On the solutions of holonomic third-order linear irreducible differential equations in terms of hypergeometric functions

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Abstract

In this paper we present algorithms that combine change of variables, exp-product and gauge transformation to represent solutions of a given irreducible third-order linear differential operator L, with rational function coefficients and without Liouvillian solutions, in terms of functions $S \in \{{}_0F_2,\ {}_1F_2,\ {}_2F_2, \check{\mathrm{B}}_\nu^2\}$ where ${}_pF_q$ with $p \in \{0, 1, 2\}$, $q = 2$, is the generalized hypergeometric function, and $\check{\mathrm{B}}_\nu^2(x) = \left(\mathrm{B}_\nu(\sqrt{x})\right)^2$ with B_ν a Bessel function (see [2]). That means we find rational functions r, r_0, r_1, r_2, f such that the solution of L will be of the form

$$y = \exp\left(\int r\,dx\right)\left(r_0 S(f(x)) + r_1 (S(f(x)))' + r_2 (S(f(x)))''\right).$$

An implementation of those algorithms in Maple is available.

Keywords: Bessel functions, Bessel square functions, composition of Bessel square functions with square roots, hypergeometric functions, operators, transformations, change of variables, exp-product, gauge transformation, singularities, generalized exponents, exponent differences, rational functions, zeroes, poles.
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1. Introduction

Consider a differential operator $\mathrm{L} = \sum_{i=0}^{n} a_i \partial^i$ with $n \in \mathbb{N}_{\geq 0}$ and the coefficients a_i in some differential field K, e.g. $\mathrm{K} = \mathbb{Q}(x)$ or $\mathrm{K} = \mathbb{C}(x)$ and $\partial = \frac{d}{dx}$. Information on the solutions of the differential equation $\mathrm{L}(y) = 0$ can be obtained by studying algebraic properties of the operator L, see e.g. van der Put and Singer [32].

If the order of L is one, then solutions y of the holonomic differential equation $\mathrm{L}(y) = 0$ can be easily computed and are called hyperexponential functions.

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If the order of L is two, then the situation is quite different. Either L is reducible, then L has a nontrivial (non-commutative) factorization. In this case the right factor is of first order leading to a hyperexponential solution of $L(y) = 0$, again. To check the reducibility of an operator, we can use some known algorithms like *Beke's algorithm* or the algorithm in [33]. Beke's algorithm was extended by Mark van Hoeij (see [18]) in his PhD thesis on factorization of linear differential operators. However, if L is irreducible, it is more difficult to find solutions of $L(y) = 0$. There are some algorithms which try to find them in some particular forms. *Kovacic's algorithm* [27] (which finds Liouvillian solutions) is an example. Some complete algorithms which solve $L(y) = 0$ in terms of special functions are given by Mark van Hoeij, Wolfram Koepf, Ruben Debeerst and Quan Yuan ([11], [12], [21], [43]).

Let us assume now that the order of L is larger than two. If L is reducible, solutions of $L(y) = 0$ can be in some case easily computed, since we know how to solve first-order holonomic differential equations and also, in some particular cases, second-order holonomic differential equations. Michael Singer described (see [36]) in which situation L has so-called *Eulerian solutions* (solutions which can be expressed as products of second-order operators using sums, products, field operations, algebraic extensions, integrals, differentiations, exponentials, logarithms and change of variables). He showed that solving such an operator L can be reduced to solving second-order operators through factoring operators (see [17], [16], [18] and [22]), or reducing operators to tensor products of lower order operators. However, if L is irreducible, it is difficult to solve the equation $L(y) = 0$. In addition to being irreducible, if Liouvillian or Eulerian solutions of L are not allowed, then no algorithm for this case is yet published. That is the case for example when L is of order three and comes from certain special and useful functions such as the hypergeometric functions ${}_0F_2$, ${}_1F_2$ and ${}_2F_2$, and the composition of Bessel square functions with square roots $\check{\mathrm{B}}_\nu^2(x) = \left(\mathrm{B}_\nu(\sqrt{x})\right)^2$ where B_ν are Bessel functions with parameter not in $1/2 + \mathbb{Z}$. That is also the reason why we focus on those operators in this paper and specially those of order three.

Let k be an extension field of $\mathbb{Q}$ which is algebraically closed and has characteristic zero, and $k(x)[\partial]$ be the ring of differential operators with coefficients in $k(x)$. Let $L \in k(x)[\partial]$ be an irreducible linear differential operator of order n without Liouvillian solutions, and $S(x)$ a special function that satisfies the linear differential equation of order n with L_S as its associated operator.

Definition 1. *A function y is called a* linear *S*-expression *if there exist rational functions $f, r, r_0, r_1, \ldots, r_{n-1}$ such that*

$$y = \exp\left(\int r\,dx\right)\left(r_0 S(f(x)) + r_1(S(f(x)))^{(1)} + r_2(S(f(x)))^{(2)} + \cdots + r_{n-1}(S(f(x)))^{(n-1)}\right) \qquad (1)$$

We say that y can be expressed in terms of S.

The form (1) is closed under the three following *transformations* that send irreducible order n operators in $k(x)[\partial]$ to linear order n irreducible operators:

(i) *change of variables*: $y(x) \to y(f(x)),\ f \in k(x) \setminus k$,

(ii) *exp-product*: $y \to \exp\left(\int r\,dx\right) y,\ r \in k(x)$, and

(iii) *gauge transformation*: $y \longrightarrow r_0 y + r_1 y' + \cdots + r_{n-1} y^{(n-1)}$, $\quad r_0, r_1, \ldots, r_{n-1} \in k(x)$.

where $y^{(j)}$ represents the j^{th} derivative of y according to the variable x. The function f in (i) above is called pullback function. These transformations are denoted by $\xrightarrow{f}_C, \xrightarrow{r}_E, \xrightarrow{r_0, r_1, \ldots, r_{n-1}}_G$ respectively. Hence, finding a solution y of L in terms of S corresponds to finding a sequence of those transformations that sends L_S to L. So every *complete solver* for finding solutions in terms of S must be able to deal with all those transformations, i.e. it must be able to find any solution of the form (1) if it exists.

We are interested here in the case of order $n = 3$ such that y in the definition above can be a solution of our given operator L with $S \in \{{}_0F_2,\ {}_1F_2,\ {}_2F_2, \check{\mathrm{B}}^2_\nu\}$. The goal of this paper is the following: given $\mathrm{L} \in k(x)[\partial]$, an irreducible third-order linear differential operator without Liouvillian solutions and L_S the differential operator associated to the function $S \in \{{}_0F_2,\ {}_1F_2,\ {}_2F_2, \check{\mathrm{B}}^2_\nu\}$, our task is to find

$$\mathrm{L}_S \xrightarrow{f}_C \mathrm{M} \xrightarrow{r}_E \mathrm{L}_1 \xrightarrow{r_0, r_1, r_2}_G \mathrm{L} \tag{2}$$

with $r, r_0, r_1, r_2, f \in k(x)$ and $\mathrm{M}, \mathrm{L}_1 \in k(x)[\partial]$. A solution y of L in terms of $S \in \{{}_0F_2,\ {}_1F_2,\ {}_2F_2, \check{\mathrm{B}}^2_\nu\}$ will be

$$y = \exp\left(\int r\,dx\right)\left(r_0 S(f(x)) + r_1 (S(f(x)))' + r_2 (S(f(x)))''\right). \tag{3}$$

We compute those transformations (r, r_0, r_1, r_2, f) from the singularities of M which are not apparent (see Definition 3). Since we don't yet know M, the only singularities of M that we know are those singularities of M that cannot disappear (turn into regular points) under transformations $\xrightarrow{r_0, r_1, r_2}_G$ and $\xrightarrow{r}_E$.

Definition 2. *A singularity of an operator is called non-removable if it stays singular under any combination of* $\xrightarrow{r_0, r_1, r_2}_G$ *and* $\xrightarrow{r}_E$. *Otherwise, it is called removable singularity.*

In order to find f, our approach is the following:

1. For $S = {}_2F_2$, the ramification index (see [33] and [16]) of L_S at ∞ is 1. Hence, we can compute the polar part of f (see Definition 5) from the generalized exponents (see also [33] and [16]) at irregular singularities of L, and then f by using the regular singularities of L or some information related to the degree of the numerator that f can have.
2. For $S \in \{{}_0F_2,\ {}_1F_2\}$, the ramification index is $n_e \geq 2$ ($n_e = 2$ for ${}_1F_2$, and $n_e = 3$ for ${}_0F_2$). We put f in the form $f = \dfrac{\mathrm{A}}{\mathrm{B}}$ with $\mathrm{A}, \mathrm{B} \in k[x]$, B monic and $\gcd(\mathrm{A}, \mathrm{B}) = 1$. Using the generalized exponents at the irregular singularities of L (see [43]), we can compute B and a bound for the degree of A. Hence, we can get the truncated series for f and some linear equations for the coefficients of A. By comparing the number of linear equations for the coefficients of A and the degree of A, we will deal with some cases which will help us to find A.
3. For $S = \check{\mathrm{B}}^2_\nu$, we will deal with $\sqrt{f}$ where $f \in k(x)$ is the change of variable parameter. Therefore, we will consider two cases: when $f = g^2$ with $g \in k(x)$, and when $f \neq g^2$ with $g \in k(x)$.

(a) Let us assume $f = g^2$ with $g \in k(x)$. Using the fact that the operator $L_{\breve{B}_\nu^2}$ can be derived from the operator $L_{B_\nu^2}$ associated to the function $B_\nu^2(x) = (B_\nu(x))^2$ by the transformation $x \to \sqrt{x}$, the transformation $L_{\breve{B}_\nu^2} \xrightarrow{f}_C M \longrightarrow_{EG} L$ is equivalent to the transformation $L_{B_\nu^2} \xrightarrow{g}_C M \longrightarrow_{EG} L$. Therefore, we just have to solve in terms of solutions of $L_{B_\nu^2}$ that means find Bessel square type solutions. To get the composition of Bessel square functions with square roots solutions (solutions in terms of $\breve{B}_\nu^2(x) = \left(B_\nu(\sqrt{x})\right)^2$), we will just replace the change of variable parameter f, in the solutions that we have found in terms of B_ν^2, by its square. We will not treat this case here, and refer for the details to the first author's PhD thesis (see [28]) which is available from http://www.mathematik.uni-kassel.de/~merlin/.

(b) If we assume $f \neq g^2$ with $g \in k(x)$, then we cannot write $\sqrt{f}$ as a quotient of polynomials. The combinatorial problem is more difficult since the ramification index of $L_{B_\nu^2}$ will now be 2. Unfortunately, by squaring $\sqrt{f}$ to make it a rational function, we double the degree of the numerator of $\sqrt{f}$, but we do not get more linear equations satisfied by the coefficient of the numerator of f, which means that the number of linear equations is only half of the degree of the numerator of f. In order to find f, we put f in the form $f = \frac{A}{B}$ with $A, B \in k[x]$, B monic and $\gcd(A, B) = 1$. Then we use the same technique as in the case $S \in \{{}_0F_2,\ {}_1F_2\}$ and also the fact that the ramification index of $L_{\breve{B}_\nu^2}$ is 2.

Since finding f is equivalent to finding M, we also get M. Our basic strategy to find r, r_0, r_1 and r_2 is to study for L and M the solution behaviour in the neighbourhood of singularities, the ramification indices, the generalized exponents and exponent differences at their non-removable singularities. That will help us to find a good way to increase the speed of our algorithms and to avoid the p-curvature test (see [24], [31], [10] and [4]) between M and L, since this test uses one of the Grothendieck's conjectures which is not yet proved.

We have implemented the methods of the given paper in a Maple package called `Solver3` which can be downloaded from http://www.mathematik.uni-kassel.de/~merlin/. The first author's PhD thesis [28] explains the algorithms in more detail. The examples in this paper are included in a Maple worksheet and are also available on the above website.

The first author's PhD thesis [28] has also treated other cases which are not covered in this paper: the Bessel square $\left(B_\nu^2\right)$-type solutions and the ${}_1F_1{}^2$-type solutions.

2. Preliminaries

Let K be a differential field of rational functions, C_K its constant field ($K = C_K(x)$) and $\overline{C_K}$ its algebraic closure.

2.1. *Differentials operators*

Let $L \in K[\partial]$. By the solutions of L we mean the solutions of the homogeneous linear differential equation $L(y) = 0$. The set of all solutions of L is called its *solution space*. It is a vector space of dimension deg(L) denoted by V(L).

We call a point $p \in \overline{C_K} \cup \{\infty\}$ a *singularity* of L if p is a zero of the leading coefficient of L or p is a pole of one of the other coefficients. All other points are called regular points of L. At all regular points of L we can find a fundamental system of power series solutions $\sum_{i=0}^{\infty} a_i t_p^i$, $a_i \in C_K$ where t_p denotes the local parameter defined as follows

$$t_p = \begin{cases} x - p & \text{if } p \neq \infty, \\ \frac{1}{x} & \text{if } p = \infty \,. \end{cases}$$

Let $L_{1/x}$ denote the operator coming from L by the change of variables $x \to \frac{1}{x}$. The following definition about singularities of an operator is the same as in [11, Definition 1.12] and [43, Definition 7].

Definition 3. *Let* $L = \sum_{i=0}^{n} a_i \partial^i \in K[\partial]$ *where* a_i *are polynomials. A singularity* p *of* L *is called*

1. *apparent singularity if all solutions of* L *are regular at* p*,*
2. *regular singularity* ($p \neq \infty$) *if* $t_p^i \dfrac{a_{n-i}}{a_n}$ *is regular at* p *for* $1 \leq i \leq n$*,*
3. *regular singularity* ($p = \infty$) *if* $L_{1/x}$ *has a regular singularity at* $x = 0$*, and*
4. *irregular singularity otherwise.*

2.2. Formal solutions and generalized exponents

Let us first define the universal extension of K, see [11, Definition 1.24] and [43, Definition 10].

Definition 4. *A* universal extension U *of* K *is a minimal (simple) differential ring in which every operator* $L \in K[\partial]$ *has precisely* deg(L) $\overline{C_K}$*-linear independent solutions. It exists if* K *has an algebraically closed field* C_K *of constants of characteristic zero.*

There exists a universal extension U of $\mathbb{C}((x))$, so V(L) has dimension deg(L) for every nonzero operator $L \in \mathbb{C}((x))[\partial]$. The construction of U can be found in [33].

In order to have the solutions in some particular form, let us define what we call a generalized exponent of an operator at a point, see [11, Definition 1.28] and [43, Definition 12].

Definition 5. *Let* $L \in \mathbb{C}(x)[\partial]$ *and* p *a point with local parameter* t_p*. An element* $e \in \mathbb{C}[{t_p}^{-1/r_e}]$*,* $r_e \in \mathbb{N}_{>0}$ *is called a* generalized exponent *of* L *at the point* p *if there exists a* formal solution *of* L *of the form*

$$y(x) = \exp\left(\int \frac{e}{t_p}\, dt_p\right) S, \quad S \in \mathbb{C}(({t_p}^{1/r_e}))[\ln(t_p)], \tag{4}$$

where the constant term of the Puiseux series S *is non-zero. For a given solution this representation is unique and* $r_e \in \mathbb{N}$ *is called the* ramification index *of* e*.*

If $e \in \mathbb{C}$ we just get a solution $x^e S$, in this case e is called an exponent. If $r_e = 1$, then e is unramified, otherwise it is ramified. Since we only consider third-order differential operators, r_e in the definition can be only 1 or 2 or 3.

Solutions that involve a logarithm will be called logarithmic solutions.

For an n-th order linear differential operator $L \in K[\partial]$, at any point p there are n generalized exponents $e_1, \ldots, e_n$ corresponding to a basis $\exp\left(\int \frac{e_i}{t_p}\, dt_p\right) S_i(p)$, $i = 1, \ldots, n$ of V(L).

Remarks 6. *If the order of* L *is n, then*

1. *at every point p we have* $y_1, \ldots, y_n \in \mathrm{V(L)}$ *linear independent solutions of* L*, that means at every point p there are n generalized exponents* $e_1, \ldots, e_n$ *such that the solution space* V(L) *is generated by the functions* $\exp\left(\int \frac{e_i}{t_p}\, dt_p\right) S_i$ *with* $S_i \in \mathbb{C}(({t_p}^{1/r_{e_i}}))[\ln(t_p)]$ *and* $1 \le i \le n$.
2. *p is an irregular singularity of* L *if* L *has at p at least one non-constant generalized exponent.*
3. *p is a regular point of* L *if the generalized exponents of* L *at p are* $0, 1, 2, \ldots, n-1$.
4. *If p is an apparent singularity of* L*, then all the generalized exponents of* L *at p are non-negative integers.*
5. *If p is a regular singularity of* L*, then all the generalized exponents of* L *at p are constants.*

Generalized exponents can be computed in Maple with Mark van Hoeij's command `gen_exp`, which belongs to the package DEtools.

Remarks 7.
- *The square of a solution of differential equation of order two satisfies a differential equation of order three.*
- *The change of variables transformation preserves the order of the differential operator.*

2.3. Bessel and composition of Bessel square functions with square roots

The solutions of the operators $\mathrm{L}_{B_1} = x^2\partial^2 + x\partial + \left(x^2 - \nu^2\right)$ and $\mathrm{L}_{B_2} = x^2\partial^2 + x\partial - \left(x^2 + \nu^2\right)$ are called Bessel functions and modified Bessel functions, respectively.

L_{B_1} has two linearly independent solutions $J_\nu(x)$ and $Y_\nu(x)$ (see [28]) called Bessel functions of the first and second kind, respectively. Similarly, the two linearly independent solutions $I_\nu(x)$ and $K_\nu(x)$ (see [28]) are called the modified Bessel functions of the first and second kind, respectively.

Remark 8. *When* $\nu \in \frac{1}{2} + \mathbb{Z}$, L_{B_1} *and* L_{B_2} *are reducible and their solutions are hyperexponential functions.*

Since we only consider irreducible operators, we will exclude the case $\nu \in \frac{1}{2} + \mathbb{Z}$ from this paper.

The change of variables $x \to ix$ (where $i^2 = -1$) sends $\mathrm{V}(\mathrm{L}_{B_1})$ to $\mathrm{V}(\mathrm{L}_{B_2})$ and vice versa. Since in our algorithm we have to deal with change of variables, we only need one of the two operators. We choose the modified Bessel case and we denote $\mathrm{L}_B := \mathrm{L}_{B_2}$. B_ν will refer to one of the functions I_ν and K_ν, the solutions of L_B.

Using Remarks 7 above, the composition of Bessel square functions with square roots $\check{\mathrm{B}}^2_\nu(x) = \left(\mathrm{B}_\nu(\sqrt{x})\right)^2$ is solution of a third-order linear differential equation. Its associated differential operator is called the Bessel square root operator and is given by

$$\mathrm{L}_{\check{B}^2_\nu} = 2x^2\partial^3 + 6x\partial^2 + \left(2 - 2x - 2\nu^2\right)\partial - 1. \tag{5}$$

Remark 9. *The* $\mathbb{C}(x)$ *vector space* $\mathrm{E} := \mathbb{C}(x)\left(\check{\mathrm{B}}^2_\nu(x)\right)'' + \mathbb{C}(x)\left(\check{\mathrm{B}}^2_\nu(x)\right)' + \mathbb{C}(x)\check{\mathrm{B}}^2_\nu(x)$ *is invariant under* $\nu \mapsto -\nu$, $\nu \mapsto \nu + 1$ *and* $\nu \mapsto \nu - 1$.

For $\nu \notin \frac{1}{2} + \mathbb{Z}$, $\mathrm{L}_{\check{B}^2_\nu}$ is irreducible and has two singularities: one regular at 0 and the other irregular at ∞.

$\mathrm{L}_{\check{B}^2_\nu}$ has 0, $-\nu$ and ν as generalized exponents at $x = 0$, and $\frac{1}{2}$, $\frac{1}{t} + \frac{1}{2}$ (with $t^2 = \frac{1}{x}$) at $x = \infty$. Hence the ramification index of $\mathrm{L}_{\check{B}^2_\nu}$ is 2 at $x = \infty$ and 1 at all the other points.

Remark 10. *The necessary and sufficient condition for $\mathrm{L}_{\check{B}^2_\nu}$ to have a logarithmic solution at $x = 0$ is that the Bessel parameter is an integer ($\nu \in \mathbb{Z}$).*

2.4. Hypergeometric functions

The generalized hypergeometric series ${}_pF_q$ is defined by

$${}_pF_q\left(\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q \end{array} \middle| x\right) = \sum_{k=0}^{+\infty} \frac{(\alpha_1)_k \cdot (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdot (\beta_2)_k \cdots (\beta_q)_k \cdot k!} x^k ,$$

where $(\lambda)_k$ denotes the Pochhammer symbol

$$(\lambda)_k := \begin{cases} 1 & \text{if } k = 0 , \\ \lambda \cdot (\lambda + 1) \cdots (\lambda + k - 1) & \text{if } k > 0 . \end{cases}$$

It satisfies the following differential equation

$$\theta(\theta + \beta_1 - 1) \cdots (\theta + \beta_q - 1) y(x) = x(\theta + \alpha_1) \cdots (\theta + \alpha_p) y(x)$$

where $\theta = x\frac{d}{dx}$. This equation has order $\max(p, q + 1)$. For $p \leq q$ the series ${}_pF_q$ is convergent for all x. For $p > q + 1$ the radius of convergence is zero, and for $p = q + 1$ the series converges for $|x| < 1$. For $p \leq q + 1$ the series and its analytic continuation is called hypergeometric function.

Theorem 11. *Let θ be the differential operator given by $\theta(f(x)) = xf'(x) = x\dfrac{d}{dx}f(x)$. The generalized hypergeometric function ${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x\right)$ satisfies the derivative rule*

$$\theta(f_n(x)) = n\,(f_{n+1}(x) - f_n(x))$$

for any of its numerator parameters $n := a_i$ ($i = 1, \ldots, p$), and

$$\theta(f_n(x)) = (n - 1)\,(f_{n-1}(x) - f_n(x))$$

for any of its denominator parameters $n := b_i$ ($i = 1, \ldots, q$).

Proof. See [25]. □

Theorem 12. *The generalized hypergeometric function ${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x\right)$ has this property:*

$$\frac{d}{dx}{}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x\right) = \frac{\prod_{i=1}^{p} a_i}{\prod_{i=1}^{p} b_i}\,{}_pF_q\left(\begin{array}{c} a_1 + 1, \ldots, a_p + 1 \\ b_1 + 1, \ldots, b_q + 1 \end{array} \middle| x\right).$$

Proof. See [25]. □

Corollary 13. *The linear space over $\mathbb{C}(x)$ spanned by*

$$ {}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right) \text{ and } x\frac{d}{dx}{}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right) $$

contains each of ${}_pF_q\left(\begin{array}{c} a_1,\dots,a_j+1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right)$, ${}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_j-1,\dots,b_q \end{array} \middle| x \right)$, $x{}_pF_q\left(\begin{array}{c} a_1+1,\dots,a_p+1 \\ b_1+1,\dots,b_q+1 \end{array} \middle| x \right)$ *and* ${}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right)$.

Similarly, by applying the differentiation formulas twice, there are $\left(\begin{array}{c} p+q+3 \\ 2 \end{array} \right)$ *such functions contained in* $\left\{1, x\frac{d}{dx}, \left(x\frac{d}{dx}\right)^2\right\}{}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right)$, *which has dimension three (any four are linearly dependent).*

Proof. We use Theorem 11 and Theorem 12. □

Definition 14. *A function obtained by adding ± 1 to exactly one of the parameters a_i in b_j, $i \in \{1,\dots,p\}$ and $j \in \{1,\dots,q\}$, is called contiguous to* ${}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right)$.

For a given ${}_pF_q$ hypergeometric function, the following corollary shows us the existence of a linear space over $\mathbb{C}(x)$, with dimension equal to the order of the differential equation for ${}_pF_q$, which contains all the functions coming from ${}_pF_q$ and all its derivatives by any integer parameter-shift.

Corollary 15. *Let us consider a generalized hypergeometric function* ${}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right)$ *and the $\mathbb{C}(x)$ vector space*

$$ \mathrm{E} := \mathbb{C}(x){}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right) + \cdots + \mathbb{C}(x)\frac{d^{n-1}}{dx^{n-1}}{}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right) $$

where $n = \max\{p, q+1\}$ is the order of the differential equation for ${}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right)$. *Hence, E is invariant under substitution $\nu \mapsto \nu+1$ and $\nu \mapsto \nu-1$ of upper and lower parameter(s) of* ${}_pF_q\left(\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array} \middle| x \right)$.

Proof. We use Theorem 11 and Theorem 12. □

Remark 16. *In our case, when ${}_pF_q$ satisfies a third-order differential equation, we have* $n = \max\{p, q+1\} = 3$ *and*

$$\mathrm{E} := \mathbb{C}(x)\,{}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\,\middle|\, x\right) + \mathbb{C}(x)\frac{d}{dx}\,{}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\,\middle|\, x\right) + \mathbb{C}(x)\frac{d^2}{dx^2}\,{}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\,\middle|\, x\right).$$

Hence, this space S *is invariant under any integer parameter-shift of ${}_pF_q$. Therefore, we will assume that the upper and lower parameter(s) of ${}_pF_q$ belong to* $[0, 1]$.

All our functions ${}_0F_2$, ${}_1F_2$, and ${}_2F_2$ satisfy third-order linear differential equations. Their associated differential operators are

- for ${}_2F_2$: $\mathrm{L}_{22} = x^2\partial^3 + x(1 + b_1 + b_2 - x)\,\partial^2 + (b_1b_2 - x(1 + a_1 + a_2))\,\partial - a_1a_2$,
- for ${}_1F_2$: $\mathrm{L}_{12} = x^2\partial^3 + x(1 + b_1 + b_2)\,\partial^2 + (b_1b_2 - x)\,\partial - a_1$,
- for ${}_0F_2$: $\mathrm{L}_{02} = x^2\partial^3 + x(1 + b_1 + b_2)\,\partial^2 + b_1b_2\partial - 1$.

The operators L_{22}, L_{12} and L_{02} have two singularities: one regular at 0 and the other irregular at ∞.

- L_{22} has $0,\ 1 - b_1$ and $1 - b_2$ as generalized exponents at $x = 0$, and $a_1,\ a_2$ and $-t^{-1} + b_1 + b_2 - (a_1 + a_2)$ (with $t = \frac{1}{x}$) at $x = \infty$.
- L_{12} has $0,\ 1-b_1$ and $1-b_2$ as generalized exponents at $x = 0$, and $a_1,\ t^{-1}+1/2\,(b_1 + b_2 - a_1 - 1/2)$ (with $t^2 = \frac{1}{x}$) at $x = \infty$.
- L_{02} has $0,\ 1 - b_1$ and $1 - b_2$ as generalized exponents at $x = 0$, and $t^{-1} + 1/3\,(b_1 + b_2 - 1)$ (with $t^3 = \frac{1}{x}$) at $x = \infty$.

Hence the ramification index of $\mathrm{L}_0 \in \{\mathrm{L}_{22}, \mathrm{L}_{12}, \mathrm{L}_{02}\}$ at $x = 0$ is 1. At $x = \infty$, L_{22} has ramification index 1, but L_{12} and L_{02} have 2 and 3 as ramification indices, respectively.

Remark 17. *The necessary and sufficient condition for* $\mathrm{L}_0 \in \{\mathrm{L}_{22}, \mathrm{L}_{12}, \mathrm{L}_{02}\}$ *to have a logarithmic solution at $x = 0$ is that the lower parameter(s) of $F_0 \in \{{}_2F_2,\ {}_1F_2,\ {}_0F_2\}$ satisfy $b_1 \in \mathbb{Z}$ or $b_2 \in \mathbb{Z}$ or $b_1 - b_2 \in \mathbb{Z}$ or $(2b_1,\ 2b_2 \in \mathbb{Z}$ with $b1 \cdot b2 < 0)$.*

3. Transformations

Let K be a differential field of rational functions, $\mathrm{C_K}$ its constant field ($\mathrm{K} = \mathrm{C_K}(x)$) and $\overline{\mathrm{C_K}}$ its algebraic closure.

The following definition about transformation between two differential operators can be found in [11, Definition 2.1] and [43, Definition 18].

Definition 18. *A transformation between two differential operators* $\mathrm{L}_1, \mathrm{L}_2 \in \mathrm{K}[\partial]$ *is a map from the solution space* $\mathrm{V}(\mathrm{L}_1)$ *onto the solution space* $\mathrm{V}(\mathrm{L}_2)$.

There are three known types of transformations that preserve the differential field and preserve order three. For $L_1 \in K[\partial]$ and $y = y(x) \in V(L_1)$ we have:

1. change of variables: $y(x) \to y(f(x)), \; f \in K \setminus k$,
2. exp-product: $y \to \exp\left(\int r\,dx\right) y, \; r \in K$, and
3. gauge transformation: $y \to r_0 y + r_1 y' + r_2 y'', \quad r_0, r_1, r_2 \in K$.

They are denoted, as in [11] and [43] for second-order operators, by $\longrightarrow_C, \longrightarrow_E, \longrightarrow_G$, respectively and for the resulting operator $L_2 \in K[\partial]$ we write $L_1 \xrightarrow{f}_C L_2, L_1 \xrightarrow{r}_E L_2, L_1 \xrightarrow{r_0,r_1,r_2}_G L_2$, respectively. Furthermore, we write $L_1 \longrightarrow_{EG} L_2$ if some combination of 2. and 3. sends L_1 to L_2, and $L_1 \longrightarrow_{CEG} L_2$ if there exists a sequence of those three transformations that sends L_1 to L_2. The rational functions f, r, r_0, r_1 and r_2 will be called parameters of the transformations, and in case 2. the function $\exp\left(\int r\,dx\right)$ is a hyperexponential function.

Remark 19. *We can consider $\longrightarrow_C$, $\longrightarrow_E$ and $\longrightarrow_G$ as binary relations on $\mathbb{C}(x)[\partial]$. Hence, $\longrightarrow_E$, $\longrightarrow_G$ and $\longrightarrow_{EG}$ are equivalence relations, but $\longrightarrow_C$ is not: the symmetry of $\longrightarrow_C$ would require algebraic functions as parameter. For example, to cancel the operation $x \mapsto x^3$, we would need $x \mapsto x^{1/3}$.*

The following definition about equivalence of two operators can be found in [43, Definition 20].

Definition 20. *We say $L_1 \in K[\partial]$ is*

1. *gauge equivalent to L_2 if and only if $L_1 \longrightarrow_G L_2$,*
2. *exp-product equivalent to L_2 if and only if $L_1 \longrightarrow_E L_2$,*
3. *projectively equivalent to L_2 if and only if $L_1 \longrightarrow_{EG} L_2$.*

We will see now how we can interchange the order of those three transformations. Numbers 3. and 4. in this lemma are the same as in [11, Lemma 2.7] and [43, Lemma 6]. Numbers 1. and 2. are as in the proof of [11, Theorem 2.10].

Lemma 21. *Let $L_1, L_2, L_3 \in K[\partial]$ be three irreducible third-order linear differential operators. The following holds:*

1. $L_1 \longrightarrow_E L_2 \longrightarrow_C L_3 \Longrightarrow \exists M \in K[\partial]$*:* $L_1 \longrightarrow_C M \longrightarrow_E L_3$,
2. $L_1 \longrightarrow_G L_2 \longrightarrow_C L_3 \Longrightarrow \exists M \in K[\partial]$*:* $L_1 \longrightarrow_C M \longrightarrow_G L_3$,
3. $L_1 \longrightarrow_E L_2 \longrightarrow_G L_3 \Longrightarrow \exists M \in K[\partial]$*:* $L_1 \longrightarrow_G M \longrightarrow_E L_3$,
4. $L_1 \longrightarrow_G L_2 \longrightarrow_E L_3 \Longrightarrow \exists M \in K[\partial]$*:* $L_1 \longrightarrow_E M \longrightarrow_G L_3$.

Proof. For 1. and 2. the proof is similar as in [11, Theorem 2.10], and for 3. and 4. it is similar as in [11, Lemma 2.7] □

Note that the converse of 1. and 2. is not generally true since $\longrightarrow_C$ is not symmetric.

By the lemma above, we can deduce the following statement, compare [11, Theorem 2.10] and [43, Theorem 4].

Lemma 22. *Let* $\mathrm{L}_1, \mathrm{L}_2 \in \mathrm{K}[\partial]$ *be two irreducible third-order linear differential operators such that* $\mathrm{L}_1 \longrightarrow_{CEG} \mathrm{L}_2$. *Then there exists an operator* $\mathrm{M} \in \mathrm{K}[\partial]$ *such that* $\mathrm{L}_1 \longrightarrow_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}_2$.

Proof. We use the fact that $\mathrm{L}_1 \longrightarrow_{CEG} \mathrm{L}_2$ if there exists a sequence of the three transformations $\longrightarrow_C, \longrightarrow_E$ and $\longrightarrow_G$ that sends L_1 to L_2. The rest follows from Lemma 21. □

The following lemma states how the generalized exponent varies after an exp-product transformation:

Lemma 23. *Let* $\mathrm{L}, \mathrm{M} \in \mathrm{K}[\partial]$ *be two irreducible third-order linear differential operators such that* $\mathrm{M} \xrightarrow{r}_E \mathrm{L}$ *and let* e_1 *and* e_2 *be the generalized exponents of* M *and the operator* $\partial - r$ *at the point* $p \in k \cup \{\infty\}$, *respectively, with the ramification index* n_{e_1} *and* n_{e_2} *in* $\mathbb{N}^*$, *respectively. Then the generalized exponent of* L *at* p *is* $e = e_1 + e_2$.

Proof. Since e_1 and e_2 are the generalized exponents of M and the operator $\partial - r$ at $p \in k \cup \{\infty\}$, respectively, M and $\partial - r$ have solutions of the form

$$y_1 = \exp\left(\int \frac{e_1}{t_p}\, dt_p\right) S_1 \quad \text{and} \quad y_2 = \exp\left(\int \frac{e_2}{t_p}\, dt_p\right) S_2,$$

respectively, for some Puiseux series $S_1 \in k((t_p^{1/n_{e_1}}))[\ln(t_p)]$ and $S_2 \in k((t_p^{1/n_{e_2}}))[\ln(t_p)]$ with non-zero constant terms. $\mathrm{M} \xrightarrow{r}_E \mathrm{L}$ means $\mathrm{L} = \mathrm{M} \circledS (\partial - r)$ where $\circledS$ is the symmetric product (see [20], [19] and [26]). Then a solution of L at p is

$$y = y_1 y_2 = \exp\left(\int \frac{e_1}{t_p}\, dt_p\right) S_1 \cdot \exp\left(\int \frac{e_2}{t_p}\, dt_p\right) S_2 = \exp\left(\int \frac{e_1 + e_2}{t_p}\, dt_p\right) S_1 S_2 = \exp\left(\int \frac{e}{t_p}\, dt_p\right) S$$

where $e = e_1 + e_2$ and $S = S_1 S_2 \in k((t_p^{1/n_{e_1} \cdot n_{e_2}}))[\ln(t_p)]$ is a Puiseux series with non-zero constant term. □

Remarks 24. *Let* $r \in k(x)$ *and* $p \in k \cup \{\infty\}$. *consider the operator* $\partial - r$. *Let us assume that* r *has at* p *the series representation* $r = \sum_{i=m_p}^{+\infty} r_i t_p^i$ *with* $m_p \in \mathbb{Z}$, $r_i \in k$ *and* $r_{m_p} \neq 0$. *It follows from [28, Lemma 2.9] that:*

1. *If* p *is not a pole of* r *then* $m_p \geq 0$ *and the generalized exponent of* $\partial - r$ *at* p *is*

$$e = \begin{cases} 0 & \text{if } p \neq \infty, \\ -r_0 t_\infty^{-1} - r_1 & \text{otherwise.} \end{cases}$$

2. *If* p *is a pole of* r *then we will have* $m_p \leq -1$, *where* $-m_p$ *is the multiplicity order of* r *at* p, *and the generalized exponent of* $\partial - r$ *at* p *will be given by*

$$e = \begin{cases} \sum_{i=m_p}^{-1} r_i t_p^{i+1} & \text{if } p \neq \infty, \\ -\sum_{i=m_\infty}^{1} r_i t_\infty^{i-1} & \text{otherwise.} \end{cases}$$

The following lemma states how the generalized exponent varies after a gauge transformation:

Lemma 25. *Let* $\mathrm{L}, \mathrm{M} \in \mathbb{C}(x)[\partial]$ *be two irreducible third-order linear differential operators such that* $\mathrm{M} \longrightarrow_G \mathrm{L}$ *and let* e *be a generalized exponent of* M *at the point* p*. The operator* L *has at* p *a generalized exponent* $\overline{e}$ *such that* $\overline{e} = e \bmod \frac{1}{n_e}\mathbb{Z}$*, where* $n_e \in \mathbb{N}^*$ *is the ramification index of* e*.*

Proof. This lemma is the same as [11, Lemma 2.14] which was stated for second-order differential operators just in the unramified case ($n_e = 1$). In order to prove our lemma, we can just repeat the process of the proof of this lemma ([11, Lemma 2.14]) by considering $z = r_2y'' + r_1y' + r_0y$ instead of $z = r_1y' + r_0y$. That means we take into account the second derivative of y. For more details see the first author's PhD thesis [28]. □

The following theorem states how the generalized exponents look like after a change of variables f at the point p such that $f(p) = 0$ and $f(p) = \infty$ (i.e. at the zeroes and poles of f) since our differential operator $\mathrm{L}_{B_\nu^2}$ that we want to solve in terms of its solutions has only two singularities: 0 and ∞.

Theorem 26. *Let* $\mathrm{L}_0, \mathrm{M} \in \mathrm{K}[\partial]$ *be two irreducible third-order linear differential operators such that* $\mathrm{L}_0 \xrightarrow{f}_C \mathrm{M},\ f \in \mathrm{K} \setminus k$*. Let* p *be a zero or a pole of* f *with multiplicity* $m_p \in \mathbb{N}_{>0}$ *and* e *a generalized exponent of* L_0 *at* $x = a$, $a \in \{0, \infty\}$*, with ramification index* $n_e \in \mathbb{N}^*$*. We take* $a = 0$ *when* p *is a zero of* f *and* $a = \infty$ *when* p *is a pole of* f*. Then the generalized exponent of* M *at* p *related to* e *is*

$$m_p \cdot e_0 - t_p \cdot \bar{e}(f) \tag{6}$$

where

$$\begin{cases} e = \sum_{i=0}^{n} e_i t_a^{-i/n_e} & \text{with } n \in \mathbb{N} \text{ and } e_i \in k, \\ \bar{e} = -\sum_{i=1}^{n} \frac{e_i \cdot n_e}{i} t_a^{-i/n_e} & \text{when } n > 0 \text{ or } \bar{e} = 0 \text{ when } n = 0. \end{cases}$$

Furthermore,

1. *if* p *is a zero of* f*, then* p *is a regular singularity of* M,
2. *if* p *is a pole of* f*, then* p *is an irregular singularity of* M.

Proof. The proof can be found in the first author's PhD thesis [28]. □

4. Steps to find solutions

Let $F_0 \in \left\{{}_2F_2,\ {}_1F_2,\ {}_0F_2, \check{\mathrm{B}}_\nu^2\right\}$ with L_0 its associated operator. We say that we can solve a differential operator L in terms of F_0 when we can find the transformations

$$\mathrm{L}_0 \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L} \tag{7}$$

where M is a differential operator. That is: the solutions of L can be written in the following form

$$\exp\left(\int r\, dx\right)\left(r_2F_0(f(x))'' + r_1F_0(f(x))' + r_0F_0(f(x))\right) \tag{8}$$

with $r, r_2, r_1, r_0, f \in \mathrm{K}$ (parameters of transformations). Finding these transformations is equivalent to find their parameter(s). We proceed as follows:

1. first we find the change of variable parameter f and the upper and lower parameter(s) associated to the function F_0 when $F_0 \in \{{}_2F_2,\ {}_1F_2,\ {}_0F_2\}$ or the Bessel parameter ν associated to the function F_0 when $F_0 = \check{\mathrm{B}}_\nu^2$,
2. then we find the parameters r, r_0, r_1 and r_2 for the exp-product and gauge transformations.

Let K be a differential field of rational functions, $\mathrm{C_K}$ its constant field ($\mathrm{K} = \mathrm{C_K}(x)$) and $\overline{\mathrm{C_K}}$ its algebraic closure.

For $F_0 = \check{\mathrm{B}}_\nu^2$ we will deal with $\sqrt{f}$ where $f \in k(x)$ is the change of variable parameter. Therefore, we will consider two cases: when $f = g^2$ with $g \in k(x)$, and when $f \neq g^2$ with $g \in k(x)$. In this paper, we will not treat the case $f = g^2$ with $g \in k(x)$ and refer to the first author's PhD thesis (see [28]). Hence, we will assume $f \neq g^2$ with $g \in k(x)$ when $F_0 = \check{\mathrm{B}}_\nu^2$.

In the next sections, for $n \in \mathbb{N} \setminus \{0\}$ and a an element of $\mathrm{K} = k(x)$, when we will talk about a modulo $\frac{1}{n}\mathbb{Z}$, that will mean a modulo an additive element of $\frac{1}{n}\mathbb{Z}$.

5. Exponent differences

The following definition is given in [11, Definition 2.13] and [43, Definition 25].

Definition 27. *Let* $\mathrm{L} \in \mathrm{K}[\partial]$ *be a linear differential operator of order greater than one, let p be any point, and e_1, e_2 be two generalized exponents of* L *at p. Then the difference $e_2 - e_1$ is called an exponent difference of* L *at p.*

If $\deg(\mathrm{L}) = 3$ there exist just three generalized exponents at each point p: e_1, e_2 and e_3, and we define $\Delta_1(\mathrm{L}, p) = \pm(e_2 - e_1)$, $\Delta_2(\mathrm{L}, p) = \pm(e_3 - e_1)$, and $\Delta_3(\mathrm{L}, p) = \pm(e_3 - e_2)$. We define Δ modulo a factor -1 to make it well-defined because we have no ordering in the generalized exponents.

Let us see how the exponent difference varies under $\longrightarrow_{EG}$. This result can also be found in [11, Corollary 2.15] and [43, Lemma 8].

Corollary 28. *Let* $\mathrm{L} \in \mathrm{K}[\partial]$ *be an irreducible third-order linear differential operator. Let e_1 and e_2 be two generalized exponents of* L *at p with ramification index $n_{e_i}, i = 1, 2$. The exponent difference $e_2 - e_1$ is invariant modulo $\frac{1}{n}\mathbb{Z}$ under $\longrightarrow_{EG}$, where $n \in \mathbb{N}^*$ is the smallest positive integer such that $\left\{\frac{1}{n_{e_1}}\mathbb{Z}, \frac{1}{n_{e_2}}\mathbb{Z}\right\} \subseteq \frac{1}{n}\mathbb{Z}$.*

Proof. We just use Lemma 23 and Lemma 25. □

Definition 29. *Let* $\mathrm{L} \in \mathrm{K}[\partial]$ *be a irreducible third-order linear differential operator. We define*

$$\mathrm{S}_{log}(\mathrm{L}) = \{p \mid \mathrm{L} \textit{ has a logarithmic solution at } p\}, \tag{9}$$

$$\mathrm{S}_{reg}(\mathrm{L}) = \{p \mid \{\Delta_i(\mathrm{L}, p), i = 1, 2, 3\} \subset k \textit{ and } (\{\Delta_i(\mathrm{L}, p), i = 1, 2, 3\} \not\subset \mathbb{N} \textit{ or } p \in \mathrm{S}_{log}(\mathrm{L}))\}, \tag{10}$$

$$\mathrm{S}_{irr}(\mathrm{L}) = \left\{p \mid \exists i \in \{1, 2, 3\},\ \Delta_i(\mathrm{L}, p) \textit{ contains } t_p\right\}, \tag{11}$$

$$\overline{\mathrm{S}_{reg}(\mathrm{L})} = \left\{P_s \in k[x] \mid P_s \textit{ is the minimal polynomial at } s \in \mathrm{S}_{reg}(\mathrm{L}) \textit{ over } k\right\}, \tag{12}$$

$$\textit{and } \overline{\mathrm{S}_{irr}(\mathrm{L})} = \{P_s \in k[x] \mid P_s \textit{ is the minimal polynomial at } s \in \mathrm{S}_{irr}(\mathrm{L}) \textit{ over } k\}. \tag{13}$$

We define the minimal polynomial at ∞ as $P_\infty \equiv 1$ and its degree $\deg(P_\infty) \equiv 1$.

Note that these names S_{reg} *and* S_{irr} *appeared in [11, Definition 3.3] and [43, Definition 26] where they were defined and covered in a similar way related to second-order linear differential operators.*

Corollary 30. *Let* L_0 *be the associated differential operator of the function* $F_0 \in \left\{{}_2F_2,\ {}_1F_2,\ {}_0F_2, \check{\mathrm{B}}^2_\nu\right\}$, *and let* $\mathrm{L}, \mathrm{M} \in \mathrm{K}[\partial]$ *be irreducible third-order linear differential operators such that* $\mathrm{L}_0 \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$. *The following holds:*

(a) $p \in \mathrm{S}_{irr}(\mathrm{L}) \iff$ *p is a pole of f, and*

(b) $p \in \mathrm{S}_{reg}(\mathrm{L}) \implies$ *p is a zero of f.*

Proof. The proof can be found in [28, Corollary 3.13]. □

This Corollary extends [11, Corollary 3.4] and [43, Corollary 1].
Using Theorem 26, the generalized exponents of M at

- a zero p of f (with multiplicity order m_p) are
  - for $\mathrm{L}_0 \in \{\mathrm{L}_{22}, \mathrm{L}_{12}, \mathrm{L}_{02}\}$: $0,\ \ m_p(1-b_1),\ \ 2m_p(1-b_2)$
  - for $\mathrm{L}_0 = \mathrm{L}_{\check{B}^2_\nu}$: $0,\ \ -m_p\nu,\ \ m_p\nu$
- a pole p of f (with multiplicity order m_p) are
  - for $\mathrm{L}_0 = \mathrm{L}_{\check{B}^2_\nu}$: $\dfrac{m_p}{2},\ \dfrac{m_p}{2} + \sum_{j=-m_p}^{-1} \frac{j}{\varepsilon}\overline{f}_{1,j+m_p} t_p^{j/2}$ with ε solution of $X^2-1=0$ and $f^{1/2} = \sum_{j=-m_p}^{+\infty} \overline{f}_{1,j+m_p} t_p^{j/2}$ with $\overline{f}_{1,j+m_p} \in k$,
  - for $\mathrm{L}_0 = \mathrm{L}_{22}$: $m_p a_1,\ \ m_p a_2,\ \ m_p\left[b_1+b_2-(a_1+a_2)\right] + \sum_{j=-m_p}^{-1} j f_j t_p^j$
  - for $\mathrm{L}_0 = \mathrm{L}_{12}$: $m_p a_1,\ \ \frac{m_p}{2}\left(b_1+b_2-a_1-\frac{1}{2}\right) + \sum_{j=-m_p}^{-1} \frac{j}{\varepsilon}\overline{f}_{1,j+m_p} t_p^{j/2}$ with ε solution of $X^2-1=0$ and $f^{1/2} = \sum_{j=-m_p}^{+\infty} \overline{f}_{1,j+m_p} t_p^{j/2}$ with $\overline{f}_{1,j+m_p} \in k$,
  - for $\mathrm{L}_0 = \mathrm{L}_{02}$: $\frac{m_p}{3}(b_1+b_2-1) + \sum_{j=-m_p}^{-1} \frac{j}{\varepsilon}\overline{f}_{1,j+m_p} t_p^{j/3}$ with ε solution of $X^3+1=0$ and $f^{1/3} = \sum_{j=-m_p}^{+\infty} \overline{f}_{1,j+m_p} t_p^{j/3}$ with $\overline{f}_{1,j+m_p} \in k$,

  where $f = t_p^{-m_p} \sum_{j=0}^{+\infty} f_{j-m_p} t_p^j$ with $f_{j-m_p} \in k$ and $f_{-m_p} \neq 0$.

Hence the ramification index of M at all the zeroes of f is 1. At the poles of f, it is 1 for L_{22}, belongs to $\{1,2\}$ for L_{12} and $L_{\breve{B}_\nu^2}$, and belongs to $\{1,3\}$ for L_{02}. Since the exp-product and gauge transformation don't change the ramification index, for a given point p, the index of L at this point is the same like the ramification index of M also at this point.

Using the generalized exponents of M at p, we get the exponent differences of M, modulo a factor -1, at p.

- When p is a zero of f, they are
  * for $L_0 = L_{\breve{B}_\nu^2}$: $-m_p\nu, \; m_p\nu, \; 2m_p\nu$
  * for $L_0 \in \{L_{22}, L_{12}, L_{02}\}$: $m_p(1-b_1), \; m_p(1-b_2), \; m_p(b_1-b_2)$.
- When p is a pole of f, they are
  * for $L_0 = L_{\breve{B}_\nu^2}$

$$\left[\sum_{j=-m_p}^{-1} \pm j\overline{f}_{1,j+m_p} t_p^{j/2}, \quad \sum_{j=-m_p}^{-1} \mp j\overline{f}_{1,j+m_p} t_p^{j/2}, \quad -\sum_{j=-m_p}^{-1} 2j\overline{f}_{1,j+m_p} t_p^{j/2}\right],$$

* for $L_0 = L_{22}$:

$$\left[m_p(a_2-a_1), \; m_p[b_1+b_2-(2a_1+a_2)] + \sum_{j=-m_p}^{-1} jf_j t_p^j, \right.$$
$$\left. m_p[b_1+b_2-(a_1+2a_2)] + \sum_{j=-m_p}^{-1} jf_j t_p^j\right],$$

* for $L_0 = L_{12}$:

$$\left[\frac{m_p}{2}\left(b_1+b_2-3a_1-\frac{1}{2}\right) + \sum_{j=-m_p}^{-1} \pm j\overline{f}_{1,j+m_p} t_p^{j/2}, \; \frac{m_p}{2}\left(b_1+b_2-3a_1-\frac{1}{2}\right)\right.$$
$$\left. + \sum_{j=-m_p}^{-1} \mp j\overline{f}_{1,j+m_p} t_p^{j/2}, \quad -\sum_{j=-m_p}^{-1} 2j\overline{f}_{1,j+m_p} t_p^{j/2}\right],$$

* for $L_0 = L_{02}$:

$$\left[\sum_{j=-m_p}^{-1} j\left(\varepsilon_2^{-1}-\varepsilon_1^{-1}\right)\overline{f}_{1,j+m_p} t_p^{j/3}, \quad \sum_{j=-m_p}^{-1} j\left(\varepsilon_3^{-1}-\varepsilon_1^{-1}\right)\overline{f}_{1,j+m_p} t_p^{j/3},\right.$$
$$\left. \sum_{j=-m_p}^{-1} j\left(\varepsilon_3^{-1}-\varepsilon_2^{-1}\right)\overline{f}_{1,j+m_p} t_p^{j/3}\right],$$

where $\varepsilon_1, \varepsilon_2$ and ε_3 are the three distinct solutions of $X^3 + 1 = 0$.

By using the fact that the exponent difference at any point p is invariant modulo $\frac{1}{n_p}\mathbb{Z}$ (n_p is the ramification index of M at p, here $n_p \in \{1,2,3\}$) under the exp-product and gauge transformations, the exponent differences of L, modulo a factor -1, at

- a zero p of f, for $\mathrm{L}_0 = \mathrm{L}_{B^2_\nu}$ are: $-m_p\nu + \alpha_1,\ m_p\nu + \alpha_2,\ 2m_p\nu + \alpha_3$ with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$.
- a zero p of f, for $\mathrm{L}_0 \in \{\mathrm{L}_{22}, \mathrm{L}_{12}, \mathrm{L}_{02}\}$ are $\left[m_p(1-b_1) + \alpha_1,\ m_p(1-b_2) + \alpha_2,\ m_p(b_1 - b_2) + \alpha_3\right]$ with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$.
- a pole p of f for $\mathrm{L}_0 = \mathrm{L}_{B^2_\nu}$ are:

$$\left[\sum_{j=-m_p}^{-1} \pm j\,\overline{f}_{1,j+m_p} t_p^{j/2} + \beta_1,\quad \sum_{j=-m_p}^{-1} \mp j\,\overline{f}_{1,j+m_p} t_p^{j/2} + \beta_2,\quad -\sum_{j=-m_p}^{-1} 2j\,\overline{f}_{1,j+m_p} t_p^{j/2} + \beta_3\right],$$

  where $\beta_1, \beta_2, \beta_3 \in \mathbb{Z}$.
- a pole p of f for $\mathrm{L}_0 = \mathrm{L}_{22}$ are:

$$\Bigg[m_p(a_2 - a_1) + \beta_1,\ m_p[b_1 + b_2 - (2a_1 + a_2)] + \sum_{j=-m_p}^{-1} j f_j t_p^j + \beta_2,$$
$$m_p[b_1 + b_2 - (a_1 + 2a_2)] + \sum_{j=-m_p}^{-1} j f_j t_p^j + \beta_3\Bigg],$$

  where $\beta_1, \beta_2, \beta_3 \in \mathbb{Z}$.
- a pole p of f for $\mathrm{L}_0 = \mathrm{L}_{12}$ are:

$$\Bigg[\frac{m_p}{2}\left(b_1 + b_2 - 3a_1 - \frac{1}{2}\right) + \sum_{j=-m_p}^{-1} \pm j\,\overline{f}_{1,j+m_p} t_p^{j/2} + \beta_1,\ \frac{m_p}{2}\Big(b_1 + b_2$$
$$-3a_1 - \frac{1}{2}\Big) + \sum_{j=-m_p}^{-1} \mp j\,\overline{f}_{1,j+m_p} t_p^{j/2} + \beta_2,\quad -\sum_{j=-m_p}^{-1} 2j\,\overline{f}_{1,j+m_p} t_p^{j/2} + \beta_3\Bigg],$$

  where $\beta_1, \beta_2, \beta_3 \in \mathbb{Z}$.
- a pole p of f for $\mathrm{L}_0 = \mathrm{L}_{02}$ are:

$$\Bigg[\sum_{j=-m_p}^{-1} j\left(\varepsilon_2^{-1} - \varepsilon_1^{-1}\right)\overline{f}_{1,j+m_p} t_p^{j/3} + \beta_1,\quad \sum_{j=-m_p}^{-1} j\left(\varepsilon_3^{-1} - \varepsilon_1^{-1}\right)\overline{f}_{1,j+m_p} t_p^{j/3} + \beta_2,$$
$$\sum_{j=-m_p}^{-1} j\left(\varepsilon_3^{-1} - \varepsilon_2^{-1}\right)\overline{f}_{1,j+m_p} t_p^{j/3} + \beta_3\Bigg],$$

  where $\varepsilon_1, \varepsilon_2$ and ε_3 are the three distinct solutions of $X^3 + 1 = 0$ and $\beta_1, \beta_2, \beta_3 \in \mathbb{Z}$.

For computing f and the parameter(s) of $F_0 \in \left\{{}_2F_2,\ {}_1F_2,\ {}_0F_2, \check{\mathrm{B}}_\nu^2\right\}$, the only information retrieved from L that we can use is the information on invariance of exponent differences of L under projective equivalence. The poles and zeroes of f are the main points at which we can really use those exponent differences.

6. Change of variable parameter f and the parameters of F_0 when $F_0 = {}_2F_2$

6.1. *Parameter f up to a constant*

Since, by Corollary 30, $\mathrm{S_{irr}(L)}$ contains all the poles of f, we can always find candidates for the parameter f up to a constant using the set $\mathrm{S_{irr}(L)}$ and the exponent differences at its elements: that is the polar part of f. The polar part of f at a pole p is given by $\sum_{j=-m_p}^{-1} f_j t_p^j$ where m_p is the multiplicity order of p and $f = \sum_{j=-m_p}^{\infty} f_j t_p^j$.

Let us have a look at the exponent differences of L at a pole p of f. They can be brought into the form

$$\alpha,\ \beta + \sum_{j=-m_p}^{-1} j f_j t_p^j,\ \gamma + \sum_{j=-m_p}^{-1} j f_j t_p^j$$

with $\alpha, \beta, \gamma \in k$. We choose a non-constant exponent difference of L at p. We take its non-constant part: it will be $\sum_{j=-m_p}^{-1} j f_j t_p^j$. By dividing any coefficient of this series by the power of its associated parameter t_p, we get the polar part $Polar(f)_p$ of f, modulo a factor -1, at p: $Polar(f)_p = \sum_{j=-m_p}^{-1} f_j t_p^j$.

Maple's output for the generalized exponents is not ordered. Hence it will be difficult to order the exponent differences. They will be defined modulo a factor -1. So the polar part of f at a pole p will appear modulo a factor -1 using Maple. Therefore, we will have two candidates for the polar part of f at any of its poles. By choosing at each pole one candidate and doing the summation of all of them, we obtain a candidate $Polar(f)$ for f up to a constant (candidate of the polar part of f): $Polar(f) = \sum_{p \in \mathrm{S_{irr}(L)}} Polar(f)_p$.

We have implemented in Maple an algorithm called `Hyp2F2Subst` for finding the candidates for f up to a constant.

The problem now is how to find this constant. Knowing a zero of f (element of $\mathrm{S_{reg}(L)}$) can be helpful. Hence we will distinguish two cases: when we know at least one zero of f $\left(\mathrm{S_{reg}(L)} \neq \emptyset\right)$ and when we don't know any zero of f $\left(\mathrm{S_{reg}(L)} = \emptyset\right)$.

6.2. *No zero of f is known ($\mathrm{S}_{reg}(\mathrm{L}) = \emptyset$)*

We call this case the integer case. Here we have absolutely no information about the zeroes of f and all the exponent differences of L at $p \notin \mathrm{S_{irr}(L)}$ are integers. What we know is just the candidates for f modulo a constant $c \in k$.

Definition 31. *For $m \in \mathbb{N}$, we can define*

$$\mathrm{N}(m) = \left\{ \frac{j}{m} \middle| \, j \in \mathbb{Z}, j \neq 0, |j| \leq m-1 \right\}. \tag{14}$$

The name $\mathrm{N}(m)$ appeared in [11, Section 3.3.2] where it was defined with $2m$ instead of m.

Remark 32. *By using Definition 31 in [28] and taking in [28, Section 4.1], instead of 2ν where ν is the Bessel square parameter, respectively $1-b_2$ and b_1-b_2 for $\mathrm{L}_0 = \mathrm{L}_{22}$ and $1-b_1$ for $\mathrm{L}_0 = \mathrm{L}_{11}^2$, [28, Lemma 4.4], [28, Lemma 4.5] and [28, Corollary 4.6] hold also here (in this case). Let n be the degree of the numerator of f and for $i \in \{1,2\}$, $m_i = denom(b_i)$ and $m_3 = denom(b_1-b_2)$. That means we know how to find n and the candidates for m_i belong to the set of divisors of n which divide all the multiplicity orders of the zeroes of f. Therefore, the monic part of the numerator of f will be a m_i-th power, $i \in \{1,2,3\}$.*

Let g be a candidate for the polar part of f and m a divisor of the degree n of the numerator of f. If there exists $c \in k$ such that the monic part of the numerator of $g+c$ becomes an m-th power, then $g+c$ is a candidate for f. For all such $c \in k$, we will have a list $Cand(f)_{g,m}$ of candidates $g+c$ for f related to m and g. Using Remark 16 and Remark 17, we will assume $b_1, b_2 \in]0,1[$ and $b_1 - b_2 \in]-1,1[\backslash\{0\}$ (we will see later that b_1 or b_2 or b_1-b_2 integer implies $\mathrm{S}_{\mathrm{reg}}(\mathrm{L}) \neq \emptyset$). Let us define

$$\mathrm{A}_{g,m} = \left\{ \frac{i}{m} \middle| \, i \in \mathbb{N}, |i| \leq m-1, \; i \neq 0 \right\} \text{ and } \mathrm{R}_{g,m} = \left\{ \{i,j\} \in \mathrm{A}_{g,m} \middle| \, i-j \in \mathrm{N}(m) \text{ or } j-i \in \mathrm{N}(m) \right\}.$$

If $\mathrm{R}_{g,m} \neq \emptyset$ then $\mathrm{R}_{g,m}$ is the list of candidates for the lower parameters b_1 and b_2 of our function $F_0 = {}_2F_2$. We have implemented in Maple an algorithm called `findbi2F2` to get candidates for $[b_1, b_2]$. Hence, $\mathrm{E}_{g,m} = \left[Cand(f)_{g,m}, \; \mathrm{R}_{g,m} \right]$ is a list related to g and m such that its first element is the list of candidates for f and its second element are their associated candidates for b_1 and b_2. All those elements $\mathrm{E}_{g,m}$ generate a set $\mathcal{W}$ which will represent all the candidates for $(f, \{b_1, b_2\})$.

Let us assume that we know candidates for f and the lower parameters of $F_0 = {}_2F_2$. To have candidates for the upper parameters, we proceed as follows:

1. We take a candidate g for the change of variable parameter f and its associated set B_g of candidates for the lower parameters b_1 and b_2 of $F_0 = {}_2F_2$.
2. We take an candidate $\{b_1, b_2\}$ in B_g.
3. Let $p \in \mathrm{S}_{\mathrm{irr}}(\mathrm{L})$ with m_p its multiplicity order as a pole of g.
   (a) We call C_p^1 the constant exponent difference of L at p: $\mathrm{C}_p^1 = m_p(a_2 - a_1)$ modulo $\mathbb{Z}$. We call C_p^2 and C_p^3 the constant part of the other exponent differences: $\mathrm{C}_p^2 = m_p\left[b_1 + b_2 - (2a_1 + a_2)\right]$ modulo $\mathbb{Z}$ and $\mathrm{C}_p^3 = m_p\left[b_1 + b_2 - (a_1 + 2a_2)\right]$ modulo $\mathbb{Z}$.
   (b) We compute the sets N_p^1, N_p^2 and N_p^3 of candidates modulo $\mathbb{Z}$ for $\pm(a_2 - a_1)$, $\pm(2a_1 + a_2)$ and $\pm(a_1 + 2a_2)$, respectively:

$$\mathrm{N}_p^1 = \left\{ \frac{j \pm \mathrm{C}_p^1}{m_p} \middle| \, j \in \mathbb{Z}, |j| \leq m_p - 1 \right\}, \quad \mathrm{N}_p^2 = \left\{ b_1 + b_2 - \frac{j \pm \mathrm{C}_p^2}{m_p} \middle| \, j \in \mathbb{Z}, |j| \leq m_p - 1 \right\}$$

$$\text{and} \quad \mathrm{N}_p^3 = \left\{ b_1 + b_2 - \frac{j \pm \mathrm{C}_p^3}{m_p} \middle| \, j \in \mathbb{Z}, |j| \leq m_p - 1 \right\}.$$

(c) We compute $N_p^4 = \left\{ \frac{i+j+s}{3} \middle| i \in N_p^2 \text{ and } j \in N_p^3, |s| \le 2 \right\}$. N_p^4 is the set of candidates modulo $\mathbb{Z}$ for $\pm(a_1 + a_2)$. Hence, $N_p^5 = \left\{ \frac{i+j+s}{2} \middle| i \in N_p^4 \text{ and } j \in N_p^1, |s| \le 1 \right\}$ is the set of candidates modulo $\mathbb{Z}$ for $\pm a_2$ and $N_p = \left\{ (i-j, i) | \, i \in N_p^5 \text{ and } j \in N_p^1 \right\}$ is the set of candidates modulo $\mathbb{Z} \times \mathbb{Z}$ for (a_1, a_2). We replace in N_p all the rational parts of the elements in the pair by their representants modulo $\mathbb{Z}$ in $[0, 1]$.

4. $E = \bigcap_{p \in S_{irr}(L)} N_p$ is the set of candidates for (a_1, a_2). If $E = \emptyset$, the value of b_1 or b_2 does not lead to a solution of our differential operator L. So we take another candidate for the lower parameters $\{b_1, b_2\}$ of in B_g and we repeat the process. If no candidate in B_g leads to a solution, then g is not a good candidate for f. Then we will return and take another candidate g for f and its associated set B_g of candidates for the lower parameters b_1 and b_2.

We have implemented in Maple an algorithm called `findcandai2F2` to find candidates modulo $\mathbb{Z}$ for the upper parameters $\{a_1, a_2\}$ of $F_0 = {}_2F_2$, if they exist.

To find at same time candidates for f, lower and upper parameters of $F_0 = {}_2F_2$, if they exist, we have implemented in Maple an algorithm called `find2F2Int`.

6.3. Some zeroes of f are known $\left(S_{reg}(L) \neq \emptyset\right)$

We will distinguish here two cases:

- the logarithmic case: when $b_1 \in \mathbb{Z}$ or $b_2 \in \mathbb{Z}$ or $b_1 - b_2 \in \mathbb{Z}$ or $(2b_1,\ 2b_2 \in \mathbb{Z}$ with $b1 \cdot b2 < 0)$,
- the non-logarithmic case: when we are not in the logarithmic case.

The same name ”logarithmic case” appeared in the cases given by [11, Lemma 3.9] and [43, Theorem 9].

Logarithmic Case

Here $S_{reg}(L)$ represents all the zeroes of f. So $S_{reg}(L) \neq \emptyset$ in this case

Let g be a candidate for the polar part of f. If there exists a constant c such that all the elements in $S_{reg}(L)$ are zeroes of $\overline{g} = g + c$ then $\overline{g}$ is a candidate for f. If not, g is not a good candidate for the polar part of f. We have implemented in Maple an algorithm called `Candichangvar2F2` to find $\overline{g} = g + c$. All those elements g for which this is true generate a set $\overline{F}$. If $\overline{F} = \emptyset$ then L has not $F_0 = {}_2F_2$ type solutions.

Let us assume that we know a candidate $\overline{g}$ for the change of variable parameter f and consider for $\alpha, \beta \in k$ the following assertion:

$$A(\{\alpha, \beta\}) : \quad \alpha \in \mathbb{Z} \ \text{ or } \ \beta \in \mathbb{Z} \ \text{ or } \ \alpha - \beta \in \mathbb{Z} \ \text{ or } \ (2\alpha,\ 2\beta \in \mathbb{Z} \text{ with } \alpha \cdot \beta < 0)\,. \qquad (15)$$

To find the candidates $\{b_1, b_2\}$ for the lower parameters of $F_0 = {}_2F_2$ related to $\overline{g}$, we proceed as follows:

1. For $p \in \mathrm{S}_{\mathrm{reg}}(\mathrm{L})$ with m_p its multiplicity order as a zero of $\overline{g}$, the three exponent differences modulo $\mathbb{Z}$ of L at p are: $\Delta_1(\mathrm{L}, p) = m_p(1 - b_1)$, $\Delta_2(\mathrm{L}, p) = m_p(1 - b_2)$ and $\Delta_3(\mathrm{L}, p) = m_p(b_1 - b_2)$. We compute the sets

$$\mathrm{N}_p^1 = \left\{ \frac{\pm\Delta_1(\mathrm{L}, p) + j}{m_p} \middle| j \in \mathbb{Z}, |j| \leq m_p - 1 \right\}, \quad \mathrm{N}_p^2 = \left\{ \frac{\pm\Delta_2(\mathrm{L}, p) + j}{m_p} \middle| j \in \mathbb{Z}, |j| \leq m_p - 1 \right\},$$
$$\text{and} \quad \mathrm{N}_p^3 = \left\{ \frac{\pm\Delta_3(\mathrm{L}, p) + j}{m_p} \middle| j \in \mathbb{Z}, |j| \leq m_p - 1 \right\}.$$

   (a) For $i \in \{1, 2, 3\}$, we take N_p^i as the set of candidates for $b_1 - b_2$. Then the other N_p^j, $j \in \{1, 2, 3\} \setminus \{i\}$ will be the sets of candidates for $1 - b_1$ and $1 - b_2$, respectively, since by permuting the values of b_1 and b_2 our function F_0 doesn't change. Hence we can get the sets N_p^{j1} and N_p^{j2} of candidates for b_1 and b_2. Using Remark 16, we replace in N_p^i all the rational parts of the elements by their representant modulo $\mathbb{Z}$ in $[-1, 1]$, and in N_p^{j1} and N_p^{j2} all the rational parts of the elements by their representant modulo $\mathbb{Z}$ in $]0, 1]$. We define

$$\mathrm{E}_p^i = \left\{ \{b_1, b_2\} | \, b_1 \in \mathrm{N}_p^{j1}, b_2 \in \mathrm{N}_p^{j2}, A(\{b_1, b_2\}) \text{ is true, and } b_1 - b_2 \in \mathrm{N}_p^i \text{ or } b_2 - b_1 \in \mathrm{N}_p^i \right\}.$$

   (b) $\mathrm{E}_p = \bigcup_{i\in\{1,2,3\}} \mathrm{E}_p^i$ is the set of candidates for $\{b_1, b_2\}$ associated to p. If $\mathrm{E}_p = \emptyset$ then b_1 and b_2 don't exist. That means $\overline{g}$ is not a good candidate for f.
2. $\mathrm{E} = \bigcap_{p\in \mathrm{S}_{\mathrm{reg}}(\mathrm{L})} \mathrm{E}_p$ is the set of candidates for $\{b_1, b_2\}$. If $\mathrm{E} = \emptyset$ then b_1 and b_2 don't exist. That means $\overline{g}$ is not a good candidate for f.

We have implemented in Maple an algorithm called `findbi2F2ln` to get candidates for $[b_1, b_2]$.

By knowing candidates for f and the lower parameters of $F_0 = {}_2F_2$, we compute the set of candidates for the upper parameters of $F_0 = {}_2F_2$ using the method in the case $\mathrm{S}_{\mathrm{reg}}(\mathrm{L}) = \emptyset$ above.

To find at same time candidates for f, lower and upper parameters of $F_0 = {}_2F_2$, if they exist, we have implemented in Maple an algorithm called `find2F2ln`.

Non-Logarithmic Case

In this case we know at least one zero of f ($\mathrm{S}_{\mathrm{reg}}(\mathrm{L}) \neq \emptyset$), and we have the following conditions on the lower parameter(s) of F_0: $b_1 \notin \mathbb{Z}$ and $b_2 \notin \mathbb{Z}$ and $b_1 - b_2 \notin \mathbb{Z}$ and ($2b_1$ or $2b_2 \notin \mathbb{Z}$ or $b1 \cdot b2 < 0$).

To find candidates for $(\{a_1, a_2\}, [b_1, b_2], f)$ we proceed as in the logarithmic case, but with one modification and new considerations in the computations:

1. we replace the assertion $A(\{\alpha,\beta\})$ (see (15)) by

$$A(\{\alpha,\beta\}) : \quad \alpha \notin \mathbb{Z} \text{ and } \beta \notin \mathbb{Z} \text{ and } \alpha - \beta \notin \mathbb{Z} \text{ and } (2\alpha \text{ or } 2\beta \notin \mathbb{Z} \text{ with } \alpha \cdot \beta \geq 0).$$

2. for L, some elements of $\mathrm{S}_{\mathrm{reg}}(\mathrm{L})$ can have exponent differences not in $\mathbb{Q}$ or k,
3. the constant parts of the exponent differences of L at some elements of $\mathrm{S}_{\mathrm{irr}}(\mathrm{L})$ can be not in $\mathbb{Q}$ or k.

So, we need sometimes to work in an extension field of $\mathbb{Q}$, and in some cases in an extension field of k ($\mathbb{Q} \subset k$). Hence, we will take into consideration all the extension fields of $\mathbb{Q}$ or k coming from the exponent differences of L at every element of $S_{reg}(L)$ and $S_{irr}(L)$. For $S_{irr}(L)$, we need just to consider the extension field of $\mathbb{Q}$ or k coming from the constant part of the exponent differences of L at all its elements. This case combines similar cases as the rational, basefield and irrational cases from the Bessel square type solutions (see [28, Section 4.1]).

For this case, we have implemented in Maple algorithms called `find2F2RatIrr` to find at same time candidates for f, lower and upper parameter(s) of $F_0 = {}_2F_2$, if they exist.

7. Change of variable parameter f and the parameter(s) of F_0 when $F_0 \in \left\{ {}_1F_2,\ {}_0F_2, \check{B}_\nu^2 \right\}$

Taking $f \in k(x)$ means we can assume $f = \dfrac{A}{B}$ with $A, B \in k[x]$, B monic and $\gcd(A, B) = 1$.

The problem now is how to get information about A and B from L (our input operator).

Since we know how to find all the poles of f (using Corollary 30) which are elements of the set $S_{irr}(L)$, and also their multiplicity orders by using their exponent differences, we can start in order to find $f = \dfrac{A}{B}$, to first find the truncated series for f at all its poles by raising to power n_e the truncated series for f^{1/n_e} at all those poles, where n_e is the ramification index of $L_0 \in \{L_{12}, L_{02}, L_{\check{B}_\nu^2}\}$.

7.1. *Truncated series for f*

Let $L_0 \in \left\{ L_{12}, L_{02}, L_{\check{B}_\nu^2} \right\}$ with $n_e \in \{2, 3\}$ its ramification index at ∞. We know all the poles of f, but the exponent differences give us just information about the polar part of f^{1/n_e}.

Lemma 33. *Let p be a point in k. If $f \in k(x)$ and $f^{1/2} = \sum_i a_i t_p^i$, where $i \in \frac{1}{2}\mathbb{Z}$, $a_i \in k$ and t_p is the local parameter at $x = p$, then the set $\{i \,|a_i \neq 0\}$ is either a subset of $\mathbb{Z}$ or a subset of $\frac{1}{2} + \mathbb{Z}$.*

Proof. Similar to the proof of Lemma 9 in [43]. □

The following definition and remark about the n-term truncated series for f^{1/n_e} are the same as [43, Definition 28] and [43, Remark 12], respectively.

Definition 34. *Let $p \in k$ and $f^{1/n_e} = \sum_{i=N}^{\infty} a_i t_p^{i/n_e}$, $a_i \in k$ and $a_N \neq 0$. We say that we have an n-term truncated power series for f^{1/n_e} if the coefficients of $t_p^{N/2}, \ldots, t_p^{(N+n-1)/n_e}$ are known.*

Remark 35. *If an n-term truncated series for f^{1/n_e} is known, then we can compute an n-term truncated series for f.*

Let us have a look at the exponent differences of L at a pole p of f in section 5. They can be brought in the form

$$\alpha + \sum_{j=-m_p}^{-1} a_j t_p^{j/n_e},\ \ \beta + \sum_{j=-m_p}^{-1} b_j t_p^{j/n_e},\ \ \gamma + \sum_{j=-m_p}^{-1} c_j t_p^{j/n_e}$$

with $\alpha, \beta, \gamma \in \overline{k}$ ($\overline{k}$ is the algebraic closure of k) and

1. for $F_0 = \check{B}_\nu^2$: $a_j = \pm j\overline{f}_{1,j+m_p}$, $b_j = \pm j\overline{f}_{1,j+m_p}$ and $c_j = \pm 2j\overline{f}_{1,j+m_p}$. That means the truncated series for $f^{1/2}$, modulo a factor -1, at a pole p of f is

$$\sum_{j=-m_p}^{-1} \frac{a_j}{j} t_p^{j/2} = \sum_{j=-m_p}^{-1} \frac{b_j}{j} t_p^{j/2} = \sum_{j=-m_p}^{-1} \frac{c_j}{2j} t_p^{j/2}$$

2. for $F_0 = {}_1F_2$: $a_j = \pm j\overline{f}_{1,j+m_p}$, $b_j = \pm j\overline{f}_{1,j+m_p}$ and $c_j = \pm 2j\overline{f}_{1,j+m_p}$. That means the truncated series for $f^{1/2}$, modulo a factor -1, at a pole p of f is

$$\sum_{j=-m_p}^{-1} \frac{a_j}{j} t_p^{j/2} = \sum_{j=-m_p}^{-1} \frac{b_j}{j} t_p^{j/2} = \sum_{j=-m_p}^{-1} \frac{c_j}{2j} t_p^{j/2}$$

3. for $F_0 = {}_0F_2$: $a_j = \pm j\left(\varepsilon_2^{-1} - \varepsilon_1^{-1}\right)\overline{f}_{1,j+m_p}$, $b_j = \pm j\left(\varepsilon_3^{-1} - \varepsilon_1^{-1}\right)\overline{f}_{1,j+m_p}$ and $c_j = \pm j\left(\varepsilon_3^{-1} - \varepsilon_2^{-1}\right)\overline{f}_{1,j+m_p}$ where $\varepsilon_1, \varepsilon_2$ and ε_3 are the three distinct solutions of $X^3 + 1 = 0$. That means the truncated series for $f^{1/2}$, modulo a factor -1, at a pole p of f is

$$\sum_{j=-m_p}^{-1} \frac{a_j}{j\left(\varepsilon_2^{-1} - \varepsilon_1^{-1}\right)} t_p^{j/3} = \sum_{j=-m_p}^{-1} \frac{b_j}{j\left(\varepsilon_3^{-1} - \varepsilon_1^{-1}\right)} t_p^{j/3} = \sum_{j=-m_p}^{-1} \frac{c_j}{j\left(\varepsilon_3^{-1} - \varepsilon_2^{-1}\right)} t_p^{j/3}.$$

Let $p \in \mathrm{S}_{\mathrm{irr}}(\mathrm{L})$ and $f^{1/n_e} = \sum_{i=-m_p}^{\infty} a_i t_p^{i/n_e}$, $a_i \in k$ with m_p the multiplicity order of p as a zero of f. By what we have seen above and using Lemma 33, we have a truncated series for f^{1/n_e} with $\lceil m_p/n_e \rceil$ terms. We raise it to power n_e in order to obtain a truncated series of f at p. But this truncated series for f has $\lceil m_p/n_e \rceil$ terms (see Remark 35) which is only $1/n_e$ (rounded up) of the polar part of f.

We have implemented in Maple algorithms called `SirrBesSqRootinfo1` when $\mathrm{L}_0 = \mathrm{L}_{\check{B}_\nu^2}$, `Sirr1F2info1` when $\mathrm{L}_0 = \mathrm{L}_{12}$ and `Sirr0F2info1` when $\mathrm{L}_0 = \mathrm{L}_{02}$, to find all those truncated series for f related to the elements of $\mathrm{S}_{\mathrm{irr}}(\mathrm{L})$.

7.2. *How to compute the denominator* B *of* f

We retrieve B from $\mathrm{S}_{\mathrm{irr}}(\mathrm{L})$ as follows:

Lemma 36. *Consider the situation* (7) *and* $f = \mathrm{A}/\mathrm{B}$ *with* $\mathrm{A}, \mathrm{B} \in k[x]$, B *monic and gcd*$(\mathrm{A}, \mathrm{B}) = 1$. *Then*

$$\mathrm{B} = \prod_{p \in \mathrm{S}_{irr}(\mathrm{L})} (x-p)^{m_p} = \prod_{p \in \mathrm{S}_{irr}(\mathrm{L})} t_p^{m_p} = \prod_{P_s \in \overline{\mathrm{S}_{irr}(\mathrm{L})} \setminus \{1\}} P_s^{\ m_s} \tag{16}$$

where $\forall p \in \mathrm{S}_{irr}(\mathrm{L})$, $f = \sum_{i=-m_p}^{\infty} a_i t_p^i$ *(m_p is the multiplicity order of p as a pole of f).*

Proof. We just use, by Corollary 30, the fact that $\mathrm{S}_{\mathrm{irr}}(\mathrm{L})$ represents all the poles of f. □

The way to compute this denominator B is similar to the one in [43, Lemma 10].

Our implemented algorithms `SirrBesSqRootinfo1` when $L_0 = L_{\breve{B}_\nu^2}$, `Sirr1F2info1` when $L_0 = L_{12}$ and `Sirr0F2info1` when $L_0 = L_{02}$, that we have implemented in Maple compute this denominator B of f.

We know how to find B (denominator of f). The next problem is now to find the numerator A of f. In order to solve this problem, we will need to find a bound for the degree of A.

7.3. How to get a bound for the degree of the numerator A *of* f

The following remarks and lemma about the bound for the degree of the numerator A of f, denoted by d_A, are the same as [43, Remark 13] and [43, Lemma 11], respectively.

Remarks 37. *(i) If* $\infty \in S_{irr}(L)$ *we will have* $deg(A) > deg(B)$.

(ii) If $\infty \in S_{reg}(L)$ *we will have* $deg(A) < deg(B)$.

(iii) If ∞ *is an apparent singularity of* L, ∞ *can be a zero of* f *but never a pole of* f. *So* $deg(A) \leq deg(B)$ *(if* ∞ *is not a zero of* f *then* $deg(A) = deg(B)$*).*

The following lemma gives us a bound for the degree of A.

Lemma 38. *Let*

$$d_A = \begin{cases} deg(B) + m_\infty & \text{if } \infty \in S_{irr}(L), \\ deg(B) & \text{otherwise} \end{cases} \tag{17}$$

where m_∞ *is the multiplicity order of* ∞ *as a pole of* f *(*$\infty \in S_{irr}(L)$*).*

(i) If $\infty \in S_{irr}(L)$ *then* $deg(A) = d_A$,

(ii) if $\infty \in S_{reg}(L)$ *then* $deg(A) < d_A$,

(iii) otherwise $deg(A) \leq d_A$ *(*$\infty \notin S_{irr}(L) \cup S_{reg}(L)$*).*

Proof. This follows from Remarks 37. □

Hence, d_A given by (17) is a bound for the degree of the numerator A of f.

Corollary 39. *For* $s \in S_{irr}(L)$, *let* m_s *be its multiplicity order as a pole of* f. *Then* $d_A = \sum_{s \in S_{irr}(L)} m_s$.

Proof. By Lemma 38 we have

$$d_A = \begin{cases} \deg(B) + m_\infty & \text{if } \infty \in S_{irr}(L), \\ \deg(B) & \text{otherwise} \end{cases} \Rightarrow d_A = \begin{cases} \sum\limits_{s \in S_{irr}(L) \setminus \{\infty\}} m_s + m_\infty & \text{if } \infty \in S_{irr}(L), \\ \sum\limits_{s \in S_{irr}(L)} m_s & \text{otherwise .} \end{cases}$$

Hence $d_A = \sum\limits_{s \in S_{irr}(L)} m_s$. □

Our implemented algorithms `SirrBesSqRootinfo1` when $L_0 = L_{\breve{B}_\nu^2}$, `Sirr1F2info1` when $L_0 = L_{12}$ and `Sirr0F2info1` when $L_0 = L_{02}$ compute d_A.

Now we know a bound d_A for the degree of A. The next step will be to find its coefficients. In order to achieve this, we will first see how to get linear equations for those coefficients of A.

7.4. How to get linear equations for the coefficients of the numerator A *of* f

Since we know a bound for the degree of A, denoted by d_{A}, we can write $\mathrm{A} = \sum_{i=0}^{d_{\mathrm{A}}} a_i x^i$, with $a_i \in k$. So we have $d_{\mathrm{A}} + 1$ unknowns: $a_0, \ldots, a_{d_{\mathrm{A}}}$. The equations for those coefficients of A will come from the set $\mathrm{S}_{\mathrm{irr}}(\mathrm{L}) \cup \mathrm{S}_{\mathrm{reg}}(\mathrm{L})$. For a point $s \in \mathrm{S}_{\mathrm{irr}}(\mathrm{L}) \cup \mathrm{S}_{\mathrm{reg}}(\mathrm{L})$, we will deal with two cases: when $s \in k$, and when $s \in \overline{k}$ but $s \notin k$ ($\overline{k}$ is the algebraic closure of k). That is done by taking instead of $x - s$ the minimal polynomial P_s of s over k and working on $k(s)$ instead of k.

Lemma 40. *Let us assume* $\mathrm{S}_{reg}(\mathrm{L}) \neq \emptyset$. *Then the remainder of the Euclidean division of* A *by* $\prod_{P_s \in \overline{\mathrm{S}_{reg}(\mathrm{L})}} P_s$ *will give us* $\sum_{P_s \in \overline{\mathrm{S}_{reg}(\mathrm{L})}} deg(P_s)$ *linear equations for the coefficients of* A.

Proof. For $s \in \mathrm{S}_{\mathrm{reg}}(\mathrm{L})$, let m_s be its multiplicity order as a zero of f. We can write A in the form $\mathrm{A} = c \prod_{P_s \in \overline{\mathrm{S}_{\mathrm{reg}}(\mathrm{L})}} P_s^{m_s}$ with $c \in k$. Since $m_s \geq 1 \ \ \forall s \in \mathrm{S}_{\mathrm{reg}}(\mathrm{L})$, let R be the remainder of A divided by $\prod_{P_s \in \overline{\mathrm{S}_{\mathrm{reg}}(\mathrm{L})}} P_s$. Hence R is a polynomial of degree $-1 + \sum_{P_s \in \overline{\mathrm{S}_{\mathrm{reg}}(\mathrm{L})}} \deg(P_s)$. The fact that $m_s \in \mathbb{N} \setminus \{0\}$ $\forall s \in \mathrm{S}_{\mathrm{reg}}(\mathrm{L})$ implies $\mathrm{R} = 0$ and therefore we will have $\sum_{P_s \in \overline{\mathrm{S}_{\mathrm{reg}}(\mathrm{L})}} \deg(P_s)$ linear equations for the coefficients of A. □

Remark 41. *By the proof of Lemma 40,* $\infty \in \mathrm{S}_{reg}(\mathrm{L})$ *gives us one equation:* $a_{d_{\mathrm{A}}} = 0$. *That means we have* $deg(\mathrm{A}) < d_{\mathrm{A}}$.

Lemma 42. *For* $s \in \mathrm{S}_{irr}(\mathrm{L})$ *let* m_s *be its multiplicity order as a zero of* f, g_s *the polar part of* f *at* s *and* $\overline{g}_s$ *the* $\lceil m_s/n_e \rceil$*-truncated series of* f *at* s. *Let*

$$u(x) = \left[f - \sum_{P_s \in \overline{\mathrm{S}_{irr}(\mathrm{L})}} \overline{g}_s \right] \cdot \prod_{P_s \in \overline{\mathrm{S}_{irr}(\mathrm{L})} \setminus \{1\}} P_s^{\lfloor m_s * (n_e - 1)/n_e \rfloor}.$$

Then $u(x) \in k[x]$ *and the remainder of the Euclidean division of numer*$(u(x))$ *by denom*$(u(x))$ *will give us* $\sum_{P_s \in \overline{\mathrm{S}_{irr}(\mathrm{L})} \setminus \{1\}} deg(P_s) \cdot \lceil m_s/n_e \rceil$ *linear equations for the coefficients of* A.

Furthermore, if $\infty \in \mathrm{S}_{irr}(\mathrm{L})$, *the quotient of the Euclidean division of numer*$(u(x))$ *by denom*$(u(x))$ *will give us, in addition,* $\lceil m_\infty/n_e \rceil$ *linear equations.*

Proof. The proof can be found in the first author's PhD thesis [28]. □

Remarks 43. 1. *By Lemma 40 and Lemma 42, we have*

$$\sum_{P_s \in \overline{\mathrm{S}_{reg}(\mathrm{L})}} deg(P_s) + \sum_{P_s \in \overline{\mathrm{S}_{irr}(\mathrm{L})}} deg(P_s) \cdot \lceil m_s/n_e \rceil \ \ \textit{linear equations for the coefficients of } \mathrm{A}$$

where m_s *is the multiplicity order of* $s \in \mathrm{S}_{irr}(\mathrm{L})$ *as a pole of* f. *That means we have*

$$\sum_{s \in \mathrm{S}_{reg}(\mathrm{L})} 1 + \sum_{s \in \mathrm{S}_{irr}(\mathrm{L})} \lceil m_s/n_e \rceil \ \ \textit{linear equations.}$$

2. *Since* $\mathrm{S}_{reg}(\mathrm{L})$ *can be an empty set but* $\mathrm{S}_{irr}(\mathrm{L})$ *not, we have at least*

$$\sum_{P_s \in \mathrm{S}_{irr}(\mathrm{L})} deg(P_s) \cdot \lceil m_s/n_e \rceil \text{ linear equations for the coefficients of } \mathrm{A},$$

that means $\sum_{s \in \mathrm{S}_{irr}(\mathrm{L})} \lceil m_s/n_e \rceil$ *linear equations.*

Lemma 44. *The number of linear equations for the coefficients of* A *is greater or equal to* $\frac{1}{n_e} d_{\mathrm{A}} + \sum_{p \in \mathrm{S}_{reg}(\mathrm{L})} 1$.

Proof. By Corollary 39 we have

$$d_{\mathrm{A}} = \sum_{s \in \mathrm{S}_{\mathrm{irr}}(\mathrm{L})} m_s \Rightarrow \frac{1}{n_e} d_{\mathrm{A}} = \sum_{s \in \mathrm{S}_{\mathrm{irr}}(\mathrm{L})} \frac{m_s}{n_e} \leq \sum_{s \in \mathrm{S}_{\mathrm{irr}}(\mathrm{L})} \left\lceil \frac{m_s}{n_e} \right\rceil.$$

Hence, by the part 1. of Remarks 43, the number of linear equations for the coefficients of A is greater or equal to $\frac{1}{n_e} d_{\mathrm{A}} + \sum_{p \in \mathrm{S}_{\mathrm{reg}}(\mathrm{L})} 1$. □

Our implemented algorithms `SirrBesSqRootinfo1` when $\mathrm{L}_0 = \mathrm{L}_{\check{B}^2_\nu}$, `Sirr1F2info1` when $\mathrm{L}_0 = \mathrm{L}_{12}$ and `Sirr0F2info1` when $\mathrm{L}_0 = \mathrm{L}_{02}$ compute d_{A} and the number of linear equations satisfied by the coefficients of A.

We have now the linear equations satisfied by the coefficients of the numerator A of f. Let n_b be the number of those equations. If this number is greater than the degree of A then we can solve those equations to get the coefficients of A. If this is not the case then later we will discuss further methods using the zeroes of f (elements of $\mathrm{S}_{\mathrm{reg}}(\mathrm{L})$) and the exponent differences of L at those zeroes. At the same time, we can see how to find candidates for the parameter(s) of $F_0 \in \left\{ {}_1F_2,\ {}_0F_2, \check{\mathrm{B}}^2_\nu \right\}$.

7.5. *How to compute the numerator* A *of* f *and the parameter(s) of* $F_0 \in \left\{ {}_1F_2,\ {}_0F_2, \check{\mathrm{B}}^2_\nu \right\}$

By comparing d_{A} and the number n_b of linear equations satisfied by the coefficients of A, we have:

1. if $n > d_{\mathrm{A}}$ then we can solve those equations and get A: that is the "Easy case". This name appeared similarly in [43, Section 4.2].
2. if $n \leq d_{\mathrm{A}}$, we have to distinguish between two cases:
   (a) when all the zeroes of f are known: "Logarithmic case"
      i. "Logarithmic case"
         * $\nu \in \mathbb{Z}$ for $F_0 = \check{\mathrm{B}}^2_\nu$
         * $b_1 \in \mathbb{Z}$ or $b_2 \in \mathbb{Z}$ or $b_1 - b_2 \in \mathbb{Z}$ or $(2b_1,\ 2b_2 \in \mathbb{Z}$ with $b1 \cdot b2 < 0)$ for $F_0 \in \{ {}_1F_2,\ {}_0F_2 \}$.
      ii. "Irrational case"
         * $\nu \in k \setminus \mathbb{Q}$ for $F_0 = \check{\mathrm{B}}^2_\nu$

* we are not in the logarithmic case and b_1 or b_2 is not a rational number, for $F_0 \in \{{}_1F_2,\ {}_0F_2\}$.

The name "Irrational case" appeared in the similar way in [11, Lemma 3.9] and [43, Section 4.4] related to F_0 which is a Bessel function and a composition of a Bessel function with square roots, respectively.

(b) when we are not sure that we know all the zeroes of f: "Rational case"

i. $\nu \in \mathbb{Q} \setminus \mathbb{Z}$ for $F_0 = \check{\mathrm{B}}_\nu^2$

ii. we are not in the logarithmic case and $b_1, b_2 \in \mathbb{Q}$, for $F_0 \in \{{}_1F_2,\ {}_0F_2\}$.

Also the name "Rational case" appeared similarly in [11, Lemma 3.9] and [43, Section 4.4].

Normally, we just have three cases: "Logarithmic case", "Irrational case" and "Rational case". The "Easy case" just helps us to find candidates for f easier. To find candidates for the parameter(s) of $F_0 \in \left\{{}_1F_2,\ {}_0F_2, \check{\mathrm{B}}_\nu^2\right\}$ we have to search whether we are in the "Logarithmic case", "Irrational case" or "Rational case" and use a particular technique, too.

7.5.1. *Easy Case*

Lemma 45. *In the Easy case,* $\mathrm{S}_{reg}(\mathrm{L}) \neq \emptyset$.

Proof. For $s \in \mathrm{S}_{\mathrm{irr}}(\mathrm{L})$, let m_s be its multiplicity order as a pole of f.

Since $\left\lceil \frac{m_s}{2} \right\rceil \leq m_s\ \forall s \in \mathrm{S}_{\mathrm{irr}}(\mathrm{L})$, we get $\sum\limits_{s\in\mathrm{S}_{\mathrm{irr}}(\mathrm{L})} \left\lceil \frac{m_s}{2} \right\rceil \leq \sum\limits_{s\in\mathrm{S}_{\mathrm{irr}}(\mathrm{L})} m_s$, and by Corollary 39 we have

$$\sum_{s\in\mathrm{S}_{\mathrm{irr}}(\mathrm{L})} \left\lceil \frac{m_s}{2} \right\rceil \leq d_{\mathrm{A}}\ . \tag{18}$$

If $\mathrm{S}_{\mathrm{reg}}(\mathrm{L}) = \emptyset$ then, using the part (a) of Remarks 43, we will have $n = \sum\limits_{s\in\mathrm{S}_{\mathrm{irr}}(\mathrm{L})} \left\lceil \frac{m_s}{2} \right\rceil$ linear equations for the coefficients of A. So by (18), $n \leq d_{\mathrm{A}}$, and that means we are not in the Easy case ($n > d_{\mathrm{A}}$). □

This case help us to find just the candidates for the parameter f. To find candidates for the parameter(s) of $F_0 \in \left\{{}_1F_2,\ {}_0F_2, \check{\mathrm{B}}_\nu^2\right\}$ we have to search whether we are in the "Logarithmic case", "Irrational case" or "Rational case" and use a particular technique (those cases will be treated and explained in the next sections).

We solve our linear equations for the coefficients of A using Lemma 40 and Lemma 42. If we find solutions then we already have A and therefore $f = \mathrm{A}/\mathrm{B}$ because we know B. If not, then we cannot find $F_0 \in \left\{{}_1F_2,\ {}_0F_2, \check{\mathrm{B}}_\nu^2\right\}$ type solutions for L using the Easy case.

We have implemented in Maple, for this case, algorithms called

1. `easyBesSqRoot` to find candidates for (ν, f) when $\mathrm{L}_0 = \mathrm{L}_{\check{B}_\nu^2}$, if they exist.
2. `easy1F2` to find candidates for $[\{a_1\}, \{b_1, b_2\}, f]$ when $\mathrm{L}_0 = \mathrm{L}_{12}$, if they exist.
3. `easy0F2` to find candidates for $[\{b_1, b_2\}, f]$ when $\mathrm{L}_0 = \mathrm{L}_{02}$, if they exist.

7.5.2. Logarithmic case

In this case we know all the zeroes of f: the set $\mathrm{S_{reg}(L)}$. They have logarithmic solutions. So we have now to do a combinatorial search to find their multiplicities as zeroes of A: try all possible combinations of multiplicities of zeroes of A. For a zero s of f, let m_s be its multiplicity order. We will have

$$\deg(\mathrm{A}) = \sum_{P_s \in \overline{\mathrm{S_{reg}(L)}}} \deg(P_s) \cdot m_s. \tag{19}$$

To find the list of combinations of multiplicities m_s of $s \in \mathrm{S_{reg}(L)}$ as zeroes of A we proceed as follows:

1. We take one element of $\mathrm{S_{reg}(L)}$ and call it s_0.
2. We put deg(A) in the form

$$\deg(\mathrm{A}) = Q_{s_0} \cdot \deg(P_{s_0}) + R_{s_0} \text{ with } Q_{s_0}, R_{s_0} \in \mathbb{N} \text{ and } 0 \leq R_{s_0} < \deg(P_{s_0}).$$

   By using (19) and the fact that $m_s \geq 1\ \forall s \in \mathrm{S_{reg}(L)}$, we have $m_{s_0} \in \{1, \ldots, Q_{s_0}\}$.
3. For $m_{s_0} \in \{1, \ldots, Q_{s_0}\}$, we repeat the process by considering $\overline{\mathrm{S_{reg}(L)}} = \overline{\mathrm{S_{reg}(L)}} \setminus \{P_{s_0}\}$ and $\deg(\mathrm{A}) = \deg(\mathrm{A}) - \deg(P_{s_0}) \cdot m_{s_0}$.
4. At the end, we will have the list of combinations of multiplicities m_s of $s \in \mathrm{S_{reg}(L)}$. The only unknown will be the leading coefficient of A. By Remarks 43 we have enough equations to find it.

We have implemented in Maple an algorithm called `SearchKnLog` to find candidates for A up to a multiplicative constant.

Once we get candidates for A, we have also candidates for f because $f = \mathrm{A/B}$ and we know B.

For $F_0 = \check{\mathrm{B}}_\nu^2$, we use the same technique as in the logarithmic case with $f = g^2$ and $g \in k(x)$ (see [28]) to find candidates for ν related to any f.

For $F_0 \in \{{}_1F_2,\ {}_0F_2\}$, we use the same technique as in the logarithmic case for $F_0 = {}_2F_2$ to find the candidates for the lower parameter of F_0.

For $F_0 = {}_1F_2$, let us assume that we know candidates for f and the lower parameters of F_0. To get candidates for the upper parameters, we proceed as follows:

1. We take a candidate g for the change of variable parameter f and its associated set B_g of candidates for the lower parameters b_1 and b_2 of $F_0 = {}_1F_2$.
2. We take a candidate $\{b_1, b_2\}$ in B_g.
3. Let $p \in \mathrm{S_{irr}(L)}$ with m_p its multiplicity order as a pole of g.
   (a) We choose the two exponent differences of L at p which have the same, up to a factor -1 and modulo $\frac{1}{2}\mathbb{Z}$, non-constant part. Let C_p^1 and C_p^2 be the constant part of those exponent differences:

$$\begin{cases} \mathrm{C}_p^1 = \pm \dfrac{m_p}{2}\left(b_1 + b_2 - 3a_1 - \dfrac{1}{2}\right) \text{ modulo } \dfrac{1}{2}\mathbb{Z}, \\ \mathrm{C}_p^2 = \pm \dfrac{m_p}{2}\left(b_1 + b_2 - 3a_1 - \dfrac{1}{2}\right) \text{ modulo } \dfrac{1}{2}\mathbb{Z}. \end{cases}$$

(b) We compute the set N_p of candidates modulo $\mathbb{Z}$ for $\pm a_1$ associated to p

$$\mathrm{N}_p = \left\{ \left(b_1 + b_2 - \frac{1}{2} - \frac{j \pm 2\mathrm{C}_p^1}{m_p} \right) \cdot \frac{1}{3} \,\middle|\, j \in \mathbb{Z}, |j| \leq 3m_p - 1 \right\}.$$

We replace in N_p all the rational parts of elements by their representant modulo $\mathbb{Z}$ in $[0, 1]$.

4. $\mathrm{E} = \bigcap_{p \in \mathrm{S}_{irr}(\mathrm{L})} \mathrm{N}_p$ is the set of candidates modulo $\mathbb{Z}$ for $\pm a_1$. If $\mathrm{E} = \emptyset$ then a_1 doesn't exist. That means $\{b_1, b_2\}$ is not a good candidate for the lower parameters of ${}_1F_2$. We take another $\{b_1, b_2\}$ and repeat the process. If no candidate in B_g leads to a solution of our differential operator L, then g is not a good candidate for f. Then we will return and take another candidate g for f and its associated set B_g of candidates for the lower parameters b_1 and b_2.

We have implemented in Maple an algorithm called `findcandai1F2` to find candidates modulo $\mathbb{Z}$ for the upper parameter $\{a_1, a_2\}$ of $F_0 = {}_1F_2$, if they exist.

To find at the same time candidates for f and parameter(s) for $F_0 \in \left\{ {}_1F_2,\ {}_0F_2, \check{\mathrm{B}}_\nu^2 \right\}$, we have implemented in Maple algorithms called

1. `findBesSqRootln` which gives us candidates for (ν, f) when $\mathrm{L}_0 = \mathrm{L}_{\check{B}_\nu^2}$, if they exist.
2. `find1F2ln` which gives us candidates for $[\{a_1\}, \{b_1, b_2\}, f]$ when $\mathrm{L}_0 = \mathrm{L}_{12}$, if they exist.
3. `find0F2ln` which gives us candidates for $[\{b_1, b_2\}, f]$ when $\mathrm{L}_0 = \mathrm{L}_{02}$, if they exist.

7.5.3. Irrational case

The following lemma about the zeroes of f and their multiplicity order is the same as in [43, Lemma 16].

Lemma 46. *In the irrational case, we know all zeroes of f and their multiplicity as well.*

Proof. The proof can be found in the author's PhD thesis [28]. □

To find candidates for the numerator A of f we proceed as follows:

1. We find all the zeroes of f by computing the set $\mathrm{S}_{reg}(\mathrm{L})$.
2. For every zero of f, we find its multiplicity order by using the proof of Lemma 46 in [28]. So we have A up to a multiplicative constant.
3. Now there is only one unknown coefficient of A that we have to find: the leading coefficient. By Remarks 43 we have enough equations to find it.

Once we get candidates for A, we have also candidates for f because $f = \mathrm{A}/\mathrm{B}$ and we know B.

For $F_0 \in \{{}_1F_2,\ {}_0F_2\}$, we use the same technique as in the logarithmic case to find the candidates for the parameters for F_0.

For $F_0 = \check{\mathrm{B}}_\nu^2$, let us assume that we know candidates for f. Therefore, we have also all the multiplicity orders of the elements of $\mathrm{S}_{reg}(\mathrm{L})$ as zeroes of f. Using now the exponent differences of L at the elements of $\mathrm{S}_{reg}(\mathrm{L})$, we can get ν modulo an integer. The following definition, lemma and remark will be very useful in that case.

Definition 47. *Consider* (7) *and* $s \in S_{reg}(L)$. *Let* m_s *be the multiplicity order of* s *as a zero of* f. *We define*

$$N_s = \left\{ \frac{\Delta(L, s) + j}{m_s} \middle| 0 \leq j \leq m_s - 1 \right\} \tag{20}$$

$$\text{and } N = \left\{ \nu \in k \setminus \mathbb{Z} \middle| \forall s \in S_{reg}(L), \ \exists \alpha_s \in \mathbb{Z} : \quad \nu + \alpha_s \in N_s \right\} . \tag{21}$$

The name N_s appeared similarly in [11, Definition 3.10] and [43, Lemma 12], and also the name N in [11, Definition 3.10].

For every singularity $s \in S_{\text{reg}}(L)$ the Bessel parameter ν appears in N_s modulo some integer.

We will see now how to find the candidates for ν modulo $\mathbb{Z}$. This lemma can also be found in [11, Lemma 3.11].

Lemma 48. *Consider* (7) *and assume* $S_{reg}(L) \neq \emptyset$. *Then there exists some integer* $z \in \mathbb{Z}$ *such that* $\nu + z \in N$. *That means* N *is the set of candidates for* ν *modulo* $\mathbb{Z}$.

Since $\mathbb{C}(x)\check{B}_\nu^{2''} + \mathbb{C}(x)\check{B}_\nu^{2'} + \mathbb{C}(x)\check{B}_\nu^2$ is invariant under $\nu \mapsto \nu + 1$ and $\nu \mapsto -\nu$ we only need to find ν modulo an integer and a factor -1. So, in the non-logarithmic case, we can regard N as a set of candidates for ν. By the definition of N we have

$$N = \bigcap_{s \in S_{\text{reg}}(L)} [N_s \text{ modulo } \mathbb{Z} \text{ or a factor } -1]. \tag{22}$$

Remark 49. *The improved set of candidates for* ν *will be the new set* N *after the following modifications:*

1. *because of the invariance of* $\mathbb{C}(x)\check{B}_\nu^{2''} + \mathbb{C}(x)\check{B}_\nu^{2'} + \mathbb{C}(x)\check{B}_\nu^2$ *under* $\nu \mapsto -\nu$, *we remove in* N *all the negative elements.*
2. *because of the invariance of* $\mathbb{C}(x)\check{B}_\nu^{2''} + \mathbb{C}(x)\check{B}_\nu^{2'} + \mathbb{C}(x)\check{B}_\nu^2$ *under* $\nu \mapsto \nu + 1$, *we remove in* N *all the modulo an integer equivalents of its elements.*
3. *because we are not in the logarithmic case, we remove all the integers in* N.
4. *because* $L_{\check{B}_\nu^2}$ *has to be an irreducible operator, we also remove in* N *all the elements of the form* $\frac{1}{2} + z$ *with* $z \in \mathbb{Z}$.

Since $\nu \in k \setminus \mathbb{Q}$ in this case, we obtain the set of candidates for ν by removing in N all the elements which belong to $\mathbb{Q}$.

We have implemented in Maple, for this case, algorithms called

1. `findBesSqRootIrr` to find candidates for (ν, f) when $L_0 = L_{\check{B}_\nu^2}$, if they exist.
2. `find1F2Irr` to find candidates for $[\{a_1\}, \{b_1, b_2\}, f]$ when $L_0 = L_{12}$, if they exist.
3. `find0F2Irr` to find candidates for $[\{b_1, b_2\}, f]$ when $L_0 = L_{02}$, if they exist.

7.5.4. Rational case

Here we are not sure that we know all the zeroes of f. The denominator of an element in $\{\nu, 1-b_1, 1-b_2, b_1-b_2\}$ (denoted by d) will be very important in that case because d along with the multiplicities will determine whether the singularities disappear. Let s be a root of A with multiplicity m_s, then $\Delta(\mathrm{L}, s) \equiv \pm m_s a \bmod \mathbb{Z}$ with $a \in \{\nu, 1-b_1, 1-b_2, b_1-b_2\}$. If $d\,|m_s$, the change of variables $x \mapsto f$ can send s to a removable singularity, and then not all the roots of A are known (not all the roots of A are in $\mathrm{S_{reg}(L)}$). We can conclude that if a zero of A becomes a removable singularity, then m_s must be a multiple of d.

For $s \in \mathrm{S_{reg}(L)}$ the exponent differences of L at s in the case $F_0 \in \{{}_1F_2,\ {}_0F_2\}$ are the same as for $F_0 = \check{\mathrm{B}}^2_\nu$ by replacing ν by $1-b_1, 1-b_2$ and b_1-b_2. Since our technique to find f will be based on the set $\mathrm{S_{reg}(L)}$, we will just treat the case $F_0 = \check{\mathrm{B}}^2_\nu$. The other cases $F_0 \in \{{}_1F_2,\ {}_0F_2\}$ will be treated using a similar method. Therefore, in this section, we will assume $F_0 = \check{\mathrm{B}}^2_\nu$ and $\mathrm{L}_0 = \mathrm{L}_{\check{B}^2_\nu}$.

Definition 50. *Let us define for* $\mathrm{S}_{reg}(\mathrm{L})$ *the following sets:*

$$\begin{cases} \mathrm{NS}_{reg}(\mathrm{L}) = \{s \in \mathrm{S}_{reg}(\mathrm{L})\,|\exists i \in \{1,2,3\},\ \ \Delta_i(\mathrm{L}, s) \in \mathbb{Q}\setminus\mathbb{Z}\}, \\ \mathrm{RemS}_{reg}(\mathrm{L}) = \{s \in \mathrm{S}_{reg}(\mathrm{L})\,|\Delta_i(\mathrm{L}, s) \in \mathbb{Z}\ \ \forall i \in \{1,2,3\}\}, \\ \mathcal{A} = \ \textit{set of all the zeroes of } \mathrm{A}, \\ \overline{\mathrm{NS}_{reg}(\mathrm{L})} = \left\{P_s \in k[x] \mid P_s \textit{ is the minimal polynomial of } s \in \mathrm{NS}_{reg}(\mathrm{L}) \textit{ over } k\right\}, \\ \overline{\mathrm{RemS}_{reg}(\mathrm{L})} = \left\{P_s \in k[x] \mid P_s \textit{ is the minimal polynomial of } s \in \mathrm{RemS}_{reg}(\mathrm{L}) \textit{ over } k\right\}, \\ \overline{\mathcal{A}} = \{P_s \in k[x] \mid P_s \textit{ is the minimal polynomial of } s \in \mathcal{A} \textit{ over } k\}. \end{cases}$$

Remark 51. *Every time when we will take an element s of* $\mathrm{NS}_{reg}(\mathrm{L})$, $\Delta(\mathrm{L}, s)$ *will represent one of* $\Delta_i(\mathrm{L}, s)$, $i \in \{1,2,3\}$, *which belongs to* $\mathbb{Q}\setminus\mathbb{Z}$.

Remark 52. $\mathrm{RemS}_{reg}(\mathrm{L})$ *and* $\mathrm{NS}_{reg}(\mathrm{L})$ *are, respectively, the sets of removable and non-removable singularities of* L *in* $\mathrm{S}_{reg}(\mathrm{L})$, *and we have* $\mathrm{S}_{reg}(\mathrm{L}) = \mathrm{NS}_{reg}(\mathrm{L}) \bigcup \mathrm{RemS}_{reg}(\mathrm{L})$.

Lemma 53. A *can be written in the form* $\mathrm{A} = c\mathrm{A}_1\mathrm{A}_2^d$ *where*

$$\begin{cases} c \in k \ \ \textit{and } d = \textit{denom}(\nu), \\ \mathrm{A}_1 = \prod_{P_s \in \overline{\mathcal{A}}} P_s^{\beta_s}, \\ \mathrm{A}_2 = a \prod_{P_s \in \overline{\mathcal{A}}} P_s^{\alpha_s}, \ \ \textit{with } a \in k, \\ m_s = \ \textit{the multiplicity order of } s \in \mathcal{A} \textit{ as zero of } f, \\ m_s = \alpha_s \cdot d + \beta_s \ \ \textit{with } s \in \mathcal{A}, (\alpha_s,\beta_s) \in \mathbb{N}^2,\ 0 \le \beta_s < d. \end{cases}$$

Proof. For $s \in \mathcal{A}$, let m_s be the multiplicity order of s as a zero of f. We can put m_s in the form: $m_s = \alpha_s \cdot d + \beta_s$ with $(\alpha_s,\beta_s) \in \mathbb{N}^2$ and $0 \le \beta_s < d$.

$$\mathrm{A} = b\prod_{P_s\in\overline{\mathcal{A}}} P_s^{m_s} = b\prod_{P_s\in\overline{\mathcal{A}}} P_s^{\beta_s} \cdot \prod_{P_s\in\overline{\mathcal{A}}} P_s^{\alpha_s \cdot d} = c\prod_{P_s\in\overline{\mathcal{A}}} P_s^{\beta_s} \cdot \left[a\prod_{P_s\in\overline{\mathcal{A}}} P_s^{\alpha_s}\right]^d,$$

with $a, b, c \in k$ and $b = c \cdot a^d$. □

The form $\mathrm{A} = c\mathrm{A}_1\mathrm{A}_2^d$ is the same as in [43, Section 4.5] for the rational case.

Remark 54. *Since we are not in the logarithmic case and* L_{B^2} *is irreducible, we will have* $d = denom(\nu) > 2$.

Lemma 55. *Let* $d = denom(\nu)$.

1. *If* $\mathrm{NS}_{reg}(\mathrm{L}) = \emptyset$ *then* $\mathrm{A}_1 = 1$, $d \,|deg(\mathrm{A})$ *and* $deg(\mathrm{A}_2) = \dfrac{deg(\mathrm{A})}{d}$.
2. *If* $\mathrm{NS}_{reg}(\mathrm{L}) \neq \emptyset$ *then* $\mathrm{A}_1 = \prod_{P_s \in \overline{\mathrm{NS}_{reg}(\mathrm{L})}} P_s^{\beta_s}$ *where* $m_s = \alpha_s \cdot d + \beta_s$ *with* $(\alpha_s, \beta_s) \in \mathbb{N}^2$, $1 \leq \beta_s < d$ *for* $s \in \mathrm{NS}_{reg}(\mathrm{L})$ *and* m_s *its multiplicity order as zero of* f.

Proof. The proof can be found in [28]. □

By the previous lemma, we can conclude that

Corollary 56. A *can be written in the form* $\mathrm{A} = c\mathrm{A}_1\mathrm{A}_2^d$ *where*

$$\begin{cases} c \in k \;\; \text{and } d = denom(\nu), \\ \mathrm{A}_1 = \begin{cases} 1 & \text{if } \mathrm{NS}_{reg}(\mathrm{L}) = \emptyset, \\ \prod_{P_s \in \overline{\mathrm{NS}_{reg}(\mathrm{L})}} P_s^{\beta_s} & \text{otherwise}, \end{cases} \\ \mathrm{A}_2 = a \prod_{P_s \in \overline{\mathcal{A}}} P_s^{\alpha_s}, \;\; \text{with } a \in k, \\ m_s = \;\; \text{the multiplicity order of } s \in \mathcal{A} \text{ as zero of } f, \\ m_s = \begin{cases} \alpha_s \cdot d + \beta_s \;\; \text{with } (\alpha_s, \beta_s) \in \mathbb{N}^2, \; 1 \leq \beta_s < d & \text{if } \; s \in \mathrm{NS}_{reg}(\mathrm{L}), \\ \alpha_s \cdot d & \text{otherwise.} \end{cases} \end{cases}$$

Proof. Just use Lemma 53 and Lemma 55. □

Lemma 57. *Let* $s \in \mathcal{A}$ *and* $d_s = denom(\Delta(\mathrm{L}, s))$. *Then we have* $d_s \mid d$ *and therefore, if* $\mathrm{NS}_{reg}(\mathrm{L}) \neq \emptyset$, $\; l \mid d$ *where* $l = lcm_{s \in \mathrm{NS}_{reg}(\mathrm{L})}(d_s)$.

Proof. Let $\nu = \dfrac{a}{d}$, $\; a \in \mathbb{Z}$, $\gcd(a, d) = 1$. Let $s \in \mathcal{A}$ and m_s be its multiplicity as a zero of f. Then

$$\Delta(\mathrm{L}, s) = \; m_s \nu + z_s = m_s \left(\frac{a}{d}\right) + z_s = \; \frac{\sigma}{d} \;\; \text{with } z_s \in \mathbb{Z} \text{ and } \sigma = m_s \cdot a + d \cdot z_s \in \mathbb{Z}.$$

Therefore, $\mathrm{denom}(\Delta(\mathrm{L}, s)) \mid d$ i.e $\; d_s \mid d$.
Let us assume $\mathrm{NS}_{\mathrm{reg}}(\mathrm{L}) \neq \emptyset$. Since $\mathrm{NS}_{\mathrm{reg}}(\mathrm{L}) \subseteq \mathcal{A}$ then $\mathrm{denom}(\Delta(\mathrm{L}, s)) \mid d$ for all $s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$. That means $\mathrm{lcm}_{s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})}(d_s) \big| d$ i.e. $\; l \mid d$. □

Lemma 58. *Let us assume* $\mathrm{NS}_{reg}(\mathrm{L}) \neq \emptyset$ *and* $l = lcm_{s \in \mathrm{NS}_{reg}(\mathrm{L})}(d_s)$. *Then* $\forall s \in \mathrm{NS}_{reg}(\mathrm{L})$ *with* β_s *as in Corollary 56,* $\dfrac{d}{d_s} \Big| \beta_s$, *and therefore* $deg(\mathrm{A}_1) \geq \dfrac{d}{d_s} \cdot deg(P_s)$ *for all* $P_s \in \overline{\mathrm{NS}_{reg}(\mathrm{L})}$ *and* $deg(\mathrm{A}_1) \geq \dfrac{d}{l}$.

Proof. Let $\nu = \frac{a}{d}$, $a \in \mathbb{Z}$, $\gcd(a,d) = 1$. Let $s \in \mathrm{NS_{reg}(L)}$ and m_s its multiplicity as a zero of f such that $m_s = \alpha_s \cdot d + \beta_s$ with $(\alpha_s, \beta_s) \in \mathbb{N}^2$, $0 \leq \beta_s < d$. Then we have $\Delta(\mathrm{L}, s) = m_s \nu + z_s \notin \mathbb{Z}$, $z_s \in \mathbb{Z}$

$$\Rightarrow (\alpha_s \cdot d + \beta_s)\left(\frac{a}{d}\right) + z_s \notin \mathbb{Z} \Rightarrow \beta_s \frac{a}{d} \notin \mathbb{Z} \text{ and } d_s = \mathrm{denom}(\Delta(\mathrm{L}, s)) = \mathrm{denom}\left(\beta_s \frac{a}{d}\right)$$

since $\alpha_s a + z_s \in \mathbb{Z}$. By Lemma 57, we have $d_s \mid d$. Therefore $d = \varepsilon_s \cdot d_s$ with $\varepsilon_s \in \mathbb{N}$ i.e $\varepsilon_s = \frac{d}{d_s}$.

$$\begin{cases} \beta_s \frac{a}{d} \notin \mathbb{Z}, \quad d = \varepsilon_s \cdot d_s, \\ \gcd(a,d) = 1, \quad d_s = \mathrm{denom}\left(\beta_s \frac{a}{d}\right) \end{cases} \Rightarrow \begin{cases} \varepsilon_s \nmid a, \quad d_s \nmid a, \\ \varepsilon_s \mid \beta_s. \end{cases}$$

Hence $\frac{d}{d_s} = \varepsilon_s \mid \beta_s$. So $\forall s \in \mathrm{NS_{reg}(L)}$, $\left.\frac{d}{d_s}\right| \beta_s$. Let $l = \mathrm{lcm}_{s \in \mathrm{NS_{reg}(L)}}(d_s)$ and $\forall s \in \mathrm{NS_{reg}(L)}$, $\beta_s = \sigma_s \cdot \frac{d}{d_s}$ with $\sigma_s \in \mathbb{N}$. Using Corollary 56, $\beta_s \geq 1$ and therefore $\sigma_s \geq 1$. Again by Corollary 56, we have $\mathrm{A}_1 = \prod_{P_s \in \overline{\mathrm{NS_{reg}(L)}}} P_s^{\beta_s}$

$$\begin{aligned}
&\Rightarrow \deg(\mathrm{A}_1) = \sum_{P_s \in \overline{\mathrm{NS_{reg}(L)}}} \deg(P_s) \cdot \beta_s = \sum_{P_s \in \overline{\mathrm{NS_{reg}(L)}}} \deg(P_s) \cdot \sigma_s \cdot \frac{d}{d_s} \\
&\Rightarrow \deg(\mathrm{A}_1) \geq \deg(P_s) \cdot \sigma_s \cdot \frac{d}{d_s} \quad \forall s \in \mathrm{NS_{reg}(L)} \text{ since } \deg(P_s) \cdot \sigma_s \cdot \frac{d}{d_s} \geq 0 \quad \forall s \in \mathrm{NS_{reg}(L)} \\
&\Rightarrow \deg(\mathrm{A}_1) \geq \deg(P_s) \cdot \frac{d}{d_s} \quad \forall s \in \mathrm{NS_{reg}(L)} \text{ since } \sigma_s \geq 1 \\
&\Rightarrow \deg(\mathrm{A}_1) \geq \frac{d}{d_s} \quad \forall s \in \mathrm{NS_{reg}(L)} \text{ since } \deg(P_s) \geq 1 \\
&\Rightarrow \deg(\mathrm{A}_1) \geq \frac{d}{l} \quad \text{since } l \geq d_s \quad \forall s \in \mathrm{NS_{reg}(L)}.
\end{aligned}$$

□

Lemma 59. *Assume* $\mathrm{NS}_{reg}(\mathrm{L}) \neq \emptyset$ *and* $l = lcm_{s \in \mathrm{NS}_{reg}(\mathrm{L})}(d_s)$. *If* $l \geq deg(\mathrm{A})$ *then we have* $deg(\mathrm{A}) = deg(\mathrm{A}_1)$ *and* $deg(\mathrm{A}_2) = 0$. *That means* $\mathrm{A}_2 = \alpha$ *with* $\alpha \in k$.

Proof. Let $\mathrm{NS_{reg}(L)} \neq \emptyset$ and $l = \mathrm{lcm}_{s \in \mathrm{NS_{reg}(L)}}(d_s)$. Then $\deg(\mathrm{A}_1) \geq 1$ and by Lemma 58, $l \mid d$ where $d = \mathrm{denom}(\nu)$. Hence $d = \sigma \cdot l$ with $\sigma \in \mathbb{N} \setminus \{0\}$ since $d \neq 0$.

$$\deg(\mathrm{A}) = \deg(\mathrm{A}_1) + \deg(\mathrm{A}_2) \cdot d = \deg(\mathrm{A}_1) + \deg(\mathrm{A}_2) \cdot \sigma \cdot l. \tag{23}$$

Using (23) and the fact that $l \geq \deg(\mathrm{A})$ and $\deg(\mathrm{A}_1) \geq 1$ we will get $\deg(\mathrm{A}_2) = 0$. Therefore $\deg(\mathrm{A}) = \deg(\mathrm{A}_1)$ and $\mathrm{A}_2 = \alpha$ with $\alpha \in k$. □

How to find $d = denom(\nu)$

We can find a list of candidates for $d = \text{denom}(\nu)$.

Lemma 60. *Let us assume* $\text{NS}_{reg}(\text{L}) = \emptyset$*. Then the candidates for* $d = denom(\nu)$ *are*

$$\left\{ i \in \mathbb{N} \middle| 3 \leq i \leq \frac{d_{\text{A}}}{2} \ \textit{and} \ i \mid d_{\text{A}} \right\} \bigcup \{d_{\text{A}}\}.$$

Proof. Since $\text{NS}_{\text{reg}}(\text{L}) = \emptyset$, by Lemma 55, $d \mid \deg(\text{A})$ and $\deg(\text{A}) = \deg(\text{A}_2) \times d$.

1. If $\deg(\text{A}_2) = 0$, then ∞ is a zero of f, otherwise f will be a constant function and this contradicts the assumption that f should not be in k ($f^2 \in k(x) \backslash k$). Let m_∞ be the multiplicity order of ∞ as zero of f. We have $m_\infty = \deg(\text{B})$ and $\Delta(\text{L}, \infty) \in \mathbb{Z}$ since $\text{NS}_{\text{reg}}(\text{L}) = \emptyset$. So
$$d = \text{denom}(\nu) \mid m_\infty = \deg(\text{B}) \ \Rightarrow \ d \mid d_{\text{A}} = \deg(\text{B}) \ \text{by Lemma 38}$$
$$\Rightarrow \ d_{\text{A}} = d \cdot \sigma, \quad \sigma \in \mathbb{N} \setminus \{0\} \ \text{since} \ d_{\text{A}} = \deg(\text{B}) \neq 0 \ \Rightarrow \ d = d_{\text{A}} \ \text{or} \ d \leq \frac{d_{\text{A}}}{2}.$$
By Remark 54 we also know that $3 \leq d$. Hence d satisfies $\left(3 \leq d \leq \frac{d_{\text{A}}}{2} \ \text{and} \ d \mid d_{\text{A}}\right)$ or $d = d_{\text{A}}$.
2. If $\deg(\text{A}_2) = 1$, then $\deg(\text{A}) = \deg(\text{A}_2) \cdot d = d$. Since we don't know $\deg(\text{A})$ but just its bound d_{A}, d can be taken as d_{A}: $d = d_{\text{A}}$.
3. If $\deg(\text{A}_2) \geq 2$, then we have $\deg(\text{A}) = \deg(\text{A}_2) \cdot d \ \Rightarrow \ d = \frac{\deg(\text{A})}{\deg(\text{A}_2)} \leq \frac{\deg(\text{A})}{2}$.
By Remark 54 we also know that $3 \leq d$. So d satisfies $3 \leq d \leq \frac{\deg(\text{A})}{2}$ and $d \mid \deg(\text{A})$. Since we don't know $\deg(\text{A})$ but just its bound d_{A}, d satisfies $3 \leq d \leq \frac{d_{\text{A}}}{2}$ and $d \mid d_{\text{A}}$.

□

Lemma 61. *Let us assume* $\text{NS}_{reg}(\text{L}) \neq \emptyset$ *and* $d_s = denom(\Delta(\text{L}, s))$ $\forall s \in \text{NS}_{reg}(\text{L})$*. Let* $l = lcm_{s \in \text{NS}_{reg}(\text{L})}(d_s)$.

1. *If* $l \geq d_{\text{A}}$ *then the candidates for* $d = denom(\nu)$ *are* $\{i \in \mathbb{N} \,|\, 3 \leq i \leq d_{\text{A}} \cdot l \ \textit{and} \ l \mid i\}$.
2. *If* $l < d_{\text{A}}$*, let* d_{A} *be of the form* $d_{\text{A}} = q \cdot l + r$ *with* $q, r \in \mathbb{N}$, $0 \leq r < l$*. Then the candidates for* $d = denom(\nu)$ *are*
$$\left\{ i \cdot l \,\middle|\, i \in \{1, \ldots, q\}, \ \sum_{P_s \in \text{NS}_{reg}(\text{L})} \frac{i \cdot l}{d_s} \cdot deg(P_s) \leq d_{\text{A}} \ \textit{and} \ gcd_{s \in \text{NS}_{reg}(\text{L})} \left(\frac{i \cdot l}{d_s} \right) \middle| \, d_{\text{A}} \right\}.$$

Proof. By Lemma 57 we know that $d_s \mid d \ \ \forall s \in \text{NS}_{\text{reg}}(\text{L})$ and $l \mid d$.

1. Let us assume $l \geq d_{\text{A}}$. By Lemma 58 and Lemma 59 we have, respectively, $d_{\text{A}} = \deg(\text{A}_1)$ and $d/l \leq \deg(\text{A}_1)$. So $d/l \leq d_{\text{A}}$ i.e $d \leq d_{\text{A}} \cdot l$. We also know by Remark 54 that $d \geq 3$. Therefore we have $3 \leq d \leq d_{\text{A}} \cdot l$ and $l \mid d$. So the candidates for $d = \text{denom}(\nu)$ are $\{i \in \mathbb{N} \,|\, 3 \leq i \leq d_{\text{A}} \cdot l \ \text{and} \ l \mid i\}$.

2. Let us assume $l < d_A$. $l \mid d$ and $d \geq 3$ by Remark 54 then $d = l \cdot i$, $i \in \mathbb{N}$ and $i \geq 1$. So $\deg(A) = \deg(A_2) \cdot d + \deg(A_1) = \deg(A_2) \cdot l \cdot i + \deg(A_1)$. Since we don't know deg(A) but just d_A, we can take

$$d_A = \deg(A_2) \cdot l \cdot i + \deg(A_1), \quad i \geq 1. \tag{24}$$

On the other hand, let d_A have the form

$$d_A = q \cdot l + r, \quad \text{with } q, r \in \mathbb{N},\ 0 \leq r < l. \tag{25}$$

(24) and (25) imply

$$i \in \{1, \ldots, q\}. \tag{26}$$

Let $d_1 = \sum_{P_s \in \overline{NS_{reg}(L)}} \frac{d}{d_s} \cdot \deg(P_s) = \sum_{P_s \in \overline{NS_{reg}(L)}} \frac{l \cdot i}{d_s} \cdot \deg(P_s)$. By Lemma 58 we get $\left.\frac{d}{d_s}\right| \beta_s$ $\forall s \in NS_{reg}(L)$. Then $\forall s \in NS_{reg}(L)$ $\beta_s = \sigma_s \cdot \frac{d}{d_s}$, with $\sigma_s \in \mathbb{N}$ and $\sigma_s \geq 1$ because $\beta_s \neq 0$.

$$\begin{aligned}
d_A &= \deg(A_2) \cdot l \cdot i + \deg(A_1) = \deg(A_2) \cdot l \cdot i + \sum_{P_s \in \overline{NS_{reg}(L)}} \beta_s \cdot \deg(P_s) \\
&= \deg(A_2) \cdot l \cdot i + \sum_{P_s \in \overline{NS_{reg}(L)}} \sigma_s \cdot \frac{d}{d_s} \cdot \deg(P_s) \\
&\geq \sum_{P_s \in \overline{NS_{reg}(L)}} \sigma_s \cdot \frac{d}{d_s} \cdot \deg(P_s) \ \text{ since } \deg(A_2) \cdot l \cdot i \geq 0 \\
&\geq \sum_{P_s \in \overline{NS_{reg}(L)}} \frac{d}{d_s} \cdot \deg(P_s) \ \text{ since } \sigma_s \geq 1 \ \forall s \in NS_{reg}(L) \\
&\Rightarrow d_A \geq d_1.
\end{aligned} \tag{27}$$

Let $d_2 = \gcd_{s \in NS_{reg}(L)} \left(\frac{d}{d_s}\right) = \gcd_{s \in NS_{reg}(L)} \left(\frac{i \cdot l}{d_s}\right)$. $\forall s \in NS_{reg}(L)$, Since $d_2 \left| \frac{d}{d_s} \right.$ and $\left.\frac{d}{d_s}\right| \beta_s$, we get $d_2 \mid \beta_s$. Therefore $d_2 \,|\deg(A_1)$ because $\deg(A_1) = \sum_{P_s \in \overline{NS_{reg}(L)}} \beta_s \cdot \deg(P_s)$.

$$\begin{aligned}
& d_2 \left| \frac{d}{d_s} \right. \ \forall s \in NS_{reg}(L) \Rightarrow \frac{d}{d_s} = b_s \cdot d_2 \ \text{ with } b_s \in \mathbb{N} \ \forall s \in NS_{reg}(L) \\
\Rightarrow\ & d = (d_s \cdot b_s) \cdot d_2 \ \forall s \in NS_{reg}(L) \Rightarrow d_2 \mid d.
\end{aligned}$$

We have $d_2 \,|\deg(A_1)$, $d_2 \mid d$ and $d_A = \deg(A_2) \cdot d + \deg(A_1)$, therefore

$$d_2 \,| d_A \,. \tag{28}$$

Using (25), (26), (27) and (28), the candidates for d are $i \cdot l$ such that

$$\begin{cases}
d_A = q \cdot l + r, \quad \text{with } q, r \in \mathbb{N},\ 0 \leq r < l, \\
i \in \{1, \ldots, q\}, \\
\sum_{P_s \in \overline{NS_{reg}(L)}} \frac{i \cdot l}{d_s} \cdot \deg(P_s) \leq d_A \ \text{ and } \ \gcd_{s \in NS_{reg}(L)} \left(\frac{i \cdot l}{d_s}\right) \Big| \, d_A.
\end{cases}$$

□

How to find the degrees of A_1 *and* A_2

Let us first see how to get a list of candidates for $\deg(\mathrm{A}_2)$.

Lemma 62. *Let* $d = denom(\nu)$.

1. *If* $\mathrm{NS}_{reg}(\mathrm{L}) = \emptyset$, *then we have* $deg(\mathrm{A}_2) = \dfrac{d_\mathrm{A}}{d}$.
2. *If* $\mathrm{NS}_{reg}(\mathrm{L}) \neq \emptyset$, *let* $d_\mathrm{A} = q \cdot d + r$, *with* $q, r \in \mathbb{N}$, $0 \leq r < d$. *Then we have* $deg(\mathrm{A}_2) \in \{0, \ldots, q\}$.

Proof. Let $d = \mathrm{denom}(\nu)$.

1. Let us assume $\mathrm{NS}_{\mathrm{reg}}(\mathrm{L}) = \emptyset$. We have by Lemma 55 $d_\mathrm{A} = \deg(\mathrm{A}_2) \cdot d$. So we have $\deg(\mathrm{A}_2) = \dfrac{d_\mathrm{A}}{d}$.
2. Let us assume $\mathrm{NS}_{\mathrm{reg}}(\mathrm{L}) \neq \emptyset$ and $d_\mathrm{A} = q \cdot d + r$, with $q, r \in \mathbb{N}$, $0 \leq r < d$. Since we also have $d_\mathrm{A} = \deg(\mathrm{A}_2) \cdot d + \deg(\mathrm{A}_1)$, we can take the candidates for $\deg(\mathrm{A}_2)$ in the set $\{0, \ldots, q\}$.

□

When we find a candidate for $\deg(\mathrm{A}_2)$ we can also get its associated candidate for $\deg(\mathrm{A}_1)$.

Corollary 63. *Let us assume that we know* $deg(\mathrm{A}_2)$, *then we also have* $deg(\mathrm{A}_1)$ *by using the relation* $d_\mathrm{A} = deg(\mathrm{A}_2) \cdot d + deg(\mathrm{A}_1)$.

Proof. Since we know $\deg(\mathrm{A}_2)$, we just use the relation $d_\mathrm{A} = \deg(\mathrm{A}_2) \cdot d + \deg(\mathrm{A}_1)$ and get $\deg(\mathrm{A}_1) = d_\mathrm{A} - \deg(\mathrm{A}_2) \cdot d$. □

How to find A_1

For $\mathrm{NS}_{\mathrm{reg}}(\mathrm{L}) = \emptyset$, we can use Lemma 55 to get A_1: $\mathrm{A}_1 = 1$. The problem now is what to do if $\mathrm{NS}_{\mathrm{reg}}(\mathrm{L}) \neq \emptyset$. In that case, also by Lemma 55, we know that $\mathrm{A}_1 = \prod_{P_s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} P_s^{\beta_s}$. Hence, finding A_1 is equivalent to find β_s $\forall s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$.

We will see in this lemma which technique can be used to find all the β_s with $s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$.

Lemma 64. *Let* $d = denom(\nu)$. *Let us assume* $\mathrm{NS}_{reg}(\mathrm{L}) \neq \emptyset$ *and* $d_s = denom(\Delta(\mathrm{L}, s))$ *for all* $s \in \mathrm{NS}_{reg}(\mathrm{L})$. *Then we can find* β_s $\forall s \in \mathrm{NS}_{reg}(\mathrm{L})$ *by solving the Diophantine equation:*

$$\sum_{P_s \in \mathrm{NS}_{reg}(\mathrm{L})} \beta_s \cdot deg(P_s) = deg(\mathrm{A}_1) \;\; with \;\; \begin{cases} 1 \leq \beta_s < d, \\ \dfrac{d}{d_s} \Big| \beta_s. \end{cases} \tag{29}$$

Proof. By Lemma 55 we have $\mathrm{A}_1 = \prod_{P_s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} P_s^{\beta_s}$, $1 \leq \beta_s < d$. Then $\deg(\mathrm{A}_1) = \sum_{P_s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} \beta_s \cdot \deg(P_s)$. By Lemma 58 we have $\dfrac{d}{d_s} \Big| \beta_s$ for all $s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$. Since we know d, $\deg(\mathrm{A}_1)$ and d_s $\forall s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$, all the β_s with $s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$ will be solutions of this Diophantine equation

$$\sum_{P_s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} \beta_s \cdot \deg(P_s) = \deg(\mathrm{A}_1) \;\; \text{with} \;\; \begin{cases} 1 \leq \beta_s < d, \\ \dfrac{d}{d_s} \Big| \beta_s. \end{cases}$$

□

Let us see now how to solve our Diophantine equation and find all $\beta_s,\ s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$. This will lead to find candidates for A_1. The following lemma appeared similarly in [43, Lemma 18].

Lemma 65. *Let us assume* $\mathrm{NS}_{reg}(\mathrm{L}) \neq \emptyset$. *Then we can solve the Diophantine equation of Lemma 64 and get a list of candidates for* A_1.

Proof. Let $d = \mathrm{denom}(\nu)$. For $s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$, let $d_s = \mathrm{denom}(\Delta(\mathrm{L}, s)),\ m_s$ the multiplicity order of s as a zero of f and put m_s in the form

$$m_s = \alpha_s \cdot d + \beta_s \ \text{ with } (\alpha_s, \beta_s) \in \mathbb{N}^2,\ 1 \leq \beta_s < d. \tag{30}$$

By Lemma 58, $\forall s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$

$$\frac{d}{d_s}\Big| \beta_s \ \Rightarrow\ \beta_s = \frac{d}{d_s} \cdot \sigma_s \ \text{ with } \ \sigma_s \in \mathbb{N} \ \Rightarrow\ 1 \leq \frac{d}{d_s} \cdot \sigma_s < d \ \text{ by (30).}$$

Hence $\sigma_s \geq 1$.

$$\begin{aligned}
\mathrm{A}_1 &= \prod_{P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})}} P_s^{\beta_s} = \prod_{P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})}} P_s^{(d/d_s)\cdot \sigma_s} \\
\Rightarrow \deg(\mathrm{A}_1) &= \sum_{P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})}} \frac{d}{d_s} \cdot \sigma_s \cdot \deg(P_s) \geq \sum_{P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})}} \frac{d}{d_s} \cdot \deg(P_s) \ \left(\sigma_s \geq 1\ \forall s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})\right) \\
\Rightarrow \deg(\mathrm{A}_1) &\geq \frac{d}{d_s} \cdot \deg(P_s) \ \ \forall P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} \ \text{ since } \ \frac{d}{d_s} \cdot \deg(P_s) \geq 0 \ \ \forall P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} \\
\Rightarrow \deg(\mathrm{A}_1) &- \frac{d}{d_s} \cdot \deg(P_s) \geq 0 \ \ \forall P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})}.
\end{aligned} \tag{31}$$

Let us take one element $s_1 \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$. By (31) we have

$$\deg(\mathrm{A}_1) - \frac{d}{d_{s_1}} \cdot \deg(P_{s_1}) \geq 0. \tag{32}$$

$$\begin{aligned}
\deg(\mathrm{A}_1) &= \sum_{P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})}} \beta_s \cdot \deg(P_s) \Rightarrow \ \deg(\mathrm{A}_1) - \sum_{P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} \setminus \{P_{s_1}\}} \beta_s \cdot \deg(P_s) = \beta_{s_1} \cdot \deg(P_{s_1}) \\
\Rightarrow \ \deg(\mathrm{A}_1) &- \sum_{P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} \setminus \{P_{s_1}\}} \deg(P_s) \geq \beta_{s_1} \cdot \deg(P_{s_1}) \ \text{ since } \beta_s \geq 1 \ \ \forall s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L}) \\
\Rightarrow \ \deg(\mathrm{A}_1) &- \sum_{P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} \setminus \{P_{s_1}\}} \deg(P_s) \geq \sigma_{s_1} \cdot \left[\frac{d}{d_{s_1}} \cdot \deg(P_{s_1})\right].
\end{aligned} \tag{33}$$

On the other hand, let

$$\begin{aligned}
\deg(\mathrm{A}_1) - \sum_{P_s \in \overline{\mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} \setminus \{P_{s_1}\}} \deg(P_s) = q \cdot \left[\frac{d}{d_{s_1}} \cdot \deg(P_{s_1})\right] + r \ \text{ with } \ q, r \in \mathbb{N} \\
\text{and } \ 0 \leq r < \frac{d}{d_{s_1}} \cdot \deg(P_{s_1}).
\end{aligned} \tag{34}$$

Then we can take, because of (33), $\sigma_{s_1} \in \{0, \ldots, q\}$. The fact that $\sigma_{s_1} \geq 1$ reduces this set to

$$\sigma_{s_1} \in \{1, \ldots, q\} \tag{35}$$

$$\beta_{s_1} < d \Rightarrow \frac{d}{d_{s_1}} \cdot \sigma_{s_1} < d. \tag{36}$$

Since we know d, $\deg(\mathrm{A}_1)$ and $d_s \quad \forall s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$, using (32), (34), (35) and (36) we can find candidates for σ_{s_1} by solving

$$\begin{cases} \deg(\mathrm{A}_1) - \sum\limits_{P_s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L}) \setminus \{P_{s_1}\}} \deg(P_s) = q \cdot \left[\frac{d}{d_{s_1}} \cdot \deg(P_{s_1}) \right] + r \text{ with } q, r \in \mathbb{N} \\ \qquad\qquad \text{and } 0 \leq r < \frac{d}{d_{s_1}} \cdot \deg(P_{s_1}), \\ \sigma_{s_1} \in \{1, \ldots, q\}, \ \sigma_{s_1} \cdot \frac{d}{d_{s_1}} < d, \text{ and } \deg(\mathrm{A}_1) - \frac{d}{d_{s_1}} \cdot \deg(P_{s_1}) \geq 0. \end{cases} \tag{37}$$

Therefore, by using the relation $\beta_{s_1} = \frac{d}{d_{s_1}} \cdot \sigma_{s_1}$, we get also the candidates for β_{s_1}.

We continue the process by considering $\mathrm{A}_1 = \prod\limits_{P_s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L}) \setminus \{P_{s_1}\}} P_s^{\beta_s}, \quad 1 \leq \beta_s < d$. That means $\deg(\mathrm{A}_1) = \deg(\mathrm{A}_1) - \frac{d}{d_{s_1}} \cdot \sigma_{s_1} \cdot \deg(P_{s_1})$, and we take another $s_2 \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$. At the end, we will find all the candidates for $\beta_s, \quad \forall s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})$.

Since $\mathrm{A}_1 = \prod\limits_{P_s \in \mathrm{NS}_{\mathrm{reg}}(\mathrm{L})} P_s^{\beta_s}$, we have also the candidates for A_1. □

How to find c *such that* $\mathrm{A} = c\mathrm{A}_1\mathrm{A}_2^d$

We have seen how to find $\mathrm{A}_1, \mathrm{B} \in k[x]$. That means we have $\mathrm{A}_1/\mathrm{B} \in k(x)$ and $f \cdot \mathrm{B}/\mathrm{A}_1 \in k[x]$. So $c\mathrm{A}_2^d \in k[x]$. This doesn't imply that $\mathrm{A}_2 \in k[x]$. To get $\mathrm{A}_2 \in k[x]$, as in our assumption, we choose $c \in k$ by the following method.

Case 1: $(k \cup \{\infty\}) \cap \mathrm{S}_{\mathrm{irr}}(\mathrm{L}) \neq \emptyset$.

This case is similar as the *case* 1 in [43, Section 4.5]. We have here $\Delta(\mathrm{L}, s) \in k(t_s) \quad \forall s \in (k \cup \{\infty\}) \cap \mathrm{S}_{\mathrm{irr}}(\mathrm{L})$ and we can compute a truncated series for $f = \left(c\mathrm{A}_1\mathrm{A}_2^d\right)/\mathrm{B}$ at $x = s$. Therefore, we have a truncated series for $f \cdot \mathrm{B}/\mathrm{A}_1$ (which equals $c\mathrm{A}_2^d$) at $x = s$. If we want to have A_2^d monic then we have to take c as the coefficient of the first term of this series (the truncated series of $f \cdot \mathrm{B}/\mathrm{A}_1$ at $x = s$). That implies the truncated series of A_2 at $x = s$ has first term 1 (or another d-th root of 1 in k). Hence we can construct other terms of A_2 by Hensel Lifting (see [43]). By the construction method, A_2 will be in $k[x]$.

The problem we face now is the following: if we choose other $\widetilde{c}$ such that $\mathrm{A}_2 \in k[x]$, will that lead to the same candidates for f?

The following definition and lemma appeared similarly in [43, Definition 29] and [43, Lemma 19], respectively.

Definition 66. *We say that c_1 and c_2 are equivalent ($c_1 \sim c_2$), if $c_1 = c_2 \cdot c^d$ where $c \in k$.*

Lemma 67. *Assume $k \cup \{\infty\}) \cap \mathrm{S}_{irr}(\mathrm{L}) \neq \emptyset$. A_1 and B, which are monic and in $k[x]$, are fixed. Let $s \in k \cup \{\infty\}) \cap \mathrm{S}_{irr}(\mathrm{L})$ and c_1, c_2 and A_2 be computed by the method we introduced in case 1 above and Theorem 70 below such that $\left(c_1\mathrm{A}_1\mathrm{A}_2^d\right)/\mathrm{B}$ and $\left(c_2\mathrm{A}_1\widetilde{\mathrm{A}}_2^d\right)/\mathrm{B}$ are candidates for f with $\mathrm{A}_2, \widetilde{\mathrm{A}}_2 \in k[x]$. Then c_1 and c_2 lead to the same candidates if and only if $c_1 \sim c_2$.*

Proof. 1. Assume that c_1 and c_2 lead to the same candidates for f. That means we will have

$$\frac{c_1\mathrm{A}_1\mathrm{A}_2^d}{\mathrm{B}} = \frac{c_2\mathrm{A}_1\widetilde{\mathrm{A}}_2^d}{\mathrm{B}} \Rightarrow c_1\mathrm{A}_2^d = c_2\widetilde{\mathrm{A}}_2^d \Rightarrow \frac{c_1}{c_2} = \frac{\widetilde{\mathrm{A}}_2^d}{\mathrm{A}_2^d} \in k \Rightarrow \mathrm{A}_2 = c\widetilde{\mathrm{A}}_2 \text{ with } c \in k,$$

such that $c^d = c_1/c_2$. So $c_2 \cdot c^d = c_1$ and then $c_1 \sim c_2$.

2. Assume that $c_1 \sim c_2$. That means $c_1 = c_2 \cdot c^d$.

$$\frac{c_1\mathrm{A}_1\mathrm{A}_2^d}{\mathrm{B}} = \frac{c_2 \cdot c^d\mathrm{A}_1\mathrm{A}_2^d}{\mathrm{B}} = \frac{c_2\mathrm{A}_1\left(c\mathrm{A}_2\right)^d}{\mathrm{B}} = \frac{c_2\mathrm{A}_1\widetilde{\mathrm{A}}_2^d}{\mathrm{B}} \text{ with } \widetilde{\mathrm{A}}_2 = c\mathrm{A}_2 \in k[x].$$

So $\left(c_1\mathrm{A}_1\mathrm{A}_2^d\right)/\mathrm{B}$ gotten from c_1 can also be gotten from c_2. Therefore c_1 and c_2 lead to the same candidates for f.

□

Conclusion : If we choose other $\widetilde{c}$ such that $\mathrm{A}_2 \in k[x]$, then it will lead to the same candidates of f. So our method is sufficient in this case.

Case 2 : $(k \cup \{\infty\}) \cap \mathrm{S}_{\mathrm{irr}}(\mathrm{L}) = \emptyset$

This case is similar as the *case* 2 in [43, Section 4.5]. For some $s \in \mathrm{S}_{\mathrm{irr}}(\mathrm{L})$, we can temporarily extend the field k to $k(s)$ and recompute the duplicate local data over the new field. Then we compute all candidates $g \in k(s, x)$ for f as in case 1. Any candidate g for f that is defined over $k(x)$ will be discarded without further computation (see the last part of chapter 4 in [28] called "Algebraic Extension").

The following remark appeared similarly in [43, Remark 19].

Remark 68. *In the case 2, sometimes we can still use the way in case 1 to guess the value of c. But it might not lead to the correct candidates because c might not be unique (up to multiplication by a d-th power) over k. So we need to introduce algebraic extensions (see the last part of chapter 4 in [28] called "Algebraic Extension").*

How to find A_2

In order to find A in $f = \dfrac{c\mathrm{A}_1\mathrm{A}_2^d}{\mathrm{B}}$, the only unknown part is now A_2. This lemma gives us the minimum number of equations, related to the degree of A, satisfied by the coefficients of A_2 that we need to get A_2. It appeared in the same way in [43, Lemma 20].

Lemma 69. *To recover A_2 we only need $\frac{1}{3}d_{\mathrm{A}} + 1$ linear equations for its coefficients.*

Proof. Let $d = \text{denom}(\nu)$. By Remark 54 $d \geq 3$.

$$d_A = \deg(A_1) + \deg(A_2)\cdot d \;\Rightarrow\; \deg(A_2)\cdot d \leq d_A \;\Rightarrow\; \deg(A_2) \leq \frac{d_A}{d} \;\Rightarrow\; \deg(A_2) \leq \frac{d_A}{3}$$

since $d \geq 3$. Hence to get A_2 we only need $\frac{1}{3}d_A + 1$ linear equations for its coefficients. □

Let $A_2 = \sum_{i=0}^{\deg(A_2)} b_i x^i,\; b_i \in k$. By using the same methods as in Lemma 48 and Lemma 53, the equations we get for $\{b_i\}$ will not be linear because we need to evaluate the d-th power. We use here the method of Hensel Lifting (see [43]) to find linear equations for the coefficients $\{b_i\}$ of A_2.

The following theorem and remark appeared similarly in [43, Theorem 10] and [43, Remark 20], respectively.

Theorem 70. *Let* A_2 *in* $A = cA_1A_2^d$ *be on the form* $A_2 = \sum_{i=0}^{deg(A_2)} b_i x^i,\; b_i \in k$. *For each* $s \in S_{irr}(L)$ *with* m_s *as its multiplicity order as a pole of* f*, we will get* $\lceil m_s \rceil$ *linear equations for* $\{b_i\}$.

Proof. The proof is similar as the proof of theorem 10 in [43]. □

Remark 71. *If* $s \notin k$*, we can use the results from Lemma 40 and Lemma 42 to get equations. Therefore, we can always obtain* $\geq \frac{1}{2}d_A$ *linear equations, while* $\left\lfloor \frac{1}{3}d_A \right\rfloor + 1$ *equations are sufficient. So we always get enough linear equations.*

How to find the Bessel parameter ν

We take $\nu = a/d,\; a \in \mathbb{Z},\; d \in \mathbb{N}\setminus\{0\}$ and $\gcd(a,d) = 1$. We know how to find $d = \text{denom}(\nu)$. What remains now is how to find a.

Since we just have to take ν modulo $\mathbb{Z}$, then $a \in [-d,d] \cap \mathbb{Z}$. That is also equivalent to $a \in [-d/2, d/2] \cap \mathbb{Z}$. We also know that $a/d \notin \mathbb{Z}$, if not, we are in the logarithmic case. Therefore $a \in (]-d/2,-1] \cup [1,d/2[) \cap \mathbb{Z}$. Since taking ν or $-\nu$ is the same for our operator $L_{\tilde{B}_\nu^2}$, $a \in [1,d/2[\cap \mathbb{N}$. Let $V_1 = \left\{ \frac{a}{d} \middle| a \in \left[1, \frac{d}{2}\right[\cap \mathbb{N} \right\}$. If $S_{reg}(L) \neq \emptyset$, using Definition 47 and Lemma 48, we can find N, the set of candidates for ν modulo $\mathbb{Z}$. Therefore, the reduced set of candidates for ν is $V = V_1 \cap N$ where "$\cap$" represents the intersection modulo $\mathbb{Z}$.

We have implemented in Maple an algorithm called `BesSqRootRat` to find candidates for (ν, f) in the rational case, if they exist.

The case $F_0 \in \{{}_1F_2,\; {}_0F_2\}$

Here the numerator A of f can be written in the form $A = cA_1A_2^d$ as in Corollary 56.

The technique to find candidates for f is similar as for the Bessel square root case with $f \neq g^2$ where $g \in k(x)$.

Once we get candidates for $f = A/B = \dfrac{cA_1A_2^d}{B}$, we use the same technique as in the case $F_0 = {}_2F_2$ to find candidates for the lower parameters of $F_0 \in \{{}_1F_2,\; {}_0F_2\}$, and the same technique as in the logarithmic case to find the candidates for the upper parameters for $F_0 = {}_1F_2$ related to any candidate for f.

We have implemented in Maple, for this case, one algorithm called `find1F2Rat` to find candidates for $[\{a_1\}, \{b_1, b_2\}, f]$ when $\mathrm{L}_0 = \mathrm{L}_{12}$ and another one called `find0F2Rat` to find candidates for $[\{b_1, b_2\}, f]$ when $\mathrm{L}_0 = \mathrm{L}_{02}$.

8. exp-product and gauge transformation parameters

This theorem finds the exp-product parameter in the projective equivalence ($\longrightarrow_{EG}$).

Theorem 72. *Let $\mathrm{L}_1, \mathrm{L}_2 \in k(x)[\partial]$ be two irreducible third-order linear differential operators such that $\mathrm{L}_1 \longrightarrow_{EG} \mathrm{L}_2$ and r the parameter of the exp-product transformation. Let $\mathbb{S}$ be the set of all non-apparent singularities of L_2 and $\mathbb{P}_0$ the set of all the poles of r. For $p \in \mathbb{P}_0 \cup \mathbb{S}$, let us set*

$$e_p^i = e_p^i(\mathrm{L}_2) - e_p^i(\mathrm{L}_1), \quad i = 1, 2, 3$$

where $e_p^i(\mathrm{L}_j)$ is the i-th generalized exponent of L_j at p, $j \in \{1, 2\}$, and r has series representation

$$r = \sum_{i=-m_p}^{+\infty} r_{p,i} t_p^i, \quad m_p \in \mathbb{N} \quad with \quad r_{p,i} \in k \quad and \quad r_{p,-m_p} \neq 0.$$

If we assume that

1. $\mathbb{P}_0^{11} = \{p \in \mathbb{P}_0 | \; \{e_p^1(\mathrm{L}_2), e_p^2(\mathrm{L}_2), e_p^3(\mathrm{L}_2)\} \subseteq \mathbb{Z} \text{ and } r_{p,-1} \notin \mathbb{Z}\Big\} = \emptyset,$
2. *L_1 is not the image of an exp-product transformation with rational function $-r + a_p t_p^{-1}$ with $a_p \in k$ and $p \in \mathbb{P}_0$ such that $m_p \geq 2$ if $p \neq \infty$,*

then

$$\sum_{p \in \mathbb{S} \setminus \{\infty\}} \frac{e_p^i}{t_p} - t_\infty \cdot \overline{e_\infty^i} = r + \sum_{p \in \mathbb{S} \setminus \{\infty\}} \frac{b_p}{n_p} t_p^{-1} - \sum_{p \in \mathbb{P}_0^{12} \setminus (\mathbb{S} \cup \{\infty\})} r_{p,-1} t_p^{-1} \tag{38}$$

where $\overline{e_\infty^i} = e_\infty^i - const(e_\infty^i)$ with $const(e_\infty^i)$ the constant term of e_∞^i, $b_p \in \mathbb{Z}$, $n_p = \max\left\{n_{e_p^i(\mathrm{L}_2)}, i = 1, 2, 3\right\}$ with $n_{e_p^i(\mathrm{L}_2)}$ the ramification index of $e_p^i(\mathrm{L}_2)$, and $\mathbb{P}_0^{12} = \{p \in \mathbb{P}_0 | \; \{e_p^1(\mathrm{L}_2), e_p^2(\mathrm{L}_2), e_p^3(\mathrm{L}_2), r_{p,-1}\} \subseteq \mathbb{Z}\Big\}$.

Proof. The proof can be found in [28, Theorem 3.31]. □

Lemma 73. *Let us consider the hypothesis and notations of Theorem 72, and assume that all the conditions of Theorem 72 are satisfied. Then the parameter r of the exp-product transformation is given by*

$$r = \sum_{p \in \mathbb{S} \setminus \{\infty\}} \frac{e_p^i}{t_p} - t_\infty \cdot \overline{e_\infty^i} + \sum_{p \in \mathbb{S} \setminus \{\infty\}} \frac{c_p}{n_p} t_p^{-1} \tag{39}$$

with $c_p \in \mathbb{Z}$ and $|c_p| < n_p$.

Proof. By Theorem 72 we have

$$\begin{aligned} r &= \sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{e_p^i}{t_p} - t_\infty \cdot \overline{e_\infty^i} - \sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{b_p}{n_p} t_p^{-1} + \sum_{p\in\mathbb{P}_0^{12}\setminus(\mathbb{S}\cup\{\infty\})} r_{p,-1} t_p^{-1} \\ &= \sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{e_p^i}{t_p} - t_\infty \cdot \overline{e_\infty^i} + \sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{c_p}{n_p} t_p^{-1} + \sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{d_p}{n_p} t_p^{-1} + \sum_{p\in\mathbb{P}_0^{12}\setminus(\mathbb{S}\cup\{\infty\})} r_{p,-1} t_p^{-1} \end{aligned}$$

where $-b_p = c_p + d_p$ with $c_p, d_p \in \mathbb{Z}$ and $\left|c_p\right| < n_p$. Let y be a solution of L_1. Since we are searching solutions $\overline{y}$ of L_2 of the form

$$\overline{y} = \exp\left(\int r\,dx\right)\left(r_0 y + r_1 y^{'} + r_2 y^{''}\right),$$

the term $\sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{d_p}{n_p} t_p^{-1} + \sum_{p\in\mathbb{P}_0^{12}\setminus(\mathbb{S}\cup\{\infty\})} r_{p,-1} t_p^{-1}$ in the expression of r will be transformed as follows:

$$\begin{aligned} &\exp\left(\int \left(\sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{d_p}{n_p} t_p^{-1} + \sum_{p\in\mathbb{P}_0^{12}\setminus(\mathbb{S}\cup\{\infty\})} r_{p,-1} t_p^{-1}\right) dx\right)\left(r_0 y + r_1 y^{'} + r_2 y^{''}\right) \\ =& x^{\sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{d_p}{n_p} + \sum_{p\in\mathbb{P}_0^{12}\setminus(\mathbb{S}\cup\{\infty\})} r_{p,-1}} \left(r_0 y + r_1 y^{'} + r_2 y^{''}\right) \\ =& \overline{r_0} y + \overline{r_1} y^{'} + \overline{r_2} y^{''} \end{aligned}$$

with $\overline{r_i} = r_i \cdot x^{\sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{d_p}{n_p} + \sum_{p\in\mathbb{P}_0^{12}\setminus(\mathbb{S}\cup\{\infty\})} r_{p,-1}}$. Using the fact that $r_{p,-1} \in \mathbb{Z}\ \forall p \in \mathbb{P}_0^{12} \setminus (\mathbb{S} \cup \{\infty\})$, and also $d_p \in \mathbb{Z}\ \forall p \in \mathbb{S} \setminus \{\infty\}$, we will have $\overline{r_0}, \overline{r_1}, \overline{r_2} \in k(x)$. Therefore, we can take the exp-product parameter r as

$$r = \sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{e_p^i}{t_p} - t_\infty \cdot \overline{e_\infty^i} + \sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{c_p}{n_p} t_p^{-1},$$

and that will just change the gauge transformation parameters r_1, r_2 and r_3. □

To find the gauge parameters $r_0, \ldots, r_{n-1}$ (gauge equivalence problem) between two operators L_1 and L_2 of order n such that $\mathrm{L}_1 \overset{r_0,\ldots,r_{n-1}}{\longrightarrow}_G \mathrm{L}_2$, we use Mark van Hoeij's gauge equivalence test (see [17]) which gives us those parameters as the coefficients of an $n-1$-st order linear differential operator. There already exists in Maple an implementation for this purpose called `Homomorphisms`, implemented by Mark van Hoeij (see [17]), which takes as input two linear differential operators L_1 and L_2 of order n and returns a basis of all the operators of order $n-1$ satisfying the equation $\mathrm{L}_1 = \mathrm{L}_2\mathrm{X}$ if the first input is L_1, and $\mathrm{L}_2 = \mathrm{L}_1\mathrm{X}$ if the first input is L_2 (X is the unknown of this equation). Otherwise it returns an empty list. Let $n = 3$ and $\mathrm{X} = a_2(x)\partial^2 + a_1(x)\partial + a_0(x)\partial$, we deduce the gauge parameters r_0, r_1 and r_2 by taking

$$\begin{cases} r_0 = a_0(x), \\ r_1 = a_1(x), \\ r_2 = a_2(x). \end{cases} \tag{40}$$

Conclusion

Let us consider two irreducible third-order linear differential operators $\mathrm{L}, \mathrm{M} \in k(x)[\partial]$ such that $\mathrm{M} \longrightarrow_{EG} \mathrm{L}$. To find the exp-product and gauge transformation parameters we proceed as follows:

1. We compute the set $\mathbb{S}$ of non-apparent singularities of L.
2. We compute for all $p \in \mathbb{S},\ i = 1,2,3$ the difference $e_p^i = e_p^i(\mathrm{L}) - e_p^i(\mathrm{M})$ where $e_p^i(\mathrm{L})$ and $e_p^i(\mathrm{M})$ are the i^{th}generalized exponent of L and M at p, respectively.
3. We compute the exp-product parameter r using the relation

$$r = \sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{e_p^i}{t_p} - t_\infty \cdot \overline{e_\infty^i} + \sum_{p\in\mathbb{S}\setminus\{\infty\}} \frac{c_p}{n_p} t_p^{-1}$$

   where $\overline{e_\infty^i} = e_\infty^i - \mathrm{const}(e_\infty^i)$ with $\mathrm{const}(e_\infty^i)$ the constant term of e_∞^i, $n_p = \max\left\{n_{e_p^i(\mathrm{L})}, i = 1,2,3\right\}$ with $n_{e_p^i(\mathrm{L})}$ the ramification index of $e_p^i(\mathrm{L})$, and $c_p \in \mathbb{Z}$ such that $\left|c_p\right| < n_p$.
4. We compute the middle operator $\mathrm{L}_1 \in k(x)[\partial]$ such that $\mathrm{M} \xrightarrow{r}_E \mathrm{L}_1 \longrightarrow_G \mathrm{L}$ which is given by $\mathrm{L}_1 = \mathrm{M} ⓢ (\partial - r)$. There exists in Maple an algorithm called `symmetry` (see [20], [19] and [26]) to compute the symmetric product between two differential operators.
5. We compute the operator $\mathrm{G} \in k(x)[\partial]$ of order two such that $\mathrm{L} = \mathrm{G}\mathrm{L}_1$. That can be done in Maple by the algorithm called `Homomorphisms` (see [17] and [33]).
6. Let $\mathrm{G} = a_2\partial^2 + a_1\partial + a_0$ with $a_0, a_1, a_2 \in k(x)$. We deduce the gauge parameters $r_0, r_1, r_2 \in k(x)$ by taking

$$\begin{cases} r_0 = a_0, \\ r_1 = a_1, \\ r_2 = a_2. \end{cases}$$

We have succeeded by writing and implementing in Maple algorithms for the projective equivalence in our cases called

1. `EquivExpgauge2F2` when $\mathrm{L}_0 = \mathrm{L}_{22}$,
2. `EquivExpgaugeBesSqRoot1` when $\mathrm{L}_0 = \mathrm{L}_{\breve{B}_\nu^2}$,
3. `EquivExpgauge1F2` when $\mathrm{L}_0 = \mathrm{L}_{12}$,
4. `EquivExpgauge0F2` when $\mathrm{L}_0 = \mathrm{L}_{02}$.

9. Example

We have implemented the methods of the given paper in Maple. The Maple functions are called `Hyp2F2Solutions` when $\mathrm{L}_0 = \mathrm{L}_{22}$, `BesSqRootSolutions` when $\mathrm{L}_0 = \mathrm{L}_{\breve{B}_\nu^2}$, `Hyp1F2Solutions` when $L_0 = L_{12}$ and `Hyp0F2Solutions` when $L_0 = L_{02}$. They take as input any irreducible third-order linear differential operator L and return, if they exist, all the parameters of transformations $(r, r_2, r_1, r_0, f \in k(x))$ and also the parameter(s) of $F_0 \in \left\{{}_2F_2,\ {}_1F_2,\ {}_0F_2, \check{B}_\nu^2\right\}$ such that we are in situation (7)

$$L_0 \xrightarrow{f}_C M \longrightarrow_{EG} L$$

where $L_0 \in \{L_{22}, L_{12}, L_{02}, L_{\check{B}_\nu^2}\}$ is the operator associated to F_0. If not, we return "`No` F_0 `type solutions`". Those algorithms deal with all the cases.

We will take here just one example and show how some of our algorithms work.
Let us consider the differential operator L_{02} associated to the ${}_0F_2$ hypergeometric function with lower parameters $b_1 = 1/3$ and $b_2 = 1/7$.

```
> read "Solver3.mpl":
```

To get L_{02} with Maple we proceed as follows:[1]

```
> eq:=sumdiffeq(hyperterm([],[b1,b2],x,i),i,J(x));
```

$$eq := x^2 \frac{d^3}{dx^3} J(x) + (b2 + 1 + b1)\, x \frac{d^2}{dx^2} J(x) + b2\, b1 \frac{d}{dx} J(x) - J(x) = 0$$

```
> LA:=de2diffop(eq,J(x));
```

$$LA := x^2 Dx^3 + (xb2 + x + xb1)\, Dx^2 + b2\, b1\, Dx - 1$$

```
> L02:=subs({b1=1/3,b2=1/7},LA);
```

$$L02 := x^2 Dx^3 + \frac{31}{21} xDx^2 + \frac{1}{21} Dx - 1$$

Let us apply to L_{02} the change of variables transformation with parameter f given by

```
> f:=(2*(x-1)^2*(x-3)*(x-7)^3)/((x-9)^2*(x-12)^3);
```

$$f := 2 \frac{(x-1)^2 (x-3)(x-7)^3}{(x-9)^2 (x-12)^3}$$

That gives us the following operator called L:

```
> L:=ChangeOfVariables(L02,f);
```

$$L := 21\, (x-12)^6 (x-9)^5 (x-7)^2 (x-3)^2 (x-1)^2 \left(5193 - 3852\,x - 60\,x^3 + 830\,x^2 + x^4\right)^2 Dx^3 + \left(-173735685 + 300375864\,x + 32319144\,x^3 - 159987168\,x^2 + 31\,x^8 - 635762\,x^4 - 3720\,x^7 - 649176\,x^5 + 86200\,x^6\right) Dx^2 (x-1)(x-3)(x-7)(x-9)^4 \times (x-12)^5 \left(5193 - 3852\,x - 60\,x^3 + x^4 + 830\,x^2\right) + \left(-5387188885607952\,x + x^{16} - 6395117622870960\,x^3 + 3388591949109444\,x^4 + 7748517717658728\,x^2 - 240\,x^{15} + 54803306488\,x^{10} - 3065641808\,x^{11} - 3904976\,x^{13} + 63960\,x^{14} - 746349293552\,x^9 + 130520372\,x^{12} - 56984531313168\,x^7 + 7599004335182\,x^8 - 1223256100618800\,x^5 + 311300882943048\,x^6 + 1632102637284153\right) Dx\, (x-9)^3 (x-12)^4 - 42\, (-3852\,x$$

[1] To generate the initial differential equation, we use the `hsum` package from Wolfram Koepf (see [26]).

$$+5193 - 60\,x^3 + x^4 + 830\,x^2\Big)^5 (x-7)^2 (x-1)$$

Let us assume that we have as input this operator L. Now we will show how we can find its ${}_0F_2$ type solutions with our codes if they exist. That means how we can find the hypergeometric parameters b_1 and b_2 of ${}_0F_2$ and the transformation parameters such that

$$\mathrm{L}_{02} \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}.$$

Using our code we get:

```
> Hyp0F2Solutions(L);
```

$$\left\{\left[\left\{\left[\left[\frac{1}{3}, \frac{1}{7}\right], [0], [1]\right]\right\}, 2\,\frac{(x-1)^2 (x-3) (x-7)^3}{(x-9)^2 (x-12)^3}\right]\right\}$$

Hence, L has ${}_0F_2$ type solutions with hypergeometric parameters $b_1 = 1/3$ and $b_2 = 1/7$, and transformation parameters: $r = 0$, $r_0 = 1$, $r_1 = 0$, $r_2 = 0$ and $f = \dfrac{2(x-1)^2(x-3)(x-7)^3}{(x-9)^2(x-12)^3}$. All explanations of the functioning of this algorithm are available in [28]. The Maple package containing it is called `Solver3` and can be downloaded from http://www.mathematik.uni-kassel.de/~merlin/.

10. Conclusion

We gave an algorithm to find the $S \in \{{}_0F_2,\ {}_1F_2,\ {}_2F_2, \check{\mathrm{B}}_\nu^2\}$ type solutions of an irreducible third-order linear differential operator without Liouvillian solutions and with rational function coefficients, where ${}_pF_q$ with $p \in \{0, 1, 2\}$, $q = 2$, is the generalized hypergeometric function, and $\check{\mathrm{B}}_\nu^2(x) = \left(\mathrm{B}_\nu(\sqrt{x})\right)^2$ with B_ν the Bessel function. We extended the algorithm described in [28] about Bessel square solutions which already solved the problem in the case $f = g^2$ with $g \in k(x)$. We have also implemented those algorithms in Maple (available from http://www.mathematik.uni-kassel.de/~merlin/). Putting together the algorithms developed in this paper and those developed in [28] for $\left(\mathrm{B}_\nu^2\right)$-type solutions and ${}_1F_1^2$-type solutions, we have an extended algorithm to find all hypergeometric type solutions for holonomic third-order linear irreducible differential equations with radius of convergence ∞. The only cases which are not covered since they require different methods, are the cases ${}_2F_1^2$ and ${}_3F_2$ with radius of convergence 1. Those cases are part of our future work.

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References

[1] S. Abramov, M. Barkatou, and M. van Hoeij. Apparent singularities of linear differential equations with polynomial coefficients. In ISSAC'06 Proceedings, pages 117–133, 2006.

[2] M. Abramowitz and I. A. Stegun. Handbook of Mathematical Functions, volume 55 of Applied Mathematics Series. Tenth Printing, 1972.

[3] M. A. Barkatou and E. Pflügel. On the equivalence problem of linear differential systems and its application for factoring completely reducible systems. In ISSAC'98 Proceedings, pages 268–275, 1998.

[4] A. Bostan, X. Caruso, and E. Schost. A fast algorithm for computing the characteristic polynomial of the p-curvature. arXiv: 1405.5341v1 [cs.SC], France, 21 May 2014.

[5] A. Bostan, T. Cluzeau, and B. Salvy. Fast algorithms for polynomial solutions of linear differential equations. In ISSAC'05 Proceedings, pages 45–52, 2005.

[6] M. Bronstein. An improved algorithm for factoring linear ordinary differential operators. In ISSAC'94 Proceedings, pages 336–340, 1994.

[7] M. Bronstein and S. Lafaille. Solutions of linear ordinary differential equations in terms of special functions. In ISSAC'02 Proceedings, pages 23–28, 2002.

[8] B. C. Carlson. Special Function of Applied Mathematics. Academic Press, London, 1977.

[9] L. Chan and E. S. Cheb-Terrab. Non-liouvillian solutions for second order linear odes. ISSAC'04 Proceedings, pages 80–86, 2004.

[10] T. Cluzeau and M. van Hoeij. A modular algorithm to compute the exponential solutions of a linear differential operator. J. Symb. Comput. 38, pages 1043–1076, 2004.

[11] R. Debeerst. Solving differential equations in terms of Bessel functions. Master's thesis, Universität Kassel, 2007.

[12] R. Debeerst, M. van Hoeij, and W. Koepf. Solving differential equations in terms of Bessel functions. ISSAC'08 Proceedings, pages 39–46, 2008.

[13] W. N. Everitt and C. Markett. On a generalization of Bessel functions satisfying higher–order differential equations. J. Comput. Appl. Math. 54, 3, pages 325–349, 1994.

[14] W. N. Everitt, D. J. Smith, and M. van Hoeij. The fourth–order type linear ordinary differential equations. arXiv:math/0603516, 2006.

[15] T. Fang and M. van Hoeij. 2–descent for second order linear differential equations. ISSAC'11 Proceedings, pages 107–114, 2011.

[16] M. van Hoeij. Factorization of linear differential operators. PhD thesis, Universitijt Nijmegen, 1996.

[17] M. van Hoeij. Rational solutions of the mixed differential equation and its application to factorization of differential operators. ISSAC'96 Proceedings, pages 219–225, 1996.

[18] M. van Hoeij. Factorization of linear differential operators with rational functions coefficients. J. Symbolic Computation 24, pages 237–561, 1997.

[19] M. van Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. Journal of Pure and Applied Algebra 139, pages 109–131, 1999.

[20] M. van Hoeij. Solving third order linear differential equations in terms of second order equations. ISSAC'07 Proceedings, pages 355–360, 2007.

[21] M. van Hoeij and Q. Yuan. Finding all Bessel type solutions for linear differential equations with rational function coefficients. ISSAC'10 Proceedings, pages 37–44, 2010.

[22] P. Horn. Faktorisierung in Schief-Polynomringen. PhD thesis, Universität Kassel, 2008. https://kobra.bibliothek.uni-kassel.de/handle/urn:nbn:de:hebis:34-2009030226513

[23] E. L. Ince. Ordinary Differential Equations. Dover Publications Inc., 1956.

[24] N. M. Katz. A conjecture in the arithmetic theory of differential equations. Bulletin de la Soc. Math., France, tome 110, pages 203–239, 1982.

[25] W. Koepf. Identities for families of orthogonal polynomials and special functions. Integral Transforms and Special Functions 5, 69-102, 1997.

[26] W. Koepf. Hypergeometric Summation—An Algorithmic Approach to Summation and Special Function Identities. Springer, 2014.

[27] J. Kovacic. An algorithm for solving second-order linear homogeneous equations. J. Symb. Comp. 2, pages 3–43, 1986.

[28] M. Mouafo Wouodjié. On the solutions of holonomic third-order linear irreducible differential equations in terms of hypergeometric functions. PhD thesis, Universität Kassel, 2018. https://kobra.bibliothek.uni-kassel.de/handle/urn:nbn:de:hebis:34-2018060655613

[29] E. Pflügel. An algorithm for computing exponential solutions of first order linear differential system. ISSAC'97 Proceedings, pages 164–171, 1997.

[30] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. Integrals and Series, vol. 3: More Special Functions. Gordon and Breach Science, 1990.

[31] M. van der Put. Differential equations in characteristic p. Compositio Mathematica 97, pages 227–251, 1996.

[32] M. van der Put. Reduction modulo p of differential equations. Indag. Math N.S 7(3), pages 367–387, 1996.

[33] M. van der Put and M. F. Singer. Galois Theory of Linear Differential Equations, volume 328. Comprehensive Studies in Mathematics, Springer, Berlin, 2003.

[34] Salvy, B. and Zimmermann, P.: GFUN: A Maple package for the manipulation of generating and holonomic functions in one variable. ACM Transactions on Mathematical Software 20, 163–177, 1994.

[35] A. V. Shannin and V. Craster. Removing false singular points as a method of solving ordinary differential equations. Eur. J. Appl. Math. 13, pages 617–639, 2002.

[36] M. F. Singer. Solving homogeneous linear differential equations in terms of second order linear differential equations. Am. J. of Math. 107, pages 663–696, 1985.

[37] M. F. Singer and F. Ulmer. Galois groups for second and third order linear differential equations. J. Symb. Comp. 16, pages 9–36, 1993.

[38] M. F. Singer and F. Ulmer. Liouvillian and algebraic solutions of second and third order linear differential equations. J. Symb. Comp. 16, pages 37–73, 1993.

[39] S. Y. Slavyanov and W. Lay. Special Functions, A Unified Theory Based on Singularities. Oxford Mathematical Monographs, 2000.

[40] R. P. Stanley. Differentiably finite power series. Europ. J. Combinatorics 1, pages 175–188, 1980.

[41] Z. X. Wang and D. R. Guo. Special Functions. World Scientific Publishing, Singapore, 1989.

[42] B. L. Willis. An extensible differential equation solver. SIGSAM Bulletin 35, 1, pages 3–7, March 2001.

[43] Q. Yuan. Finding all Bessel type solutions for linear differential equations with rational function coefficients. PhD thesis, Florida State University, 2012.