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A Unified Representation for Some Interpolation Formulas

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Summary: As an extension of Lagrange interpolation, we introduce a class of interpolation formulae and study its existence and uniqueness. In the sequel, we consider some particular cases of it and construct the corresponding weighted quadrature rules. Numerical examples are finally given and compared.

1 Introduction

Let $\{x_j\}_{j=0}^n \in [a,b]$ and $\{f_j\}_{j=0}^n$, which may be samples of a function, say f , be given. If $\Psi(x; a_0, \dots, a_n)$ is a family of functions of a single variable x with $n+1$ free parameters $\{a_j\}_{j=0}^n$, then the interpolation problem for Ψ consists of determining $\{a_j\}_{j=0}^n$ so that for $n+1$ given real or complex pairs of distinct numbers $\{(x_j, f_j)\}_{j=0}^n$ we have

$$\Psi(x_j; a_0, \dots, a_n) = f_j, \qquad j = 0, 1, \dots, n. \tag{1.1}$$

Relation (1.1) leads to a linear interpolation problem if Ψ depends linearly on the parameters a_i, i.e.

$$\Psi(x; a_0, \dots, a_n) = a_0\Psi_0(x) + \cdots + a_n\Psi_n(x).$$

For a comprehensive discussion see [1, 4, 7, 13, 19, 20].

For a polynomial type interpolation problem, various classical methods such as Lagrange, Newton and Hermite interpolations are used to determine the associated parameters $\{a_j\}_{j=0}^n$. Lagrange's interpolation which is a classical method for approximating a continuous function $f : [a,b] \to \mathbb{R}$ at $n+1$ distinct nodes $a \le x_0 < \cdots < x_n \le b$ is applied in several branches of numerical analysis and approximation theory [2, 14, 16, 17]. Let

$$L_n(f;x) = \sum_{j=0}^{n} f(x_j)\ell_j^{(n)}(x),$$

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be the Lagrange polynomial of degree n which interpolates a given function f at the knots $\{x_j\}_{j=0}^n$, where

$$\ell_j^{(n)}(x) = \frac{w_n(x)}{(x-x_j)w_n'(x_j)}, \tag{1.2}$$

are the fundamental Lagrange polynomials with the node polynomial

$$w_n(x) = \prod_{j=0}^{n}(x-x_j).$$

$L_n(f;x)$ is a unique element in the space of all polynomials of degree at most n, say $\mathcal{P}_n$, which solves the interpolation problem

$$L_n(f;x_j) = f(x_j), \qquad j = 0,1,2,\ldots,n.$$

Lagrange interpolation polynomials are easily computable and therefore they are useful tools for approximating smooth functions and their derivatives, numerical integration and projection methods for numerical treatment of functional equations.

For non-polynomial type interpolation problems, Sloan introduced an interpolating function of the form [18]

$$\psi_n(x) = \sum_{j=0}^{n} a_j u_j(x),$$

where $\{u_j(x)\}_{j=0}^n$ is a set of linearly independent real-valued continuous functions on $[a,b]$ and $\{a_j\}_{j=0}^n$ are determined by the interpolation conditions

$$\psi_n(x_j) = f(x_j), \qquad j = 0,1,\ldots,n.$$

The function $\psi_n(x)$ exists and is unique in the space of span$\{u_j\}_{j=0}^n$ for all $f \in C[a,b]$ if and only if the matrix $\{u_j(x_k)\}_{j,k=0}^n$ is nonsingular. It is clear that the problem is reduced to a polynomial interpolation if $u_j(x) = x^j$.

In [3, p. 62] the authors discuss a more general definition. With prescribed nodes $x_0,\ldots,x_n$ in X, where X is an arbitrary set, they define

$$v_j(x) = \prod_{\substack{k=0\\k\neq j}}^{n} \frac{\varphi(x,x_k)}{\varphi(x_j,x_k)}, \qquad j = 0,1,2,\ldots,n,$$

where $\varphi : X \times X \to \mathbb{R}$ is a function that

$$\varphi(x,y) = 0 \quad \Leftrightarrow \quad x = y.$$

Therefore

$$v_j(x_k) = \delta_{j,k} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases}$$

and the interpolating function is defined by $p(x;f) = \sum\limits_{j=0}^{n} f(x_j)v_j(x)$.

As a special case of the above-mentioned extension, Dişibüyük has recently considered an interpolation problem to the space $\pi_n(f_1, f_2)$, which is spanned by the basis $\left\{f_1^{n-k} f_2^k\right\}_{k=0}^n$, where f_1 and f_2 are two linearly independent functions. Then, he introduces the function [6]

$$l_j(x) = \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{d(x_k, x)}{d(x_k, x_j)}, \qquad j = 0, 1, 2, \ldots, n,$$

as a generalization of Lagrange polynomial (1.2) in which

$$d(x, y) = f_1(x) f_2(y) - f_1(y) f_2(x).$$

By noting these assumptions, the interpolating function takes the form

$$p(x) = \sum_{j=0}^{n} f(x_j) l_j(x),$$

so that $l_j(x_k) = \delta_{j,k}$.

The aim of this paper is to introduce a unified class of interpolating functions, which generalizes Lagrange interpolating polynomials. The paper is organized as follows. Hence, next Section contains the definition and the main properties of such interpolating systems with an estimate for the remainder term. Section 2.1 is devoted to study some special cases. In Section 3, we construct weighted quadrature rules corresponding to the introduced interpolation formulas and in Section 4 we present several numerical examples and compare them with standard cases.

2 A Class of Interpolation Formulas

Let $\Lambda_{n-1} := \{\lambda_l(x)\}_{l=1}^{n-1}$ be a sequence of continuous functions defined on $[a, b]$ such that $\lambda_l(x_i) \neq \lambda_l(x_j)$ for any $i \neq j$ and $\{x_k\}_{k=1}^n$ are n distinct points in $[a, b]$. We define the functions $\Phi_k(x, \Lambda_{n-1})$ for $k = 1, 2, \ldots, n$ respectively as follows

$$\Phi_1(x, \Lambda_{n-1}) := \frac{(\lambda_1(x) - \lambda_1(x_2))(\lambda_2(x) - \lambda_2(x_3)) \ldots (\lambda_{n-1}(x) - \lambda_{n-1}(x_n))}{(\lambda_1(x_1) - \lambda_1(x_2))(\lambda_2(x_1) - \lambda_2(x_3)) \ldots (\lambda_{n-1}(x_1) - \lambda_{n-1}(x_n))},$$

$$\Phi_2(x, \Lambda_{n-1}) := \frac{(\lambda_1(x) - \lambda_1(x_1))(\lambda_2(x) - \lambda_2(x_3)) \ldots (\lambda_{n-1}(x) - \lambda_{n-1}(x_n))}{(\lambda_1(x_2) - \lambda_1(x_1))(\lambda_2(x_2) - \lambda_2(x_3)) \ldots (\lambda_{n-1}(x_2) - \lambda_{n-1}(x_n))},$$

$$\vdots \qquad\qquad \vdots$$

$$\Phi_n(x, \Lambda_{n-1}) := \frac{(\lambda_1(x) - \lambda_1(x_1))(\lambda_2(x) - \lambda_2(x_2)) \ldots (\lambda_{n-1}(x) - \lambda_{n-1}(x_{n-1}))}{(\lambda_1(x_n) - \lambda_1(x_1))(\lambda_2(x_n) - \lambda_2(x_2)) \ldots (\lambda_{n-1}(x_n) - \lambda_{n-1}(x_{n-1}))}.$$

They can also be represented as

$$\Phi_k(x, \Lambda_{n-1}) := \prod_{(i,j)\in I^*_{n,k}} \frac{\lambda_i(x) - \lambda_i(x_j)}{\lambda_i(x_k) - \lambda_i(x_j)}, \qquad k = 1, \ldots, n, \tag{2.1}$$

where

$$I_{n,k}^* := \{(r,r)|\, r = 1, \ldots, k-1,\; k > 1\} \cup \{(r, r+1)|\, r = k, \ldots, n-1,\; k < n\}\,.$$

One can directly verify that the functions introduced in (2.1) satisfy the relations

$$\Phi_k(x_l, \Lambda_{n-1}) = \delta_{k,l}, \qquad l, k = 1, 2, \ldots, n.$$

$\{\Phi_k(x, \Lambda_{n-1})\}_{k=1}^n$ is a linear independent system on $[a, b]$. Now, by considering the space

$$\Pi_n := \text{span}\ \{\Phi_1(x, \Lambda_{n-1}), \ldots, \Phi_n(x, \Lambda_{n-1})\}\,,$$

denote by $G(x; f; \Lambda_{n-1})$ the unique element in Π_n interpolating f for which the following interpolation conditions at n given points (interpolation nodes) are satisfied:

$$G(x_k; f; \Lambda_{n-1}) = f(x_k), \qquad k = 1, 2, \ldots, n.$$

It is easy to see that the mentioned interpolation function exists uniquely, because

$$\begin{vmatrix} \Phi_1(x_1, \Lambda_{n-1}) & \cdots & \Phi_n(x_1, \Lambda_{n-1}) \\ \vdots & & \vdots \\ \Phi_1(x_n, \Lambda_{n-1}) & \cdots & \Phi_n(x_n, \Lambda_{n-1}) \end{vmatrix} = \begin{vmatrix} 1 & & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & & 1 \end{vmatrix} \neq 0.$$

Indeed, $\{\Phi_k(x, \Lambda_{n-1})\}_{k=1}^n$ is a Chebyshev system [10].

Evidently, the interpolation function can be represented as

$$G(x; f; \Lambda_{n-1}) := \sum_{k=1}^{n} f(x_k)\Phi_k(x, \Lambda_{n-1}). \tag{2.2}$$

Note that to construct the basis functions $\{\Phi_k(x, \Lambda_{n-1})\}_{k=1}^n$, we first need to have a sequence of numbers $\{\lambda_i(x_k)\}$ for $i = 1, 2, \ldots, n-1$ and $k = 1, 2, \ldots, n$. These quantities and subsequently the basis functions $\Phi_k(x, \Lambda_{n-1})$ do not depend on the data $f(x_k)$ and it is not required to recompute the basis functions to interpolate each new function.

Remark 2.1 Let $\Lambda_{n,1} = \{\lambda_l(x)\}_{l=1}^n$ and $\Lambda_{n,2} = \{a_l\lambda_l(x) + b_l\}_{l=1}^n$ be two given sequences of continuous functions, where a_l, b_l are real numbers and $a_l \neq 0$ for every l. It can be verified that $\Lambda_{n,1}$ and $\Lambda_{n,2}$ eventually lead to one result with respect to distinct points $\{x_k\}_{k=1}^{n+1}$ provided that $\lambda_l(x_i) \neq \lambda_l(x_j)$, $\forall\, i \neq j$.

Remark 2.2 Given n distinct points $\{x_k\}_{k=1}^n$ in $[a, b]$, assume that $\Lambda_{n-1} = \{lx\}_{l=1}^{n-1}$ and then consider the functions

$$\Phi_k\big(x, \{lx\}_{l=1}^{n-1}\big) = \prod_{(h,j)\in I_{n,k}^*} \frac{hx - hx_j}{hx_k - hx_j}.$$

Now assume that $\Delta_{n-1}=\{x+l\}_{l=1}^{n-1}$ and consider the functions

$$\Psi_k\big(x,\{x+l\}_{l=1}^{n-1}\big)=\prod_{(h,j)\in I^*_{n,k}}\frac{(x+h)-(x_j+h)}{(x_k+h)-(x_j+h)}.$$

We observe that both Λ_{n-1} and Δ_{n-1} have the same result as

$$\Phi_k\big(x,\{lx\}_{l=1}^{n-1}\big)=\Psi_k\big(x,\{x+l\}_{l=1}^{n-1}\big)=\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{x-x_j}{x_k-x_j},$$

and therefore the corresponding interpolating function is the Lagrange polynomial of degree $n-1$ as

$$G\big(x;f;\{lx\}_{l=1}^{n-1}\big)=G\big(x;f;\{x+l\}_{l=1}^{n-1}\big)=\sum_{k=1}^{n}f(x_k)\,\ell_k^{(n)}(x).$$

Remark 2.3 If $\lambda_l(x)=\lambda(x)$ for every $l=1,\ldots,n$, with $\lambda(x_i)\neq\lambda(x_j)$ for $i\neq j$, then

$$G(x;f;\{\lambda(x)\}_{l=1}^{n})=\sum_{k=1}^{n+1}f(x_k)\Phi_k(x,\{\lambda(x)\}_{l=1}^{n}),$$

interpolates f at $\{x_k\}_{k=1}^{n+1}$, where

$$\Phi_k(x,\{\lambda(x)\}_{l=1}^{n})=\prod_{\substack{j=1\\ j\neq k}}^{n+1}\frac{\lambda(x)-\lambda(x_j)}{\lambda(x_k)-\lambda(x_j)}.$$

In this case, we have

$$\text{span}\,\{\Phi_k(x,\{\lambda(x)\}_{l=1}^{n})\}_{k=1}^{n+1}\;=\;\text{span}\{1,\lambda(x),\lambda^2(x),\ldots,\lambda^n(x)\},$$

and $G(x;f;\{\lambda(x)\}_{l=1}^{n})$ can be therefore represented as

$$G(x;f;\{\lambda(x)\}_{l=1}^{n})=A_0+A_1\lambda(x)+A_2\left(\lambda(x)\right)^2+\cdots+A_n\left(\lambda(x)\right)^n,$$

in which $\{A_j\}_{j=0}^{n}$ are obtained by the given interpolation conditions. Also, the generalized Newton representation of $G(x;f;\{\lambda(x)\}_{l=1}^{n})$ is given in [12], where a generalization of divided differences is introduced for constructing some interpolation formulas.

Theorem 2.4 *Consider a sequence of continuous functions $\Lambda_{n-1}=\{\lambda_k\}_{k=1}^{n-1}$ on $[a,b]$ and n distinct points $\{x_k\}_{k=1}^{n}$such that $\lambda_k(x_i)\neq\lambda_k(x_j)$ when $i\neq j$. Let $f\in C^{n-1}([a,b])$ and $f^{(n)}$ exist at any point of (a,b). Suppose that $G(\cdot;f;\Lambda_{n-1})$ interpolates f at $\{x_k\}_{k=1}^{n}$, $G(\cdot;f;\Lambda_{n-1})\in C^{n-1}([a,b])$, and $G^{(n)}(\cdot;f;\Lambda_{n-1})$ exist at any point of (a,b). Then, there exists a point $\xi_x\in(a,b)$ depending on x such that*

$$f(x)-G(x;f;\Lambda_{n-1})=\frac{f^{(n)}(\xi_x)-G^{(n)}(\xi_x;f;\Lambda_{n-1})}{n\,!}\prod_{k=1}^{n}(x-x_k). \tag{2.3}$$

Proof: Define

$$F(x) = f(x) - G(x; f; \Lambda_{n-1}) - \gamma(\bar{x}) \prod_{k=1}^{n} (x - x_k),$$

where $\gamma(\bar{x})$ is defined in such a way that $F(\bar{x}) = 0$ for any arbitrary point $\bar{x} \neq \{x_k\}_{k=1}^{n}$ in $[a, b]$. In this case, we have

$$\gamma(\bar{x}) = \frac{f(\bar{x}) - G(\bar{x}; f; \Lambda_{n-1})}{\prod\limits_{k=1}^{n} (\bar{x} - x_k)}.$$

Since $F(x)$ has at least $n+1$ distinct roots in $[a, b]$, i.e. $x_1, \ldots, x_n, \bar{x}$, applying Rolle's theorem successively n times, we find out that the function $F^{(n)}(x)$ has at least one zero in (a, b), say ξ_x. Hence $F^{(n)}(\xi_x) = 0$ and

$$\gamma(\bar{x}) = \frac{f^{(n)}(\xi_x) - G^{(n)}(\xi_x; f; \Lambda_{n-1})}{n\,!}.$$

This means that

$$f(\bar{x}) - G(\bar{x}; f; \Lambda_{n-1}) = \frac{f^{(n)}(\xi_x) - G^{(n)}(\xi_x; f; \Lambda_{n-1})}{n\,!} \prod_{k=1}^{n} (\bar{x} - x_k),$$

which also holds for any $\bar{x} = x$. □

Remark 2.5 Notice that if $\lambda_k(x) = a_k x + b_k$, $1 \leq k \leq n-1$, with $a_k \neq 0$, then

$$G(x; f; \{a_k x + b_k\}_{k=1}^{n-1}),$$

is a polynomial of degree at most $n-1$ and $G^{(n)}(x; f; \{a_k x + b_k\}_{k=1}^{n-1}) = 0$. Therefore

$$f(x) - G(x; f; \{a_k x + b_k\}_{k=1}^{n-1}) = \frac{f^{(n)}(\xi_x)}{n\,!} \prod_{k=1}^{n} (x - x_k),$$

which is the remainder term of Lagrange interpolation for functions $f \in C^n([a, b])$.

Remark 2.6 Since $\Lambda_{n-1} = \{\lambda_k(x)\}_{k=1}^{n-1}$ are not fixed, it is clear that we cannot discuss about the convergence and stability of the introduced interpolations in a general case unless $\{\lambda_k(x)\}_{k=1}^{n-1}$ are explicitly specified.

2.1 Some Particular Examples of Formula (2.2)

Example 2.7 Let

$$\lambda_l(x) = x^{b_l} \qquad \text{for} \qquad l = 1, \ldots, n-1,$$

where $b_1 < b_2 < \cdots < b_{n-1}$ are nonzero real values. For given distinct points $\{x_k\}_{k=1}^n$ in $(0,\infty)$, and distinct values $\{w_k\}_{k=1}^n$ there exists a Müntz polynomial of order $\leq \sum_{l=1}^{n-1} b_l$, which is the unique solution of the interpolation problem

$$f(x_k) = w_k, \qquad k = 1, \ldots, n, \tag{2.4}$$

and can be represented as

$$G\Big(x; f; \{x^{b_l}\}_{l=1}^{n-1}\Big) = \sum_{k=1}^{n} w_k \Phi_k\Big(x, \{x^{b_l}\}_{l=1}^{n-1}\Big),$$

in which

$$\Phi_k\Big(x, \{x^{b_l}\}_{l=1}^{n-1}\Big) = \prod_{(i,j)\in I_{n,k}^*} \frac{x^{b_i} - x_j^{b_i}}{x_k^{b_i} - x_j^{b_i}}.$$

For instance, for $n = 4$ and the parameters $b_1 = \frac{1}{3}$, $b_2 = \frac{1}{2}$, $b_3 = 1$, the basis functions $\Phi_k\Big(x, \{x^{b_l}\}_{l=1}^{3}\Big)$ with nodes $\{1,2,3,4\}$ are displayed in figure 2.1 and labeled as $\mathrm{phi1}$, $\mathrm{phi2}$, $\mathrm{phi3}$, $\mathrm{phi4}$, respectively.

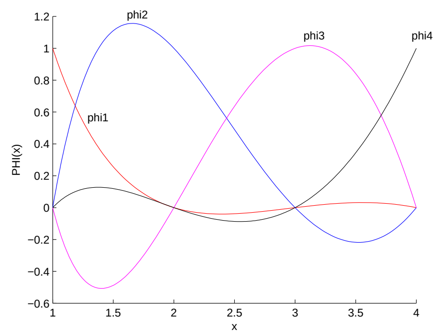


Figure 2.1 The basis functions introduced in example 2.7 ($n = 4$)

Example 2.8 Let

$$\lambda_l(x) = \frac{1}{x - \beta_l} \qquad \text{for} \qquad l = 1, \ldots, n-1,$$

where $\beta_1 < \beta_2 < \cdots < \beta_{n-1}$ are real values. For given distinct points $\{x_k\}_{k=1}^n$ in (β_{n-1}, ∞) first we have

$$\Phi_k\big(x, \{\frac{1}{x-\beta_l}\}_{l=1}^{n-1}\big) = \prod_{(i,j)\in I_{n,k}^*} \frac{\frac{1}{x-\beta_i} - \frac{1}{x_j-\beta_i}}{\frac{1}{x_k-\beta_i} - \frac{1}{x_j-\beta_i}} = \prod_{(i,j)\in I_{n,k}^*} \frac{x_k - \beta_i}{x_k - x_j}\,\frac{x - x_j}{x - \beta_i}\,.$$

Then, the unique solution of the interpolation problem (2.4) in the space generated by span $\left\{\Phi_k(x,\{\frac{1}{x-\beta_l}\}_{l=1}^{n-1})\right\}_{k=1}^{n}$ can be represented as

$$G\Big(x;f;\{\frac{1}{x-\beta_l}\}_{l=1}^{n-1}\Big)=\sum_{k=1}^{n}w_k\Phi_k\Big(x,\{\frac{1}{x-\beta_l}\}_{l=1}^{n-1}\Big).$$

For $n=4$ and the parameters $\beta_1=-4$, $\beta_2=-3$, $\beta_3=-2$, the basis functions $\Phi_k(x,\{\frac{1}{x-\beta_l}\}_{l=1}^{3})$ with nodes $x_k=\cos\frac{(2k-1)\pi}{8}$, $k=1,2,3,4$, are displayed in figure 2.2.

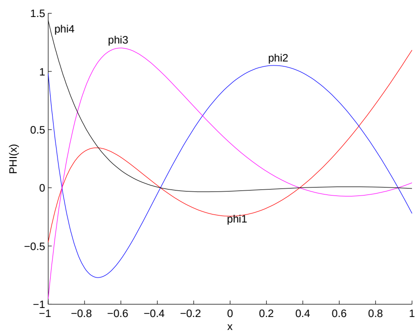


Figure 2.2 The basis functions introduced in example 2.8 ($n=4$)

Example 2.9 Given n distinct points $-\frac{\pi}{2}\le x_1<\ldots<x_n\le\frac{\pi}{2}$, assume that

$$\lambda_l(x)=\sin\frac{x}{l}\qquad\text{for}\qquad l=1,\ldots,n-1,$$

and then construct the functions

$$\Phi_k\Big(x,\{\sin\frac{x}{l}\}_{l=1}^{n-1}\Big)=\prod_{(h,j)\in I^*_{n,k}}\frac{\sin\frac{x}{h}-\sin\frac{x_j}{h}}{\sin\frac{x_k}{h}-\sin\frac{x_j}{h}}.$$

As before, the final interpolating function is given by

$$G\Big(x;f;\{\sin\frac{x}{l}\}_{l=1}^{n-1}\Big)=\sum_{k=1}^{n}f(x_k)\Phi_k\Big(x,\{\sin\frac{x}{l}\}_{l=1}^{n-1}\Big).$$

For $n=4$ the basis functions $\Phi_k\Big(x,\{\sin\frac{x}{l}\}_{l=1}^{3}\Big)$ with equally spaced nodes $\{x_k\}_{k=1}^{4}$ on $[-\frac{\pi}{2},\frac{\pi}{2}]$ are displayed in figure 2.3.

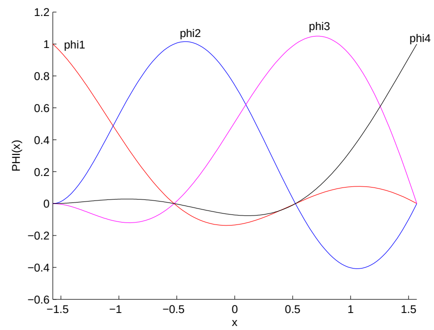


Figure 2.3 The basis functions introduced in example 2.9 ($n = 4$)

The next case emphasizes that it is not necessary for $\{\lambda_l(x)\}$ to be just a homogeneous sequence of functions. In other words, for given distinct points $\{x_k\}_{k=1}^n \in [a, b]$, one can choose any arbitrary set of continuous functions $\Lambda_{n-1} = \{\lambda_l(x)\}_{l=1}^{n-1}$ provided that $\lambda_l(x_i) \neq \lambda_l(x_j), \ \forall\, i \neq j$.

Example 2.10 Consider the non-homogeneous set

$$\Lambda_5 = \left\{x,\, e^{-x},\, e^{x},\, \cos x,\, \sin x\right\}.$$

For 6 distinct points $0 \leq x_1 < \cdots < x_6 \leq \frac{\pi}{2}$ and given values $w_1, \ldots, w_6$, the unique solution of the interpolation problem (2.4) is represented by

$$G(x; f; \Lambda_5) = \sum_{k=1}^{6} w_k \Phi_k(x, \Lambda_5),$$

where

$$\Phi_1(x, \Lambda_5) = \frac{(x - x_2)(e^{-x} - e^{-x_3})(e^{x} - e^{x_4})(\cos x - \cos x_5)(\sin x - \sin x_6)}{(x_1 - x_2)(e^{-x_1} - e^{-x_3})(e^{x_1} - e^{x_4})(\cos x_1 - \cos x_5)(\sin x_1 - \sin x_6)},$$

$$\Phi_2(x, \Lambda_5) = \frac{(x - x_1)(e^{-x} - e^{-x_3})(e^{x} - e^{x_4})(\cos x - \cos x_5)(\sin x - \sin x_6)}{(x_2 - x_1)(e^{-x_2} - e^{-x_3})(e^{x_2} - e^{x_4})(\cos x_2 - \cos x_5)(\sin x_2 - \sin x_6)},$$

$$\vdots \qquad\qquad \vdots$$

$$\Phi_6(x, \Lambda_5) = \frac{(x - x_1)(e^{-x} - e^{-x_2})(e^{x} - e^{x_3})(\cos x - \cos x_4)(\sin x - \sin x_5)}{(x_6 - x_1)(e^{-x_6} - e^{-x_2})(e^{x_6} - e^{x_3})(\cos x_6 - \cos x_4)(\sin x_6 - \sin x_5)}.$$

The basis functions $\Phi_k\big(x, \Lambda_5\big)$ with equally spaced nodes $\{x_k\}_{k=1}^6$ on $[0, \frac{\pi}{2}]$ are displayed in figure 2.4.

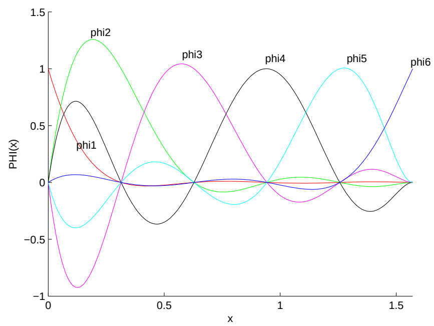


Figure 2.4 The basis functions introduced in example 2.10 ($n = 6$)

3 Quadrature Rules Based on the Introduced Interpolation Formulas

A general *n*-point quadrature formula is denoted by

$$\int_a^b f(x)\omega(x)dx = \sum_{k=1}^{n} w_{k,n} f(x_k) + R_n[f], \tag{3.1}$$

where $\{x_k\}_{k=1}^n$ and $\{w_{k,n}\}_{k=1}^n$ are respectively nodes and weight coefficients and $R_n[f]$ denotes the corresponding error ([5, 8]). Here $\omega(x)$ is also a nonnegative weight function on the interval $[a,b]$. This formula has degree of exactness *d* if for every polynomial $p \in \mathcal{P}_d$ we have $R_n[p] = 0$. In addition, if $R_n[p] \neq 0$ for some $\mathcal{P}_{d+1}$, formula (3.1) has precise degree of exactness *d*.

The convergence order of formula (3.1) depends on the smoothness of the function f as well as on its degree of exactness ([11]). It is well known that for given mutually different nodes $\{x_k\}_{k=1}^n$ we can always achieve a degree of exactness $d = n-1$ by interpolating at these nodes and integrating the interpolated polynomial instead of f. Namely, taking the node polynomial $w_n(x) = \prod_{k=1}^{n}(x - x_k)$, multiplying the Lagrange interpolation formula by $\omega(x)$ and then integrating gives (3.1) with

$$w_{k,n} = \frac{1}{w_n'(x_k)} \int_a^b \frac{w_n(x)}{x - x_k}\omega(x)dx, \qquad k = 1, 2, \ldots, n, \tag{3.2}$$

and

$$R_n[f] = \int_a^b r_n(f;x)\omega(x)dx,$$

where $r_n(f;x)$ denotes the error function of the Lagrange interpolation formula.

In this sense, if $f \in C^n[a,b]$, then (see, e.g., [5])

$$R_n[f] = \frac{1}{n!}\int_a^b f^{(n)}\big(\xi(x)\big) w_n(x)\omega(x)dx, \qquad \text{where } \xi(x) \in (a,b).$$

Now, by considering the interpolating function (2.2), we can follow the above approach and construct a quadrature rule as

$$\int_a^b f(x)\omega(x)dx = \sum_{k=1}^{n} w^*_{k,n} f(x_k) + R^*_n[f],$$

where

$$\begin{aligned} w^*_{k,n} &= \int_a^b \Phi_k(x,\Lambda_{n-1})\omega(x)dx \\ &= \frac{1}{\prod_{(i,j)\in I^*_{n,k}} (\lambda_i(x_k)-\lambda_i(x_j))} \int_a^b \omega(x) \prod_{(i,j)\in I^*_{n,k}} \big(\lambda_i(x)-\lambda_i(x_j)\big)dx, \end{aligned} \tag{3.3}$$

and

$$R^*_n[f] = \frac{1}{n\,!}\int_a^b \Big(f^{(n)}(\zeta(x)) - G^{(n)}(\zeta(x);f;\Lambda_{n-1})\Big)\, w_n(x)\omega(x)dx, \quad a<\zeta(x)<b. \tag{3.4}$$

Remark 3.1 Relation (3.4) shows that there might be a sequence of functions for $\Lambda_{n-1} = \{\lambda_l(x)\}_{l=1}^{n-1}$ such that $f^{(n)}(\zeta(x)) - G^{(n)}(\zeta(x);f;\Lambda_{n-1}) \to 0$ for any $\zeta(x)\in(a,b)$.

Example 3.2 Let

$$\Lambda_4 = \{\lambda_l(x)\}_{l=1}^{4} = \big\{x+1\,,\ x^2+5x\,,\ x^3+5.1774\,x\,,\ x^3-0.4851x^2-6x-3\big\}\,,$$

be a sequence of polynomial functions which is not homogeneous and

$$\{x_k\}_{k=1}^{5} = \{0.1\,,\ 0.3\,,\ 0.5\,,\ 0.7\,,\ 0.9\} \in [0,1].$$

By noting remark 3.1, we can construct a five-point interpolatory quadrature with e.g. the constant weight function $\omega(x)=1$ on $[0,1]$ as follows

$$\int_0^1 f(x)\,dx \approx \sum_{k=1}^{5} w^*_{k,5} f(x_k),$$

where according to (3.3) the coefficients are computed as

$$w^*_{1,5} = 0.23810506, \quad w^*_{2,5} = 0.08965330, \quad w^*_{3,5} = 0.34395356,$$
$$w^*_{4,5} = 0.09043601, \quad w^*_{5,5} = 0.23792545.$$

For instance, substituting $f(x)=\frac{1}{x+1}$ in the above quadrature gives

$$\ln 2 = \int_0^1 \frac{1}{x+1}\,dx \approx \sum_{k=1}^{5} \frac{1}{x_k+1}\, w^*_{k,5} \cong 0.693147180511734,$$

and the corresponding error (3.4) is estimated as $R_n^*[f] = 4.8210 \times 10^{-11}$.

On the other hand, by considering the classical Lagrange case $\Lambda_4 = \{\lambda_l(x)\}_{l=1}^4 = \{x\}$, which leads to the quadrature formula with weighting coefficients (3.2) in which $\omega(x) = 1$, we have

$$w_{1,5} = 0.23871527, \quad w_{2,5} = 0.08680555, \quad w_{3,5} = 0.34895833,$$
$$w_{4,5} = 0.08680555, \quad w_{5,5} = 0.23871527,$$

and therefore

$$\int_0^1 \frac{1}{x+1}\,dx \approx \sum_{k=1}^{5} \frac{1}{x_k+1}\,w_{k,5} \cong 0.693127993437590.$$

In latter case, the corresponding error is estimated as $R_n[f] = 1.9187 \times 10^{-5}$.

4 Numerical Examples

Example 4.1 Let us choose a sequence of polynomial functions as

$$\Lambda_4 = \{2x^3 + 3x^2 - 6x + 6,\ x^2 + 2x - 2,\ x^2 + 1,\ x + 10\},$$

and interpolating points as

$$\{x_k\}_{k=1}^5 = \{\frac{1}{2}(1+\cos\frac{(11-2k)\pi}{10})\}_{k=1}^5 \cong \{0.0245, 0.2061, 0.5000, 0.7939, 0.9755\}\,.$$

According to formula (2.2), the function $f(x) = \frac{\log(1+x)}{(1+x^2)^6}e^{x^2}$, $x \in [0,1]$ taken from [15] can be approximated by a polynomial of degree 8 as follows

$$\frac{\log(1+x)}{(1+x^2)^6}e^{x^2} \approx 0.1514\,x^8 + 0.3449\,x^7 - 1.3336\,x^6 - 1.1887\,x^5 + 4.6404\,x^4 - 2.1424\,x^3 - 1.5700\,x^2 + 1.1280\,x - 0.0025.$$

In this sense, the error norm

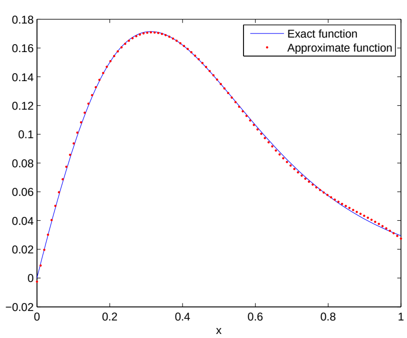
$$E_n = \max_{t_j} |f(t_j) - G(t_j; f; \Lambda_{n-1})|\,, \tag{4.1}$$

is estimated as $E_5 = 0.0025$ in which $\{t_j\}$ have been considered as 100 equidistant points of $[0,1]$. Moreover, figure 4.1(b) shows the interpolation error for $n = 5$ and the graph of exact and approximate functions are displayed in figure 4.1(a).

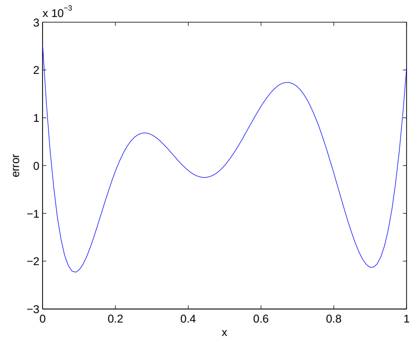
Now, let us consider the Lagrange case $\Delta_4 = \{\delta_l(x) = x\}_{l=1}^4$ to construct the interpolating function with the previous given points. In this case, the function is approximated by a polynomial of degree 4 as follows

$$\frac{\log(1+x)}{(1+x^2)^6}e^{x^2} \approx -0.8577\,x^4 + 2.9169\,x^3 - 3.3745\,x^2 + 1.3522\,x - 0.0070,$$

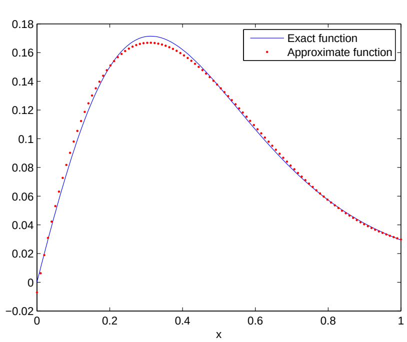
which is indeed the Lagrange interpolating polynomial of f, and the error norm is estimated as $E_5 = 0.0070$. Figure 4.1(d) shows the interpolation error when Δ_4 is applied and the graph of exact and approximate functions are displayed in figure 4.1(c).



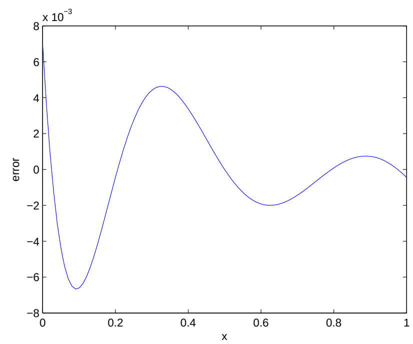
(a) Approximate function is obtained by considering Λ_4 (new approach)



(b)



(c) Approximate function is obtained by considering Δ_4 (classical approach)



(d)

Figure 4.1 Plots of example 4.1

Example 4.2 We consider the function $f(x) = \frac{1}{1+25x^2}$ given by Runge. Figure 4.3 shows interpolation errors on $[-1, 1]$, where the interpolating points are roots of the first kind of Chebyshev polynomials up to degree 80, i.e.

$$x_k = \cos\frac{(2k-1)\pi}{2n}, \qquad k = 1, 2, \ldots, n = 80,$$

and interpolating functions (2.2) are computed in different ways by considering various sequences of functions $\Lambda_{n-1} = \{\lambda_l(x)\}_{l=1}^{n-1}$, which are specified in table 4.1. In this table, we have computed E_n where t_j are 200 equidistant points of $[-1, 1]$. Finally, graphs of exact and approximate functions are displayed in figure 4.2.

Table 4.1 Interpolation errors for $f(x) = \frac{1}{1+25x^2}$

case	$\lambda_l(x),\ \ l = 1, 2, \ldots, n-1$	E_n
(a)	$\sin \frac{x}{l}$	2.1873×10^{-7}
(b)	$\exp \frac{x}{l}$	2.0590×10^{-7}
(c)	$\frac{1}{x-(2l-81)}$	3.0918×10^{-7}
(d)	x	2.2986×10^{-7}

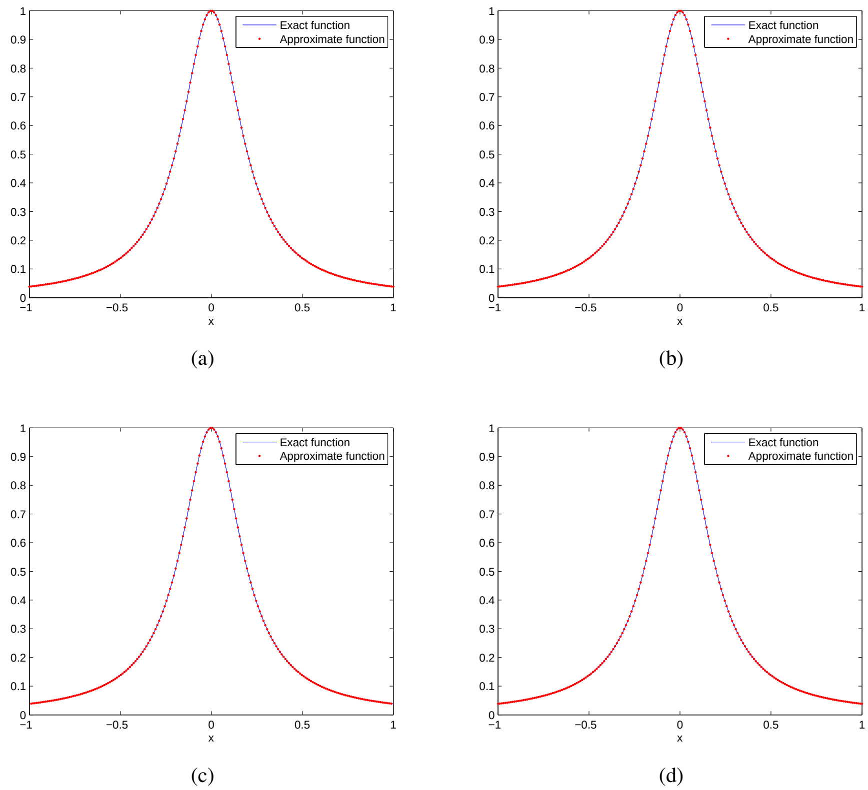


Figure 4.2 Comparison of exact and approximate functions for $n = 80$

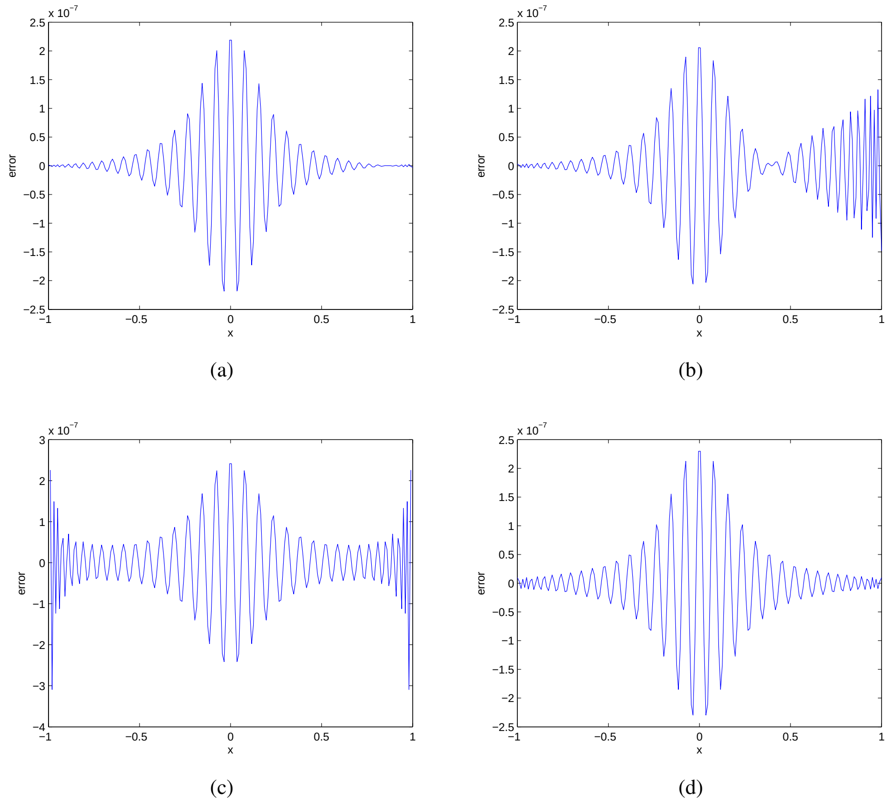


Figure 4.3 Interpolation errors for $f(x) = \frac{1}{1+25x^2}$ and $n = 80$

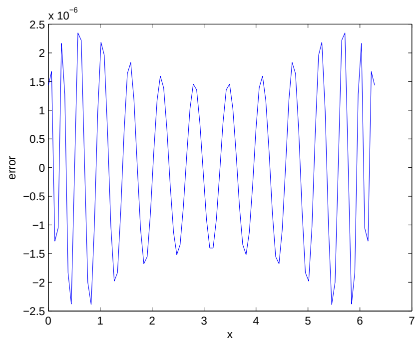
Example 4.3 We consider the function $f(x) = \frac{3}{5-4\cos x}$ taken from [9]. By choosing the roots of Chebyshev polynomials on $[0, 2\pi]$ as interpolating points, i.e.

$$x_k = \pi\big(\cos\frac{(2k-1)\pi}{2n} + 1\big), \qquad k = 1, 2, \dots, n,$$

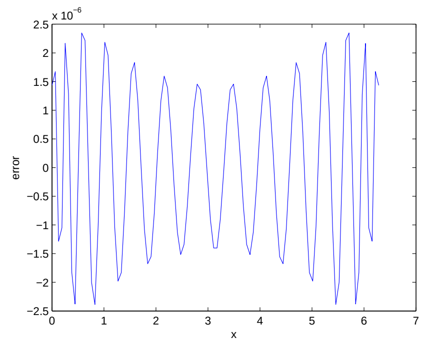
interpolating functions (2.2) are computed in different ways by considering various sequences of functions $\Lambda_{n-1} = \{\lambda_l(x)\}_{l=1}^{n-1}$, which are specified in table 4.2. In this table, E_n defined in (4.1) is computed for t_j which are 100 equidistant points of $[0, 2\pi]$. Also, figure 4.4 shows interpolation errors for $n = 30$.

Table 4.2 Interpolation errors for $f(x) = \frac{3}{5-4\cos x}$

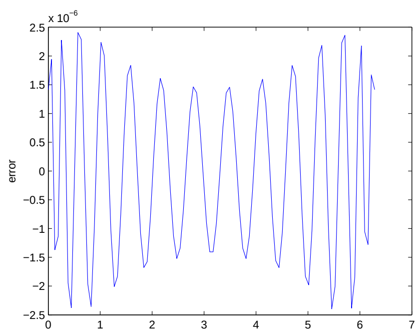
case	(a)	(b)	(c)	(d)
n \ $\lambda_l(x)$	x	$\frac{1}{0.1x+10nl}$	$\exp(\frac{1}{x+10nl})$	$\left(\exp(\frac{0.1x}{l})\right)^{\frac{0.1}{l}}$
5	0.3957	0.3979	0.4195	0.3996
10	0.0322	0.0322	0.0322	0.0322
15	0.0036	0.0036	0.0037	0.0036
20	3.9404×10^{-4}	3.9404×10^{-4}	3.9395×10^{-4}	3.9404×10^{-4}
25	2.1924×10^{-5}	2.1992×10^{-5}	2.2365×10^{-5}	2.2280×10^{-5}
30	2.3871×10^{-6}	2.3871×10^{-6}	2.4084×10^{-6}	2.3838×10^{-6}



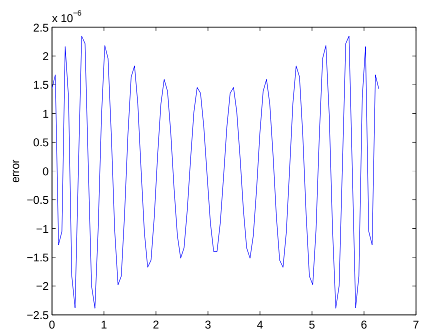
(a)



(b)



(c)



(d)

Figure 4.4 Interpolation errors for $f(x) = \frac{3}{5-4\cos x}$ and $n = 30$

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