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Analysis of Non-Linear Recurrence Relations for the Recurrence Coefficients of Generalized Charlier Polynomials

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Abstract

The recurrence coefficients of generalized Charlier polynomials satisfy a system of non-linear recurrence relations. We simplify the recurrence relations, show that they are related to certain discrete Painlevé equations, and analyze the asymptotic behaviour.

1 Introduction

Orthogonal polynomials on the real line always satisfy a three-term recurrence relation. For monic polynomials P_n this is of the form

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \tag{1.1}$$

with initial values $P_0 = 1$ and $P_{-1} = 0$. For the orthonormal polynomials p_n the recurrence relation is

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \tag{1.2}$$

where $a_n^2 = \gamma_n$ and $b_n = \beta_n$. The recurrence coefficients are given by the integrals

$$a_n = \int xp_n(x)p_{n-1}(x)\, d\mu(x), \qquad b_n = \int xp_n^2(x)\, d\mu(x)$$

and can also be expressed in terms of determinants containing the moments of the orthogonality measure μ [11]. For classical orthogonal polynomials one knows these recurrence coefficients explicitly, but when one uses non-classical weights, then often one does not know the recurrence coefficients explicitly. When the weight satisfies a first order differential equation with polynomial coefficients, then one can use a technique developped by Laguerre, Shohat and Freud [1, 4] to obtain non-linear recurrence relations for the recurrence coefficients. The most famous example is the Freud weight $w(x) = \exp(-x^4)$ on the real line [4, 9], for which $\beta_n = 0$ and γ_n satisfies

$$\gamma_{n+1} + \gamma_n + \gamma_{n-1} = \frac{n}{4\gamma_n},$$

which is a special case of the discrete Painlevé equation d-$\mathrm{P_I}$ [11]. In this paper we will investigate similar non-linear recurrence relations for discrete orthogonal polynomials which are non-classical extensions of the Charlier polynomials.

2 Generalized Charlier polynomials

Charlier polynomials [2, 12] are the orthogonal polynomials with respect to the Poisson distribution on the non-negative integers:

$$w_k = \frac{a^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

with $a > 0$. They are usually denoted by $C_n(x; a)$ and the orthogonality conditions are

$$\sum_{k=0}^{\infty} C_n(k; a) C_m(k; a) \frac{a^k}{k!} = n! a^{-n} e^a \delta_{m,n}.$$

The three-term recurrence relation for these polynomials is

$$-x C_n(x; a) = a C_{n+1}(x; a) - (n + a) C_n(x; a) + n C_{n-1}(x; a).$$

For the monic polynomials $P_n(x) = (-a)^n C_n(x; a)$ the three-term recurrence relation is

$$x P_n(x) = P_{n+1}(x) + (n + a) P_n(x) + an P_{n-1}(x),$$

so that

$$\beta_n = n + a, \qquad \gamma_n = an.$$

Generalized Charlier polynomials were introduced in [6]. These are discrete orthogonal polynomials on $\mathbb{N}$ with weights

$$w_k = \frac{a^k}{(k!)^N}, \qquad k = 0, 1, 2 \ldots,$$

where $N \geq 1$ and $a > 0$. For $N = 1$ one deals with the ordinary Charlier polynomials. For $N = 2$ the recurrence coefficients β_k, γ_{k+1} $(k = 0, 1, 2, \ldots)$ satisfy the Laguerre-Freud equations

$$\gamma_n + \gamma_{n+1} = -\binom{n}{2} - \beta_n^2 + n\beta_n + \sum_{i=0}^{n-1} \beta_i + a \tag{2.1}$$

$$(\beta_n + \beta_{n+1})\gamma_{n+1} = -n \sum_{i=0}^{n} \beta_i + n\gamma_{n+1} + \binom{n+1}{3} + \sum_{i=0}^{n} \beta_i^2 + 2 \sum_{i=1}^{n} \gamma_i + \gamma_{n+1} \tag{2.2}$$

with initial values

$$\beta_0 = \frac{\sqrt{a} I_1(2\sqrt{a})}{I_0(2\sqrt{a})}, \quad \gamma_0 = 0,$$

where I_0 and I_1 are the modified Bessel functions of order 0 and 1 [6].

We will now simplify this system of recurrence relations:

Proposition 1. *Let $\beta_n = n + b_n$, then the recurrence coefficients for generalized Charlier polynomials (for $N = 2$) satisfy*

$$\begin{aligned} (n + b_n + b_{n-1})\gamma_n &= na && (2.3) \\ na(b_{n-1} - b_n) &= \gamma_n(\gamma_{n+1} - \gamma_{n-1}), && (2.4) \end{aligned}$$

with initial conditions

$$b_0 = \frac{\sqrt{a} I_1(2\sqrt{a})}{I_0(2\sqrt{a})}, \qquad \gamma_0 = 0,\ \gamma_1 = a - b_0^2. \tag{2.5}$$

Proof. If we put $\beta_n = n + b_n$, then we have

$$\sum_{i=0}^{n-1} \beta_i = \binom{n}{2} + \sum_{i=0}^{n-1} b_i, \qquad \sum_{i=0}^{n} \beta_i^2 = \sum_{i=0}^{n} b_i^2 + 2\sum_{i=1}^{n} i b_i + \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

If we insert this into (2.1), then this gives

$$\gamma_{n+1} + \gamma_n = \sum_{i=0}^{n-1} b_i - b_n^2 - n b_n + a, \tag{2.6}$$

and (2.2) becomes

$$(2n+1+b_n+b_{n+1})\gamma_{n+1} = (n+1)\gamma_{n+1} - n\sum_{i=0}^{n-1} b_i + n b_n + 2\sum_{i=1}^{n-1} i b_i + \sum_{i=0}^{n} b_i^2 + 2\sum_{i=1}^{n} \gamma_i. \tag{2.7}$$

Summing both sides of (2.6) gives

$$\gamma_{n+1} + 2\sum_{i=1}^{n} \gamma_i = -2\sum_{i=1}^{n-1} i b_i - n b_n - \sum_{i=0}^{n} b_i^2 + n\sum_{i=0}^{n-1} b_i + (n+1)a, \tag{2.8}$$

and if we use this in (2.7) then we find (2.3). Next, we apply the difference operator to both sides of (2.6) to find

$$\gamma_{n+1} - \gamma_{n-1} = (b_{n-1} - b_n)(n + b_n + b_{n-1}). \tag{2.9}$$

If we now use (2.3), then this gives (2.4). ∎

The equations can be put into a somewhat more pleasing form, involving a special case of the discrete Painlevé II equation.

Proposition 2. *For every $n \geq 1$ we have $0 < \gamma_n < a$ and $b_n > 0$. If we put $\gamma_n = a(1-c_n^2)$ then (2.3)–(2.4) can be written as*

$$b_n = \sqrt{a} c_n c_{n+1} \tag{2.10}$$

$$n c_n = \sqrt{a}(c_{n+1} + c_{n-1})(1 - c_n^2), \tag{2.11}$$

with $c_1 = I_1(2\sqrt{a})/I_0(2\sqrt{a})$ and $c_0 = 1$.

Proof. Since γ_n are recurrence coefficients of orthogonal polynomials, it follows that $\gamma_n > 0$ for $n \geq 1$. Summing (2.4) gives

$$a\left(\sum_{k=0}^{n-1} b_k - n b_n\right) = \gamma_n \gamma_{n+1},$$

but on the other hand (2.6) gives

$$\sum_{k=0}^{n-1} b_k - n b_n = \gamma_{n+1} + \gamma_n + b_n^2 - a.$$

Combining both results gives

$$a(\gamma_{n+1} + \gamma_n + b_n^2 - a) = \gamma_n \gamma_{n+1}.$$

This can be written as

$$(\gamma_n - a)(\gamma_{n+1} - a) = a b_n^2,$$

hence $\gamma_n - a$ and $\gamma_{n+1} - a$ have the same sign. The initial condition $\gamma_1 = a - b_0^2$ hence implies that $\gamma_n - a < 0$ for every $n \geq 1$. Therefore we can write $\gamma_n = a - a c_n^2$ and we find (2.10). If we insert this into (2.3), then we find

$$a(1 - c_n^2)(n + \sqrt{a} c_n c_{n+1} + \sqrt{a} c_n c_{n-1}) = na,$$

which can easily be reduced to (2.11). ■

Recall that the discrete Painlevé II equation d-P$_{\mathrm{II}}$ is given by [5]

$$x_{n+1} + x_{n-1} = \frac{x_n z_n + \gamma}{1 - x_n^2},$$

where $z_n = \alpha n + \beta$. The equation (2.11) for c_n is of this form with $\beta = \gamma = 0$ and $\alpha = 1/\sqrt{a}$.

We now have the following asymptotic behaviour of the recurrence coefficients.

Theorem 1. *For the recurrence coefficients (β_n, γ_n) of generalized Charlier polynomials we set $\beta_n = n + b_n$. Then*

$$\lim_{n\to\infty} \gamma_n = a, \qquad \lim_{n\to\infty} b_n = 0.$$

Proof. If we write $\gamma_n = a(1 - c_n^2)$, then $0 < \gamma_n < a$ implies that $0 < c_n^2 < 1$ for $n \geq 1$. This mean that the right hand side of (2.11) remains bounded as $n \to \infty$. But then $n c_n$ remains bounded as well, which implies that $c_n \to 0$, and hence $\gamma_n \to a$. The asymptotic behavior of b_n then easily follows from (2.10). ■

The asymptotic behaviour given in the previous theorem was already suggested by the remarks in [6, p. 364].

3 Concluding remarks

We have found that the recurrence coefficients of generalized Charlier polynomials with $N = 2$ are given by the equations (2.10)–(2.11). The original Laguerre-Freud equations (2.1)–(2.2) are more complicated and some work was needed to find the underlying discrete Painlevé equation. It would be more convenient to have a method that gives the simple equations (2.10)–(2.11) directly.

The asymptotic behaviour of the recurrence coefficients allows us to obtain the asymptotic zero distribution of the generalized Charlier polynomials, using the results from [7]. If $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ are the zeros of P_n, then the contracted zero distribution

$$\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_{j,n}/n},$$

where δ_c is a Dirac probability measure concentrated at c, converges weakly to the uniform distribution on $[0, 1]$, just as in the case of the usual Charlier polynomials [7, p. 190]. This result can also be obtained from the behaviour of the orthogonality weights $w(k) = w_k = a^k/(k!)^2$, for which

$$\log w(k) = k \log a - 2 \log k! \sim k \log a - 2(k + \frac{1}{2}) \log k + 2k.$$

The growth of this weight is dominated by the term $k \log k$, which grows faster than the rate k at which the points in the support tend to infinity. Following remarks in [8, p. 197], we see that the asymptotic zero distribution is indeed given by the uniform distribution on $[0, 1]$, as is the case for the usual Charlier polynomials [8, p. 200]. The asymptotic behaviour of the recurrence coefficients also gives information about the largest zero $x_{n,n}$. In the formula

$$\beta_n = \int x p_n^2(x)\, d\mu(x)$$

we can compute the integral exactly using Gauss quadrature at the $n + 1$ zeros of the orthogonal polynomial P_{n+1}, so that

$$\beta_n = \sum_{j=1}^{n+1} x_{j,n+1} p_n^2(x_{j,n+1})\lambda_{j,n+1},$$

where $\lambda_{j,n+1} > 0$ are the Christoffel numbers [2]. Since $x_{j,n+1} \le x_{n+1,n+1}$ for every j, we easily find

$$\beta_n \le x_{n+1} \sum_{j=1}^{n+1} p_n^2(x_{j,n+1})\lambda_{j,n+1} = x_{n+1,n+1},$$

so that β_n is a lower bound for the largest zero $x_{n+1,n+1}$. The results in [7] and [8] show that $\lim_{n\to\infty} x_{n,n}/n = 1$.

The recurrence coefficients of generalized Meixner polynomials [13], which are the orthogonal polynomials on the linear lattice with orthogonality weight

$$w_k = \frac{(\alpha_1)_k(\alpha_2)_k}{(k!)^2} c^k, \qquad k = 0, 1, 2, 3, \ldots$$

with $\alpha_1, \alpha_2 > 0$ and $0 < c < 1$ satisfy recurrence relations [3] which are somewhat similar to (2.1)–(2.2), but now with three parameters α_1, α_2 and c:

$$\begin{aligned}(1-c)(\gamma_n+\gamma_{n+1}) &= -(1-c)\binom{n}{2} - (1-c)\beta_n^2 + [(1+c)n + c(\alpha_1+\alpha_2)]\beta_n \\ &\quad + (1+c)\sum_{i=0}^{n-1}\beta_i + c(\alpha_1+\alpha_2)n + c\alpha_1\alpha_2 \end{aligned} \tag{3.1}$$

$$\begin{aligned}(1-c)(\beta_n+\beta_{n+1})\gamma_{n+1} &= -n\sum_{i=0}^{n}\beta_i + [(1+c)n + c(\alpha_1+\alpha_2)+1]\gamma_{n+1} \\ &\quad + \binom{n+1}{3} + \sum_{i=0}^{n}\beta_i^2 + 2\sum_{i=1}^{n}\gamma_i, \end{aligned} \tag{3.2}$$

with initial values

$$\beta_0 = c\alpha_1\alpha_2 \frac{{}_2F_1(\alpha_1+1, \alpha_2+1; 2; c)}{{}_2F_1(\alpha_1, \alpha_2; 1; c)}, \qquad \gamma_0 = 0,$$

where ${}_2F_1$ is Gauss' hypergeometric function. The analysis of these equations is more involved and will be presented in another paper, where we will show that the asymptotic behaviour conjectured in [14] indeed holds.

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