```
> restart:
> read "ODE3solve.mpl":
```

Package "Solving third-order holonomic differential equations", Maple 16
Copyright 2017, Mouafo Wouodjie Merlin, University of Kassel
Package "Hypergeometric Summation", Maple V - Maple 17
Copyright 1998-2013, Wolfram Koepf, University of Kassel **(1)**

Here are the Maples examples in chapter 3.

```
>            ####### IN THE PROOF OF LEMMA 3.2 #######
```

In the proof of "Lemma 3.3" we have the following Maple implementations:

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,\nu^2 - 2\,x + 2\right)Dx - 1$$ **(2)**

```
> gen_exp(LBB,t,x=0);
```
$$\left[\,[0, t=x], [\nu, t=x], [-\nu, t=x]\,\right]$$ **(3)**

```
> gen_exp(LBB,t,x=infinity);
```
$$\left[\left[\frac{1}{2}, t=\frac{1}{x}\right], \left[\frac{1}{t}+\frac{1}{2}, t^2=\frac{1}{x}\right]\right]$$ **(4)**

```
> LBB:=subs(nu=0,LBB);
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x + 2)\,Dx - 1$$ **(5)**

```
> formal_sol(LBB,t,x=0);
```
$$\left[\left[\frac{\ln(t)^2}{2}+\left(\frac{\ln(t)^2}{4}-\frac{\ln(t)}{2}\right)t+\left(\frac{3\ln(t)^2}{64}-\frac{11\ln(t)}{64}+\frac{1}{8}\right)t^2+\left(\frac{5\ln(t)^2}{1152}\right.\right.\right.$$ **(6)**

$$\left.-\frac{73\ln(t)}{3456}+\frac{3}{128}\right)t^3+\left(\frac{35\ln(t)^2}{147456}-\frac{1217\ln(t)}{884736}+\frac{419}{221184}\right)t^4+\left(\frac{7\ln(t)^2}{819200}\right.$$

$$\left.-\frac{4127\ln(t)}{73728000}+\frac{157}{1769472}\right)t^5+\mathrm{O}(t^6), \ln(t)+\left(\frac{\ln(t)}{2}-\frac{1}{2}\right)t+\left(\frac{3\ln(t)}{32}-\frac{11}{64}\right)t^2$$

$$+\left(\frac{5\ln(t)}{576}-\frac{73}{3456}\right)t^3+\left(\frac{35\ln(t)}{73728}-\frac{1217}{884736}\right)t^4+\left(\frac{7\ln(t)}{409600}-\frac{4127}{73728000}\right)t^5$$

$$+\mathrm{O}(t^6), 1+\frac{1}{2}\,t+\frac{3}{32}\,t^2+\frac{5}{576}\,t^3+\frac{35}{73728}\,t^4+\frac{7}{409600}\,t^5+\mathrm{O}(t^6), t=x\right]$$

```
> DEtools['formal_sol'](LBB,`has logarithm?`,x=0);
```
true **(7)**

```
> L12 := x^2*Dx^3+(x*b2+x+x*b1)*Dx^2+(b2*b1-x)*Dx-a1;
```
$$L12 := x^2\,Dx^3 + (x\,b1 + x\,b2 + x)\,Dx^2 + (b2\,b1 - x)\,Dx - a1$$ **(8)**

```
> L12:=subs({a1=1,b1=1,b2=1},L12);
```

$$L12 := x^2\,Dx^3 + 3\,x\,Dx^2 + (-x+1)\,Dx - 1 \tag{9}$$

```
> formal_sol(L12,t,x=0);
```

$$\left[\left[\left(\frac{\ln(t)^2}{2} + \left(\frac{\ln(t)^2}{2} - 2\ln(t) + 3\right)t + \left(\frac{\ln(t)^2}{8} - \frac{3\ln(t)}{4} + \frac{23}{16}\right)t^2 + \left(\frac{\ln(t)^2}{72}\right.\right.\right.\right. \tag{10}$$

$$\left. - \frac{11\ln(t)}{108} + \frac{97}{432}\right)t^3 + \left(\frac{\ln(t)^2}{1152} - \frac{25\ln(t)}{3456} + \frac{485}{27648}\right)t^4 + \left(\frac{\ln(t)^2}{28800} - \frac{137\ln(t)}{432000}\right)$$

$$\left. + \frac{14269}{17280000}\right)t^5 + O(t^6),\ \ln(t) + (\ln(t) - 2)\,t + \left(\frac{\ln(t)}{4} - \frac{3}{4}\right)t^2 + \left(\frac{\ln(t)}{36}\right.$$

$$\left. - \frac{11}{108}\right)t^3 + \left(\frac{\ln(t)}{576} - \frac{25}{3456}\right)t^4 + \left(\frac{\ln(t)}{14400} - \frac{137}{432000}\right)t^5 + O(t^6),\ 1 + t + \frac{1}{4}t^2$$

$$\left. + \frac{1}{36}t^3 + \frac{1}{576}t^4 + \frac{1}{14400}t^5 + O(t^6),\ t=x\right]\right]$$

```
> DEtools['formal_sol'](L12,`has logarithm?`,x=0);
```

$$true \tag{11}$$

Here are the Maples examples in section 3.2 called "The first step to solve our problem"

```
>              ####### CASE1: L=M #######
```

In the part called "Observations" we have the following Maple implementations for the case 1 where L=M:

In part (a)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```

$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,v^2 - 2\,x + 2\right)Dx - 1 \tag{12}$$

```
> LBB:=subs(nu=3,LBB);
```

$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{13}$$

```
> LBB:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```

$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{14}$$

```
> f:=(x-1)^8*x^6/(x-12)^10;
```

$$f := \frac{(x-1)^8\,x^6}{(x-12)^{10}} \tag{15}$$

```
> L:=ChangeOfVariables(LBB,f);
```

$$L := Dx^3\,(x-1)^2\,x^2\,(x-12)^{13}\left(x^2 - 41\,x + 18\right)^2 + 3\left(x^4 - 82\,x^3 + 575\,x^2 - 468\,x\right. \tag{16}$$

$$\left. + 216\right)Dx^2\,(x-1)\,x\,(x-12)^{12}\left(x^2 - 41\,x + 18\right) - \left(16\,x^{22} - 2752\,x^{21} + 183968\,x^{20}\right.$$

$$\left. - 5927232\,x^{19} + 92172927\,x^{18} - 627170724\,x^{17} + 2409383746\,x^{16} - 6314702496\,x^{15}\right.$$

$$\left. + 30279851895\,x^{14} - 744912611952\,x^{13} + 19429306967672\,x^{12} - 384289295082784\,x^{11}\right.$$

$$+ 5730368146930000\, x^{10} - 6440964148306304\, x^{9} + 541073854638635904\, x^{8}$$
$$- 3332740360025590272\, x^{7} + 14553041176441463040\, x^{6} - 42594780648286715904\, x^{5}$$
$$+ 76208058878030512128\, x^{4} - 72791577911076323328\, x^{3} + 36716049153820459008\, x^{2}$$
$$- 9299742418343559168\, x + 933087744110886912 \big)\, Dx\, (x - 12) - 32\, (x$$
$$- 1)^{7} x^{5} (x^{2} - 41\, x + 18)^{5}$$

```
> {solve(coeff(L,Dx,3),x)};
```
$$\left\{ 0,\, 1,\, 12,\, \frac{41}{2} - \frac{\sqrt{1609}}{2},\, \frac{41}{2} + \frac{\sqrt{1609}}{2} \right\} \tag{17}$$

```
> gen_exp(L,t,x=0);
```
$$[[-18, 0, 18, t = x]] \tag{18}$$

```
> gen_exp(L,t,x=1);
```
$$[[-24, 0, 24, t = x - 1]] \tag{19}$$

```
> gen_exp(L,t,x=12);
```
$$\left[[5, t = x - 12],\, \left[-\frac{252996480}{t^{5}} - \frac{124198272}{t^{4}} - \frac{24489432}{t^{3}} - \frac{2383876}{t^{2}} - \frac{104392}{t} + 5, t \right. \right. \tag{20}$$
$$\left. = x - 12 \right],\, \left[\frac{252996480}{t^{5}} + \frac{124198272}{t^{4}} + \frac{24489432}{t^{3}} + \frac{2383876}{t^{2}} + \frac{104392}{t} + 5, t = x \right.$$
$$\left. \left. - 12 \right] \right]$$

```
> gen_exp(L,t,x=41/2-(1/2)*sqrt(1609));
```
$$\left[\left[0, 2, 4, t = x - \frac{41}{2} + \frac{\sqrt{1609}}{2} \right] \right] \tag{21}$$

```
> gen_exp(L,t,x=41/2+(1/2)*sqrt(1609));
```
$$\left[\left[0, 2, 4, t = x - \frac{41}{2} - \frac{\sqrt{1609}}{2} \right] \right] \tag{22}$$

```
> gen_exp(L,t,x=infinity);
```
$$\left[\left[2, t = \frac{1}{x} \right],\, \left[-\frac{4}{t^{2}} - \frac{112}{t} + 2, t = \frac{1}{x} \right],\, \left[\frac{4}{t^{2}} + \frac{112}{t} + 2, t = \frac{1}{x} \right] \right] \tag{23}$$

In part (b)

```
> L02 := x^2*Dx^3+(x*b2+x+x*b1)*Dx^2+b2*b1*Dx-1;
```
$$L02 := x^{2}\, Dx^{3} + (x\, b1 + x\, b2 + x)\, Dx^{2} + b2\, b1\, Dx - 1 \tag{24}$$

```
> L02:=subs({b1=3/4,b2=1/2},L02);
```
$$L02 := x^{2}\, Dx^{3} + \frac{9}{4}\, x\, Dx^{2} + \frac{3}{8}\, Dx - 1 \tag{25}$$

```
> L02:= x^2*Dx^3+(9/4)*x*Dx^2+(3/8)*Dx-1;
```
$$L02 := x^{2}\, Dx^{3} + \frac{9}{4}\, x\, Dx^{2} + \frac{3}{8}\, Dx - 1 \tag{26}$$

```
> f:=2*(x-1)^8;
```

$$f := 2\,(x-1)^8 \tag{27}$$

```
> L:=ChangeOfVariables(L02,f);
```
$$L := Dx^3\,(x-1)^2 + (-3\,x+3)\,Dx^2 + 3\,Dx - 1024\,(x-1)^7 \tag{28}$$

```
> {solve(coeff(L,Dx,3),x)};
```
$$\{1\} \tag{29}$$

```
> gen_exp(L,t,x=1);
```
$$[[0,2,4,t=x-1]] \tag{30}$$

```
> formal_sol(L,t,x=1);
```
$$\left[\left[\frac{1}{8}+O(t^6),\ -\frac{1}{2}\,t^2+O(t^6),\ t^4+O(t^6),\ t=x-1\right]\right] \tag{31}$$

In part (c)

```
> L02 := x^2*Dx^3+(x*b2+x+x*b1)*Dx^2+b2*b1*Dx-1;
```
$$L02 := x^2\,Dx^3 + (x\,b1 + x\,b2 + x)\,Dx^2 + b2\,b1\,Dx - 1 \tag{32}$$

```
> L02:=subs({b1=3/4,b2=1/2},L02);
```
$$L02 := x^2\,Dx^3 + \frac{9}{4}\,x\,Dx^2 + \frac{3}{8}\,Dx - 1 \tag{33}$$

```
> L02:= x^2*Dx^3+(9/4)*x*Dx^2+(3/8)*Dx-1;
```
$$L02 := x^2\,Dx^3 + \frac{9}{4}\,x\,Dx^2 + \frac{3}{8}\,Dx - 1 \tag{34}$$

```
> f:=2*(x-1)^4;
```
$$f := 2\,(x-1)^4 \tag{35}$$

```
> L:=ChangeOfVariables(L02,f);
```
$$L := Dx^3 - 128\,x + 128 \tag{36}$$

```
> {solve(coeff(L,Dx,3),x)};
```
$$\varnothing \tag{37}$$

```
> gen_exp(L,t,x=1);
```
$$[[0,1,2,t=x-1]] \tag{38}$$

```
>            ####### CASE2: L<>M #######
```

In the part called "Observations" we have the following Maple implementations for the case 2 where L<>M:

In part (a)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,v^2 - 2\,x + 2\right)Dx - 1 \tag{39}$$

```
> LBB:=subs(nu=3,LBB);
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{40}$$

```
> LBB:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```

$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{41}$$

```
> f:=4*(x-1)^2;
```

$$f := 4\,(x-1)^2 \tag{42}$$

```
> M:=ChangeOfVariables(LBB,f);
```

$$M := Dx^3\,(x-1)^2 + (3\,x - 3)\,Dx^2 + (-16\,x^2 + 32\,x - 51)\,Dx - 16\,x + 16 \tag{43}$$

```
> r:=(x-3)^5/(x-7)^2;
```

$$r := \frac{(x-3)^5}{(x-7)^2} \tag{44}$$

```
> L1:=ExpProduct(M,r);
```

$$\begin{aligned}
L1 := &\, Dx^3\,(x-7)^6\,(x-1)^2 - 3\,(x^6 - 16\,x^5 + 105\,x^4 - 360\,x^3 + 674\,x^2 - 634\,x \\
&+ 194)\,Dx^2\,(x-7)^4\,(x-1) + (3\,x^{12} - 96\,x^{11} + 1398\,x^{10} - 12240\,x^9 + 71670\,x^8 \\
&- 295032\,x^7 + 872012\,x^6 - 1850976\,x^5 + 2763364\,x^4 - 2756132\,x^3 + 1637220\,x^2 \\
&- 410476\,x - 66075)\,Dx\,(x-7)^2 - x^{17} + 47\,x^{16} - 1036\,x^{15} + 14220\,x^{14} - 136068\,x^{13} \\
&+ 962598\,x^{12} - 5207528\,x^{11} + 21964884\,x^{10} - 72915454\,x^9 + 190709494\,x^8 \\
&- 390079508\,x^7 + 612617908\,x^6 - 712520204\,x^5 + 569424702\,x^4 - 253119160\,x^3 \\
&- 6765252\,x^2 + 74560847\,x - 30962009
\end{aligned} \tag{45}$$

```
> r0:=1;
```

$$r0 := 1 \tag{46}$$

```
> r1:=0;
```

$$r1 := 0 \tag{47}$$

```
> r2:=0;
```

$$r2 := 0 \tag{48}$$

```
> L:=GaugeTransf(L1,r0,r1,r2);
```

$$\begin{aligned}
L := &\, (x-1)^5\,(x-7)^{12}\,Dx^3 - 3\,(x^6 - 16\,x^5 + 105\,x^4 - 360\,x^3 + 674\,x^2 - 634\,x + 194)\,(x \\
&- 1)^4\,(x-7)^{10}\,Dx^2 + (3\,x^{12} - 96\,x^{11} + 1398\,x^{10} - 12240\,x^9 + 71670\,x^8 - 295032\,x^7 \\
&+ 872012\,x^6 - 1850976\,x^5 + 2763364\,x^4 - 2756132\,x^3 + 1637220\,x^2 - 410476\,x \\
&- 66075)\,(x-1)^3\,(x-7)^8\,Dx - (x^{17} - 47\,x^{16} + 1036\,x^{15} - 14220\,x^{14} + 136068\,x^{13} \\
&- 962598\,x^{12} + 5207528\,x^{11} - 21964884\,x^{10} + 72915454\,x^9 - 190709494\,x^8 \\
&+ 390079508\,x^7 - 612617908\,x^6 + 712520204\,x^5 - 569424702\,x^4 + 253119160\,x^3 \\
&+ 6765252\,x^2 - 74560847\,x + 30962009)\,(x-1)^3\,(x-7)^6
\end{aligned} \tag{49}$$

```
> gen_exp(M,t,x=1);
```

$$[[-6, 0, 6, t = x - 1]] \tag{50}$$

```
> gen_exp(L,t,x=1);
```

$$[[-6, 0, 6, t = x - 1]] \tag{51}$$

In part (b)

```
> L02 := x^2*Dx^3+(x*b2+x+x*b1)*Dx^2+b2*b1*Dx-1;
```

$$L02 := x^2\, Dx^3 + (x\, b1 + x\, b2 + x)\, Dx^2 + b2\, b1\, Dx - 1 \tag{52}$$

```
> L02:=subs({b1=3/4,b2=1/2},L02);
```

$$L02 := x^2\, Dx^3 + \frac{9}{4}\, x\, Dx^2 + \frac{3}{8}\, Dx - 1 \tag{53}$$

```
> L02:= x^2*Dx^3+(9/4)*x*Dx^2+(3/8)*Dx-1;
```

$$L02 := x^2\, Dx^3 + \frac{9}{4}\, x\, Dx^2 + \frac{3}{8}\, Dx - 1 \tag{54}$$

```
> f:=2/(x-1)^4;
```

$$f := \frac{2}{(x-1)^4} \tag{55}$$

```
> M:=ChangeOfVariables(LBB,f);
```

$$M := Dx^3\,(x-1)^7 + 3\, Dx^2\,(x-1)^6 - \left(143\, x^4 - 572\, x^3 + 858\, x^2 - 572\, x + 175\right) Dx\,(x-1) + 64 \tag{56}$$

```
> r:=(3/7)/x;
```

$$r := \frac{3}{7\, x} \tag{57}$$

```
> L1:=ExpProduct(M,r);
```

$$L1 := 343\, Dx^3\, x^3\,(x-1)^7 + 147\,(4\, x + 3)\, Dx^2\, x^2\,(x-1)^6 - 7\left(7043\, x^6 - 28118\, x^5\right.$$
$$\left. + 41952\, x^4 - 27488\, x^3 + 7855\, x^2 + 414\, x - 90\right) Dx\, x\,(x-1) + 21141\, x^7 - 105315\, x^6$$
$$+ 208950\, x^5 - 204960\, x^4 + 123361\, x^3 - 18795\, x^2 - 2940\, x + 510 \tag{58}$$

```
> r0:=x-1;
```

$$r0 := x - 1 \tag{59}$$

```
> r1:=0;
```

$$r1 := 0 \tag{60}$$

```
> r2:=0;
```

$$r2 := 0 \tag{61}$$

```
> L:=GaugeTransf(L1,r0,r1,r2);
```

$$L := 343\, x^3\,(x-1)^{13}\, Dx^3 - 441\, x^2\,(x-1)^{13}\, Dx^2 - 7\, x\left(6917\, x^6 - 27488\, x^5 + 40692\, x^4\right.$$
$$\left. - 26228\, x^3 + 7225\, x^2 + 540\, x - 90\right)(x-1)^7\, Dx + \left(69560\, x^7 - 297731\, x^6 + 493794\, x^5\right.$$
$$\left. - 388556\, x^4 + 173936\, x^3 - 15015\, x^2 - 3570\, x + 510\right)(x-1)^6 \tag{62}$$

```
> gen_exp(M,t,x=infinity);
```

$$\left[\left[-12, 0, 12, t = \frac{1}{x}\right]\right] \tag{63}$$

```
> gen_exp(L,t,x=infinity);
```

$$\left[\left[-\frac{94}{7}, -\frac{10}{7}, \frac{74}{7}, t = \frac{1}{x}\right]\right] \tag{64}$$

In part (c)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```

$$LBB := 2\, x^2\, Dx^3 + 6\, x\, Dx^2 + \left(-2\, v^2 - 2\, x + 2\right) Dx - 1 \tag{65}$$

```
> LBB:=subs(nu=1/3,LBB);
```

$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,x + \frac{16}{9}\right)Dx - 1 \tag{66}$$

```
> LBB:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x+16/9)*Dx-1;
```

$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,x + \frac{16}{9}\right)Dx - 1 \tag{67}$$

```
> f:=4/(x-1)^6;
```

$$f := \frac{4}{(x-1)^6} \tag{68}$$

```
> M:=ChangeOfVariables(LBB,f);
```

$$M := Dx^3\,(x-1)^9 + 3\,Dx^2\,(x-1)^8 - 3\,\left(x^6 - 6\,x^5 + 15\,x^4 - 20\,x^3 + 15\,x^2 - 6\,x + 49\right)Dx\,(x \tag{69}$$
$$-\,1) + 432$$

```
> r:=-3/x;
```

$$r := -\frac{3}{x} \tag{70}$$

```
> L1:=ExpProduct(M,r);
```

$$L1 := Dx^3\,x^3\,(x-1)^9 + 3\,(4\,x - 3)\,Dx^2\,x^2\,(x-1)^8 + 3\,\left(11\,x^8 - 84\,x^7 + 279\,x^6 - 526\,x^5\right. \tag{71}$$
$$+\,615\,x^4 - 456\,x^3 + 161\,x^2 - 54\,x + 6\bigr)\,Dx\,x\,(x-1) + 15\,x^9 - 135\,x^8 + 531\,x^7 - 1197\,x^6$$
$$+\,1701\,x^5 - 1575\,x^4 + 945\,x^3 + 81\,x^2 + 72\,x - 6$$

```
> r0:=1;
```

$$r0 := 1 \tag{72}$$

```
> r1:=0;
```

$$r1 := 0 \tag{73}$$

```
> r2:=0;
```

$$r2 := 0 \tag{74}$$

```
> L:=GaugeTransf(L1,r0,r1,r2);
```

$$L := x^3\,(x-1)^{18}\,Dx^3 + 3\,x^2\,(4\,x - 3)\,(x-1)^{17}\,Dx^2 + 3\,x\,\left(11\,x^8 - 84\,x^7 + 279\,x^6 - 526\,x^5\right. \tag{75}$$
$$+\,615\,x^4 - 456\,x^3 + 161\,x^2 - 54\,x + 6\bigr)\,(x-1)^{10}\,Dx + 3\,\left(5\,x^9 - 45\,x^8 + 177\,x^7 - 399\,x^6\right.$$
$$+\,567\,x^5 - 525\,x^4 + 315\,x^3 + 27\,x^2 + 24\,x - 2\bigr)\,(x-1)^9$$

```
> gen_exp(M,t,x=infinity);
```

$$\left[\left[-2, 0, 2, t = \frac{1}{x}\right]\right] \tag{76}$$

```
> gen_exp(L,t,x=infinity);
```

$$\left[\left[1, 3, 5, t = \frac{1}{x}\right]\right] \tag{77}$$

```
> formal_sol(L,t,x=infinity);
```

$$\left[\left[t\left(\frac{1}{8} - \frac{1}{4}\,t - \frac{1}{8}\,t^2 + \frac{1}{8}\,t^4 + \frac{1}{4}\,t^5 + O(t^6)\right), t\left(-\frac{1}{2}\,t^2 + \frac{1}{2}\,t^4 + t^5 + O(t^6)\right), t\left(t^4 + 2\,t^5\right.\right.\right. \tag{78}$$
$$\left.\left.\left. +\,O(t^6)\right), t = \frac{1}{x}\right]\right]$$

In part (d)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```

$$LBB := 2\, x^2\, Dx^3 + 6\, x\, Dx^2 + \left(-2\, v^2 - 2\, x + 2 \right) Dx - 1 \qquad \text{(79)}$$

```
> LBB:=subs(nu=3,LBB);
```

$$LBB := 2\, x^2\, Dx^3 + 6\, x\, Dx^2 + \left(-2\, x - 16 \right) Dx - 1 \qquad \text{(80)}$$

```
> LBB:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```

$$LBB := 2\, x^2\, Dx^3 + 6\, x\, Dx^2 + \left(-2\, x - 16 \right) Dx - 1 \qquad \text{(81)}$$

```
> f:=4*(x-1)^2;
```

$$f := 4\,(x-1)^2 \qquad \text{(82)}$$

```
> M:=ChangeOfVariables(LBB,f);
```

$$M := Dx^3\,(x-1)^2 + (3\,x - 3)\, Dx^2 + \left(-16\, x^2 + 32\, x - 51 \right) Dx - 16\, x + 16 \qquad \text{(83)}$$

```
> r:=(x-3)^5/(x-1)^2;
```

$$r := \frac{(x-3)^5}{(x-1)^2} \qquad \text{(84)}$$

```
> L1:=ExpProduct(M,r);
```

$$\begin{aligned} L1 := {}& Dx^3\,(x-1)^6 - 3\left(x^5 - 15\, x^4 + 90\, x^3 - 270\, x^2 + 404\, x - 242 \right) Dx^2\,(x-1)^4 + \big(3\, x^{10} \\ & - 90\, x^9 + 1215\, x^8 - 9720\, x^7 + 51015\, x^6 - 183498\, x^5 + 458069\, x^4 - 783836\, x^3 \\ & + 880339\, x^2 - 586306\, x + 175881 \big)\, Dx\,(x-1)^2 - x^{15} + 45\, x^{14} - 945\, x^{13} + 12285\, x^{12} \\ & - 110553\, x^{11} + 729387\, x^{10} - 3644249\, x^9 + 14039831\, x^8 - 42052279\, x^7 + 97936689\, x^6 \\ & - 175940275\, x^5 + 239499317\, x^4 - 239206503\, x^3 + 165535207\, x^2 - 70983755\, x \\ & + 14218567 \end{aligned} \qquad \text{(85)}$$

```
> r0:=1;
```

$$r0 := 1 \qquad \text{(86)}$$

```
> r1:=0;
```

$$r1 := 0 \qquad \text{(87)}$$

```
> r2:=0;
```

$$r2 := 0 \qquad \text{(88)}$$

```
> L:=GaugeTransf(L1,r0,r1,r2);
```

$$\begin{aligned} L := {}& (x-1)^{12}\, Dx^3 - 3\left(x^5 - 15\, x^4 + 90\, x^3 - 270\, x^2 + 404\, x - 242 \right)(x-1)^{10}\, Dx^2 + \big(3\, x^{10} \\ & - 90\, x^9 + 1215\, x^8 - 9720\, x^7 + 51015\, x^6 - 183498\, x^5 + 458069\, x^4 - 783836\, x^3 \\ & + 880339\, x^2 - 586306\, x + 175881 \big)\,(x-1)^8\, Dx - \big(x^{15} - 45\, x^{14} + 945\, x^{13} - 12285\, x^{12} \\ & + 110553\, x^{11} - 729387\, x^{10} + 3644249\, x^9 - 14039831\, x^8 + 42052279\, x^7 - 97936689\, x^6 \\ & + 175940275\, x^5 - 239499317\, x^4 + 239206503\, x^3 - 165535207\, x^2 + 70983755\, x \\ & - 14218567 \big)\,(x-1)^6 \end{aligned} \qquad \text{(89)}$$

```
> gen_exp(M,t,x=1);
```

$$[[-6, 0, 6, t = x - 1]] \qquad \text{(90)}$$

```
> gen_exp(L,t,x=1);
```

$$\left[\left[74 - \frac{32}{t}, 80 - \frac{32}{t}, 86 - \frac{32}{t}, t = x - 1 \right] \right] \qquad \text{(91)}$$

[In part (e)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,\nu^2 - 2\,x + 2\right)Dx - 1 \tag{92}$$

```
> LBB:=subs(nu=3,LBB);
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{93}$$

```
> LBB:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{94}$$

```
> f:=4*(x-1)^2;
```
$$f := 4\,(x-1)^2 \tag{95}$$

```
> M:=ChangeOfVariables(LBB,f);
```
$$M := Dx^3\,(x-1)^2 + (3\,x - 3)\,Dx^2 + \left(-16\,x^2 + 32\,x - 51\right)Dx - 16\,x + 16 \tag{96}$$

```
> r:=(x-3)^5/(x-7)^2;
```
$$r := \frac{(x-3)^5}{(x-7)^2} \tag{97}$$

```
> L1:=ExpProduct(M,r);
```
$$
\begin{aligned}
L1 := {} & Dx^3\,(x-7)^6\,(x-1)^2 - 3\,\bigl(x^6 - 16\,x^5 + 105\,x^4 - 360\,x^3 + 674\,x^2 - 634\,x \\
& + 194\bigr)\,Dx^2\,(x-7)^4\,(x-1) + \bigl(3\,x^{12} - 96\,x^{11} + 1398\,x^{10} - 12240\,x^9 + 71670\,x^8 \\
& - 295032\,x^7 + 872012\,x^6 - 1850976\,x^5 + 2763364\,x^4 - 2756132\,x^3 + 1637220\,x^2 \\
& - 410476\,x - 66075\bigr)\,Dx\,(x-7)^2 - x^{17} + 47\,x^{16} - 1036\,x^{15} + 14220\,x^{14} - 136068\,x^{13} \\
& + 962598\,x^{12} - 5207528\,x^{11} + 21964884\,x^{10} - 72915454\,x^9 + 190709494\,x^8 \\
& - 390079508\,x^7 + 612617908\,x^6 - 712520204\,x^5 + 569424702\,x^4 - 253119160\,x^3 \\
& - 6765252\,x^2 + 74560847\,x - 30962009
\end{aligned} \tag{98}
$$

```
> r0:=1;
```
$$r0 := 1 \tag{99}$$

```
> r1:=0;
```
$$r1 := 0 \tag{100}$$

```
> r2:=0;
```
$$r2 := 0 \tag{101}$$

```
> L:=GaugeTransf(L1,r0,r1,r2);
```
$$
\begin{aligned}
L := {} & (x-1)^5\,(x-7)^{12}\,Dx^3 - 3\,\bigl(x^6 - 16\,x^5 + 105\,x^4 - 360\,x^3 + 674\,x^2 - 634\,x + 194\bigr)\,(x \\
& -1)^4\,(x-7)^{10}\,Dx^2 + \bigl(3\,x^{12} - 96\,x^{11} + 1398\,x^{10} - 12240\,x^9 + 71670\,x^8 - 295032\,x^7 \\
& + 872012\,x^6 - 1850976\,x^5 + 2763364\,x^4 - 2756132\,x^3 + 1637220\,x^2 - 410476\,x \\
& - 66075\bigr)\,(x-1)^3\,(x-7)^8\,Dx - \bigl(x^{17} - 47\,x^{16} + 1036\,x^{15} - 14220\,x^{14} + 136068\,x^{13} \\
& - 962598\,x^{12} + 5207528\,x^{11} - 21964884\,x^{10} + 72915454\,x^9 - 190709494\,x^8 \\
& + 390079508\,x^7 - 612617908\,x^6 + 712520204\,x^5 - 569424702\,x^4 + 253119160\,x^3 \\
& + 6765252\,x^2 - 74560847\,x + 30962009\bigr)\,(x-1)^3\,(x-7)^6
\end{aligned} \tag{102}
$$

```
> {solve(coeff(M,Dx,3),x)};
```
$$\{1\} \tag{103}$$

```
> gen_exp(M,t,x=7);
```
$$[[0, 1, 2, t = x - 7]] \tag{104}$$

```
> gen_exp(L,t,x=7);
```
$$\left[\left[1280 + \frac{1024}{t}, 1281 + \frac{1024}{t}, 1282 + \frac{1024}{t}, t = x - 7\right]\right] \tag{105}$$

In part (f)

```
> L02 := x^2*Dx^3+(x*b2+x+x*b1)*Dx^2+b2*b1*Dx-1;
```
$$L02 := x^2 Dx^3 + (x\,b1 + x\,b2 + x)\,Dx^2 + b2\,b1\,Dx - 1 \tag{106}$$

```
> L02:=subs({b1=3/4,b2=1/2},L02);
```
$$L02 := x^2 Dx^3 + \frac{9}{4}\,x\,Dx^2 + \frac{3}{8}\,Dx - 1 \tag{107}$$

```
> L02:= x^2*Dx^3+(9/4)*x*Dx^2+(3/8)*Dx-1;
```
$$L02 := x^2 Dx^3 + \frac{9}{4}\,x\,Dx^2 + \frac{3}{8}\,Dx - 1 \tag{108}$$

```
> f:=2*(x-1)^4;
```
$$f := 2\,(x-1)^4 \tag{109}$$

```
> M:=ChangeOfVariables(L02,f);
```
$$M := Dx^3 - 128\,x + 128 \tag{110}$$

```
> r:=x-3;
```
$$r := x - 3 \tag{111}$$

```
> L1:=ExpProduct(M,r);
```
$$L1 := Dx^3 + (9 - 3\,x)\,Dx^2 + 3\,(x-2)\,(x-4)\,Dx - x^3 + 9\,x^2 - 152\,x + 146 \tag{112}$$

```
> r0:=x-1;
```
$$r0 := x - 1 \tag{113}$$

```
> r1:=0;
```
$$r1 := 0 \tag{114}$$

```
> r2:=0;
```
$$r2 := 0 \tag{115}$$

```
> L:=GaugeTransf(L1,r0,r1,r2);
```
$$L := Dx^3\,(x-1)^6 - 3\,(x-2)^2\,(x-1)^5 Dx^2 + 3\,(x-2)\,(x^3 - 6\,x^2 + 11\,x - 8)\,(x-1)^4 Dx \tag{116}$$
$$- \left(x^6 - 12\,x^5 + 185\,x^4 - 654\,x^3 + 972\,x^2 - 680\,x + 194\right)\,(x-1)^3$$

```
> {solve(coeff(M,Dx,3),x)};
```
$$\varnothing \tag{117}$$

```
> gen_exp(M,t,x=1);
```
$$[[0, 1, 2, t = x - 1]] \tag{118}$$

```
> gen_exp(L,t,x=1);
```
$$[[1, 2, 3, t = x - 1]] \tag{119}$$

```
> formal_sol(L,t,x=1);
```
$$\left[\left[t\left(\frac{1}{2} - t + \frac{5}{4}\,t^2 - \frac{7}{6}\,t^3 + \frac{57}{16}\,t^4 - \frac{237}{40}\,t^5 + O(t^6)\right), t\left(-t + 2\,t^2 - \frac{5}{2}\,t^3 + \frac{7}{3}\,t^4\right.\right.\right. \tag{120}$$

$$-\frac{157}{40}\,t^5 + \mathrm{O}(t^6)\Big),\, t\left(t^2 - 2\,t^3 + \frac{5}{2}\,t^4 - \frac{7}{3}\,t^5 + \mathrm{O}(t^6)\right),\, t = x - 1\Big]\Big]$$

In part (g)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,v^2 - 2\,x + 2\right)Dx - 1 \tag{121}$$

```
> LBB:=subs(nu=3,LBB);
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{122}$$

```
> LBB:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{123}$$

```
> f:=4*(x-1)^2;
```
$$f := 4\,(x-1)^2 \tag{124}$$

```
> M:=ChangeOfVariables(LBB,f);
```
$$M := Dx^3\,(x-1)^2 + (3\,x - 3)\,Dx^2 + \left(-16\,x^2 + 32\,x - 51\right)Dx - 16\,x + 16 \tag{125}$$

```
> r:=(x-3)^5/(x-7)^2;
```
$$r := \frac{(x-3)^5}{(x-7)^2} \tag{126}$$

```
> L1:=ExpProduct(M,r);
```
$$
\begin{aligned}
L1 := {} & Dx^3\,(x-7)^6\,(x-1)^2 - 3\,\big(x^6 - 16\,x^5 + 105\,x^4 - 360\,x^3 + 674\,x^2 - 634\,x \\
& + 194\big)\,Dx^2\,(x-7)^4\,(x-1) + \big(3\,x^{12} - 96\,x^{11} + 1398\,x^{10} - 12240\,x^9 + 71670\,x^8 \\
& - 295032\,x^7 + 872012\,x^6 - 1850976\,x^5 + 2763364\,x^4 - 2756132\,x^3 + 1637220\,x^2 \\
& - 410476\,x - 66075\big)\,Dx\,(x-7)^2 - x^{17} + 47\,x^{16} - 1036\,x^{15} + 14220\,x^{14} - 136068\,x^{13} \\
& + 962598\,x^{12} - 5207528\,x^{11} + 21964884\,x^{10} - 72915454\,x^9 + 190709494\,x^8 \\
& - 390079508\,x^7 + 612617908\,x^6 - 712520204\,x^5 + 569424702\,x^4 - 253119160\,x^3 \\
& - 6765252\,x^2 + 74560847\,x - 30962009
\end{aligned}
\tag{127}
$$

```
> r0:=1;
```
$$r0 := 1 \tag{128}$$

```
> r1:=0;
```
$$r1 := 0 \tag{129}$$

```
> r2:=0;
```
$$r2 := 0 \tag{130}$$

```
> L:=GaugeTransf(L1,r0,r1,r2);
```
$$
\begin{aligned}
L := {} & (x-1)^5\,(x-7)^{12}\,Dx^3 - 3\,\big(x^6 - 16\,x^5 + 105\,x^4 - 360\,x^3 + 674\,x^2 - 634\,x + 194\big)\,(x \\
& - 1)^4\,(x-7)^{10}\,Dx^2 + \big(3\,x^{12} - 96\,x^{11} + 1398\,x^{10} - 12240\,x^9 + 71670\,x^8 - 295032\,x^7 \\
& + 872012\,x^6 - 1850976\,x^5 + 2763364\,x^4 - 2756132\,x^3 + 1637220\,x^2 - 410476\,x \\
& - 66075\big)\,(x-1)^3\,(x-7)^8\,Dx - \big(x^{17} - 47\,x^{16} + 1036\,x^{15} - 14220\,x^{14} + 136068\,x^{13} \\
& - 962598\,x^{12} + 5207528\,x^{11} - 21964884\,x^{10} + 72915454\,x^9 - 190709494\,x^8 \\
& + 390079508\,x^7 - 612617908\,x^6 + 712520204\,x^5 - 569424702\,x^4 + 253119160\,x^3
\end{aligned}
\tag{131}
$$

$$+ 6765252\, x^2 - 74560847\, x + 30962009 \big)\, (x-1)^3\, (x-7)^6$$

```
> gen_exp(M,t,x=infinity);
```

$$\left[\left[1, t = \frac{1}{x} \right], \left[-\frac{4}{t} + 1, t = \frac{1}{x} \right], \left[\frac{4}{t} + 1, t = \frac{1}{x} \right] \right] \tag{132}$$

```
> gen_exp(L,t,x=infinity);
```

$$\left[\left[-\frac{157}{t} - 1279 - \frac{27}{t^2} + \frac{1}{t^3} - \frac{1}{t^4}, t = \frac{1}{x} \right], \left[-\frac{153}{t} - 1279 - \frac{27}{t^2} + \frac{1}{t^3} - \frac{1}{t^4}, t = \frac{1}{x} \right], \right. \tag{133}$$
$$\left. -\frac{161}{t} - 1279 - \frac{27}{t^2} + \frac{1}{t^3} - \frac{1}{t^4}, t = \frac{1}{x} \right]$$

Here are the Maples examples in section 3.3 called "The second step to solve our problem"

```
>                   ####### THE EXP-PRODUCT EQUIVALENCE #######
```

In the part called "The Exp-product Equivalence" we have the following Maple implementations for the examples:

(1)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```

$$LBB := 2\, x^2\, Dx^3 + 6\, x\, Dx^2 + \left(-2\, v^2 - 2\, x + 2 \right) Dx - 1 \tag{134}$$

```
> L1:=subs(nu=3,LBB);
```

$$L1 := 2\, x^2\, Dx^3 + 6\, x\, Dx^2 + (-2\, x - 16)\, Dx - 1 \tag{135}$$

```
> L1:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```

$$L1 := 2\, x^2\, Dx^3 + 6\, x\, Dx^2 + (-2\, x - 16)\, Dx - 1 \tag{136}$$

```
> r:= (x-3)^5/(x-7)^2;
```

$$r := \frac{(x-3)^5}{(x-7)^2} \tag{137}$$

```
> L2:=ExpProduct(L1,r);
```

$$L2 := 2\, Dx^3\, (x-7)^6\, x^2 - 6\, \left(x^6 - 15\, x^5 + 90\, x^4 - 270\, x^3 + 404\, x^2 - 229\, x - 49 \right) Dx^2\, (x \tag{138}$$
$$- 7)^4\, x + 2\, \left(3\, x^{12} - 90\, x^{11} + 1215\, x^{10} - 9720\, x^9 + 51015\, x^8 - 183276\, x^7 + 454281\, x^6 \right.$$
$$- 757351\, x^5 + 784370\, x^4 - 397780\, x^3 - 12644\, x^2 + 80017\, x - 19208 \big)\, Dx\, (x-7)^2$$
$$- 2\, x^{17} + 90\, x^{16} - 1890\, x^{15} + 24570\, x^{14} - 221106\, x^{13} + 1458408\, x^{12} - 7276896\, x^{11}$$
$$+ 27913502\, x^{10} - 82688140\, x^9 + 187904862\, x^8 - 320893598\, x^7 + 394295049\, x^6$$
$$- 316407588\, x^5 + 119726385\, x^4 + 37324028\, x^3 - 65292369\, x^2 + 32352936\, x - 9452737$$

```
> ExpProdEquiv(L1,L2);
```

$$\left[\frac{(x-3)^5}{(x-7)^2} \right] \tag{139}$$

[(2)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,v^2 - 2\,x + 2\right)Dx - 1 \tag{140}$$

```
> L1:=subs(nu=3,LBB);
```
$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{141}$$

```
> L1:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```
$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{142}$$

```
> r:= (x-7)*(x-12);
```
$$r := (x - 7)\,(x - 12) \tag{143}$$

```
> L2:=ExpProduct(L1,r);
```
$$
\begin{aligned}
L2 := {}& 2\,x^2\,Dx^3 - 6\left(x^3 - 19\,x^2 + 84\,x - 1\right)Dx^2\,x + \left(6\,x^6 - 228\,x^5 + 3174\,x^4 - 19176\,x^3\right. \\
& + 42678\,x^2 - 1010\,x - 16\left.\right)Dx - 2\,x^8 + 114\,x^7 - 2670\,x^6 + 32888\,x^5 - 224850\,x^4 \\
& + 810734\,x^3 - 1214174\,x^2 + 42314\,x + 1343
\end{aligned} \tag{144}
$$

```
> ExpProdEquiv(L1,L2);
```
$$[\,(x - 7)\,(x - 12)\,] \tag{145}$$

[(3)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,v^2 - 2\,x + 2\right)Dx - 1 \tag{146}$$

```
> L1:=subs(nu=3,LBB);
```
$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{147}$$

```
> L1:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```
$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{148}$$

```
> r0:=x-1;
```
$$r0 := x - 1 \tag{149}$$

```
> r1:=x-9;
```
$$r1 := x - 9 \tag{150}$$

```
> r2:=x-11;
```
$$r2 := x - 11 \tag{151}$$

```
> L2:=GaugeTransf(L1,r0,r1,r2);
```
$$
\begin{aligned}
L2 := {}& 2\,Dx^3\,x^2\left(4\,x^7 - 20\,x^6 + 308\,x^5 - 1892\,x^4 - 1535\,x^3 + 17719\,x^2 - 11033\,x - 12463\right) \\
& - 2\left(20\,x^6 - 616\,x^5 + 5676\,x^4 + 6140\,x^3 - 88595\,x^2 + 66198\,x + 87241\right)Dx^2\,x - 2\left(4\,x^7\right. \\
& + 8\,x^6 + 184\,x^5 + 2474\,x^4 - 38935\,x^3 + 15413\,x^2 + 158511\,x - 89155\left.\right)Dx\,x + 4\,x^7 \\
& + 52\,x^6 - 160\,x^5 + 9156\,x^4 - 80677\,x^3 - 11893\,x^2 + 299673\,x + 19965
\end{aligned} \tag{152}
$$

```
> ExpProdEquiv(L1,L2);
```
$$0 \tag{153}$$

[(4)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,\nu^2 - 2\,x + 2\right)Dx - 1 \tag{154}$$

```
> L1:=subs(nu=3,LBB);
```
$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,x - 16\right)Dx - 1 \tag{155}$$

```
> L1:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```
$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,x - 16\right)Dx - 1 \tag{156}$$

```
> r:= x-1;
```
$$r := x - 1 \tag{157}$$

```
> M1:=ExpProduct(L1,r);
```
$$M1 := 2\,x^2\,Dx^3 - 6\left(x^2 - x - 1\right)Dx^2\,x + \left(6\,x^4 - 12\,x^3 - 12\,x^2 + 10\,x - 16\right)Dx - 2\,x^5 + 6\,x^4 \tag{158}$$
$$+ 6\,x^3 - 14\,x^2 + 14\,x - 17$$

```
> r0:=0;
```
$$r0 := 0 \tag{159}$$

```
> r1:=x;
```
$$r1 := x \tag{160}$$

```
> r2:=0;
```
$$r2 := 0 \tag{161}$$

```
> L2:=GaugeTransf(M1,r0,r1,r2);
```
$$L2 := 2\,Dx^3\,x^3\left(2\,x^5 - 6\,x^4 - 6\,x^3 + 14\,x^2 - 14\,x + 17\right) - 2\left(6\,x^7 - 24\,x^6 + 6\,x^5 + 48\,x^4\right. \tag{162}$$
$$\left. - 90\,x^3 + 93\,x^2 - 37\,x - 34\right)Dx^2\,x^2 + 2\left(6\,x^9 - 30\,x^8 + 30\,x^7 + 40\,x^6 - 112\,x^5 + 195\,x^4\right.$$
$$\left. + 40\,x^3 - 335\,x^2 + 225\,x - 153\right)Dx\,x - 4\,x^{11} + 24\,x^{10} - 36\,x^9 - 20\,x^8 + 92\,x^7 - 204\,x^6$$
$$- 88\,x^5 + 686\,x^4 - 1624\,x^3 + 1186\,x^2 - 793\,x + 306$$

```
> ExpProdEquiv(L1,L2);
```
$$0 \tag{163}$$

```
>                 ####### THE GAUGE EQUIVALENCE #######
```

In the part called "The Gauge Equivalence" we have the following Maple implementations for the examples:

[(1)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,\nu^2 - 2\,x + 2\right)Dx - 1 \tag{164}$$

```
> L1:=subs(nu=3,LBB);
```
$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,x - 16\right)Dx - 1 \tag{165}$$

```
> L1:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```

(166)

$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{166}$$

```
> r0:=x-7;
```
$$r0 := x - 7 \tag{167}$$

```
> r1:=x-9;
```
$$r1 := x - 9 \tag{168}$$

```
> r2:=x-11;
```
$$r2 := x - 11 \tag{169}$$

```
> L2:=GaugeTransf(L1,r0,r1,r2);
```
$$
\begin{aligned}
L2 := {} & 2\,Dx^3\,x^2\,(4\,x^7 - 92\,x^6 + 884\,x^5 - 5216\,x^4 + 18133\,x^3 - 1505\,x^2 - 132473\,x - 128623) \\
& - 2\,(92\,x^6 - 1768\,x^5 + 15648\,x^4 - 72532\,x^3 + 7525\,x^2 + 794838\,x + 900361)\,Dx^2\,x \\
& - 2\,(4\,x^7 - 64\,x^6 + 40\,x^5 + 4454\,x^4 - 57991\,x^3 + 327005\,x^2 - 322917\,x - 1013155)\,Dx\,x \\
& + 4\,x^7 + 28\,x^6 - 1576\,x^5 + 14544\,x^4 - 111253\,x^3 + 179279\,x^2 + 445665\,x + 19965
\end{aligned} \tag{170}
$$

```
> Homomorphisms(L1,L2);
```
$$\left[\,(x-11)\,Dx^2 + (x-9)\,Dx + x - 7\,\right] \tag{171}$$

(2)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,v^2 - 2\,x + 2)\,Dx - 1 \tag{172}$$

```
> LBB:=subs(nu=3,LBB);
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{173}$$

```
> LBB:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{174}$$

```
> r:= x-1;
```
$$r := x - 1 \tag{175}$$

```
> L1:=ExpProduct(LBB,r);
```
$$
\begin{aligned}
L1 := {} & 2\,x^2\,Dx^3 - 6\,(x^2 - x - 1)\,Dx^2\,x + (6\,x^4 - 12\,x^3 - 12\,x^2 + 10\,x - 16)\,Dx - 2\,x^5 + 6\,x^4 \\
& + 6\,x^3 - 14\,x^2 + 14\,x - 17
\end{aligned} \tag{176}
$$

```
> r0:=0;
```
$$r0 := 0 \tag{177}$$

```
> r1:=x;
```
$$r1 := x \tag{178}$$

```
> r2:=0;
```
$$r2 := 0 \tag{179}$$

```
> L2:=GaugeTransf(L1,r0,r1,r2);
```
$$
\begin{aligned}
L2 := {} & 2\,Dx^3\,x^3\,(2\,x^5 - 6\,x^4 - 6\,x^3 + 14\,x^2 - 14\,x + 17) - 2\,(6\,x^7 - 24\,x^6 + 6\,x^5 + 48\,x^4 \\
& - 90\,x^3 + 93\,x^2 - 37\,x - 34)\,Dx^2\,x^2 + 2\,(6\,x^9 - 30\,x^8 + 30\,x^7 + 40\,x^6 - 112\,x^5 + 195\,x^4 \\
& + 40\,x^3 - 335\,x^2 + 225\,x - 153)\,Dx\,x - 4\,x^{11} + 24\,x^{10} - 36\,x^9 - 20\,x^8 + 92\,x^7 - 204\,x^6 \\
& - 88\,x^5 + 686\,x^4 - 1624\,x^3 + 1186\,x^2 - 793\,x + 306
\end{aligned} \tag{180}
$$

```
> Homomorphisms(L1,L2);
```

$$[x\,Dx] \tag{181}$$

> ####### THE PROJECTIVE EQUIVALENCE #######

In the part called "The Projective Equivalence" we have the following Maple implementations for the examples:

(1)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```
$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,v^2 - 2\,x + 2\right)Dx - 1 \tag{182}$$

```
> L1:=subs(nu=3,LBB);
```
$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{183}$$

```
> L1:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```
$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1 \tag{184}$$

```
> r:= x-1;
```
$$r := x - 1 \tag{185}$$

```
> L2:=ExpProduct(L1,r);
```
$$L2 := 2\,x^2\,Dx^3 - 6\,(x^2 - x - 1)\,Dx^2\,x + (6\,x^4 - 12\,x^3 - 12\,x^2 + 10\,x - 16)\,Dx - 2\,x^5 + 6\,x^4 \tag{186}$$
$$+ 6\,x^3 - 14\,x^2 + 14\,x - 17$$

```
> r0:=0;
```
$$r0 := 0 \tag{187}$$

```
> r1:=x;
```
$$r1 := x \tag{188}$$

```
> r2:=0;
```
$$r2 := 0 \tag{189}$$

```
> L3:=GaugeTransf(L2,r0,r1,r2);
```
$$L3 := 2\,Dx^3\,x^3\,(2\,x^5 - 6\,x^4 - 6\,x^3 + 14\,x^2 - 14\,x + 17) - 2\,(6\,x^7 - 24\,x^6 + 6\,x^5 + 48\,x^4 \tag{190}$$
$$- 90\,x^3 + 93\,x^2 - 37\,x - 34)\,Dx^2\,x^2 + 2\,(6\,x^9 - 30\,x^8 + 30\,x^7 + 40\,x^6 - 112\,x^5 + 195\,x^4$$
$$+ 40\,x^3 - 335\,x^2 + 225\,x - 153)\,Dx\,x - 4\,x^{11} + 24\,x^{10} - 36\,x^9 - 20\,x^8 + 92\,x^7 - 204\,x^6$$
$$- 88\,x^5 + 686\,x^4 - 1624\,x^3 + 1186\,x^2 - 793\,x + 306$$

```
> B2:= Singular(L2,{});
```
$$B2 := [[x, 0], [\infty, \infty]] \tag{191}$$

```
> S2:= NotAppSing(L2,B2,{});
```
$$S2 := [[x, 0], [\infty, \infty]] \tag{192}$$

```
> C2:=IrrRegAppsingBesSqRoot(L2,t,B2,{}):
> EquivExpgaugeBesSqRoot1(L1,L2,C2,x,t,T,{});
```
$$\left\{\left[\left[\frac{x^2 - x - 3}{x}\right], [x^3]\right], \left[\left[\frac{x^2 - x + 3}{x}\right], \left[\frac{1}{x^3}\right]\right], \left[\left[\frac{x^2 - x + 6}{x}\right], \left[\frac{1}{x^6}\right]\right],\right. \tag{193}$$
$$\left.\left[\left[\frac{(x+2)\,(x-3)}{x}\right], [x^6]\right], [[x-1], [1]]\right\}$$

```
> B3:= Singular(L3,{});
```

$$B3 := \left[\left[x^5 - 3\,x^4 - 3\,x^3 + 7\,x^2 - 7\,x + \frac{17}{2}, RootOf\left(2\,\_Z^5 - 6\,\_Z^4 - 6\,\_Z^3 + 14\,\_Z^2 - 14\,\_Z\right.\right.\right.$$ $$\left.\left.\left. + 17\right)\right], [x, 0], [\infty, \infty]\right]$$ **(194)**

```
> S3:= NotAppSing(L3,B3,{});
```

$$S3 := [[x, 0], [\infty, \infty]]$$ **(195)**

```
> C3:= IrrRegAppsingBesSqRoot(L3,t,B3,{}):
> EquivExpgaugeBesSqRoot1(L2,L3,C3,x,t,T,{});
```

$$\left\{\left[\left[\frac{1}{x}\right], \left[Dx - \frac{1}{x}\right]\right], \left[\left[-\frac{6}{x}\right], \left[x^7\,Dx + 6\,x^6\right]\right], \left[\left[-\frac{3}{x}\right], \left[Dx\,x^4 + 3\,x^3\right]\right], \left[\left[-\frac{2}{x}\right], \left[Dx\,x^3\right.\right.\right.$$ **(196)** $$\left.\left.+ 2\,x^2\right]\right], \left[\left[\frac{3}{x}\right], \left[\frac{Dx}{x^2} - \frac{3}{x^3}\right]\right], \left[\left[\frac{4}{x}\right], \left[\frac{Dx}{x^3} - \frac{4}{x^4}\right]\right], \left[\left[\frac{6}{x}\right], \left[\frac{Dx}{x^5} - \frac{6}{x^6}\right]\right]\right\}$$

```
> EquivExpgaugeBesSqRoot1(L1,L3,C3,x,t,T,{});
```

$$\left\{\left[\left[\frac{x^2 - x + 1}{x}\right], \left[Dx - \frac{1}{x}\right]\right], \left[\left[\frac{x^2 - x - 3}{x}\right], \left[x^4\,Dx + 3\,x^3\right]\right], \left[\left[\frac{x^2 - x + 3}{x}\right], \left[\frac{Dx}{x^2}\right.\right.\right.$$ **(197)** $$\left.\left.- \frac{3}{x^3}\right]\right], \left[\left[\frac{x^2 - x + 4}{x}\right], \left[\frac{Dx}{x^3} - \frac{4}{x^4}\right]\right], \left[\left[\frac{x^2 - x + 6}{x}\right], \left[\frac{Dx}{x^5} - \frac{6}{x^6}\right]\right],$$ $$\left[\left[\frac{(x + 1)\,(x - 2)}{x}\right], \left[x^3\,Dx + 2\,x^2\right]\right], \left[\left[\frac{(x + 2)\,(x - 3)}{x}\right], \left[x^7\,Dx + 6\,x^6\right]\right], [[x - 1],$$ $$[x\,Dx]]\right\}$$

(2)

```
> LBB:=2*x^2*Dx^3+6*x*Dx^2+(2-2*x-2*nu^2)*Dx-1;
```

$$LBB := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + \left(-2\,v^2 - 2\,x + 2\right)Dx - 1$$ **(198)**

```
> L1:=subs(nu=3,LBB);
```

$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1$$ **(199)**

```
> L1:= 2*x^2*Dx^3+6*x*Dx^2+(-2*x-16)*Dx-1;
```

$$L1 := 2\,x^2\,Dx^3 + 6\,x\,Dx^2 + (-2\,x - 16)\,Dx - 1$$ **(200)**

```
> r:= (x-1)*(x-7);
```

$$r := (x - 1)\,(x - 7)$$ **(201)**

```
> L2:=ExpProduct(L1,r);
```

$$L2 := 2\,x^2\,Dx^3 - 6\left(x^3 - 8\,x^2 + 7\,x - 1\right)Dx^2\,x + \left(6\,x^6 - 96\,x^5 + 468\,x^4 - 696\,x^3 + 438\,x^2\right.$$ **(202)** $$\left.- 86\,x - 16\right)Dx - 2\,x^8 + 48\,x^7 - 426\,x^6 + 1714\,x^5 - 3222\,x^4 + 3290\,x^3 - 1710\,x^2$$ $$+ 228\,x + 111$$

```
> r0:=x-9;
```

$$r0 := x - 9$$ **(203)**

```
> r1:=x;
```

$$r1 := x$$ **(204)**

```
> r2:=0;
```

$$r2 := 0 \qquad (205)$$

```
> L3:=GaugeTransf(L2,r0,r1,r2);
```

$$L3 := 2\,Dx^3\,x^2\left(2\,x^9 - 48\,x^8 + 432\,x^7 - 1864\,x^6 + 4560\,x^5 - 8354\,x^4 + 9752\,x^3 - 8528\,x^2 \qquad (206)\right.$$
$$\left. + 4211\,x - 1296\right) - 2\left(6\,x^{12} - 192\,x^{11} + 2490\,x^{10} - 16956\,x^9 + 67248\,x^8 - 171918\,x^7\right.$$
$$\left. + 319920\,x^6 - 425946\,x^5 + 413743\,x^4 - 284040\,x^3 + 128063\,x^2 - 35638\,x + 3888\right)Dx^2\,x$$
$$+ \left(12\,x^{15} - 480\,x^{14} + 8136\,x^{13} - 76440\,x^{12} + 440652\,x^{11} - 1657216\,x^{10} + 4333048\,x^9\right.$$
$$- 8391000\,x^8 + 12203078\,x^7 - 13401932\,x^6 + 11005272\,x^5 - 6540700\,x^4 + 2623026\,x^3$$
$$\left. - 593486\,x^2 + 22068\,x + 20736\right)Dx - 4\,x^{17} + 192\,x^{16} - 4020\,x^{15} + 48304\,x^{14}$$
$$- 369996\,x^{13} + 1910192\,x^{12} - 6928124\,x^{11} + 18406312\,x^{10} - 37207068\,x^9 + 58247804\,x^8$$
$$- 71057310\,x^7 + 67146884\,x^6 - 48140538\,x^5 + 24996944\,x^4 - 8411426\,x^3 + 1220934\,x^2$$
$$+ 242081\,x - 125568$$

```
> B2:= Singular(L2,{});
```

$$B2 := [\,[x, 0], [\infty, \infty]\,] \qquad (207)$$

```
> S2:= NotAppSing(L2,B2,{});
```

$$S2 := [\,[x, 0], [\infty, \infty]\,] \qquad (208)$$

```
> C2:= IrrRegAppsingBesSqRoot(L2,t,B2,{}):
> EquivExpgaugeBesSqRoot1(L1,L2,C2,x,t,T,{});
```

$$\left\{[\,[(x-1)\,(x-7)\,], [1]\,], \left[\left[\frac{x^3 - 8\,x^2 + 7\,x - 6}{x}\right], [x^6]\right], \left[\left[\frac{x^3 - 8\,x^2 + 7\,x - 3}{x}\right], [x^3]\right], \qquad (209)\right.$$
$$\left.\left[\left[\frac{x^3 - 8\,x^2 + 7\,x + 3}{x}\right], \left[\frac{1}{x^3}\right]\right], \left[\left[\frac{x^3 - 8\,x^2 + 7\,x + 6}{x}\right], \left[\frac{1}{x^6}\right]\right]\right\}$$

```
> B3:= Singular(L3,{});
```

$$B3 := \left[\,[x, 0], [\infty, \infty], \left[x^9 - 24\,x^8 + 216\,x^7 - 932\,x^6 + 2280\,x^5 - 4177\,x^4 + 4876\,x^3 - 4264\,x^2 \quad (210)\right.\right.$$
$$+ \frac{4211}{2}\,x - 648, RootOf\left(2\,\_Z^9 - 48\,\_Z^8 + 432\,\_Z^7 - 1864\,\_Z^6 + 4560\,\_Z^5 - 8354\,\_Z^4\right.$$
$$\left.\left.\left. + 9752\,\_Z^3 - 8528\,\_Z^2 + 4211\,\_Z - 1296\right)\right]\right]$$

```
> S3:= NotAppSing(L3,B3,{});
```

$$S3 := [\,[x, 0], [\infty, \infty]\,] \qquad (211)$$

```
> C3:= IrrRegAppsingBesSqRoot(L3,t,B3,{}):
> EquivExpgaugeBesSqRoot1(L2,L3,C3,x,t,T,{});
```

$$\left\{\left[\left[-\frac{6}{x}\right], [x^7\,Dx + x^7 - 3\,x^6]\right], \left[\left[-\frac{3}{x}\right], [x^4\,Dx + x^4 - 6\,x^3]\right], \left[\left[\frac{3}{x}\right], \left[\frac{Dx}{x^2} + \frac{x-12}{x^3}\right]\right], \qquad (212)\right.$$
$$\left.\left[\left[\frac{6}{x}\right], \left[\frac{Dx}{x^5} + \frac{x-15}{x^6}\right]\right]\right\}$$

```
> EquivExpgaugeBesSqRoot1(L1,L3,C3,x,t,T,{});
```

$$\left\{[\,[(x-1)\,(x-7)\,], [x\,Dx + x - 9]\,], \left[\left[\frac{x^3 - 8\,x^2 + 7\,x - 6}{x}\right], [x^7\,Dx + x^7 - 3\,x^6]\right], \qquad (213)\right.$$
$$\left[\left[\frac{x^3 - 8\,x^2 + 7\,x - 3}{x}\right], [x^4\,Dx + x^4 - 6\,x^3]\right], \left[\left[\frac{x^3 - 8\,x^2 + 7\,x + 3}{x}\right], \left[\frac{Dx}{x^2}\right.\right.$$

$$\left. + \frac{x-12}{x^3} \right]\right], \left[\left[\frac{x^3 - 8x^2 + 7x + 6}{x}\right], \left[\frac{Dx}{x^5} + \frac{x-15}{x^6}\right]\right]\right\}$$