Available at: http://www.ictp.trieste.it/~pub_off IC/2003/119

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

LAGUERRE-FREUD EQUATIONS FOR THE RECURRENCE COEFFICIENTS OF THE LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS ON SPECIAL NONUNIFORM LATTICES

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Abstract

We give an algorithmic derivation of the Laguerre-Freud equations for the recurrence coefficients β_n and γ_n of the Laguerre-Hahn orthogonal polynomials on special nonuniform lattices. This algorithm is the most general one since it is valid for the Laguerre-Hahn orthogonal polynomials of any class k, on the special nonuniform lattices including the continuous (limiting cases), linear, q-linear and the q-nonlinear ones. Moreover, the algorithm allows to deduce an upper bound for the order of the equations in β_n and γ_n, which is respectively $2k+2$ and $2k+3$ when k is even, or $2k+3$ and $2k+2$ when k is odd. Finally, as applications, we discuss explicitly these equations for $k=1$ in the continuous and linear cases, and $k=2$ in the continuous symmetric one.

MIRAMARE – TRIESTE

October 2003

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1 Introduction

In [20], A. P. Magnus introduced a class of polynomials orthogonal with respect to a positive measure $\mu(x)$, consisting of those for which the corresponding Stieltjes function S

$$S(x) = \int_{Supp.\,\mu} \frac{d\mu(t)}{x-t} \tag{1}$$

satisfies a general Riccati equation

$$\begin{aligned} A_0(x(s))\,\frac{S\left(x\left(s+\frac{1}{2}\right)\right)-S\left(x\left(s-\frac{1}{2}\right)\right)}{x\left(s+\frac{1}{2}\right)-x\left(s-\frac{1}{2}\right)} &= B_0(x(s))\,S\left(x\left(s+\frac{1}{2}\right)\right)\,S\left(x\left(s-\frac{1}{2}\right)\right) \\ &+ C_0(x(s))\,\frac{S\left(x\left(s+\frac{1}{2}\right)\right)+S\left(x\left(s-\frac{1}{2}\right)\right)}{2} + D_0(x(s)). \end{aligned} \tag{2}$$

Here A_0, B_0, C_0 and D_0 are polynomials of maximum degree $k+2$, $k+2$, $k+1$ and k ($k \in \mathbb{Z}_+$), respectively, and $x(s)$ is a complex-valued discrete variable function satisfying the relation

$$F\left(x(s), x\left(s-\frac{1}{2}\right)\right) = F\left(x(s), x\left(s+\frac{1}{2}\right)\right) = 0,\ s \in \mathbb{Z}_+, \tag{3}$$

where F is a two variable quadratic polynomial

$$F(x,y) = a\,x^2 + 2\,b\,x\,y + c\,y^2 + 2\,d\,x + 2\,e\,y + f \tag{4}$$

with a, b, c, d, e, f $\in \mathbb{C}$.

From (3) and (4) it follows that

$$x\left(s+\frac{1}{2}\right) = P(x(s)) + \sqrt{Q(x(s))},\ x\left(s-\frac{1}{2}\right) = P(x(s)) - \sqrt{Q(x(s))}, \tag{5}$$

where P and Q are polynomials of degree at most 1 and 2 respectively.

From (5) one derives the following most important canonical forms for $x(s)$ by order of increasing complexity:

$$x(s) = x(0); \tag{6}$$

$$x(s) = s; \tag{7}$$

$$x(s) = q^s; \tag{8}$$

$$x(s) = \frac{q^s+q^{-s}}{2}. \tag{9}$$

They correspond to

$$\begin{aligned} Q(x) &= 0,\ P(x)=x;\ Q(x)=\frac{1}{4},\ P(x)=x;\ Q(x)=\frac{(q-1)^2}{4\,q}\,x^2,\ P(x)=\frac{(q+1)}{2\sqrt{q}}\,x; \\ & Q(x)=\frac{(q-1)^2}{4\,q}\,(x^2-1),\ P(x)=\frac{(q+1)}{2\sqrt{q}}\,x \end{aligned} \tag{10}$$

respectively.

This class of orthogonal polynomials is called *Laguerre-Hahn* (LH) orthogonal polynomials (OP) of *class* k on *special nonuniform lattices* [20] (to mean here the discrete set of points $\left(x(s), x\left(s-\frac{1}{2}\right)\right)$, $\left(x(s), x\left(s+\frac{1}{2}\right)\right)$, $s \in \mathbb{Z}_+$, lying on the conic $F(x,y)=0$).

According to that the form of $x(s)$ is given by (6)-(9), one distinguishes the *continuous* LH polynomials and the LH polynomials on *linear* (uniform), *q-linear* and *q-nonlinear* lattices respectively. Clearly, in the continuous case, corresponding to (6), the Riccati equation (2) reads [19] (see also [7])

$$A_0(x)\,\frac{d}{dx}S_0(x) = B_0(x)\,S_0^2(x) + C_0(x)\,S_0(x) + D_0(x). \tag{11}$$

The class of LH polynomials contains as a particular case, the important subclass of the *semi-classical* orthogonal polynomials when $B_0 \equiv 0$ [20, 22, 23, 24].

The Laguerre-Hahn orthogonal polynomials on the special nonuniform lattices appear to be a natural generalization of the "classical" orthogonal polynomials (from the "very classical" orthogonal polynomials -Hermite, Laguerre and Jacobi - up to the Askey-Wilson polynomials [1]). More precisely, when $B_0 \equiv 0$ and $k = 0$, the LH polynomials are essentially the polynomials introduced by R. Askey and J. Wilson [1] and their particular limiting cases in the Nikiforov-Suslov-Uvarov tableau [27]: Classical orthogonal polynomials of continuous, discrete and q-discrete variables.

Nowadays most of known orthogonal polynomials are classified in the LH group. Let us note that despite the undeniable importance of this class of orthogonal polynomials, not much analytic properties are known for them.

Among known properties, we can firstly state the invariance of the class in rapport with the r-association operation as was proved by A. P. Magnus [20]. Difference-recurrence relations for the LH polynomials were also derived in [20].

The fourth-order difference equation (FODE) satisfied by the polynomials of the LH class and the polynomials r-associated to them can be found in [2] (see also [10, 11] for the particular cases $x(s) = s$ and $x(s) = q^s$). Also, the factorization and the solution of the fourth-order differential, difference and q-difference equations satisfied by the LH orthogonal polynomials obtained by the association operation or the finite modification of the recurrence coefficients of classical orthogonal polynomials were recently obtained [12, 14, 15].

The so-called *Laguerre-Freud* (LF) equations for the recurrence coefficients (that is two nonlinear difference equations for those coefficients), were given for the semi-classical orthogonal polynomials of class one in [5, 8, 9] for continuous, discrete and q-discrete variables respectively. Also, these equations for the LH orthogonal polynomials were given in [6] for $k = 0$ and for continuous variable, and more recently in [2] for $k = 1$, $x(s) = s$.

As far as we are aware of, all contributions in deriving the LF equations for the LH polynomials, existing in the literature, are limited to the cases $k = 1$ and $x(s) = x(0)$, $x(s) = s$ or $x(s) = q^s$.

In this work, we derive the LF equations for the LH orthogonal polynomials in the most general cases that is for k general and $x(s)$ general. More precisely, in section 2, we give an algorithm which allows to derive the equations for any nonnegative integer k and any function $x(s)$ satisfying (3) and (5) and then we deduce an upper bound for the order of these equations and finally in section 3, we give illustrative applications.

The Laguerre-Freud equations provide a systematic way to compute recursively the recurrence coefficients and can be used to analyze the asymptotic behavior of these coefficients [16, 30]. From the asymptotic behavior of the coefficients, one can deduce the asymptotic zero distribution of the corresponding orthogonal polynomials using for example results from [18] and can also obtain information about the largest zero of these polynomials.

2 The Laguerre-Freud equations

Let $\{P_n(x)\}$ be a family of orthogonal polynomials. They satisfy a three-term recurrence relation

$$x\, P_n(x) = a_{n+1}\, P_{n+1}(x) + b_n\, P_n(x) + a_n\, P_{n-1}(x),\ n \geq 1,\ P_{-1}(x) = 0,\ P_0(x) = 1, \tag{12}$$

where b_n and a_n are complex numbers with $a_n \neq 0$, $n \geq 1$.

For the corresponding monic orthogonal polynomials (i.e. $\tilde{P}_n(x) = x^n + \ldots$), the recurrence relation is

$$x\, \tilde{P}_n(x) = \tilde{P}_{n+1}(x) + \beta_n\, \tilde{P}_n(x) + \gamma_n\, \tilde{P}_{n-1}(x),\ n \geq 1,\ \tilde{P}_{-1}(x) = 0,\ \tilde{P}_0(x) = 1, \tag{13}$$

where $\beta_n = b_n,\ \gamma_n = a_n^2$ and $\tilde{P}_n(x) = a_1\, a_2\, \ldots\, a_n\, P_n(x)$.

We assume that $\{P_n(x)\}$ belongs to the LH class, that is, its formal Stieltjes function $S(x)$ given in (1) satisfies (2). The family of polynomials r-associated to $\{P_n(x)\}$ is the family denoted by $\{P_n^{(r)}(x)\}$ and satisfying

$$x\, P_n^{(r)}(x) = a_{n+r+1}\, P_{n+1}^{(r)}(x) + b_{n+r}\, P_n^{(r)}(x) + a_{n+r}\, P_{n-1}^{(r)}(x),\ n \geq 1,\ P_{-1}^{(r)}(x) = 0,\ P_0^{(r)}(x) = 1. \tag{14}$$

The polynomials $\{P_n^{(r)}(x)\}$, according to Favard's Theorem are orthogonal. Let $S_r(x)$ be its corresponding Stieltjes function. One verifies easily that $\{P_n^{(r)}(x)\}$ is of LH class if $\{P_n(x)\}$ is. In fact, let's first recall that $\{P_n(x)\}$ which is the 0-associated of $\{P_n(x)\}$ is a LH polynomials family. Next we assume that for given nonnegative integer r, $\{P_n^{(r)}(x)\}$ is a LH polynomials family; therefore, $S_r(x)$ satisfies

$$\begin{aligned} A_r(x(s))\, \frac{S_r\left(x\left(s+\frac{1}{2}\right)\right) - S_r\left(x\left(s-\frac{1}{2}\right)\right)}{x\left(s+\frac{1}{2}\right) - x\left(s-\frac{1}{2}\right)} &= B_r(x(s))\, S_r\left(x\left(s+\frac{1}{2}\right)\right)\, S_r\left(x\left(s-\frac{1}{2}\right)\right) \\ &+ C_r(x(s))\, \frac{S_r\left(x\left(s+\frac{1}{2}\right)\right) + S_r\left(x\left(s-\frac{1}{2}\right)\right)}{2} + D_r(x(s)), \end{aligned} \tag{15}$$

where A_r, B_r, C_r and D_r are polynomials in x of maximum degree $k+2$, $k+2$, $k+1$ and k respectively, for a fixed nonnegative integer k.

Use of the relation

$$S_r(x) = \frac{1}{x - b_r - a_{r+1}^2\, S_{r+1}(x)},\ r \geq 0, \tag{16}$$

as well as (5) transforms the Riccati difference equation (15) for $S_r(x)$ into a Riccati difference equation for $S_{r+1}(x)$

$$A_{r+1}(x(s))\, \frac{S_{r+1}\left(x\left(s+\frac{1}{2}\right)\right) - S_{r+1}\left(x\left(s-\frac{1}{2}\right)\right)}{x\left(s+\frac{1}{2}\right) - x\left(s-\frac{1}{2}\right)} = B_{r+1}(x(s))\, S_{r+1}\left(x\left(s+\frac{1}{2}\right)\right)\, S_{r+1}\left(x\left(s-\frac{1}{2}\right)\right)$$
$$+C_{r+1}(x(s))\, \frac{S_{r+1}\left(x\left(s+\frac{1}{2}\right)\right) + S_{r+1}\left(x\left(s-\frac{1}{2}\right)\right)}{2} + D_{r+1}(x(s)), \tag{17}$$

with

$$\begin{aligned} A_{r+1}(x) &= A_r(x) - 2\, Q(x)\, D_r(x); && (18)\\ B_{r+1}(x) &= a_{r+1}^2\, D_r(x); && (19)\\ C_{r+1}(x) &= -C_r(x) - 2\,(P(x) - b_r)\, D_r(x); && (20)\\ a_{r+1}^2 D_{r+1}(x) &= A_r(x) + (P(x) - b_r)\, C_r(x) + a_r^2 D_{r-1}(x) + ((P(x) - b_r)^2 - Q(x))\, D_r(x),\ r \geq 1. && (21) \end{aligned}$$

Note that the previous equation for $r = 0$ reads

$$a_1^2\, D_1(x) = A_0(x) + (P(x) - b_0)\, C_0(x) + B_0(x) + ((P(x) - b_0)^2 - Q(x))\, D_0(x). \tag{22}$$

From (18)-(21), it follows that, like A_r, B_r, C_r and D_r, the functions A_{r+1}, B_{r+1}, C_{r+1} and D_{r+1} are polynomials in x of degree at most $k+2$, $k+2$, $k+1$ and k respectively, which proves that the $\{P_n^{(r)}\}$ are polynomials of the Laguerre-Hahn type.

The equations (18)-(21) (obtained at first in [20]) which initially constitute an iteration relation for the association operation, play a central role in the LH theory. Starting from them, one derives the difference-recurrence relations for the LH polynomials [20] and then the fourth-order difference equation that they satisfy [2]. Interesting interconnection between the LH polynomials and the factorization method are also deduced from (18)-(21) (see [3]).

In the following, we analyze the previous equations in order to derive the two nonlinear difference equations for β_r and γ_r (the Laguerre-Freud equations).

From now on, we will use the following notations:

$$\beta_r = b_r,\ \gamma_r = a_r^2. \tag{23}$$

Moreover, for clarity, we write n instead of r since they both take the same values: 1, 2, 3, ... This allows to encounter the usual index representation for the recurrence coefficients.

2.1 Difference equations for the coefficients of A_n, C_n and D_n

First write

$$A_n(x) = \sum_{i=0}^{k+2} a_i(n)\, x^i,\ C_n(x) = \sum_{i=0}^{k+1} c_i(n)\, x^i,\ D_n(x) = \sum_{i=0}^{k} d_i(n)\, x^i,\ P(x) = p_1\, x + p_0,\ Q(x) = q_2\, x^2 + q_1\, x + q_0, \tag{24}$$

in equations (18), (20) and (21) respectively. Then we collect the coefficients of the monomials x^i in each equation and get respectively three families of difference equations $(A_i^k)_{0\le i\le k+2}$, $(C_i^k)_{0\le i\le k+1}$ and $(D_i^k)_{0\le i\le k+2}$

$$A_i^k:\ a_i(n+1)-a_i(n)+2\,q_2\,d_{i-2}(n)+2\,q_1\,d_{i-1}(n)+2\,q_0\,d_i(n)=0,\ 0\le i\le k+2; \tag{25}$$

$$C_i^k:\ c_i(n+1)+c_i(n)+2\,p_1\,d_{i-1}(n)+2\,(p_0-\beta_n)\,d_i(n)=0,\ 0\le i\le k+1; \tag{26}$$

$$\begin{aligned} D_i^k:\ & a_i(n)+p_1\,c_{i-1}(n)+(p_0-\beta_n)\,c_i(n)-\gamma_{n+1}\,d_i(n+1)+\gamma_n\,d_i(n-1)+(p_1^2-q_2)\,d_{i-2}(n)\\ & +(2\,p_1\,p_0-2\,p_1\,\beta_n-q_1)\,d_{i-1}(n)+((p_0-\beta_n)^2-q_0)\,d_i(n)=0,\ 0\le i\le k+2. \end{aligned} \tag{27}$$

Here, it is understood that

$$c_i(n)=0\ \text{ for } i<0 \text{ or } i>k+1;\ \text{ and } \gamma_n\,d_i(n-1)=\begin{cases} 0 & \text{if } n\ge 1 \text{ and } i<0;\\ 0 & \text{if } n\ge 1 \text{ and } i>k;\\ b_i(0) & \text{if } n=0 \text{ and } 0\le i\le k+2. \end{cases} \tag{28}$$

The equations (25)-(27) form a system of $3\,k+8$ equations in $3\,k+8$ unknowns which are the $3\,k+6$ coefficients of $A_n(x)$, $C_n(x)$ and $D_n(x)$ and the recurrence coefficients β_n and γ_n. The leading idea consists in eliminating successively the first $3\,k+6$ unknowns (coefficients of $A_n(x)$, $C_n(x)$ and $D_n(x)$) so that the remaining two equations, containing only the β_n and γ_n, will provide the desired Laguerre-Freud equations. But besides the algebraic character of the equations, we need to consider also the difference one (in n and i). The clue of the solution carries in a permanent combination of techniques of both kinds.

2.2 Elimination of the a_i and c_i

In the first step, we take the difference derivative of (27) (to mean here: subtract (27) from the equation obtained from it by replacing n by $n+1$) and use (25) to eliminate a_i and next (26) to eliminate $c_i(n+1)$ and $c_{i-1}(n+1)$

$$\begin{aligned} & -(-2\,p_0+\beta_{n+1}+\beta_n)\,c_i(n)+2\,p_1\,c_{i-1}(n)\\ & +(-\gamma_{n+1}+q_0+3\,{p_0}^2-4\,p_0\,\beta_n+{\beta_n}^2-2\,\beta_{n+1}\,p_0+2\,\beta_{n+1}\,\beta_n)\,d_i(n)+\gamma_n\,d_i(n-1)\\ & +(-{p_0}^2+2\,\beta_{n+1}\,p_0-{\beta_{n+1}}^2+q_0-\gamma_{n+1})\,d_i(n+1)+\gamma_{n+2}\,d_i(n+2)\\ & +(6\,p_1\,p_0-4\,p_1\,\beta_n+q_1-2\,p_1\,\beta_{n+1})\,d_{i-1}(n)+(-2\,p_1\,p_0+2\,p_1\,\beta_{n+1}+q_1)\,d_{i-1}(n+1)\\ & +(3\,{p_1}^2+q_2)\,d_{i-2}(n)+(-{p_1}^2+q_2)\,d_{i-2}(n+1)=0. \end{aligned} \tag{29}$$

In the second step, we solve the previous equation in terms of $c_i(n)$ and replace the expression of $c_i(n)$ obtained in (26) for n and $n+1$. We then get an equation without $c_i(n)$ but containing $c_{i-1}(n)$ and $c_{i-1}(n+1)$. Next, we eliminate the term $c_{i-1}(n+1)$ in this equation by using (26) for $i-1$, and get an equation which can be written as

$$(\beta_{n+2}-\beta_n)\,c_{i-1}(n)=e_i(n), \tag{30}$$

where $e_i(n)$ is function of the β_n, γ_n and the d_j.

Finally, using the previous equation for i and $i+1$ in (29), we get the following equation without the c_i (after some computations with Maple 8 [26]),

$$
\begin{aligned}
E_i^{k,0}:\ & -(2\,p_0-\beta_{n+1}-\beta_n)\,\gamma_{n+3}\,d_{i+1}(n+3)\\
& +(-2\,p_0+\beta_{n+2}+\beta_{n+1})\,\gamma_n\,d_{i+1}(n-1)+(2\,\gamma_{n+2}\,p_0\\
& -\gamma_{n+2}\,\beta_{n+1}-\gamma_{n+2}\,\beta_n+q_0\,\beta_n+\beta_{n+2}\,{\beta_{n+1}}^2+2\,\gamma_{n+1}\,p_0-\gamma_{n+1}\,\beta_{n+2}-\gamma_{n+1}\,\beta_{n+1}-4\,q_0\,p_0\\
& +q_0\,\beta_{n+2}+2\,q_0\,\beta_{n+1}+3\,{p_0}^2\,\beta_{n+2}+6\,{p_0}^2\,\beta_{n+1}+3\,{p_0}^2\,\beta_n-2\,{\beta_{n+1}}^2\,p_0+{\beta_{n+1}}^2\,\beta_n\\
& -4\,\beta_{n+1}\,p_0\,\beta_{n+2}+2\,\beta_{n+1}\,\beta_n\,\beta_{n+2}-4\,{p_0}^3-2\,p_0\,\beta_n\,\beta_{n+2}-4\,p_0\,\beta_n\,\beta_{n+1})d_{i+1}(n+1)+\\
& (2\,{p_0}^3-{p_0}^2\,\beta_{n+1}-{p_0}^2\,\beta_n-4\,{p_0}^2\,\beta_{n+2}+2\,\beta_{n+1}\,p_0\,\beta_{n+2}+2\,p_0\,\beta_n\,\beta_{n+2}+2\,{\beta_{n+2}}^2\,p_0\\
& -{\beta_{n+2}}^2\,\beta_{n+1}-{\beta_{n+2}}^2\,\beta_n-2\,q_0\,p_0+q_0\,\beta_{n+1}+q_0\,\beta_n-\gamma_{n+2}\,\beta_n+\gamma_{n+2}\,\beta_{n+2})d_{i+1}(n+2)+\\
& (-\gamma_{n+1}\,\beta_{n+2}-2\,q_0\,p_0+q_0\,\beta_{n+2}+q_0\,\beta_{n+1}-{p_0}^2\,\beta_{n+2}-{p_0}^2\,\beta_{n+1}-4\,{p_0}^2\,\beta_n+2\,{\beta_n}^2\,p_0\\
& -{\beta_n}^2\,\beta_{n+2}-{\beta_n}^2\,\beta_{n+1}+\gamma_{n+1}\,\beta_n+2\,{p_0}^3+2\,p_0\,\beta_n\,\beta_{n+2}+2\,p_0\,\beta_n\,\beta_{n+1})d_{i+1}(n)\\
& -2\,p_1\,\gamma_{n+3}\,d_i(n+3)+(6\,p_1\,{p_0}^2-8\,p_1\,p_0\,\beta_n-2\,p_1\,\beta_{n+1}\,p_0-2\,p_1\,p_0\,\beta_{n+2}-2\,q_1\,p_0\\
& +q_1\,\beta_{n+1}+2\,p_1\,\beta_n\,\beta_{n+2}-2\,p_1\,q_0+2\,p_1\,{\beta_n}^2+q_1\,\beta_{n+2}+2\,p_1\,\beta_{n+1}\,\beta_n)d_i(n)\\
& -2\,\gamma_n\,p_1\,d_i(n-1)+(-12\,p_1\,{p_0}^2+6\,p_1\,p_0\,\beta_{n+2}+12\,p_1\,\beta_{n+1}\,p_0+6\,p_1\,p_0\,\beta_n-4\,q_1\,p_0\\
& +2\,q_1\,\beta_{n+1}-2\,p_1\,\beta_n\,\beta_{n+2}-4\,p_1\,\beta_{n+1}\,\beta_n+q_1\,\beta_{n+2}-4\,p_1\,\beta_{n+1}\,\beta_{n+2}+2\,p_1\,\gamma_{n+2}-4\,p_1\,q_0\\
& -2\,p_1\,{\beta_{n+1}}^2+2\,p_1\,\gamma_{n+1}+q_1\,\beta_n)d_i(n+1)+(6\,p_1\,{p_0}^2-2\,p_1\,\beta_{n+1}\,p_0-2\,p_1\,p_0\,\beta_n-8\,p_1\,p_0\,\beta_{n+2}\\
& -2\,q_1\,p_0+q_1\,\beta_{n+1}+2\,p_1\,\beta_n\,\beta_{n+2}+2\,p_1\,\beta_{n+1}\,\beta_{n+2}-2\,p_1\,q_0+2\,p_1\,{\beta_{n+2}}^2+q_1\,\beta_n)d_i(n+2)\\
& +(6\,{p_1}^2\,p_0-2\,q_2\,p_0-{p_1}^2\,\beta_{n+1}-4\,{p_1}^2\,\beta_{n+2}+q_2\,\beta_n+q_2\,\beta_{n+1}-2\,p_1\,q_1-{p_1}^2\,\beta_n)\,d_{i-1}(n+2)\\
& +(6\,{p_1}^2\,p_0-2\,q_2\,p_0-2\,p_1\,q_1-{p_1}^2\,\beta_{n+1}-{p_1}^2\,\beta_{n+2}+q_2\,\beta_{n+1}-4\,{p_1}^2\,\beta_n+q_2\,\beta_{n+2})\,d_{i-1}(n)\\
& +(-4\,q_2\,p_0-12\,{p_1}^2\,p_0+6\,{p_1}^2\,\beta_{n+1}+2\,q_2\,\beta_{n+1}+q_2\,\beta_n+3\,{p_1}^2\,\beta_n-4\,p_1\,q_1+3\,{p_1}^2\,\beta_{n+2}\\
& +q_2\,\beta_{n+2})d_{i-1}(n+1)-2\,(-{p_1}^2+q_2)\,p_1\,d_{i-2}(n+2)-2\,(-{p_1}^2+q_2)\,p_1\,d_{i-2}(n)\\
& -4\,({p_1}^2+q_2)\,p_1\,d_{i-2}(n+1)=0.
\end{aligned}
\tag{31}
$$

The previous equation, which we call $E_i^{k,0}$ is valid for $0\le i\le k+2$ and contains only the terms

$$
\begin{aligned}
&\beta_n,\ \beta_{n+1},\ \beta_{n+2},\ \gamma_n,\ \gamma_{n+1},\ \gamma_{n+2},\ \gamma_{n+3},\ d_{i-2}(n),\ d_{i-2}(n+1),\ d_{i-2}(n+2),\\
&d_{i-1}(n),\ d_{i-1}(n+1),\ d_{i-1}(n+2),\ d_i(n-1),\ d_i(n),\ d_i(n+1),\ d_i(n+2),\ d_i(n+3),\\
&d_{i+1}(n-1),\ d_{i+1}(n),\ d_{i+1}(n+1),\ d_{i+1}(n+2),\ d_{i+1}(n+3).
\end{aligned}
$$

When i takes the values $0,\,1,\ldots,\,k+2$ in $E_i^{k,0}$, we get $k+3$ equations for $k+3$ unknowns which are β_n, γ_n and the $d_j(n)$, $0\le j\le k$.

2.3 Derivation of the Laguerre-Freud equations for $k=1$

We write and analyze the equations $E_i^{1,0}$ for $0\le i\le 3$. Taking $k=1$ in (31) and taking into account (28), equations $E_i^{1,0}$ for $0\le i\le 3$ read

$$E_3^{1,0}:\quad ({p_1}^2-q_2)\,d_1(n+2)-2\,({p_1}^2+q_2)\,d_1(n+1)+({p_1}^2-q_2)\,d_1(n)=0; \tag{32}$$

$$\begin{aligned}E_2^{1,0}:\ &(6\,{p_1}^2\,p_0-2\,q_2\,p_0-{p_1}^2\,\beta_{n+1}-4\,{p_1}^2\,\beta_{n+2}+q_2\,\beta_{n+1}-{p_1}^2\,\beta_n+q_2\,\beta_n-2\,p_1\,q_1)\,d_1(n+2)\\&+(6\,{p_1}^2\,p_0-2\,q_2\,p_0-{p_1}^2\,\beta_{n+1}+q_2\,\beta_{n+1}-{p_1}^2\,\beta_{n+2}+q_2\,\beta_{n+2}-4\,{p_1}^2\,\beta_n-2\,p_1\,q_1)d_1(n)\\&+(-4\,q_2\,p_0-12\,{p_1}^2\,p_0-4\,p_1\,q_1+3\,{p_1}^2\,\beta_{n+2}+q_2\,\beta_{n+2}+3\,{p_1}^2\,\beta_n\\&+2\,q_2\,\beta_{n+1}+q_2\,\beta_n+6\,{p_1}^2\,\beta_{n+1})d_1(n+1)-2\,(-{p_1}^2+q_2)\,p_1\,d_0(n+2)\\&-2\,(-{p_1}^2+q_2)\,p_1\,d_0(n)-4\,({p_1}^2+q_2)\,p_1\,d_0(n+1)=0;\end{aligned} \tag{33}$$

$$\begin{aligned}E_1^{1,0}:\ &-2\,p_1\,\gamma_{n+3}\,d_1(n+3)+(6\,p_1\,{p_0}^2-2\,p_1\,p_0\,\beta_{n+2}-8\,p_1\,p_0\,\beta_n-2\,q_1\,p_0-2\,p_1\,\beta_{n+1}\,p_0\\&+2\,p_1\,{\beta_n}^2+2\,p_1\,\beta_n\,\beta_{n+2}+q_1\,\beta_{n+2}+q_1\,\beta_{n+1}-2\,p_1\,q_0+2\,p_1\,\beta_{n+1}\,\beta_n)d_1(n)\\&-2\,p_1\,\gamma_n\,d_1(n-1)+(-12\,p_1\,{p_0}^2+6\,p_1\,p_0\,\beta_{n+2}+6\,p_1\,p_0\,\beta_n-4\,q_1\,p_0\\&+12\,p_1\,\beta_{n+1}\,p_0-2\,p_1\,{\beta_{n+1}}^2-2\,p_1\,\beta_n\,\beta_{n+2}+q_1\,\beta_n-4\,p_1\,\beta_{n+1}\,\beta_{n+2}\\&+2\,p_1\,\gamma_{n+1}+q_1\,\beta_{n+2}+2\,q_1\,\beta_{n+1}+2\,p_1\,\gamma_{n+2}-4\,p_1\,q_0-4\,p_1\,\beta_{n+1}\,\beta_n)d_1(n+1)\\&+(6\,p_1\,{p_0}^2-8\,p_1\,p_0\,\beta_{n+2}-2\,q_1\,p_0-2\,p_1\,\beta_{n+1}\,p_0-2\,p_1\,p_0\,\beta_n\\&+q_1\,\beta_{n+1}+q_1\,\beta_n-2\,p_1\,q_0+2\,p_1\,\beta_{n+1}\,\beta_{n+2}+2\,p_1\,\beta_n\,\beta_{n+2}+2\,p_1\,{\beta_{n+2}}^2)d_1(n+2)\\&+(6\,{p_1}^2\,p_0-2\,q_2\,p_0-{p_1}^2\,\beta_{n+1}-4\,{p_1}^2\,\beta_{n+2}+q_2\,\beta_{n+1}-{p_1}^2\,\beta_n+q_2\,\beta_n-2\,p_1\,q_1)d_0(n+2)\\&+(6\,{p_1}^2\,p_0-2\,q_2\,p_0-{p_1}^2\,\beta_{n+1}+q_2\,\beta_{n+1}-{p_1}^2\,\beta_{n+2}+q_2\,\beta_{n+2}-4\,{p_1}^2\,\beta_n-2\,p_1\,q_1)d_0(n)\\&+(-4\,q_2\,p_0-12\,{p_1}^2\,p_0-4\,p_1\,q_1+3\,{p_1}^2\,\beta_{n+2}+q_2\,\beta_{n+2}+3\,{p_1}^2\,\beta_n\\&+2\,q_2\,\beta_{n+1}+q_2\,\beta_n+6\,{p_1}^2\,\beta_{n+1})d_0(n+1)=0;\end{aligned} \tag{34}$$

$$\begin{aligned}E_0^{1,0}:\ &-\gamma_{n+3}\,(2\,p_0-\beta_{n+1}-\beta_n)\,d_1(n+3)+\gamma_n\,(-2\,p_0+\beta_{n+2}+\beta_{n+1})\,d_1(n-1)\\&+(-2\,p_0\,\beta_n\,\beta_{n+2}+2\,\beta_{n+1}\,\beta_n\,\beta_{n+2}+{\beta_{n+1}}^2\,\beta_n-2\,{\beta_{n+1}}^2\,p_0-4\,\beta_{n+1}\,p_0\,\beta_{n+2}\\&-4\,{p_0}^3+2\,\gamma_{n+1}\,p_0-\gamma_{n+1}\,\beta_{n+2}-\gamma_{n+1}\,\beta_{n+1}-4\,q_0\,p_0+q_0\,\beta_{n+2}+2\,q_0\,\beta_{n+1}\\&+3\,{p_0}^2\,\beta_{n+2}+6\,{p_0}^2\,\beta_{n+1}+3\,{p_0}^2\,\beta_n-4\,p_0\,\beta_n\,\beta_{n+1}-\gamma_{n+2}\,\beta_{n+1}+q_0\,\beta_n\\&+\beta_{n+2}\,{\beta_{n+1}}^2-\gamma_{n+2}\,\beta_n+2\,\gamma_{n+2}\,p_0)d_1(n+1)+(2\,{p_0}^3-{p_0}^2\,\beta_{n+1}-{p_0}^2\,\beta_n\\&-4\,{p_0}^2\,\beta_{n+2}+2\,\beta_{n+1}\,p_0\,\beta_{n+2}+2\,p_0\,\beta_n\,\beta_{n+2}+2\,{\beta_{n+2}}^2\,p_0-{\beta_{n+2}}^2\,\beta_{n+1}\\&-{\beta_{n+2}}^2\,\beta_n-2\,q_0\,p_0+q_0\,\beta_{n+1}+q_0\,\beta_n-\gamma_{n+2}\,\beta_n+\gamma_{n+2}\,\beta_{n+2})d_1(n+2)\\&+(2\,p_0\,\beta_n\,\beta_{n+2}+\gamma_{n+1}\,\beta_n-{\beta_n}^2\,\beta_{n+1}+2\,{p_0}^3-\gamma_{n+1}\,\beta_{n+2}-2\,q_0\,p_0+q_0\,\beta_{n+2}\\&+q_0\,\beta_{n+1}-{p_0}^2\,\beta_{n+2}-{p_0}^2\,\beta_{n+1}-4\,{p_0}^2\,\beta_n+2\,{\beta_n}^2\,p_0-{\beta_n}^2\,\beta_{n+2}\\&+2\,p_0\,\beta_n\,\beta_{n+1})d_1(n)-2\,p_1\,\gamma_{n+3}\,d_0(n+3)+(6\,p_1\,{p_0}^2-2\,p_1\,p_0\,\beta_{n+2}\end{aligned} \tag{35}$$

$$
\begin{aligned}
&- 8\,p_1\,p_0\,\beta_n - 2\,q_1\,p_0 - 2\,p_1\,\beta_{n+1}\,p_0 + 2\,p_1\,{\beta_n}^2 + 2\,p_1\,\beta_n\,\beta_{n+2} + q_1\,\beta_{n+2} \\
&+ q_1\,\beta_{n+1} - 2\,p_1\,q_0 + 2\,p_1\,\beta_{n+1}\,\beta_n)d_0(n) - 2\,p_1\,\gamma_n\,d_0(n-1) + (-12\,p_1\,{p_0}^2 \\
&+ 6\,p_1\,p_0\,\beta_{n+2} + 6\,p_1\,p_0\,\beta_n - 4\,q_1\,p_0 + 12\,p_1\,\beta_{n+1}\,p_0 - 2\,p_1\,{\beta_{n+1}}^2 \\
&- 2\,p_1\,\beta_n\,\beta_{n+2} + q_1\,\beta_n - 4\,p_1\,\beta_{n+1}\,\beta_{n+2} + 2\,p_1\,\gamma_{n+1} + q_1\,\beta_{n+2} + 2\,q_1\,\beta_{n+1} \\
&+ 2\,p_1\,\gamma_{n+2} - 4\,p_1\,q_0 - 4\,p_1\,\beta_{n+1}\,\beta_n)d_0(n+1) + (6\,p_1\,{p_0}^2 - 8\,p_1\,p_0\,\beta_{n+2} \\
&- 2\,q_1\,p_0 - 2\,p_1\,\beta_{n+1}\,p_0 - 2\,p_1\,p_0\,\beta_n + q_1\,\beta_{n+1} + q_1\,\beta_n - 2\,p_1\,q_0 \\
&+ 2\,p_1\,\beta_{n+1}\,\beta_{n+2} + 2\,p_1\,\beta_n\,\beta_{n+2} + 2\,p_1\,{\beta_{n+2}}^2)d_0(n+2) = 0.
\end{aligned}
$$

2.3.1 Elimination of $d_1(n)$

From the expressions of polynomials P and Q (see (10)), one remarks that (32) determines uniquely the coefficient $d_1(n)$ in terms of the two initial values $d_1(0)$ and $d_1(1)$. The term $d_1(1)$ is obtained by taking $i = 1$, $n = 0$ in (27) and using (22) and (23)

$$
\begin{aligned}
\gamma_1\,d_1(1) &= a_1(0) + b_1(0) + p_1\,c_0(0) + (p_0 - \beta_0)\,c_1(0) + ((p_0 - \beta_0)^2 - q_0)\,d_1(0) \\
&\quad +(2\,p_1\,p_0 - 2\,p_1\,\beta_0 - q_1)\,d_0(0).
\end{aligned} \tag{36}
$$

Remark 1 *In the following, we use the notation*

$$
F_i^{k,s}\left(n,\,\{d_r(n+j)\}_{j=u_1,v_1;r=i-2,i+1},\,\{\beta_{n+j}\}_{j=u_2,v_2},\,\{\gamma_{n+j}\}_{j=u_3,v_3}\right),\ 0 \le s \le 2,\ 0 \le i \le k+2
$$

to mean that $F_i^{k,s}$ is a function of n and the variables $d_r(n+j)$, $u_1 \le j \le v_1$, $i-2 \le r \le i+1$; β_{n+j}, $u_2 \le j \le v_2$, γ_{n+j}, $u_3 \le j \le v_3$, where u_1, u_2, u_3, v_1, v_2 and v_3 are well specified integers. Also, the $F_i^{k,s}$ is supposed linear in the variables $d_r(n+j)$, $u_1 \le j \le v_1$, $i-2 \le r \le i+1$.

2.3.2 Elimination of $d_0(n-1)$ and $d_0(n+3)$

Equation (35) contains the terms $d_0(n-1)$ and $d_0(n+3)$ which we would like to eliminate in order to keep only n, $n+1$ and $n+2$ as arguments of d_0. To do so, we proceed as follows. Equation (33) can be written as (assuming that $d_1(n)$ is known)

$$
\mathbb{T}(d_0(n)) = F_2^{1,0}\left(n,\,\{\beta_{n+j}\}_{j=0,2}\right), \tag{37}
$$

where $\mathbb{T}$ is the second-order difference operator acting on a function $f(n)$ as

$$
\mathbb{T}(f(n)) = (p_1^2 - q_2)\,f(n+2) - 2\,(p_1^2 + q_2)\,f(n+1) + (p_1^2 - q_2)\,f(n). \tag{38}
$$

From (37), we express $d_0(n-1)$ and $d_0(n+3)$ in terms of the β_n, and $d_0(n+j)$, $j = 0, 1, 2$ (by replacing n by $n-1$ and $n+1$ respectively in (37))

$$
\begin{aligned}
d_0(n-1) &= F_2^{1,1}\left(n, \{d_0(n+j)\}_{j=0,1},\,\{\beta_{n+j}\}_{j=-1,1}\right); &(39)\\
d_0(n+3) &= F_2^{1,2}\left(n, \{d_0(n+j)\}_{j=1,2},\,\{\beta_{n+j}\}_{j=1,3}\right). &(40)
\end{aligned}
$$

Equation (34) can be written as

$$F_1^{1,0}\left(n,\, \{d_0(n+j)\}_{j=0,2},\, \{\beta_{n+j}\}_{j=0,2},\, \{\gamma_{n+j}\}_{j=0,3}\right) = 0, \tag{41}$$

and (35) as

$$F_0^{1,0}\left(n,\, \{d_0(n+j)\}_{j=-1,3},\, \{\beta_{n+j}\}_{j=0,2},\, \{\gamma_{n+j}\}_{j=0,3}\right) = 0. \tag{42}$$

The previous equation contains $d_0(n-1)$ and $d_0(n+3)$, terms which we eliminate by putting (39) and (40) in (42) to obtain

$$\tilde{F}_0^{1,0}\left(n,\, \{d_0(n+j)\}_{j=0,2},\, \{\beta_{n+j}\}_{j=-1,3},\, \{\gamma_{n+j}\}_{j=0,3}\right) = 0. \tag{43}$$

2.3.3 Elimination of $d_0(n)$, $d_0(n+1)$ and $d_0(n+2)$

In the final step, to eliminate the variables $d_0(n)$, $d_0(n+1)$ and $d_0(n+2)$ and obtain the desired Laguerre-Freud, we proceed as follows. We write equations (37), (41) and (43) respectively in the forms

$$f_2^2(n)\, d_0(n+2) + f_1^2(n)\, d_0(n+1) + f_0^2(n)\, d_0(n) = g_2(n), \tag{44}$$
$$f_2^1(n)\, d_0(n+2) + f_1^1(n)\, d_0(n+1) + f_0^1(n)\, d_0(n) = g_1(n), \tag{45}$$
$$f_2^0(n)\, d_0(n+2) + f_1^0(n)\, d_0(n+1) + f_0^0(n)\, d_0(n) = g_0(n), \tag{46}$$

where $f_j^i(n)$ and $g_i(n)$ are functions of the variables $\{\beta_{n+j}\}_{j=-1,3}$, and $\{\gamma_{n+j}\}_{j=0,3}$.

Next, we solve the last three equations with respect to the unknowns $d_0(n+j)$, $j = 0,\, 1,\, 2$

$$d_0(n+2) = G_2\left(n,\, \{\beta_{n+j}\}_{j=-1,3},\, \{\gamma_{n+j}\}_{j=0,3}\right); \tag{47}$$
$$d_0(n+1) = G_1\left(n,\, \{\beta_{n+j}\}_{j=-1,3},\, \{\gamma_{n+j}\}_{j=0,3}\right); \tag{48}$$
$$d_0(n) = G_0\left(n,\, \{\beta_{n+j}\}_{j=-1,3},\, \{\gamma_{n+j}\}_{j=0,3}\right). \tag{49}$$

Finally, comparison of (47) with (48), and (48) with (49) leads to the *Laguerre-Freud equations for class* $k=1$*:*

$$G_1\left(n+1,\, \{\beta_{n+j}\}_{j=0,4},\, \{\gamma_{n+j}\}_{j=1,4}\right) = G_2\left(n,\, \{\beta_{n+j}\}_{j=-1,3},\, \{\gamma_{n+j}\}_{j=0,3}\right); \tag{50}$$
$$G_0\left(n+1,\, \{\beta_{n+j}\}_{j=0,4},\, \{\gamma_{n+j}\}_{j=1,4}\right) = G_1\left(n,\, \{\beta_{n+j}\}_{j=-1,3},\, \{\gamma_{n+j}\}_{j=0,3}\right). \tag{51}$$

Remark 2 *From the two last equations and the above procedure, one remarks that the order of the difference equations (50) and (51) are at most 5 and 4 for the variables β_n and γ_n respectively.*

2.4 Derivation of the Laguerre-Freud equations for generic k

2.4.1 Formalization of the difference equations $E_i^{k,0}$, $0 \le i \le k+2$

Taking into account (28), one remarks that equation (31) takes one of the five following forms:
Form 1 : $i = k+2$

$$E_{k+2}^{k,0}: \qquad \mathbb{T}(d_k(n)) = 0, \tag{52}$$

where $\mathbb{T}$ is given by (38). Equation (52) is identical to (32); therefore, similarly to what was mentioned in subsection 2.3.1, $d_k(n)$ is uniquely determined in terms of the initial values $d_k(0)$ and $d_k(1)$. The last term being obtained by taking $i=k$, $n=0$ in (27) and using (22) and (23)

$$\begin{aligned}\gamma_1\, d_k(1) \quad &= \quad a_k(0)+b_k(0)+p_1\, c_{k-1}(0)+(p_0-\beta_0)\, c_k(0)+((p_0-\beta_0)^2-q_0)\, d_k(0) \\ &\quad +(2\,p_1\,p_0-2\,p_1\,\beta_0-q_1)\, d_{k-1}(0)+(p_1^2-q_2)\, d_{k-2}(0).\end{aligned} \tag{53}$$

Form 2: $i=k+1$

$$E_{k+1}^{k,0}: \qquad \mathbb{T}(d_{k-1}(n)) = F_{k+1}^{k,0}\,(n,\, (\beta_{n+j})_{j=0,2})\,, \tag{54}$$

where $\mathbb{T}$ is given by (38). It should be noticed that we don't mention $d_k(n)$ because it is assumed to be known.

Form 3: $2\le i\le k$

$$\begin{aligned}&E_i^{k,0}: \qquad \mathbb{T}(d_{i-2}(n)) = \\ &\quad F_i^{k,0}\,(n,\, \{d_{i-1}(n+j)\}_{j=0,2},\ \{d_i(n+j)\}_{j=-1,3},\, \{d_{i+1}(n+j)\}_{j=-1,3},\, (\beta_{n+j})_{j=0,2},\, \{\gamma_{n+j}\}_{j=0,3})\,.\end{aligned} \tag{55}$$

Form 4: $i=1$

$$E_1^{k,0}: \quad F_1^{k,0}\,(n,\, \{d_0(n+j)\}_{j=0,2},\ \{d_1(n+j)\}_{j=-1,3},\, \{d_2(n+j)\}_{j=-1,3},\, \{\beta_{n+j}\}_{j=0,2},\, \{\gamma_{n+j}\}_{j=0,3})=0. \tag{56}$$

Form 5: $i=0$

$$E_0^{k,0}: \quad F_0^{k,0}\,(n,\, \{d_0(n+j)\}_{j=-1,3},\ \{d_1(n+j)\}_{j=-1,3},\, \{\beta_{n+j}\}_{j=0,2},\, \{\gamma_{n+j}\}_{j=0,3})=0. \tag{57}$$

Remark 3 *Equations (54), (55) for $2\le i\le k$, (56) and (57) constitute a system of $k+2$ equations for $k+2$ unknowns which are $d_i(n)$, $0\le i\le k-1$ and β_n and γ_n. Also, equations (55)-(57) contain the terms $d_i(n+j)$ and $d_{i+1}(n+j)$ for $j=-1$ or $j=3,\ 0\le i\le k-1$. The next subsection is devoted at eliminating these terms.*

For illustration, below we give explicitly equations (54), and (55) for $i=k$.

$$\begin{aligned}E_{k+1}^{k,0}:\ &(-6\,{p_1}^2\,p_0+2\,q_2\,p_0+2\,p_1\,q_1+{p_1}^2\,\beta_{n+1}-q_2\,\beta_{n+1}+{p_1}^2\,\beta_{n+2}-q_2\,\beta_{n+2}+4\,{p_1}^2\,\beta_n)\,d_k(n) \\ &+(12\,{p_1}^2\,p_0+4\,q_2\,p_0-6\,{p_1}^2\,\beta_{n+1}-q_2\,\beta_{n+2}-q_2\,\beta_n-2\,q_2\,\beta_{n+1} \\ &+4\,p_1\,q_1-3\,{p_1}^2\,\beta_{n+2}-3\,{p_1}^2\,\beta_n)d_k(n+1) \\ &+(-6\,{p_1}^2\,p_0+2\,q_2\,p_0+2\,p_1\,q_1+{p_1}^2\,\beta_n+4\,{p_1}^2\,\beta_{n+2}-q_2\,\beta_{n+1}+{p_1}^2\,\beta_{n+1}-q_2\,\beta_n)\,d_k(n+2) \\ &-2\,({p_1}^2-q_2)\,p_1\,d_{k-1}(n)+4\,(q_2+{p_1}^2)\,p_1\,d_{k-1}(n+1)-2\,({p_1}^2-q_2)\,p_1\,d_{k-1}(n+2)=0;\end{aligned} \tag{58}$$

$$
\begin{aligned}
E_k^{k,0}:\ & (-6\,p_1\,p_0{}^2+8\,p_1\,p_0\,\beta_n+2\,p_1\,\beta_{n+1}\,p_0+2\,p_1\,p_0\,\beta_{n+2}+2\,q_1\,p_0-2\,p_1\,\beta_{n+1}\,\beta_n+2\,p_1\,q_0-q_1\,\beta_{n+2}\\
& -2\,p_1\,\beta_n{}^2-q_1\,\beta_{n+1}-2\,p_1\,\beta_n\,\beta_{n+2})d_k(n)+2\,\gamma_n\,p_1\,d_k(n-1)+(12\,p_1\,p_0{}^2-6\,p_1\,p_0\,\beta_{n+2}\\
& -12\,p_1\,\beta_{n+1}\,p_0+4\,q_1\,p_0-6\,p_1\,p_0\,\beta_n-q_1\,\beta_n+2\,p_1\,\beta_{n+1}{}^2-2\,q_1\,\beta_{n+1}+4\,p_1\,q_0\\
& -2\,p_1\,\gamma_{n+1}+2\,p_1\,\beta_n\,\beta_{n+2}+4\,p_1\,\beta_{n+1}\,\beta_{n+2}+4\,p_1\,\beta_{n+1}\,\beta_n-q_1\,\beta_{n+2}-2\,p_1\,\gamma_{n+2}) \qquad (59)\\
& d_k(n+1)+(-6\,p_1\,p_0{}^2+8\,p_1\,p_0\,\beta_{n+2}+2\,p_1\,\beta_{n+1}\,p_0+2\,p_1\,p_0\,\beta_n+2\,q_1\,p_0-2\,p_1\,\beta_n\,\beta_{n+2}\\
& -q_1\,\beta_n+2\,p_1\,q_0-2\,p_1\,\beta_{n+2}{}^2-q_1\,\beta_{n+1}-2\,p_1\,\beta_{n+1}\,\beta_{n+2})d_k(n+2)+2\,p_1\,\gamma_{n+3}\,d_k(n+3)\\
& +(-6\,p_1{}^2\,p_0+2\,q_2\,p_0+2\,p_1\,q_1+p_1{}^2\,\beta_{n+1}-q_2\,\beta_{n+1}+p_1{}^2\,\beta_{n+2}-q_2\,\beta_{n+2}+4\,p_1{}^2\,\beta_n)\\
& d_{k-1}(n)+(12\,p_1{}^2\,p_0+4\,q_2\,p_0-6\,p_1{}^2\,\beta_{n+1}-q_2\,\beta_{n+2}-q_2\,\beta_n-2\,q_2\,\beta_{n+1}+4\,p_1\,q_1\\
& -3\,p_1{}^2\,\beta_{n+2}-3\,p_1{}^2\,\beta_n)d_{k-1}(n+1)+\\
& +(-6\,p_1{}^2\,p_0+2\,q_2\,p_0+2\,p_1\,q_1+p_1{}^2\,\beta_n+4\,p_1{}^2\,\beta_{n+2}-q_2\,\beta_{n+1}+p_1{}^2\,\beta_{n+1}-q_2\,\beta_n)d_{k-1}(n+2)\\
& -2\,(p_1{}^2-q_2)\,p_1\,d_{k-2}(n)+4\,(q_2+p_1{}^2)\,p_1\,d_{k-2}(n+1)\\
& -2\,(p_1{}^2-q_2)\,p_1\,d_{k-2}(n+2)=0.
\end{aligned}
$$

2.4.2 Elimination of $d_i(n-1)$ and $d_i(n+3)$ for $0 \leq i \leq k-1$

Step 1 : Elimination of $d_{k-1}(n-1)$ and $d_{k-1}(n+3)$

Starting from (54), we express $d_{k-1}(n-1)$ and $d_{k-1}(n+3)$ in terms of the β_n and $d_{k-1}(n+j)$, $j = 0,1,2$ (by replacing n by $n-1$ and $n+1$ respectively in (54))

$$
\begin{aligned}
d_{k-1}(n-1) &= F_{k+1}^{k,1}\,(n, \{d_{k-1}(n+j)\}_{j=0,1},\ \{\beta_{n+j}\}_{j=-1,1})\,; \qquad (60)\\
d_{k-1}(n+3) &= F_{k+1}^{k,2}\,(n, \{d_{k+1}(n+j)\}_{j=1,2},\ \{\beta_{n+j}\}_{j=1,3})\,.
\end{aligned}
$$

Step 2 : Elimination of $d_{k-2}(n-1)$ and $d_{k-2}(n+3)$

Equation (55) for $i = k$

$$
\mathbb{T}(d_{k-2}(n)) = F_k^{k,0}\,(n,\ \{d_{k-1}(n+j)\}_{j=0,2},\ (\beta_{n+j})_{j=0,2},\ \{\gamma_{n+j}\}_{j=0,3})\,, \qquad (61)
$$

contains no term d_i with the arguments $n-1$ or $n+3$. Use of this equation with n replaced by $n-1$ and $n+1$ gives respectively

$$
\begin{aligned}
& d_{k-2}(n-1) = \qquad (62)\\
& F_k^{k,1}\,(n, \{d_{k-2}(n+j)\}_{j=0,1}, \{d_{k-1}(n+j)\}_{j=-1,1},\ \{\beta_{n+j}\}_{j=-1,1},\ \{\gamma_{n+j}\}_{j=-1,2})\,;\\
& d_{k-2}(n+3) =\\
& F_k^{k,2}\,(n, \{d_{k-2}(n+j)\}_{j=1,2}, \{d_{k-1}(n+j)\}_{j=1,3},\ \{\beta_{n+j}\}_{j=1,3},\ \{\gamma_{n+j}\}_{j=1,4})\,.
\end{aligned}
$$

We eliminate the terms $d_{k-1}(n-1)$ and $d_{k-1}(n+3)$ in the previous equations using (60) and get

$$
d_{k-2}(n-1) = \qquad (63)
$$

$\tilde{F}_k^{k,1}\left(n, \{d_{k-2}(n+j)\}_{j=0,1}, \{d_{k-1}(n+j)\}_{j=0,1}, \{\beta_{n+j}\}_{j=-1,1}, \{\gamma_{n+j}\}_{j=-1,2}\right);$
$d_{k-2}(n+3) =$
$\tilde{F}_k^{k,2}\left(n, \{d_{k-2}(n+j)\}_{j=1,2}, \{d_{k-1}(n+j)\}_{j=1,3}, \{\beta_{n+j}\}_{j=1,3}, \{\gamma_{n+j}\}_{j=1,4}\right).$

Step 3: Elimination of $d_{k-3}(n-1)$ and $d_{k-3}(n+3)$

Equation (55) for $i = k-1$

$$\mathbb{T}(d_{k-3}(n)) = F_{k-1}^{k,0}\left(n, \{d_{k-2}(n+j)\}_{j=0,2}, \{d_{k-1}(n+j)\}_{j=-1,3}, (\beta_{n+j})_{j=0,2}, \{\gamma_{n+j}\}_{j=0,3}\right), \tag{64}$$

contains the terms $d_{k-1}(n-1)$ and $d_{k-1}(n+3)$ and is transformed using (60) into

$$\mathbb{T}(d_{k-3}(n)) = \tilde{F}_{k-1}^{k,0}\left(n, \{d_{k-2}(n+j)\}_{j=0,2}, \{d_{k-1}(n+j)\}_{j=0,2}, (\beta_{n+j})_{j=-1,3}, \{\gamma_{n+j}\}_{j=0,3}\right). \tag{65}$$

In a similar way as in the step 2, we derive from the previous equation using (60) and (63)

$$\begin{aligned} &d_{k-3}(n-1) = \\ &\tilde{F}_k^{k-1,1}\left(n, \{d_{k-3}(n+j)\}_{j=0,1}, \{d_{k-2}(n+j)\}_{j=0,1}, \{d_{k-1}(n+j)\}_{j=0,1}, \{\beta_{n+j}\}_{j=-2,2}, \{\gamma_{n+j}\}_{j=-1,2}\right); \\ &d_{k-3}(n+3) = \\ &\tilde{F}_{k-1}^{k,2}\left(n, \{d_{k-3}(n+j)\}_{j=1,2}, \{d_{k-2}(n+j)\}_{j=1,2}, \{d_{k-1}(n+j)\}_{j=1,2}, \{\beta_{n+j}\}_{j=0,4}, \{\gamma_{n+j}\}_{j=1,4}\right). \end{aligned} \tag{66}$$

Step 4: Elimination of $d_{k-4}(n-1)$ and $d_{k-4}(n+3)$

Similar approach transforms equation (55) for $i = k-2$ into the equations

$$\begin{aligned} \mathbb{T}(d_{k-4}(n)) &= \tilde{F}_{k-2}^{k,0}\left(n, \{d_{k-3}(n+j)\}_{j=0,2}, \{d_{k-2}(n+j)\}_{j=0,2}, \{d_{k-1}(n+j)\}_{j=0,2},\right. \\ &\qquad \left.\{\beta_{n+j}\}_{j=-1,3}, \{\gamma_{n+j}\}_{j=-1,4}\right), \end{aligned} \tag{67}$$

which when used together with (60), (63) and (66) yields

$$\begin{aligned} d_{k-4}(n-1) &= \tilde{F}_k^{k-1,1}\left(n, \{d_{k-4}(n+j)\}_{j=0,1}, \{d_{k-3}(n+j)\}_{j=0,1}, \{d_{k-2}(n+j)\}_{j=0,1}, \{d_{k-1}(n+j)\}_{j=0,1},\right. \\ &\qquad \left.\{\beta_{n+j}\}_{j=-2,2}, \{\gamma_{n+j}\}_{j=-2,3}\right); \\ d_{k-4}(n+3) &= \tilde{F}_{k-1}^{k,2}\left(n, \{d_{k-4}(n+j)\}_{j=1,2}, \{d_{k-3}(n+j)\}_{j=1,2}, \{d_{k-2}(n+j)\}_{j=1,2}, \{d_{k-1}(n+j)\}_{j=1,2},\right. \\ &\qquad \left.\{\beta_{n+j}\}_{j=0,4}, \{\gamma_{n+j}\}_{j=0,5}\right). \end{aligned} \tag{68}$$

Step 5: Elimination of $d_i(n-1)$ and $d_i(n+3)$ for $0 \le i \le k-2$

Repeating the proceed, we get from (31) two different generalizations:

First case:

For given integer l satisfying $1 \le l \le \left[\frac{k}{2}\right]$, where $[x]$ means the integer part of x, we have

$$\begin{aligned} \mathbb{T}(d_{k-2l}(n)) &= \tilde{F}_{k-2l+2}^{k,0}\left(n, \{d_{k-2l+1}(n+j)\}_{j=0,2}, \{d_{k-2l+2}(n+j)\}_{j=0,2}, \{d_{k-2l+3}(n+j)\}_{j=0,2},\right. \\ &\qquad \left.\{\beta_{n+j}\}_{j=1-l,1+l}, \{\gamma_{n+j}\}_{j=1-l,2+l}\right), \end{aligned} \tag{69}$$

and

$$\begin{aligned} d_{k-2\,l}(n-1) &= \tilde{F}^{k,1}_{k-2\,l+2}\,(n,\,\{d_{k-2\,l}(n+j)\}_{j=0,1}\,\{d_{k-2\,l+1}(n+j)\}_{j=0,1},\ \{d_{k-2\,l+2}(n+j)\}_{j=0,1}, \\ &\qquad \{d_{k-2\,l+3}(n+j)\}_{j=0,1},\,\{\beta_{n+j}\}_{j=-l,l},\,\{\gamma_{n+j}\}_{j=-l,1+l})\,; \qquad (70) \\ d_{k-2\,l}(n+3) &= \tilde{F}^{k,2}_{k-2\,l+2}\,(n,\,\{d_{k-2\,l}(n+j)\}_{j=1,2}\,\{d_{k-2\,l+1}(n+j)\}_{j=1,2},\ \{d_{k-2\,l+2}(n+j)\}_{j=1,2}, \\ &\qquad \{d_{k-2\,l+3}(n+j)\}_{j=1,2},\,\{\beta_{n+j}\}_{j=2-l,2+l},\,\{\gamma_{n+j}\}_{j=2-l,3+l})\,. \end{aligned}$$

Second case:

For given integer l satisfying $1 \le l \le \left[\frac{k-1}{2}\right]$, we have

$$\begin{aligned} \mathbb{T}(d_{k-2\,l-1}(n)) &= \tilde{F}^{k,0}_{k-2\,l+1}\,(n,\,\{d_{k-2\,l}(n+j)\}_{j=0,2},\ \{d_{k-2\,l+1}(n+j)\}_{j=0,2},\,\{d_{k-2\,l+2}(n+j)\}_{j=0,2}, \\ &\qquad \{\beta_{n+j}\}_{j=-l,2+l},\,\{\gamma_{n+j}\}_{j=1-l,2+l})\,, \qquad (71) \end{aligned}$$

and

$$\begin{aligned} d_{k-2\,l-1}(n-1) &= \tilde{F}^{k,1}_{k-2\,l+1}\,(n,\,\{d_{k-2\,l-1}(n+j)\}_{j=0,1}\,\{d_{k-2\,l}(n+j)\}_{j=0,1},\ \{d_{k-2\,l+1}(n+j)\}_{j=0,1}, \\ &\qquad \{d_{k-2\,l+2}(n+j)\}_{j=0,1},\,\{\beta_{n+j}\}_{j=-1-l,1+l},\,\{\gamma_{n+j}\}_{j=-l,1+l})\,; \qquad (72) \\ d_{k-2\,l-1}(n+3) &= \tilde{F}^{k,2}_{k-2\,l+1}\,(n,\,\{d_{k-2\,l-1}(n+j)\}_{j=1,2}\,\{d_{k-2\,l}(n+j)\}_{j=1,2},\ \{d_{k-2\,l+1}(n+j)\}_{j=1,2}, \\ &\qquad \{d_{k-2\,l+2}(n+j)\}_{j=1,2},\,\{\beta_{n+j}\}_{j=1-l,3+l},\,\{\gamma_{n+j}\}_{j=2-l,3+l})\,. \end{aligned}$$

Step 6 : Transformation of the equations $E_1^{k,0}$ and $E_0^{k,0}$

Elimination of $d_0(n+j)$, $d_1(n+j)$ and $d_2(n+j)$ for $j=-1$ or $j=3$ in (56) and (57) using (70) and (72) yields respectively (depending on the parity of k)

$$\tilde{F}_1^{k,0}\left(n,\,\{d_0(n+j)\}_{j=0,2},\ \{d_1(n+j)\}_{j=0,2},\,\{d_2(n+j)\}_{j=0,2},\,\{\beta_{n+j}\}_{j=-\frac{k}{2},2+\frac{k}{2}},\,\{\gamma_{n+j}\}_{j=1-\frac{k}{2},2+\frac{k}{2}}\right)=0; \qquad (73)$$

$$\tilde{F}_0^{k,0}\left(n,\,\{d_0(n+j)\}_{j=0,2},\ \{d_1(n+j)\}_{j=0,2},\,\{\beta_{n+j}\}_{j=-\frac{k}{2},2+\frac{k}{2}},\,\{\gamma_{n+j}\}_{j=-\frac{k}{2},3+\frac{k}{2}}\right)=0, \qquad (74)$$

for k even, and,

$$\begin{aligned} \tilde{F}_1^{k,0}\Big(n,\,\{d_0(n+j)\}_{j=0,2},\ \{d_1(n+j)\}_{j=0,2},\,\{d_2(n+j)\}_{j=0,2},\,\{\beta_{n+j}\}_{j=-\frac{k-1}{2},2+\frac{k-1}{2}}, \qquad (75) \\ \{\gamma_{n+j}\}_{j=-\frac{k-1}{2},3+\frac{k-1}{2}}\Big)=0; \end{aligned}$$

$$\tilde{F}_0^{k,0}\left(n,\,\{d_0(n+j)\}_{j=0,2},\ \{d_1(n+j)\}_{j=0,2},\,\{\beta_{n+j}\}_{j=-1-\frac{k-1}{2},3+\frac{k-1}{2}},\,\{\gamma_{n+j}\}_{j=-\frac{k-1}{2},3+\frac{k-1}{2}}\right)=0, \qquad (76)$$

for k odd.

Remark 4 *After eliminating all $d_i(n-1)$ and $d_i(n+3)$, we obtain a system of $k+2$ equations, namely (54), (69) for $1 \le l \le \left[\frac{k}{2}\right]$, (71) for $1 \le l \le \left[\frac{k-1}{2}\right]$, (73) and (74) for k even (or (75) and (76) for k odd); for $k+2$ unknowns which are $d_i(n)$, $0 \le i \le k-1$, β_n and γ_n. This system is linear in $d_i(n+j)$, $0 \le j \le 2$, $0 \le i \le k-1$ (see Remark 1). Moreover, its order in β_n and γ_n is at most $k+2$ and $k+3$ respectively for k even, and $k+3$ and $k+2$ for k odd.*

2.4.3 Elimination of $d_i(n+j)$, $0 \le j \le 2$, $0 \le i \le k-1$

In the first step, we rewrite equations (54), (69) for $1 \le l \le \left[\frac{k}{2}\right]$, (71) for $1 \le l \le \left[\frac{k-1}{2}\right]$, (73) and (74) for k even (or (75) and (76) for k odd) respectively as

$$\begin{aligned}
&\sum_{j=0}^{2} e_j^{k+1}(n)\, d_{k-1}(n+j) = t_{k+1}(n);\\
&\sum_{j=0}^{2} e_j^{k}(n)\, d_{k-1}(n+j) + \sum_{j=0}^{2} f_j^{k}(n)\, d_{k-2}(n+j) = t_k(n);\\
&\sum_{j=0}^{2} e_j^{k-1}(n)\, d_{k-1}(n+j) + \sum_{j=0}^{2} f_j^{k-1}(n)\, d_{k-2}(n+j) + \sum_{j=0}^{2} g_j^{k-1}(n)\, d_{k-3}(n+j) = t_{k-1}(n);\\
&\sum_{j=0}^{2} e_j^{i}(n)\, d_{i+1}(n+j) + \sum_{j=0}^{2} f_j^{i}(n)\, d_{i}(n+j) + \sum_{j=0}^{2} g_j^{i}(n)\, d_{i-1}(n+j) +\\
&\qquad \sum_{j=0}^{2} g_j^{i}(n)\, d_{i-2}(n+j) = t_i(n);\quad 2 \le i \le k-2; \qquad\qquad (77)\\
&\sum_{j=0}^{2} e_j^{1}(n)\, d_{2}(n+j) + \sum_{j=0}^{2} f_j^{1}(n)\, d_{1}(n+j) + \sum_{j=0}^{2} g_j^{1}(n)\, d_{0}(n+j) = t_1(n);\\
&\sum_{j=0}^{2} e_j^{0}(n)\, d_{1}(n+j) + \sum_{j=0}^{2} f_j^{0}(n)\, d_{0}(n+j) = t_0(n),
\end{aligned}$$

where $e_j^l(x)$, $f_j^l(x)$, $g_j^l(x)$, $h_j^l(x)$ and $t_l(x)$ for $0 \le l \le k+1$, $0 \le j \le 2$ are functions of the variables

$$\beta_{n+j},\ -\frac{k}{2} \le j \le 2+\frac{k}{2},\quad \gamma_{n+j},\ -\frac{k}{2} \le j \le 3+\frac{k}{2},$$

for k even; and

$$\beta_{n+j},\ -1-\frac{k-1}{2} \le j \le 3+\frac{k-1}{2},\quad \gamma_{n+j},\ -\frac{k-1}{2} \le j \le 3+\frac{k-1}{2},$$

for k odd. Note that the two previous equations can be summarized as

$$\beta_{n+j},\ -k_1-k_2 \le j \le 2+k_1+k_2,\quad \gamma_{n+j},\ -k_1 \le j \le 3+k_1,$$

with

$$k_1 = \frac{k}{2},\ k_2 = 0,\ \text{if } k \text{ is even and } k_1 = \frac{k-1}{2},\ k_2 = 1,\ \text{if } k \text{ is odd.} \qquad (78)$$

In the second step, since our objective is to eliminate all the $d_i(n+j)$ in the previous equation, we will from now consider all $d_i(n+j)$, $0 \le i \le k-1$, $0 \le j \le 2$ as unknowns. In this case, we have a system of $k+2$ equations with $3\,k$ unknowns.

Solving (77) in terms of the unknowns $d_{k-1}(n+2)$, $d_{k-1}(n+1)$, $d_{k-1}(n)$ and $d_i(n+2)$, $0 \le i \le k-2$ we get

$$d_{k-1}(n+2) = H_2\left(n,\, \{d_l(n+j)\}_{0\le l\le k-2,\, 0\le j\le 1},\, \{\beta_{n+j}\}_{j=-k_1-k_2, 2+k_1+k_2},\, \{\gamma_{n+j}\}_{j=-k_1, 3+k_1}\right); \tag{79}$$

$$d_{k-1}(n+1) = H_1\left(n,\, \{d_l(n+j)\}_{0\le l\le k-2,\, 0\le j\le 1},\, \{\beta_{n+j}\}_{j=-k_1-k_2, 2+k_1+k_2},\, \{\gamma_{n+j}\}_{j=-k_1, 3+k_1}\right); \tag{80}$$

$$d_{k-1}(n) = H_0\left(n,\, \{d_l(n+j)\}_{0\le l\le k-2,\, 0\le j\le 1},\, \{\beta_{n+j}\}_{j=-k_1-k_2, 2+k_1+k_2},\, \{\gamma_{n+j}\}_{j=-k_1, 3+k_1}\right); \tag{81}$$

$$d_i(n+2) = J_i\left(n,\, \{d_l(n+j)\}_{0\le l\le k-2,\, 0\le j\le 1},\, \{\beta_{n+j}\}_{j=-k_1-k_2, 2+k_1+k_2},\, \{\gamma_{n+j}\}_{j=-k_1, 3+k_1}\right), \tag{82}$$

$$0 \le i \le k-2,$$

where the integers k_1 and k_2 are given by (78).

In the third step, we compare equations (79) with (80), and (80) with (81) and obtain a new system of equations without $d_{k-1}(n+j)$, $0 \le j \le 2$, namely:

$$\begin{aligned}
&H_1\left(n+1,\, \{d_l(n+j)\}_{0\le l\le k-2,\, 1\le j\le 2},\, \{\beta_{n+j}\}_{j=1-k_1-k_2, 3+k_1+k_2},\, \{\gamma_{n+j}\}_{j=1-k_1, 4+k_1}\right) = &(83)\\
&\quad H_2\left(n,\, \{d_l(n+j)\}_{0\le l\le k-2,\, 0\le j\le 1},\, \{\beta_{n+j}\}_{j=-k_1-k_2, 2+k_1+k_2},\, \{\gamma_{n+j}\}_{j=-k_1, 3+k_1}\right);\\
&H_0\left(n+1,\, \{d_l(n+j)\}_{0\le l\le k-2,\, 1\le j\le 2},\, \{\beta_{n+j}\}_{j=1-k_1-k_2, 3+k_1+k_2},\, \{\gamma_{n+j}\}_{j=1-k_1, 4+k_1}\right) = &(84)\\
&\quad H_1\left(n,\, \{d_l(n+j)\}_{0\le l\le k-2,\, 0\le j\le 1},\, \{\beta_{n+j}\}_{j=-k_1-k_2, 2+k_1+k_2},\, \{\gamma_{n+j}\}_{j=-k_1, 3+k_1}\right);\\
&d_i(n+2) = J_i\left(n,\, \{d_l(n+j)\}_{0\le l\le k-2,\, 0\le j\le 1},\, \{\beta_{n+j}\}_{j=-k_1-k_2, 2+k_1+k_2},\, \{\gamma_{n+j}\}_{j=-k_1, 3+k_1}\right), &(85)\\
&0 \le i \le k-2.
\end{aligned}$$

The previous system contains $k+1$ equations with $3(k-1)$ unknowns which are $d_i(n+j)$, $0 \le i \le k-2$, $0 \le j \le 2$ and can be rewritten as

$$\sum_{0\le j\le 2}\ \sum_{0\le i\le k-2} u_{l,j}^{i}(n)\, d_i(n+j) = v_l(n),\ 0 \le l \le k, \tag{86}$$

where $u_{l,j}^{i}(n)$ are functions of the variables $\{\beta_{n+j}\}_{j=-k_1-k_2, 3+k_1+k_2}$, $\{\gamma_{n+j}\}_{j=-k_1, 4+k_1}$. This system is similar to the one in (77) but with k replaced by $k-1$. Hence we are doing with a recursive algorithm. Repeating this operation $k-1$ times one obtains a system of three equations with three unknowns which are $d_0(n+j)$, $0 \le j \le 2$. Following the procedure presented in the Subsection 2.3.3, one deduces the two nonlinear difference equations for the β_n and γ_n which are the expected *Laguerre-Freud equations for the recurrence coefficients of the Laguerre-Hahn polynomials of generic class k.*

Also, the Laguerre-Freud equations are valid for $n \ge k_1 + k_2$ (for β_n) and $n \ge k_1$ (for γ_n). The initial values $\beta_0 \ldots \beta_{k_1+k_2-1}$ and $\gamma_1 \ldots \gamma_{k_1-1}$ are computed using (25)-(27).

Remark 5 *Since each iteration increases the order of the equations by one, one deduces from Remark 4 that the order of the Laguerre-Freud equations obtained above is at most $2\,k+2$ and $2\,k+3$ respectively for k even, and $2\,k+3$ and $2\,k+2$ for k odd.*

3 Applications

3.1 Discrete semi-classical orthogonal polynomials of class 1

Here we suppose that $x(s) = s$ (i.e. $P(x) = x$, $Q(x) = \frac{1}{4}$). Also, for this illustration, we will restrict to the cases when $k = 1$ and the polynomial $A_0(x)$ is of degree at most 2.

From equations (25)-(27), we get

$$
\begin{aligned}
&a_2(n+1) - a_2(n) = 0;\\
&2\,a_1(n+1) - 2\,a_1(n) + d_1(n) = 0;\\
&2\,a_0(n+1) - 2\,a_0(n) + d_0(n) = 0;\\
&c_2(n+1) + c_2(n) + 2\,d_1(n) = 0;\\
&c_1(n+1) + c_1(n) - 2\,\beta_n\,d_1(n) + 2\,d_0(n) = 0;\\
&c_0(n+1) + c_0(n) - 2\,\beta_n\,d_0(n) = 0;\\
&c_2(n) + d_1(n) = 0;\\
&a_2(n) - \beta_n\,c_2(n) + c_1(n) - 2\,\beta_n\,d_1(n) + d_0(n) = 0;
\end{aligned}
\tag{87}
$$

$$
d_1(n-1)\,\gamma_n + a_1(n) - \beta_n\,c_1(n) + c_0(n) - \frac{1}{4}\,d_1(n) + d_1(n)\,{\beta_n}^2 - 2\,\beta_n\,d_0(n) - d_1(n+1)\,\gamma_{n+1} = 0;
\tag{88}
$$

$$
-\beta_n\,c_0(n) + d_0(n-1)\,\gamma_n - \frac{1}{4}\,d_0(n) + d_0(n)\,{\beta_n}^2 - d_0(n+1)\,\gamma_{n+1} + a_0(n) = 0.
\tag{89}
$$

To obtain the Laguerre-Freud equations, we will have to eliminate all coefficients a_i, c_i and d_i. This elimination is always possible from the algorithm described in section 2. However, for simples cases, it may be more suitable not to proceed to the elimination of all the unknowns a_i, c_i and d_i but just to solve for part of them. By doing so, one avoids to increase the order of the final Laguerre-Freud equations, in β_n and γ_n.

First, we use equations (87) and get

$$
\begin{aligned}
a_2(n) &= a_2(0);\\
a_1(n) &= a_1(0) - \frac{n}{2}\,d_1(0);\\
c_2(n) &= -d_1(0);\\
c_1(n) &= c_1(0) + 2\,n\,a_2(0);\\
d_1(n) &= d_1(0);\\
d_0(n) &= \beta_n\,d_1(0) - c_1(0) - (2\,n+1)\,a_2(0).
\end{aligned}
$$

Next, we eliminate $a_0(n)$ and $c_0(n)$ in (88) and (89) using (87) and get respectively after taking into account the previous equations

$$
d_1(0)\,{\beta_{n+1}}^2 - (c_1(0) + 2\,a_2(0)\,n + 4\,a_2(0))\,\beta_{n+1} - d_1(0)\,{\beta_n}^2 + (c_1(0) + 2\,a_2(0)\,n)\,\beta_n
\tag{90}
$$

$$+ d_1(0)\,\gamma_{n+2} - d_1(0)\,\gamma_n - 2\,a_1(0) + d_1(0)\,n + d_1(0) = 0;$$

$$\begin{aligned}
&2\,d_1(0)\,\gamma_{n+2}\,\beta_{n+2} - 2\,a_2(0)\,{\beta_{n+1}}^2 \\
&+ (2\,d_1(0)\,\gamma_{n+2} - 4\,d_1(0)\,\gamma_{n+1} + d_1(0)\,n + 2\,d_1(0) - 2\,a_1(0))\,\beta_{n+1} + 2\,a_2(0)\,{\beta_n}^2 \\
&+ (-4\,d_1(0)\,\gamma_{n+1} + 2\,a_1(0) - d_1(0)\,n + 2\,d_1(0)\,\gamma_n)\,\beta_n + 2\,d_1(0)\,\gamma_n\,\beta_{n-1} \\
&+ (-2\,c_1(0) - 4\,a_2(0)\,n - 10\,a_2(0))\,\gamma_{n+2} + (4\,c_1(0) + 8\,a_2(0)\,n + 8\,a_2(0))\,\gamma_{n+1} \\
&+ (-2\,c_1(0) - 4\,a_2(0)\,n + 2\,a_2(0))\,\gamma_n - c_1(0) - 2\,a_2(0)\,n - 2\,a_2(0) = 0,
\end{aligned} \tag{91}$$

where $A_0(x) = a_2(0)\,x^2 + a_1(0)\,x + a_0(0)$, $C_0(x) = c_2(0)\,x^2 + c_1(0)\,x + c_0(0)$ and $D_0(x) = d_1(0)\,x + d_0(0)$.

The Laguerre-Freud equations (90) and (91) contain those of the recurrence coefficients of polynomials orthogonal with respect to the discrete weight $\rho(x)$ satisfying the discrete Pearson equation $\Delta(\sigma\,\rho) = \tau\,\rho$, where σ and τ are polynomials of degree at most 2 and Δ the forward difference operator $\Delta f(n) = f(n+1) - f(n)$. The generalized Charlier polynomials introduced in [17], which are the nonclassical extension of Charlier polynomials, contain a particular example of this type of polynomials. In fact, the generalized Charlier polynomials are discrete orthogonal polynomials with the weight

$$\rho(x) = \frac{\mu^x}{(x!)^N}, \quad x = 0,\, 1,\, 2,\, \ldots, \tag{92}$$

where $N \geq 1$ and $\mu > 0$. For $N = 1$, one deals with the ordinary Charlier polynomials. When $N = 2$, the generalized Charlier weight satisfies the discrete Pearson equation

$$\Delta(\sigma\,\rho) = \tau\,\rho, \tag{93}$$

with $\sigma(x) = x^2$ and $\tau(x) = \mu - x^2$. The previous discrete Pearson equation corresponds to the Riccati difference equation [13] (Theorem 3)

$$\sigma(x+1)\,\Delta S_0(x) = (\tau(x) - \Delta\sigma(x))\,S_0(x) + x + 1 + \beta_0, \tag{94}$$

with $\beta_0 = \frac{\mu\,{}_0F_1(2;\mu)}{{}_0F_1(1;\mu)}$.

Comparison of (94) and (2) for $x(s) = s$ allows to deduce

$$\begin{aligned}
A_0(x) &= \frac{x^2}{2} + \frac{x}{2} + \frac{2\,\mu - 1}{4}; \\
C_0(x) &= -x^2 - x + \mu - \frac{1}{4}; \\
D_0(x) &= x + \frac{1}{2} + \beta_0.
\end{aligned} \tag{95}$$

Substituting (95) into (90) and (91) produce the Laguerre-Freud equations for generalized Charlier for $N = 2$.

$${\beta_{n+1}}^2 - (n+1)\,\beta_{n+1} - {\beta_n}^2 - (1-n)\,\beta_n + \gamma_{n+2} - \gamma_n + n = 0; \tag{96}$$

$$
\begin{aligned}
2\,\gamma_{n+2}\,\beta_{n+2} \quad &- \quad {\beta_{n+1}}^2 + (2\,\gamma_{n+2} - 4\,\gamma_{n+1} + n + 1)\,\beta_{n+1} + {\beta_n}^2 + (-4\,\gamma_{n+1} + 1 - n + 2\,\gamma_n)\,\beta_n \\
&+ \quad 2\,\gamma_n\,\beta_{n-1} + (-3 - 2\,n)\,\gamma_{n+2} + 4\,n\,\gamma_{n+1} + (3 - 2\,n)\,\gamma_n - n = 0. \qquad (97)
\end{aligned}
$$

Addition of (96) to (97) gives equation

$$
(\beta_{n+2} + \beta_{n+1} - n - 1)\,\gamma_{n+2} - 2\,(\beta_{n+1} + \beta_n - n)\,\gamma_{n+1} + (\beta_n + \beta_{n-1} - n + 1)\,\gamma_n = 0, \qquad (98)
$$

which can easily be brought to

$$
(\beta_n + \beta_{n-1} - n + 1)\,\gamma_n = n\mu. \qquad (99)
$$

Notice that equations (96) and (99) were obtained in [30] after some calculations in order to simplify the initial Laguerre-Freud equations given in [17]

$$
\begin{aligned}
\gamma_{n+1} + \gamma_n \quad &= \quad -\frac{n(n-1)}{2} - \beta_n^2 + n\,\beta + \sum_{j=0}^{n-1} \beta_j + \mu; \\
(\beta_{n+1} + \beta_n)\,\gamma_{n+1} \quad &= \quad -n \sum_{j=0}^{n} \beta_j + (n+1)\,\gamma_{n+1} + \frac{(n+1)\,n\,(n-1)}{6} + \sum_{j=0}^{n} \beta_j^2 + 2 \sum_{j=1}^{n} \gamma_j.
\end{aligned}
$$

Also, these equations were used in [30] to show that the coefficients β_n and γ_n are related to certain discrete Painlevé equation and to analyze their asymptotic behavior already suggested in [9] (Conjecture 8.1, p. 112) and in [17]

$$
\lim_{n \longrightarrow \infty} (\beta_n - n) = 0, \quad \lim_{n \longrightarrow \infty} \gamma_n = \mu.
$$

Similar work [16] as the one done in [30] is under investigation using equations (90) and (91) for the generalized Meixner polynomials introduced in [28], in order to prove the asymptotic behavior of β_n and γ_n suggested in [8, 29].

3.2 Continuous semi-classical orthogonal polynomials of class 1

Here we suppose that $x(s) = x(0)$ (i.e. $P(x) = x$, $Q(x) = 0$), $k = 1$ and the polynomial $A_0(x)$ is of degree at most 2. Following the way described in the Subsection 3.1, we obtain the two Laguerre-Freud equations

$$
\begin{aligned}
{\beta_{n+1}}^2\,d_1(0) \quad &- \quad (c_1(0) + 2\,a_2(0)\,n + 4\,a_2(0))\,\beta_{n+1} - {\beta_n}^2\,d_1(0) + (c_1(0) + 2\,a_2(0)\,n)\,\beta_n \\
&-2\,a_1(0) - d_1(0)\,\gamma_n + d_1(0)\,\gamma_{n+2} = 0;
\end{aligned}
$$

$$
\begin{aligned}
&d_1(0)\,\gamma_{n+2}\,\beta_{n+2} - a_2(0)\,{\beta_{n+1}}^2 + (-2\,d_1(0)\,\gamma_{n+1} - a_1(0) + d_1(0)\,\gamma_{n+2})\,\beta_{n+1} + a_2(0)\,{\beta_n}^2 \\
&+(d_1(0)\,\gamma_n + a_1(0) - 2\,d_1(0)\,\gamma_{n+1})\,\beta_n + d_1(0)\,\gamma_n\,\beta_{n-1} + (-c_1(0) - 2\,a_2(0)\,n - 5\,a_2(0))\,\gamma_{n+2} \\
&+(2\,c_1(0) + 4\,a_2(0)\,n + 4\,a_2(0))\,\gamma_{n+1} + (-c_1(0) - 2\,a_2(0)\,n + a_2(0))\,\gamma_n = 0,
\end{aligned}
$$

where $A_0(x) = a_2(0)\,x^2 + a_1(0)\,x + a_0(0)$, $C_0(x) = c_2(0)\,x^2 + c_1(0)\,x + c_0(0)$ and $D_0(x) = d_1(0)\,x + d_0(0)$.

3.3 Continuous symmetric orthogonal polynomials

Here we assume that $P(x) = x$, $Q(x) = 0$ (continuous case) and that the polynomials are semi-classical (i.e. $B_0(x) = \sum_{j=0}^{k+2} b_j(0)\, x^j \equiv 0$) and orthogonal with respect to a *symmetric* weight function $\rho(x)$, defined on a symmetric interval $[-a, a]$ and satisfying $\rho(-x) = \rho(x)$. Therefore, $\beta_n = 0$, $n \geq 0$, and the equation (31) reduces to

$$d_{i-2}(n+2)-2\, d_{i-2}(n+1)+d_{i-2}(n)-(\gamma_{n+3}\, d_i(n+3)-\gamma_{n+2}\, d_i(n+1))+\gamma_{n+1}\, d_i(n+1)-\gamma_n\, d_i(n-1) = 0. \tag{100}$$

The previous equation can easily be transformed into

$$d_{i-2}(n+1)-d_{i-2}(n)-(\gamma_{n+2}\, d_i(n+2)-\gamma_{n+1}\, d_i(n)+\gamma_{n+1}\, d_i(n+1)-\gamma_n\, d_i(n-1)) = \alpha_i, \quad 0 \leq i \leq k+2, \tag{101}$$

where α_i is a constant with respect to n.

3.3.1 Freud weight $\rho(x) = e^{-x^4}$

For illustration, we consider that the polynomials are orthogonal with respect to the Freud weight $\rho(x) = e^{-x^4}$. This weight is symmetric, semi-classical and satisfies the Pearson equation

$$\frac{d}{dx}(\sigma(x)\, \rho(x)) = \tau(x)\, \rho(x),$$

with $\sigma(x) = 1$ and $\tau(x) = -4\, x^3$. The previous Pearson equation corresponds to the Riccati equation [24]

$$A_0(x)\, \frac{d}{dx} S_0(x) = B_0(x)\, S_0^2(x) + C_0(x)\, S_0(x) + D_0(x),$$

with

$$A_0(x) = 1,\ B_0(x) = 0,\ C_0(x) = -4\, x^3,\ D_0(x) = -4\, (x^2 + \lambda_1), \tag{102}$$

where $\lambda_1 = \frac{\Gamma^2(3/4)}{\pi\sqrt{2}}$. Therefore, one remarks that the polynomials orthogonal with respect to the Freud weight $\rho(x) = e^{-x^4}$ correspond to special case of Laguerre-Hahn orthogonal polynomials of class $k = 2$.

In order to obtain the Laguerre-Freud equation (only one in this case), we consider (101) for $0 \leq i \leq 4$ and get, taking into account (28),

$$\begin{aligned}
&d_2(n+1) - d_2(n) = \alpha_4;\\
&d_1(n+1) - d_1(n) = \alpha_3;\\
&d_0(n+1) - d_0(n) - (\gamma_{n+2}\, d_2(n+2) - \gamma_{n+1}\, d_2(n) + \gamma_{n+1}\, d_2(n+1) - \gamma_n\, d_2(n-1)) = \alpha_2;\\
&\gamma_{n+2}\, d_1(n+2) - \gamma_{n+1}\, d_1(n) + \gamma_{n+1}\, d_1(n+1) - \gamma_n\, d_1(n-1) = \alpha_1;\\
&\gamma_{n+2}\, d_0(n+2) - \gamma_{n+1}\, d_0(n) + \gamma_{n+1}\, d_0(n+1) - \gamma_n\, d_0(n-1) = \alpha_0.
\end{aligned} \tag{103}$$

First, we use equation (27) for $n = 0$ and $i = 0,\ 1,\ 2$ taking into account (28) and (102) (keeping in mind that $\beta_n = 0,\ d_j(-1) = b_j(0) = 0$) to get

$$d_2(1) = d_2(0) = -4,\ d_1(1) = d_1(0) = 0,\ d_0(1) = -\frac{1}{\gamma_1},\ d_0(0) = -4\,\gamma_1. \tag{104}$$

Next, we use the three-term recurrence relation

$$P_{n+1} = x\,P_n - \gamma_n\,P_{n-1},\ n \geq 1,\ P_0(x) = 1,\ P_1(x) = x,$$

and the orthogonality of $\{P_n\}$ with respect to the weight $\rho(x) = e^{-x^4}$ to get

$$\gamma_1 = \frac{\Gamma^2(3/4)}{\pi\sqrt{2}},\ \gamma_2 = \frac{1}{4\,\gamma_1} - \gamma_1,\ \gamma_3 = \frac{12\,\gamma_1^2 - 1}{4\,\gamma_1\,(1 - 4\,\gamma_1^2)}. \tag{105}$$

Use of equations (104) and (105) transforms (103) into equations

$$d_2(n) = -4,\ d_1(n) = 0,\ n \geq 0; \tag{106}$$

$$d_0(n+1) - d_0(n) + 4\,(\gamma_{n+2} - \gamma_n) = 0,\ n \geq 1; \tag{107}$$

$$\gamma_{n+2}\,d_0(n+2) - \gamma_{n+1}\,d_0(n) + \gamma_{n+1}\,d_0(n+1) - \gamma_n\,d_0(n-1) = 0,\ n \geq 0, \tag{108}$$

from which we derive using (104) and (105)

$$d_0(n) + 4\,(\gamma_n + \gamma_{n+1}) = 0,\ n \geq 1;$$

$$\gamma_{n+1}\,d_0(n+1) - \gamma_n\,d_0(n-1) = -1,\ n \geq 1.$$

Combination of the previous two equations lead to the equation

$$4\,(\gamma_{n+1}^2 - \gamma_n^2) + 4\,(\gamma_{n+2}\,\gamma_{n+1} - \gamma_n\,\gamma_{n-1}) = -1,\ n \geq 1,$$

which using (105) is easily transformed into the Freud equation which is a special case of the discrete Painlevé equation $\mathrm{d} - \mathrm{P_I}$ [21].

$$4\,\gamma_n\,(\gamma_{n-1} + \gamma_n + \gamma_{n+1}) = n.$$

Acknowledgments

We are very grateful to the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. This work was done within the framework of the Associateship Scheme of the Abdus Salam ICTP. Financial support from the Swedish International Development Cooperation Agency is acknowledged.

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