}}\left(1-q^{2n+\alpha+\beta+1}\right)\left(1-q^{2n+\alpha+\beta+2}\right)},$$

$$C_n = \frac{q^{\frac{1}{2}\alpha+\frac{1}{4}}\left(1-q^{n}\right)\left(1-q^{n+\beta}\right)\left(1+q^{n+\frac{1}{2}(\alpha+\beta)}\right)\left(1+q^{n+\frac{1}{2}(\alpha+\beta+1)}\right)}{\left(1-q^{2n+\alpha+\beta}\right)\left(1-q^{2n+\alpha+\beta+1}\right)}.$$

9. **Continuous Dual q-Krawtchouk polynomials**

9.1. Hypergeometric representation:

$$\tilde{K}_n(x(s);c,N|q) = \left(q^{-N};q\right)_n \, {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-s}, cq^{s-N} \\ q^{-N}, 0 \end{matrix} \,\middle|\, q;q\right),\ n=0,\,1,\,2,\ldots,N,$$

where $x(s) = q^{-s} + cq^{s-N}$.

9.2. Weight function:

$$\rho(s;c,N|q) := \rho(s) = \frac{\left(cq^{-N},q^{-N};q\right)_s}{(q,cq;q)_s}\frac{\left(1-cq^{2s-N}\right)}{\left(1-cq^{-N}\right)}\frac{c^{-S}\,q^{s(2N-s)}}{\nabla x_1(s)}.$$

9.3. Orthogonality relation:

$$\sum_{s=0}^{N}\tilde{K}_n(x(s);c,N|q)\,\tilde{K}_m(x(s);c,N|q)\,\rho(s)\,\nabla x_1(s) = \frac{(c^{-1};q)_N\,(q;q)_n\,\left(q^{-N};q\right)_n}{\left(cq^{-N}\right)^{-n}}\delta_{n,m},\ c<0.$$

9.4. The lattice:

$$x(s) = q^{-s}+cq^{s-N},\ \alpha_x = \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -c(q-1)^2\,q^{-1-N}.$$

9.5. Polynomials ϕ and ψ:

$$\phi(s) = x^2(s) - (c+1)\,q^{-N}\,x(s) - 2\,c\,(q^{-N}-q^{-2N}),\ \psi(s) = -\frac{2\,q^{\frac{1}{2}}}{q-1}x(s) + \frac{2(c+1)\,q^{\frac{1}{2}}}{(q-1)q^N}.$$

9.6. Recurrence coefficients:

$$\beta_n = (1+c)\,q^{n-N},\ \gamma_n = c(1-q^n)\,(1-q^{n-N-1})\,q^{-N}.$$

10. **Continuous big q-Hermite polynomials**

10.1. Hypergeometric representation:

$$\tilde{H}_n(x;a|q) = 2^{-n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ 0, 0 \end{matrix} \,\middle|\, q;q\right).$$

10.2. Weight function:

$$\rho(x;a|q) = \frac{1}{\sqrt{1-x^2}}\left|\frac{\left(e^{2i\theta};q\right)_\infty}{\left(ae^{i\theta};q\right)_\infty}\right|^2,\ x=\cos\theta.$$

10.3. Orthogonality relation:

If a is real and $|a| < 1$, we have

$$\frac{1}{2\,\pi}\int_{-1}^{1}\rho(x;a|q)\,\tilde{H}_n(x;a|q)\,\tilde{H}_m(x;a|q)\,dx=\frac{\delta_{n,m}}{4^n\;(q^{n+1};q)_\infty}.$$

10.4. The lattice:

$$x(s)=\frac{q^s+q^{-s}}{2},\;\alpha_x=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\;\beta_x=0,\;\delta_x=-\frac{(q-1)^2}{4q}.$$

10.5. Polynomials ϕ and ψ:

$$\phi(s)=2\,x^2(s)-a\,x(s)-1,\;\psi(s)=-\frac{4\,q^{\frac{1}{2}}}{q-1}\,x(s)+\frac{2\,a\,q^{\frac{1}{2}}}{q-1}.$$

10.6. Recurrence coefficients:

$$\beta_n=\frac{1}{2}\,a\,q^n,\;\gamma_n=\frac{1}{4}\,(1-q^n).$$

11. **Continuous q-Laguerre polynomials**

11.1. Hypergeometric representation:

$$\tilde{P}_n^{(\alpha)}(x|q)=2^{-n}\,q^{-(\frac{1}{2}\alpha+\frac{1}{4})n}\;\left(q^{\alpha+1};q\right)_n\;{}_3\phi_2\!\left(\left.\begin{matrix}q^{-n},q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta},q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta}\\ q^{\alpha+1},0\end{matrix}\,\right|\,q;q\right),\;x=\cos\theta.$$

112.2. Weight function:

$$\rho(x;q^\alpha|q)=\frac{1}{\sqrt{1-x^2}}\left|\frac{\left(e^{i\theta},e^{-i\theta};q^{\frac{1}{2}}\right)_\infty}{\left(q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta};q^{\frac{1}{2}}\right)_\infty}\right|^2,\;x=\cos\theta.$$

11.3. Orthogonality relation:

For $\alpha\ge-\frac{1}{2}$, we have

$$\frac{1}{2\,\pi}\int_{-1}^{1}\rho(x;q^\alpha|q)\,\tilde{P}_n^{(\alpha)}(x|q)\,\tilde{P}_m^{(\alpha)}(x|q)\,dx=\frac{(q;q)_n}{(q,q^{\alpha+1};q)_\infty}\frac{(q^{\alpha+1};q)_n}{4^n}\,\delta_{n,m}.$$

11.4. The lattice:

$$x(s)=\frac{q^s+q^{-s}}{2},\;\alpha_x=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\;\beta_x=0,\;\delta_x=-\frac{(q-1)^2}{4q}.$$

11.5. Polynomials ϕ and ψ:

$$\phi(s)=2\,x^2\,(s)-p^{\alpha+\frac{1}{2}}\;(p+1)\,x\,(s)+p^{2\alpha+2}-1,\;\psi\,(s)=-\frac{4p}{p^2-1}x\,(s)+\frac{2p^{\alpha+\frac{3}{2}}}{p-1}.$$

11.6. Recurrence coefficients:

$$\beta_n = \frac{1}{2} q^{n+\frac{1}{2}\alpha+\frac{1}{4}} \,(1+q^{\frac{1}{2}}),\ \gamma_n = \frac{1}{4}\,(1-q^n)\,(1-q^{n+\alpha}).$$

12. **Continuous q-Hermite polynomials**

12.1. Hypergeometric representation:

$$\tilde{H}_n(x|q) = 2^{-n}\, e^{in\theta}\, {}_2\phi_0\left(\begin{array}{c} q^{-n}, 0 \\ - \end{array} \middle|\, q; q^n\, e^{-2i\theta} \right),\ x = \cos\theta.$$

12.2. Weight function:

$$\rho(x;|q) = \frac{1}{\sqrt{1-x^2}}\, \left|\left(e^{2i\theta};q\right)_\infty\right|^2,\ x = \cos\theta.$$

12.3. Orthogonality relation:

$$\frac{1}{2\,\pi} \int_{-1}^{1} \rho(x;|q)\, \tilde{H}_n(x|q)\, \tilde{H}_m(x|q)\, dx = \frac{\delta_{n,m}}{4^n\, \left(q^{n+1};q\right)_\infty}.$$

12.4. The lattice:

$$x(s) = \frac{q^s + q^{-s}}{2},\ \alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -\frac{(q-1)^2}{4q}.$$

12.5. Polynomials ϕ and ψ:

$$\phi(s) = 2\,x^2(s) - 1,\ \psi(s) = -\frac{4\,q^{\frac{1}{2}}}{q-1}\, x(s).$$

12.6. Recurrence coefficients:

$$\beta_n = 0,\ \gamma_n = \frac{1}{4}\,(1-q^n).$$

Part II: Cases of quadratic lattices

13. **Racah polynomials**

13.1. Hypergeometric representation:

$$\tilde{R}_n(x(s);\alpha,\beta,\gamma,\delta) = \frac{(\alpha+1)_n\,(\beta+\delta+1)_n\,(\gamma+1)_n}{(n+\alpha+\beta+1)_n} \times {}_4F_3\left(\begin{array}{c} -n, n+\alpha+\beta+1, -s, s+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{array} \middle|\, 1 \right),\ n = 0,\,1,\,2,\,\ldots,\,N,$$

where $x(s) = s\,(s+\gamma+\delta+1)$ and

$$\alpha+1=-N \text{ or } \beta+\delta+1=-N \text{ or } \gamma+1=-N, \text{ with } N \text{ a nonnegative integer.}$$

13.2. Weight function:

$$\rho(s;\alpha,\beta,\gamma,\delta) := \rho(s) = \frac{(\alpha+1)_s(\beta+\delta+1)_s(\gamma+1)_s(\gamma+\delta+1)_s\left(\frac{\gamma+\delta+3}{2}\right)_s}{(-\alpha+\gamma+\delta+1)_s(-\beta+\gamma+1)_s\left(\frac{\gamma+\delta+1}{2}\right)_s(\delta+1)_s\; s!}\frac{1}{\nabla x_1(s)}.$$

13.3. Orthogonality relation:

$$\sum_{s=0}^{N} \tilde{R}_n(x(s);\alpha,\beta,\gamma,\delta)\,\tilde{R}_m(x(s);\alpha,\beta,\gamma,\delta)\,\rho(s)\,\nabla x_1(s) = h_n\,\delta_{n,m},$$

where

$$h_n = M\,\frac{(\alpha+\beta-\gamma+1)_n\,(\alpha-\delta+1)_n\,(\beta+1)_n\,n!}{(\alpha+\beta+2)_{2n}}\frac{(\alpha+1)_n\,(\beta+\delta+1)_n\,(\gamma+1)_n}{(n+\alpha+\beta+1)_n},$$

where

$$M = \begin{cases} \frac{(-\beta)_N\,(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N\,(\delta+1)_N} & \text{if } \alpha+1=-N, \\ \frac{(-\alpha+\delta)_N\,(\gamma+\delta+2)_N}{(-\alpha+\gamma+\delta+1)_N\,(\delta+1)_N} & \text{if } \beta+\delta+1=-N, \\ \frac{(\alpha+\beta+2)_N\,(-\delta)_N}{(\alpha-\delta+1)_N\,(\beta+1)_N} & \text{if } \gamma+1=-N. \end{cases}$$

13.4. The lattice:

$$x(s) = s\,(s+\gamma+\delta+1),\ \alpha_x = 1,\ \beta_x = \frac{1}{4},\ \delta_x = \frac{(\gamma+\delta+1)^2}{4}.$$

13.5. Polynomials ϕ and ψ:

$$\begin{aligned} \phi\,(s) &= 2\,x^2\,(s) + [-\beta\,\delta+\beta\,\gamma+2\,\gamma+\alpha\,\gamma+2\,\gamma\,\delta+4+\alpha\,\delta \\ &\quad +3\,\alpha+3\,\beta+2\,\beta\,\alpha+2\,\delta]\,x\,(s) + (1+\gamma)\,(1+\delta+\gamma)\,(\delta+\beta+1)\,(\alpha+1)\,, \\ \psi\,(s) &= (2\,\alpha+2\,\beta+4)\,x\,(s) + 2\,(1+\gamma)\,(\delta+\beta+1)\,(\alpha+1)\,. \end{aligned}$$

13.6. Recurrence coefficients:

$$\beta_n = -A_n - C_n,\quad \gamma_n = A_{n-1}\,C_n,$$

where

$$\begin{aligned} A_n &= \frac{(n+\alpha+1)\,(n+\alpha+\beta+1)\,(n+\beta+\delta+1)\,(n+\gamma+1)}{(2\,n+\alpha+\beta+1)\,(2\,n+\alpha+\beta+2)}, \\ C_n &= \frac{n\,(n+\alpha+\beta-\gamma)\,(n+\alpha-\delta)\,(n+\beta)}{(2\,n+\alpha+\beta)\,(2\,n+\alpha+\beta+1)}. \end{aligned}$$

14. **Dual Hahn polynomials**

14.1. Hypergeometric representation:

$$\tilde{R}_n(x(s);\gamma,\delta,N) = (\gamma+1)_n\,(-N)_n\,{}_4F_3\left(\left.\begin{matrix} -n, -s, s+\gamma+\delta+1 \\ \gamma+1, -N \end{matrix}\right| 1\right),\ n=0,\,1,\,2,\,\ldots,\,N,$$

where $x(s) = s\,(s+\gamma+\delta+1)$.

14.2. Weight function:

$$\rho(s;\gamma,\delta) := \rho(s) = \frac{(2s+\gamma+\delta+1)\,(\gamma+1)_s\,(-N)_s\,N!}{(-1)^s\,(s+\gamma+\delta+1)_{N+1}\,(\delta+1)_s\,s!}\frac{1}{\nabla x_1(s)}.$$

14.3. Orthogonality relation:

For $\gamma > -1$ and $\delta > -1$ or $\gamma < -N$ and $\delta < -N$, we have

$$\sum_{s=0}^{N} \tilde{R}_n(x(s);\gamma,\delta,N)\,\tilde{R}_m(x(s);\gamma,\delta,N)\rho(s)\,\nabla x_1(s) = \frac{\left[(\gamma+1)_n\,(-N)_n\right]^2}{\binom{\gamma+n}{n}\binom{\delta+N-n}{N-n}}\delta_{n,m}.$$

14.4. The lattice:

$$x(s) = s\,(s+\gamma+\delta+1),\ \alpha_x = 1,\ \beta_x = \frac{1}{4},\ \delta_x = \frac{(\gamma+\delta+1)^2}{4}.$$

14.5. Polynomials ϕ and ψ:

$$\phi\,(s) = (-1+2\,N+\delta-\gamma)\,x\,(s) + N\,(1+\gamma)\,(1+\delta+\gamma)\,,\ \psi\,(s) = -2\,x\,(s) + 2\,N\,(1+\gamma)\,.$$

14.6. Recurrence coefficients:

$$\beta_n = -A_n - C_n,\quad \gamma_n = C_n,$$

where

$$A_n = (n+\gamma+1)\,(n-N),\quad C_n = n\,(n-\delta-N-1).$$

3.3 Fourth-order difference equations for orthogonal polynomials

3.3.1 Intermediate relations

To derive the fourth-order difference equations for the modifications of the classical orthogonal polynomials, we shall start with the following intermediate result.

Theorem 3.7 *Let (P_n) be a sequence of classical orthogonal polynomials satisfying the second-order difference equation (3.76) with the corresponding orthogonality weight satisfying the Pearson-type equation (3.63) and the border conditions (3.65) where the functions $\sigma(s)$ and* *the polynomials $\phi(s)$ and $\psi(s)$ are related by (3.77). Then, the first associate $(P_n^{(1)})$ of (P_n) satisfies*

$$\phi(s)\,[A_1(s)\,\mathbb{F}_x + B_1(s)\,\mathbb{M}_x + C_1(n)]\,P_{n-1}^{(1)}(x(s)) = 2\,\eta\,[D_1(s)\,\mathbb{M}_x + E_1(s,n)]\,P_n(s), \quad (3.91)$$

with

$$\begin{aligned} A_1(s) &= -\phi(s) - 2\alpha^2\,\mathrm{U}_2(s)\,\phi'' + 2\,\alpha\,U_1(s)\,\psi(s) - 2\,U_1(s)\,\mathbb{M}_x\phi(s) \\ &\quad +4\,\alpha\,U_1^2(s)\,\psi' + 2\alpha\,(2\alpha^2-1)\,U_2(s)\,\psi', \\ B_1(s) &= (2\,\alpha^2-1)\,\psi(s) - 2\,\alpha\,\mathbb{M}_x\phi(s) + 2\,(4\alpha^2-1)\,U_1(s)\,\psi' - 2\,\alpha\,U_1(s)\,\phi'', \quad (3.92) \\ C_1(n) &= (2\,\alpha^2-1)\,\psi' - \alpha\,\phi'' - \lambda_n, \\ D_1(s) &= \alpha\,\phi(s) - U_1(s)\,\psi(s), \\ E_1(s,n) &= -\lambda_n\,U_1(s), \end{aligned}$$

where the polynomials U_1 and U_2 are defined by (3.34),

$$\eta = \left(\alpha_x\,\psi' - \frac{\phi''}{2}\right)\gamma_0 = (\alpha_x\,\psi_1 - \phi_2)\,\gamma_0, \quad (3.93)$$

and

$$\gamma_0 = \sum_{s=0}^{N}\rho(s)\,\nabla x_1(s) \ \text{ or } \ \gamma_0 = \int_C \rho(s)\,\nabla x_1(s)\,ds \quad (3.94)$$

for discrete and continuous orthogonality respectively.

The proof of the theorem will use the following lemma.

Lemma 3.3 *The function of second kind $Q_n(x(s))$ defined by (2.91) satisfies:*

$$(\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s))\,\rho(s)\,\frac{\nabla Q_0(x(s))}{\nabla x(s)} = \eta,\ \forall s \in (a,b),\ s \neq a,\ a+1,\ \ldots,\ b-1; \quad (3.95)$$

$$\rho(s)\,\mathbb{M}_x Q_0(x(s) = \frac{\eta\,\phi(s)}{\phi^2(s) - U_2(s)\,\psi^2(s)}, \quad (3.96)$$

where η is given by (3.93).

Proof: Since Q_n is a solution of (3.76), taking into account the fact that $\lambda_0 = 0$ from (3.80), we have

$$\phi(s)\,\mathbb{F}_x Q_0(x(s)) + \tau(s)\,\mathbb{M}_x Q_0(x(s)) = 0 \Leftrightarrow \Delta\left[(\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s))\,\rho(s)\,\frac{\nabla Q_0(x(s))}{\nabla x(s)}\right] = 0.$$

Therefore, the left-hand side of (3.95) is a periodic function of period 1. This combined with the asymptotic behavior obtained using (2.93)

$$(\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s))\,\rho(s)\,\frac{\nabla Q_0(x(s))}{\nabla x(s)} = \eta\left[1 + O\left(\frac{1}{x(s)}\right)\right],\quad x(s) \to \infty,$$

allows to deduce that the left hand-side of the previous equation is the constant η.

To derive Equation (3.96), we use an equivalent form of (3.95)

$$\frac{\nabla Q_0(x(s))}{\nabla x(s)} = \frac{\eta}{\rho(s)\left(\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s)\right)}$$

and obtain

$$\begin{aligned}\rho(s)\,\mathbb{M}_x Q_0(x(s)) &= \frac{\rho(s)}{2}\left(\frac{\Delta Q_0(x(s))}{\Delta x(s)} + \frac{\nabla Q_0(x(s))}{\nabla x(s)}\right)\\ &= \frac{1}{2}\left(\frac{\eta\,\rho(s)}{\rho(s+1)\left[\phi(s+1) - \frac{1}{2}\psi(s+1)\,\Delta x_1(s)\right]} + \frac{\eta}{\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s)}\right).\end{aligned}$$

Next, we use an equivalent form of the Pearson-type equation (3.63)

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\phi(s) + \frac{1}{2}\psi(s)\,\nabla x_1(s)}{\phi(s+1) - \frac{1}{2}\psi(s+1)\,\Delta x_1(s)} \tag{3.97}$$

to get

$$\begin{aligned}\rho(s)\,\mathbb{M}_x Q_0(x(s)) &= \frac{1}{2}\left(\frac{\eta}{\phi(s) + \frac{1}{2}\psi(s)\,\nabla x_1(s)} + \frac{\eta}{\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s)}\right)\\ &= \frac{\eta\,\phi(s)}{\phi^2(s) - \frac{1}{4}\left[\psi(s)\,\nabla x_1(s)\right]^2}\\ &= \frac{\eta\,\phi(s)}{\phi^2(s) - \psi^2(s)\,U_2(s)},\end{aligned}$$

since from Equation (3.15),

$$\frac{\left[\nabla x_1(s)\right]^2}{4} = Q(x(s)) = U_2(s).$$

□

Let us now give the proof of the Theorem 3.7.

Proof: In the first step, we use relation (2.92) and the fact that Q_n is a solution of (3.76) to get

$$\left(\phi(s)\,\mathbb{F}_x + \psi(s)\,\mathbb{M}_x + \lambda_n\right)\left[P_n(x(s))\,Q_0(x(s)) + \frac{1}{\rho(s)}\,P_{n-1}^{(1)}(x(s))\right] = 0. \tag{3.98}$$

Using the product rules (3.44) and (3.45) for $\mathbb{F}_x$ and $\mathbb{M}_x$ and Equation (3.76) in order to eliminate all occurrences of $\mathbb{F}_x P_n(x(s))$ and $\mathbb{F}_x Q_0(x(s))$ we transform the previous equation into an equation whose left-hand side is a linear combination of

$$\mathbb{F}_x P_{n-1}^{(1)}(x(s)),\ \mathbb{M}_x P_{n-1}^{(1)}(x(s)),\ P_{n-1}^{(1)}(x(s)),\ \mathbb{M}_x P_n(x(s)) \text{ and } P_n(x(s)).$$

The coefficients of this linear combination are functions of

$$x(s),\ \phi(s),\ \psi(s),\ U_1(s),\ U_2(s),\ \mathbb{M}_x Q_0(x(s)),\ \mathbb{F}_x\rho(s),\ \mathbb{M}_x\rho(s), \text{ and } \rho(s).$$

The expression $\mathbb{M}_x Q_0(x(s))$ is eliminated thanks to (3.96). It remains now to express $\mathbb{F}_x\rho(s)$ and $\mathbb{M}_x\rho(s)$ in terms of $\rho(s)$ times rational functions of $x(s)$.

For this aim, we use (3.97) to eliminate all occurrence of $\rho(s-1)$ and $\rho(s+1)$ in the equations

$$\begin{aligned} \mathbb{F}_x\rho(s) &= \frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla\rho(s)}{\nabla x(s)}, \\ \mathbb{M}_x\rho(s) &= \frac{1}{2}\left(\frac{\Delta\rho(s)}{\Delta x(s)}+\frac{\nabla\rho(s)}{\nabla x(s)}\right), \end{aligned}$$

and obtain

$$\frac{\mathbb{F}_x\rho(s)}{\rho(s)} \text{ and } \frac{\mathbb{M}_x\rho(s)}{\rho(s)}$$

as rational functions of the two variables $x(s)$ and $x(s+1)$ whose coefficients depend only on those of the polynomials $\phi(s)$ and $\psi(s)$ ((3.37) and (3.38) have been used as well). Then, assuming $\alpha \neq 1$, we combine (3.34) and (3.43) to get the system

$$\begin{aligned} U_1(s) &= (\alpha+1)[(\alpha-1)x(s)+\beta], \\ 2\,u\,\alpha\sqrt{U_2(s)} &= x(s+1)+(1-2\alpha^2)x(s)-2\beta(\alpha+1), \end{aligned}$$

where $u=\pm 1$. Solving this system in terms the unknowns $x(s)$ and $x(s+1)$ yields

$$\begin{cases} x\,(s) &= \frac{U1(s)}{(\alpha-1)(1+\alpha)}-\frac{\beta}{\alpha-1}, \\ x\,(s+1) &= \frac{\left(2\,\alpha^2-1\right)U1(s)}{(\alpha-1)(1+\alpha)}+2\,u\alpha\,\sqrt{U2\,(s)}-\frac{\beta}{\alpha-1}. \end{cases} \tag{3.99}$$

Computations with Maple 9 [50] using the previous equation allow to express

$$\frac{\mathbb{F}_x\rho(s)}{\rho(s)} \text{ and } \frac{\mathbb{M}_x\rho(s)}{\rho(s)},$$

as rational functions of the variable $x(s)$ depending only on the *integer* powers of the functions

$$U_1(s),\, U_2(s),\, \mathbb{F}_x\phi(s),\, \mathbb{M}_x\phi(s), \phi(s),\; \mathbb{M}_x\psi(s) \text{ and } \psi(s).$$

Summing up, we obtain Equation (3.91).

Notice that if $\alpha=1$, the result obtained is still valid since the singularity $\frac{1}{\alpha-1}$ appearing in (3.99) will be automatically cancelled in the expressions of

$$\frac{\mathbb{F}_x\rho(s)}{\rho(s)} \text{ and } \frac{\mathbb{M}_x\rho(s)}{\rho(s)}$$

after the computations.

□

3.3.2 Fourth-order difference equations for modifications of the classical orthogonal polynomials

Theorem 3.7 is the key to the following fundamental result.

Theorem 3.8 *Let (P_n) be a classical orthogonal polynomial system satisfying (3.76) and $(\tilde{P}_n)$ the orthogonal polynomial system related to (P_n) by*

$$\tilde{P}_n(x(s)) = I_{n,r,k}(s)\, P_{n+r}(x(s)) + J_{n,r,k}(s)\, P^{(1)}_{n+r-1}(x(s)), \tag{3.100}$$

where r and k are nonnegative integers, $I_{n,r,k}(s)$ and $J_{n,r,k}(s)$ are polynomials in the variable $x(s)$ but not depending on n for $n \geq k$, i.e.

$$I_{n,r,k}(s) := I_{r,k}(s), \quad J_{n,r,k}(s) := J_{r,k}(s) \neq 0 \ \text{ for } \ n \geq k. \tag{3.101}$$

Then, each $\tilde{P}_n$ satisfies a fourth-order difference equation of the form

$$\begin{aligned} \mathbb{G}_{n,r,k}(y(x(s))) = \left[\tilde{A}_{n,r,k}(s)\, \mathbb{F}_x + \tilde{B}_{n,r,k}(s)\, \mathbb{M}_x + \tilde{C}_{n,r,k}(s)\right] \times \\ \left[A_{r,k}(s)\, \mathbb{F}_x + B_{r,k}(s)\, \mathbb{M}_x + C_{n,r,k}(s)\right] y(s) = 0,\ n \geq k, \end{aligned} \tag{3.102}$$

where $A_{r,k}(s)$, $B_{r,k}(s)$, $C_{n,r,k}(s)$, $\tilde{A}_{n,r,k}(s)$, $\tilde{B}_{n,r,k}(s)$ and $\tilde{C}_{n,r,k}(s)$ are polynomials in the variable $x(s)$ whose degree does not depend on n.

This fourth-order difference equation can also be written as

$$\begin{aligned} \mathbb{G}_{n,r,k}(y(x(s))) &= [G_4(s;n,r,k)\, \mathbb{F}_x\mathbb{F}_x + G_3(s;n,r,k)\, \mathbb{M}_x\mathbb{F}_x + G_2(s;n,r,k)\, \mathbb{F}_x \\ &+ G_1(s;n,r,k)\, \mathbb{M}_x + G_0(s;n,r,k)]\ y(s) = 0,\ n \geq k, \end{aligned} \tag{3.103}$$

where $G_j(s;n,r,k)$, $j = 0 \ldots 4$ are polynomials in the variable $x(s)$ whose degree does not depend on n.

Proof: First, we use Equation (3.91) for $n = n + r$ and (3.100) to get

$$\begin{aligned} &\phi(s)\, [A_1(s)\, \mathbb{F}_x + B_1(s)\, \mathbb{M}_x + C_1(n+r)] \times \\ &\qquad \left[\frac{\tilde{P}_n(x(s))}{J_{r,k}(s)} - \frac{I_{r,k}(s)}{J_{r,k}(s)}\, P_{n+r}(x(s))\right] \\ &= \qquad 2\,\eta\ [D_1(s)\, \mathbb{M}_x + E_1(s,n+r)]\, P_{n+r}(s),\ n \geq k. \end{aligned} \tag{3.104}$$

Next, we combine the previous equation, the quotient rules (3.46)-(3.47) and Equation (3.76) (in order to eliminate $\mathbb{F}_x P_{n+r}(x(s))$) to obtain

$$\begin{aligned} &[A_{r,k}(s)\, \mathbb{F}_x + B_{r,k}(s)\, \mathbb{M}_x + C_{n,r,k}(s)]\, \tilde{P}_n(x(s)) \\ &\qquad = [D_{n,r,k}(s)\, \mathbb{M}_x + E_{n,r,k}(s)]\, P_{n+r}(x(s)),\ n \geq k, \end{aligned} \tag{3.105}$$

where the coefficients $A_{r,k}(s)$, $B_{r,k}(s)$, $C_{n,r,k}(s)$, $D_{n,r,k}(s)$ and $E_{n,r,k}(s)$ are polynomials, and functions of the polynomials $I_{r,k}(s)$, $J_{r,k}(s)$, $A_1(s)$, $B_1(s)$, $C_1(n)$, $D_1(s)$ and $E_1(s, n+r)$.

If we write

$$\tilde{Q}_{n+r}(x(s)) = [D_{n,r,k}(s)\, \mathbb{M}_x + E_{n,r,k}(s)]\, P_{n+r}(x(s)), \tag{3.106}$$

then using the result of Lemma 3.1 and the fact that P_{n+r} satisfies (3.91) for $n = n + r$, we get

$$\mathbb{M}_x \tilde{Q}_{n+r}(x(s)) = [G_{n,r,k}(s)\,\mathbb{M}_x + H_{n,r,k}(s)]\; P_{n+r}(x(s)), \qquad (3.107)$$

$$\mathbb{F}_x \tilde{Q}_{n+r}(x(s)) = \left[\tilde{G}_{n,r,k}(s)\,\mathbb{M}_x + \tilde{H}_{n,r,k}(s)\right] P_{n+r}(x(s)), \qquad (3.108)$$

where $G_{n,r,k}(s)$, $G_{n,r,k}(s)$, $\tilde{G}_{n,r,k}(s)$ and $\tilde{H}_{n,r,k}(s)$ are rational functions of the variable $x(s)$. Equations (3.106)-(3.108) which can be seen as a system of three linear equations with respect to the unknowns $\mathbb{M}_x P_{n+r}(x(s))$ and $P_{n+r}(x(s))$ produce the second-order difference equation for $\tilde{Q}_{n+r}(x(s))$

$$\begin{vmatrix} D_{n,r,k}(s) & E_{n,r,k}(s) & \tilde{Q}_{n+r}(x(s)) \\ G_{n,r,k}(s) & H_{n,r,k}(s) & \mathbb{M}_x \tilde{Q}_{n+r}(x(s)) \\ \tilde{G}_{n,r,k}(s) & \tilde{H}_{n,r,k}(s) & \mathbb{F}_x \tilde{Q}_{n+r}(x(s)) \end{vmatrix} = 0.$$

The previous equation after cancellation of the common denominator can be brought into the form

$$\left[\tilde{A}_{n,r,k}(s)\,\mathbb{F}_x + \tilde{B}_{n,r,k}(s)\,\mathbb{M}_x + \tilde{C}_{n,r,k}(s)\right] \tilde{Q}_{n+r}(x(s)) = 0,$$

where $\tilde{A}_{n,r,k}(s)$, $\tilde{B}_{n,r,k}(s)$ and $\tilde{C}_{n,r,k}(s)$ are polynomials in the variable $x(s)$ whose degree does not depend on n. Combination of (3.105) and the previous equation produces Equation (3.102). Finally, (3.103) is derived from (3.102) by simultaneous application of the product rules (3.32)-(3.33) and (3.44)-(3.45). □

3.3.3 Solutions of the fourth-order difference equations

In the following, we solve the fourth-order difference equation satisfied by the modifications of classical orthogonal polynomials in terms of the polynomials P_n and its corresponding function of second kind Q_n.

Theorem 3.9 *Under the hypothesis of Theorem 3.8, we have: The four linearly independent solutions of the difference equation (3.102)*

$$\mathbb{G}_{n,r,k}(y(s) = 0,\; n \geq k,$$

are

$$\begin{aligned} S_1(s;n,r,k) &= \rho(s)\, J_{r,k}(s)\, P_{n+r}(x(s)), \\ S_2(s;n,r,k) &= \rho(s)\, J_{r,k}(s)\, Q_{n+r}(x(s)), \\ S_3(s;n,r,k) &= \left[I_{r,k}(s) - \gamma_0^{-1}\rho(s)\, Q_0(x(s))\, J_{r,k}(s)\right] P_{n+r}(x(s)), \\ S_4(s;n,r,k) &= \left[I_{r,k}(s) - \gamma_0^{-1}\rho(s)\, Q_0(x(s))\, J_{r,k}(s)\right] Q_{n+r}(x(s)). \end{aligned}$$

Proof: In the first step, we observe that because of the factorization in Equation (3.102), any solution of the equation

$$[A_{r,k}(s)\,\mathbb{F}_x + B_{r,k}(s)\,\mathbb{M}_x + C_{n,r,k}(s)]\, y(s) = 0 \qquad (3.109)$$

is also solution of (3.102).

In the second step, we also observe from the procedure we have used to construct Equation (3.102) that (using (3.98) and (3.104))

$$
\begin{aligned}
[A_{r,k}(s)\,\mathbb{F}_x + B_{r,k}(s)\,\mathbb{M}_x \quad &+ \quad C_{n,r,k}(s)]\,y(s) = 0 \\
&\Updownarrow \\
[A_1(s)\,\mathbb{F}_x + B_1(s)\,\mathbb{M}_x \quad &+ \quad C_1(n+r)]\left\{\frac{y(s)}{J_{r,k}(s)}\right\} = 0,\ n \geq k \\
&\Updownarrow \\
[\phi(s)\mathbb{F}_x + \psi(s)\mathbb{M}_x \quad &+ \quad \lambda_{n+r}]\left\{\frac{y(s)}{\rho(s)\,J_{r,k}(s)}\right\} = 0,\ n \geq k.
\end{aligned}
$$

Therefore, $S_1(s;n,r,k)$ and $S_2(s;n,r,k)$ are solutions of (3.109) and therefore of (3.102).

In the third step, we use (3.100) and the relation

$$
P_n^{(r)}(x(s)) = \frac{\rho(s)[P_{r-1}(x(s))\,Q_{n+r}(x(s)) - Q_{r-1}(x(s))\,P_{n+r}(x(s))]}{\gamma_0\,\Gamma_{r-1}}
$$

obtained from the fact that P_{n+r}, Q_{n+r} and $P_n^{(r)}$ satisfy the second-order recurrence relation (see Theorem 2.12 and also [30]) to get

$$
\tilde{P}_n(x(s)) = S_3(s;n,r,k) + \gamma_0^{-1}\,S_2(s;n,r,k).
$$

Here, Γ_{r-1} and γ_0 are given respectively by (2.14) and (3.94). Then, since $\tilde{P}_n(x(s))$ and $S_2(s;n,r,k)$ are both solutions of (3.102), it turns out from the previous equation that $S_3(s;n,r,k)$ is also solution of (3.102). Also, $S_4(s;n,r,k)$ is another solution of (3.102) because it is obtained by replacing P_{n+r} by Q_{n+r} in the expression of $S_3(s;n,r,k)$ and the functions P_{n+r} and Q_{n+r} satisfy the same second-order difference equation, namely (3.76) for $n = n+r$. We complete the proof by observing that the four solutions are linearly independent since by means of (2.93), they enjoy different asymptotic behavior. □

For each of the five modifications of classical orthogonal polynomials listed in the second chapter, we get explicit expressions of the functions $S_j(s;n,r,k)$, $j = 1\ldots 4$ in terms of $\rho(s)$, $P_n(x(s))$ and $Q_n(x(s))$. As can be seen from the previous theorem, these solutions have the same structure for the difference equations satisfied by modification of any classical orthogonal polynomials. These solutions were given explicitly in our previous works for the modifications of the very classical orthogonal polynomials. We therefore refer to these three papers [29, 30, 31].

It should be mentioned that the fourth-order difference equation satisfied by the Laguerre-Hahn polynomials orthogonal on special nonuniform lattice was derived in [12]. This result which is based on the properties of the formal Stieltjes function of the corresponding functional [44] covers the modifications of the classical orthogonal polynomials. However, it does not deal with factorization nor with the solutions of the difference equation derived. Our approach, which uses the operators $\mathbb{F}_x$, $\mathbb{M}_x$, the Pearson-type equation for the orthogonality weight and the second-order divided difference equation satisfied by the initial polynomials allows us to factorize and solve the difference equations obtained and constitute a natural extension of the results obtained for the very classical orthogonal polynomials [29, 30, 31].

Chapter 4

Specializations and applications

We analyze the results obtained for some specific values of the parameters and show some applications relative to the first associated classical orthogonal polynomials, the characterization of classical orthogonal polynomials and the solutions of some difference equations involving the operators $\mathbb{F}_x$ and $\mathbb{M}_x$.

4.1 Specialization

In this section, we mainly investigate the results obtained for specific values of the parameters α_x, β_x and δ_x of the lattice $x(s)$, namely those leading to the very classical orthogonal polynomials as already mentioned in Remark 3.4.

4.1.1 The classical orthogonal polynomials of a continuous variable

For

$$\alpha_x = 1,\ \beta_x = 0 \text{ and } \delta_x = 0,$$

we have

$$\mathbb{M}_x = \frac{d}{dx},\ \mathbb{F}_x = \frac{d^2}{dx^2} \text{ and } U_1(s) = U_2(s) = 0.$$

Therefore, Relation (3.91) reads

$$\left[\phi(x)\,\frac{d^2}{dx^2} + (2\,\phi'(x) - \psi(s))\frac{d}{dx} + \lambda_n + \phi'' - \psi'\right] P_{n-1}^{(1)}(x) = (\phi'' - 2\,\psi')\,P_n'(x).$$

This relation, which was first derived by Ronveaux [55], is the key to the derivation and solutions of the fourth-order differential equations satisfied by modifications of the classical orthogonal polynomials of a continuous variable. For more details, we refer to the paper [30] which appears now to be a particular case of the results obtained in the framework of this thesis.

4.1.2 The classical orthogonal polynomials of a discrete variable on a linear lattice

For

$$\alpha_x = 1,\ \beta_x = 0 \text{ and } \delta_x = \frac{1}{4},$$

we have

$$x(s) = s,\ \mathbb{M}_x = \frac{1}{2}(\Delta + \nabla),\ U_1(s) = 0,\ \ \mathbb{F}_x = \Delta\nabla,\ \ \text{and}\ U_2(s) = \frac{1}{4}.$$

Therefore, Relation (3.91) reads

$$\begin{aligned}\{(2\,\phi(s) + \phi'' - \psi')\Delta\nabla \ &+ \ (\psi(s) + [\Delta + \nabla]\phi(s))[\Delta + \nabla] + \lambda_n + \phi'' + \psi'\}\, P_{n-1}^{(1)}(x) \\ &= \ (\phi'' - 2\,\psi')\,\Delta\nabla\, P_n(x).\end{aligned}$$

The previous relation, due to Atakishiyev, Ronveaux and Wolf [11], has been used to derive the fourth-order difference equations satisfied by the modifications of classical orthogonal polynomials of a discrete variable on a linear lattice. We refer to the paper [29] for details about these equations as well as their solutions.

4.1.3 The q-classical orthogonal polynomials

For

$$\alpha_x = \frac{q^{\frac{1}{2}} + q^{\frac{-1}{2}}}{2},\ \beta_x = \delta_x = 0,\ \text{with}\ q \neq 0,\ 1,$$

we have

$$x(s) = q^{\pm s},\ \mathbb{F}_x = q^2\, D_q\, D_{\frac{1}{q}},\ \mathbb{M}_x = \frac{1}{2}(D_q + D_{\frac{1}{q}}),\ U_1(s) = (\alpha_x^2 - 1)\, x(s),\ \text{and}\ U_2(s) = (\alpha_x^2 - 1)\, x^2(s)$$

and the corresponding families are the q-classical orthogonal polynomials.

For $x(s) = q^s$, the relation (3.91) is equivalent to Equation (16) in [31] with the polynomial ϕ replaced by

$$q^2\, \phi(x(s)) - \frac{1}{2}(q-1)q^{\frac{-1}{2}}\, x(s)\, \psi(x(s)).$$

This relation which is due to Foupouagnigni, Ronveaux and Koepf [25] allowed in [31] to derive and solve the fourth-order q-difference equation satisfied by the modifications of the q-classical orthogonal polynomials.

4.1.4 The classical (not very classical) orthogonal polynomials

The classical (not very classical) orthogonal polynomials are those orthogonal on quadratic and q-quadratic lattices $x(s)$. For such lattices, the corresponding parameters satisfy

$$\alpha_x = 1\ \ \text{and}\ \ \beta_x \neq 0,$$

or

$$\alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \ \beta_x = 0\ \ \text{and}\ \ \delta_x \neq 0.$$

By performing calculus with computer algebra software, one can obtain the coefficients of the fourth-order difference equation satisfied by the associated classical orthogonal polynomials in terms of the polynomials ϕ and ψ appearing in the Pearson-type equation (3.63).

4.2 Some applications

4.2.1 Special cases of classical orthogonal polynomials

The relation (3.91) allows to observe that when $\eta = 0$, the first associated $P_{n-1}^{(1)}$ satisfies the second-order difference equation

$$[A_1(s)\,\mathbb{F}_x + B_1(s)\,\mathbb{M}_x + C_1(n)]\,y(s) = 0,$$

where the coefficients A_1, B_1 and C_1 are those of (3.91). It turns out that the previous equations is of hypergeometric type. Under certain conditions (on the parameters involved in the definition), the first associated of the Askey-Wilson, the q-Racah, the Racah and the Wilson polynomials, remains classical. This property is not true in general because the first associated of classical orthogonal polynomials is in general not classical but belongs rather to the so-called Laguerre-Hahn class [43]-[45].

Theorem 4.1 *We have the following.*

1. *For $a\,b\,c\,d = q$, the first associated of the Askey-Wilson polynomials remains classical and is related to the Askey-Wilson polynomials by*

$$\tilde{p}_n^{(1)}\left(x(s); a, b, c, \frac{q}{a\,b\,c}|q\right) = u^n\,\tilde{p}_n\left(u\,x(s); \frac{u\,q}{a}, \frac{u\,q}{b}, \frac{u\,q}{c}, u\,a\,b\,c|q\right), \tag{4.1}$$

with $u = \pm 1$.

2. *For $\alpha\,\beta\,q = 1$, the first associated of the q-Racah polynomials remains classical and is related to the q-Racah polynomials by*

$$\tilde{R}_n^{(1)}\left(x(s); \alpha, \frac{1}{q\,\alpha}, \gamma, \delta|q\right) = (\gamma\,\delta)^n\,\tilde{R}_n\left(\frac{x(s)}{\gamma\,\delta}; \frac{1}{\alpha}, \alpha\,q, \frac{1}{\gamma}, \frac{1}{\delta}|q\right). \tag{4.2}$$

3. *For $\alpha + \beta = -1$, the first associated of the Racah polynomials remains classical and is related to the Racah polynomials by*

$$\tilde{R}_n^{(1)}(x(s); \alpha, -1-\alpha, \gamma, \delta) = \tilde{R}_n(x(s) + \gamma(\alpha+1); \delta - \alpha, 1 + \alpha - \delta, -\gamma, \delta). \tag{4.3}$$

Proof: For the Askey-Wilson polynomials, one obtains by direct computation using (3.93) and the data given in the section 3.2.3 that

$$\eta = \left(\alpha_x\,\psi' - \frac{\phi''}{2}\right) = \frac{4(q - abcd)}{1-q}$$

with $\alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}$ and

$$\begin{aligned} \gamma_{n+1}\left(a, b, c, \frac{q}{a\,b\,c}|q\right) &= \gamma_n\left(\frac{u\,q}{a}, \frac{u\,q}{b}, \frac{u\,q}{c}, u\,a\,b\,c|q\right); \\ \beta_{n+1}\left(a, b, c, \frac{q}{a\,b\,c}|q\right) &= u\,\beta_n\left(\frac{u\,q}{a}, \frac{u\,q}{b}, \frac{u\,q}{c}, u\,a\,b\,c|q\right). \end{aligned}$$

The relations (4.1) is obtained using the last two relations and the fact that $\tilde{p}_{n+1}$ and $\tilde{p}_n^{(1)}$ satisfy the same three-term recurrence relation. The proof of the identities (4.2) and (4.3) is obtained in the same way since

$$\begin{aligned}
\eta = \frac{2q(1-\alpha\beta q}{1-q},\ \alpha_x &= \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2};\\
\gamma_{n+1}\left(\alpha,\frac{1}{q\,\alpha},\gamma,\delta\right) &= \gamma^2\,\delta^2\,\gamma_n\left(\frac{1}{\alpha},\alpha q,\frac{1}{\gamma},\frac{1}{\delta}|q\right);\\
\beta_{n+1}\left(\alpha,\frac{1}{q\,\alpha},\gamma,\delta\right) &= \gamma\,\delta\,\beta_n\left(\frac{1}{\alpha},\alpha q,\frac{1}{\gamma},\frac{1}{\delta}|q\right)
\end{aligned}$$

and

$$\begin{aligned}
\eta = 2(\alpha+\beta+1),\ \alpha_x &= 1;\\
\gamma_{n+1}(\alpha,-1-\alpha,\gamma,\delta) &= \gamma_n(\delta-\alpha,1+\alpha-\delta,-\gamma,\delta);\\
\beta_{n+1}(\alpha,-1-\alpha,\gamma,\delta) &= \gamma_n(\delta-\alpha,1+\alpha-\delta,-\gamma,\delta)-\gamma(\alpha+1)
\end{aligned}$$

for the q-Racah and the Racah respectively. □

Notice that relations such as (4.1)-(4.3) exist for very classical orthogonal polynomials. They were given in references [33, 34], [5], [25] for Jacobi, Hahn and Big-q-Jacobi (and also Little-q-Jacobi) respectively.

4.2.2 Polynomial solutions of some difference equations

The product rules (3.32)-(3.33) and (3.44)-(3.45) can be used to look for polynomial solutions of difference equations with polynomial coefficients involving only the operators $\mathbb{F}_x$ and $\mathbb{M}_x$. For example, if $A(s)$, $B(s)$, $C(s)$ and $D(s)$ are polynomials in the variable $x(s)$, then the polynomial solution of the equation (when it exists)

$$A(s)\,\mathbb{F}_x y(s) + B(s)\,\mathbb{M}_x y(s) + C(s)\,y(s) = D(s)$$

can be found by writing

$$y(s) = \sum_{k=0}^{n} a_{n,k}\,x^k(s),$$

and solving the linear system obtained with respect to the unknowns $a_{n,k}$ in terms of the coefficients of the polynomials $A(s)$, $B(s)$, $C(s)$ and $D(s)$. This in the same way as for the usual differential equation since the operators $\mathbb{F}_x$ and $\mathbb{M}_x$ transform a polynomial in the variable $x(s)$ into a polynomial of the same variable. Using the quotient rules, one can look for rational solutions of some difference equations with polynomial coefficients in the same way. This approach can be used to look in general for analytic solutions (here we mean solutions which can be expanded in series with the variable $x(s)$) of difference equation with polynomial coefficients.

4.2.3 Steps forward towards the characterization of some classes of orthogonal polynomials

In this thesis, we have derived diverse results for classical orthogonal polynomials in the same line as for those of the very classical orthogonal polynomials. In the sequel, we have obtained many intermediate results such as the product and quotient rules, the coefficients $D_{n,k}$, $S_{n,k}$, $F_{n,k}$, $M_{n,k}$, β_n, $T_{n,1}$, T and γ_n. These coefficients can be used for the implementation of codes in computer algebra relative to the classical orthogonal polynomials on quadratic and q-quadratic lattices. They could also be used for the complete characterization of the classical orthogonal polynomials. As illustration, for the very classical orthogonal polynomials, the ratio $\frac{T_{n,k}}{T_{n,k-1}}$ is a rational function of n **or** q^n depending on whether the variable is continuous, linear or q linear respectively. This is an equivalent characterization property for the very classical orthogonal polynomials [2, 35]. But from the results obtained here, we observe that the ratio $\frac{T_{n,2}}{T_{n,1}}$ is a rational function of n **and** q^n (**at the same time**) for the q-quadratic lattice. This is an indication for the complexity for the formulation of the characterization theorems for the classical (not very classical) orthogonal polynomials. Summing up, we can conclude that the results obtained in this thesis constitute some important steps forward towards the complete characterization of the classical orthogonal polynomials.

4.3 Conclusion and perspectives

4.3.1 Conclusion

In this thesis, we have derived and solved the fourth-order difference equations satisfied by some modifications of the classical orthogonal polynomials. In the sequel, we have proved several important intermediate results.

Our main contribution is contained in the third chapter and can be summarized as follows:

- The derivation of the product and the quotient rules for the operators $\mathbb{D}_x$, $\mathbb{S}_x$, $\mathbb{F}_x$ and $\mathbb{M}_x$ (see Theorems 3.1-3.3);

- the derivation of the higher-order linear difference equation for products of functions satisfying each a second-order linear difference equation (see Theorem 3.4);

- the derivation of the second-order difference equation from the orthogonality relation for classical orthogonal polynomials (see Theorem 3.5);

- the derivation of the explicit expressions of the coefficients $D_{n,k-1}$, $S_{n,k}$, $F_{n,k-2}$, $M_{n,k-1}$ for $k = n,\ n-1,\ n-2$ (see Equations (3.23)-(3.30) and (3.81)-(3.83)) and the recurrence coefficients for classical orthogonal polynomials (see Theorem 3.6);

- the derivation of the relation (3.91) giving the link between the classical orthogonal polynomials and their first associated (see Theorem 3.7);

- the derivation and the solution of the fourth-order difference equation for the modifications of the classical orthogonal polynomials (see Theorems 3.8 and 3.9).

As far as we know, these results are new and constitute a generalization of several works on the derivation of the fourth-order differential or difference equations for the modification of the very classical orthogonal polynomials (see [11, 42, 54, 55], [22]-[31] and references therein).

4.3.2 Perspectives

As the continuation of this work, many investigations can be done. Among these are:

- The computations of the coefficients of the fourth-order difference equations satisfied by the associated classical orthogonal polynomials in terms of the polynomials ϕ and ψ involved in the Pearson-type equation (3.63) as was done for the very classical ones;

- the complete characterization of the classical orthogonal polynomials of a discrete variable on a quadratic and a q-quadratic lattice as was done for the very classical ones;

- the characterization of the semi-classical and the Laguerre-Hahn orthogonal polynomials of a discrete variable on a quadratic and a q-quadratic lattice;

- the development of algorithms (and their implementation in diverse computer algebra systems) aimed at solving difference equations (with polynomial coefficients) involving the operators $\mathbb{F}_x$ and $\mathbb{M}_x$.

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