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On linearization coefficients of Jacobi polynomials

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ABSTRACT

This article deals with the problem of finding closed analytical formulae for generalized linearization coefficients for Jacobi polynomials. By considering some special cases, we obtain a reduction formula using for this purpose symbolic computation, in particular Zeilberger's and Petkovsek's algorithms.

The *general linearization problem* consists in finding the coefficients $L_{ij}(k)$ in the expansion of two polynomials $Q_i(x), R_j(x)$ in terms of an arbitrary sequence $\{P_n\}_{n\geq 0}$ ($\deg P_n = n$):

$$Q_i(x)R_j(x) = \sum_{k=0}^{i+j} L_{ij}(k)P_k(x). \tag{1}$$

Particular case of this problem is the *standard linearization* or *Clebsch–Gordan type problem* ($P_n = Q_n = R_n$),

$$P_i(x)P_j(x) = \sum_{k=0}^{i+j} L_{ij}(k)P_k(x). \tag{2}$$

On the other hand, taking $R_j = 1$ in (1), this is, the so-called *connection problem*, which for $Q_i = x^i$ is known as the inversion problem for the family $\{P_n\}_n$.

The literature on linearization and connection problems is extremely vast, and a variety of methods and approaches for computing the coefficients $L_{ij}(k)$ in (1) have been developed. In the standard case (2), when $\{P_n\}_n$ is an orthogonal family (with respect to some positive measure), many results concerning the positivity of the coefficients $L_{ij}(k)$ and the recurrence relation satisfied by these coefficients are known, in some cases (classical orthogonal polynomials) the coefficients $L_{ij}(k)$ are given explicitly, very often in terms of hypergeometric functions.

We recall that ${}_pF_q$ denotes the generalized hypergeometric function with p numerator and q denominator parameters, given by

$${}_pF_q\left(\begin{matrix}(a_p)\\(b_q)\end{matrix}\,\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{x^k}{k!}, \tag{3}$$

where the contracted notation (a_p) is used to abbreviate the array of p parameters $a_1, \ldots, a_p$ and $(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)}$ denotes the well-known Pochhammer symbol.

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In this work, we consider the Jacobi polynomials defined by [1]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, \alpha+\beta+n+1\\ \alpha+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right).$$

The standard linearization problem associated to Jacobi polynomials and to establish the conditions of non-negativity of the linearization coefficients has been under considerable research for many years. Hyllareas (1962) investigated particular cases [2], Gasper (1970) found the necessary and sufficient conditions for the non-negativity of these coefficients [3,4] and Koornwinder (1978) approached the same problem from a different point of view [5]. Rahman (1981) gave an explicit representation of the standard linearization coefficients, $L_{ij}(k)$, for the Jacobi polynomials and their continuous q-analogue in terms of ${}_9F_8$ and ${}_{10}\Phi_9$ hypergeometric series, respectively, but with distinct explicit representations for even and odd values of k [6,7].

The main aim of this paper is to give a closed form of the general linearization coefficients for Jacobi polynomials in terms of the Kampé de Fériet function and to prove that in a suitable particular case these coefficients can be expressed as a product of two terminating functions. By using symbolic computation, we show that one of these two hypergeometric functions can be reduced to a simple hypergeometric term. As far as we know, the obtained reduction formula for ${}_3F_2$ is not included in any known reduction formula and appears to be new. At the end of this work, we use known connection and linearization formulae for ultraspherical polynomials to derive a reduction formula associated to a terminating double sum.
We note here that this work is motivated by a problem suggested by Dick Askey in a private discussion about linearization coefficients for Jacobi polynomials with special parameters.

The Kampé de Fériet function is the double hypergeometric function defined by: [8, p. 63]

$$F_{l:n}^{p:k}\left(\begin{matrix}(a_p): & (b_k);\ (c_k); & \\ & & x, y\\ (\alpha_l): & (\beta_n);\ (\gamma_n); & \end{matrix}\right) = \sum_{r,s=0}^{\infty} \frac{[a_p]_{r+s}[b_k]_r[c_k]_s}{[\alpha_l]_{r+s}[\beta_n]_r[\gamma_n]_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{4}$$

where $[a_p]_r = \prod_{j=1}^{p}(a_j)_r, \ldots$.
To solve the linearization problem for the Jacobi PS, we need the following result which is proved in [9].

Theorem 1. *Let $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ be three polynomial sets generated, respectively, by*

$$\begin{aligned} A_1(t)B_1\left(xC_1(t)\right) &= \sum_{n=0}^{\infty}\lambda_n^{(1)}P_n(x)t^n,\\ A_2(t)B_2\left(xC_2(t)\right) &= \sum_{n=0}^{\infty}\lambda_n^{(2)}Q_n(x)t^n,\\ A_3(t)B_3\left(xC_3(t)\right) &= \sum_{n=0}^{\infty}\lambda_n^{(3)}R_n(x)t^n, \end{aligned} \tag{5}$$

where A_p, B_p and C_p, are three formal power series satisfying $A_p(0) \neq 0, C_p(0) = 0, C_p'(0) \neq 0, B_p^{(k)}(0) \neq 0 \forall k \neq 0$ and $\lambda_n^{(p)} \neq 0; p = 1, 2, 3$.
Then, the associated linearization coefficients in (1) are given by

$$L_{ij}(k) = \frac{\lambda_k^{(1)}}{\lambda_i^{(2)}\lambda_j^{(3)}}\sum_{r=0}^{i}\sum_{s=0}^{j}\frac{\gamma_r^{(2)}\gamma_s^{(3)}}{\gamma_{r+s}^{(1)}}a_r^{(2)}(i)a_s^{(3)}(j)\psi_{r+s}(k), \quad k = 0, 1, \ldots, i+j, \tag{6}$$

where

$$A_p(t)C_p^m(t) = \sum_{i=m}^{\infty}a_m^{(p)}(i)t^i, \qquad B_p(t) = \sum_{k=0}^{\infty}\gamma_k^{(p)}t^k; \quad p = 1, 2, 3; \quad \text{and} \quad \frac{C_1^{-k}(t)}{A_1(C_1^{-1}(t))} = \sum_{n=k}^{\infty}\psi_n(k)t^n. \tag{7}$$

Recall here that a polynomial set defined by a generating function like in (5) is said to be of Boas–Buck type [10].

The Jacobi polynomial set is generated by [11]

$$(1-t)^{-\tau}{}_2F_1\left(\begin{matrix}\frac{\tau}{2}, \frac{\tau+1}{2}\\ \alpha+1\end{matrix}\,\middle|\,\frac{-2(x-1)t}{(1-t)^2}\right) = \sum_{n=0}^{\infty}\frac{(\tau)_nP_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n}t^n,$$

where $\tau = \alpha+\beta+1$.

It follows that the shifted Jacobi polynomial set is of Boas–Buck type with

$$A(t) = (1-t)^{-\tau}, \qquad C(t) = \frac{-t}{(1-t)^2} \quad \text{and} \quad B(t) = {}_2F_1\left(\begin{matrix}\frac{\tau}{2}, \frac{\tau+1}{2}\\ \alpha+1\end{matrix}\,\middle|\,t\right). \tag{8}$$

For this case, and to get the development of the formal power series in (7), we need the following lemma.

Lemma 2 (*Lagrange's Inversion Formula [8]*). *Let ξ be a function of t implicitly defined by*

$$\xi = t(1+\xi)^{s+1}, \qquad \xi(0) = 0. \tag{9}$$

Then, we have

$$(1+\xi(t))^r = \sum_{n=0}^{\infty} \frac{r}{r+(s+1)n} \binom{r+(s+1)n}{n} t^n, \tag{10}$$

where r and s are complex numbers independent of n.

In our case, we have

$$A(t) = (1-t)^{-\tau} \quad \text{and} \quad C(t) = \frac{-t}{(1-t)^2}.$$

C^{-1} is defined, implicitly, by

$$(1-C^{-1}(t))^2 t = -C^{-1}(t).$$

Using (10), with $\xi = -C^{-1}$, $s = 1$ and $r = \tau + 2k$, we obtain

$$\begin{aligned}
\frac{(C^{-1})^k(t)}{A(C^{-1}(t))} &= (-1)^k (1-C^{-1}(t))^{2k+\tau} t^k \\
&= (-1)^k \sum_{n=0}^{\infty} \frac{\tau+2k}{\tau+2n+2k} \binom{2n+2k+\tau}{n} t^{n+k} \\
&= (-1)^k \sum_{n=k}^{\infty} \frac{\tau+2k}{\tau+2n} \frac{(\tau+1+n+k)_{n-k}}{(n-k)!} t^n.
\end{aligned}$$

On the other hand, it is easy to check that

$$A(t)C^m(t) = (-1)^m \frac{t^m}{(1-t)^{2m+\tau}} = (-1)^m \sum_{n=m}^{\infty} \frac{(2m+\tau)_{n-m}}{(n-m)!} t^n. \tag{11}$$

By using Theorem 1, we deduce that the linearization coefficients in

$$P_i^{(\lambda,\delta)}(x) P_j^{(\mu,\gamma)}(x) = \sum_{k=0}^{i+j} L_{ij}(k) P_k^{(\alpha,\beta)}(x),$$

are given by

$$\begin{aligned}
L_{ij}(i+j-k) &= \frac{(\alpha+\beta+1)_{i+j-k}(\alpha+1)_{i+j}(2(i+j-k)+\alpha+\beta+1)}{(\alpha+1)_{i+j-k}(\alpha+\beta+1)_{2(i+j)-k+1}} \\
&\quad \times \frac{(-1)^k (i+j)!}{i!j!k!} \frac{(\lambda+\delta+1)_{2i}(\mu+\gamma+1)_{2j}}{(\lambda+\delta+1)_i(\mu+\gamma+1)_j} \\
&\quad \times F^{2:\,2}_{2:\,1}\left(\begin{matrix} -k, -\alpha-\beta-1-2(i+j)+k: \; -i, -\lambda-i; -j, -\mu-j; \\ -(i+j), -\alpha-(i+j): \; -2i-\lambda-\delta; -2j-\mu-\gamma; \end{matrix} \; 1, 1 \right).
\end{aligned} \tag{12}$$

In the special case $\alpha = \mu + \lambda$, $\beta = \delta + \gamma$, we get

$$\begin{aligned}
L_{ij}(i+j-k) &= \frac{(\mu+\lambda+\delta+\gamma+1)_{i+j-k}(\mu+\lambda+1)_{i+j}(2(i+j-k)+\mu+\lambda+\delta+\gamma+1)}{(\mu+\lambda+1)_{i+j-k}(\mu+\lambda+\delta+\gamma+1)_{2(i+j)-k+1}} \\
&\quad \times \frac{(-1)^k (i+j)!}{i!j!k!} \frac{(\lambda+\delta+1)_{2i}(\mu+\gamma+1)_{2j}}{(\lambda+\delta+1)_i(\mu+\gamma+1)_j} \\
&\quad \times F^{2:\,2}_{2:\,1}\left(\begin{matrix} -k, -\lambda-\mu-\delta-\gamma-1-2(i+j)+k: \; -i, -\lambda-i; -j, -\mu-j; \\ -(i+j), -\lambda-\mu-(i+j): \; -2i-\lambda-\delta; -2j-\mu-\gamma; \end{matrix} \; 1, 1 \right).
\end{aligned} \tag{13}$$

In view of the Gasper's reduction formula [12] for the product of two terminating hypergeometric functions in terms of a Kampé de Fériet function

$$ {}_3F_2\left(\begin{matrix}-n, n+a, b\\ c, d\end{matrix}\,\middle|\,1\right){}_3F_2\left(\begin{matrix}-n, n+a, e\\ c, f\end{matrix}\,\middle|\,1\right) = \frac{(-1)^n(a-c+1)_n}{(c)_n} \times F^{2:\,2}_{2:\,1}\left(\begin{matrix}-n, n+a:\ b, e; d-b, f-e;\\ d, f:\ c; a-c+1;\end{matrix}\quad 1, 1\right), \tag{14} $$

the linearization coefficient in (13) can be written as

$$ \begin{aligned} L_{ij}(i+j-k) = {} & \frac{(\alpha+\beta+1)_{i+j-k}(\alpha+1)_{i+j}(2(i+j-k)+\alpha+\beta+1)}{(\alpha+1)_{i+j-k}(\alpha+\beta+1)_{2(i+j)-k+1}} \\ & \times \frac{(i+j)!}{i!j!k!}\,\frac{(\lambda+\delta+1)_{2i}(\mu+\gamma+1)_{2j}}{(\lambda+\delta+1)_i(\mu+\gamma+1)_j}\,\frac{(-2i-\lambda-\delta)_k}{(-2j-\mu-\gamma)_k} \\ & \times {}_3F_2\left(\begin{matrix}-k, -\lambda-\mu-\delta-\gamma-1-2(i+j)+k, -i\\ -2i-\lambda-\delta, -i-j\end{matrix}\,\middle|\,1\right) \\ & \times {}_3F_2\left(\begin{matrix}-k, -\lambda-\mu-\delta-\gamma-1-2(i+j)+k, -\lambda-i\\ -2i-\lambda-\delta, -\lambda-\mu-i-j\end{matrix}\,\middle|\,1\right) \end{aligned} \tag{15} $$

Next, we consider the particular case $\lambda = \delta = \mu = \gamma$ and we will prove that one of the above terminating ${}_3F_2$ can be summed using, for this purpose, computer algebra.

Put

$$ S(k) = {}_3F_2\left(\begin{matrix}-k, -4\lambda-1-2(i+j)+k, -i\\ -2i-2\lambda, -i-j\end{matrix}\,\middle|\,1\right), $$

with Zeilberger's algorithm (see e.g. [13, Chapter 7]) via the Maple `sumrecursion` command, we obtain:

$$ \begin{aligned} 0 = {} & (1+k)(2j-k+2\lambda)(j+i+4\lambda-k)(-1+j-k+2\lambda+i)S(k) \\ & -(1-2i-2\lambda+k)(-k+i+j-1)(-k+j+i+2\lambda)(4\lambda+2i+2j-k)S(2+k) \\ & -2\lambda(-i+j)(j+2\lambda+1+i)(2j-1-2k+4\lambda+2i)S(1+k). \end{aligned} \tag{16} $$

With the `rechyper` Maple command, which is an implementation of Petkovsek's algorithm detecting all hypergeometric term solutions of a holonomic recurrence equation [13, Chapter 9][1] we obtain that 0 is the only hypergeometric solution of the recurrence relation (16), hence the first ${}_3F_2$ in the r.h.s. of relation (15) cannot be reduced to any hypergeometric term.

For the second ${}_3F_2$ of (15), consider

$$ T(k) = {}_3F_2\left(\begin{matrix}-k, -4\lambda-1-2(i+j)+k, -\lambda-i\\ -2i-2\lambda, -2\lambda-i-j\end{matrix}\,\middle|\,1\right). \tag{17} $$

Again, by Zeilberger's algorithm we obtain

$$ (2j-k+2\lambda)(1+k)T(k) - (1-2i-2\lambda+k)(4\lambda+2i+2j-k)T(2+k) = 0, \tag{18} $$

with initial conditions $T(0) = 1$ and $T(1) = 0$.

From this recurrence it follows with Petkovsek's algorithm that $T(k)$ is 0 for odd k which is also the only hypergeometric solution of relation (18).

Note here that this reduction formula can also be obtained from the Karlsson–Minton Formula [15, p. 14], with a proper choice of parameters.

For even values $k = 2m$, we get

$$ (j+\lambda-m)(2m+1)T(m) + (2i-1-2m+2\lambda)(2\lambda+i+j-m)T(m+1) = 0, \tag{19} $$

which admits the hypergeometric solution

$$ T(k) = T(2m) = \frac{(-\lambda-j)_m(2m)!}{4^m(1/2-\lambda-i)_m(-i-j-2\lambda)_m m!}. \tag{20} $$

Therefore, for integer m we obtain the following reduction formula

[1] This computation, in principle, can also be handled by Mark van Hoeij's faster algorithm [14] implemented in Maple's `LREtools[hypergeomsols]` command.

$$ {}_3F_2\left(\begin{matrix}-2m, -4\lambda-1-2(i+j)+2m, -\lambda-i\\ -2i-2\lambda, -2\lambda-i-j\end{matrix}\,\middle|\,1\right) = \frac{(-\lambda-j)_m(2m)!}{4^m(1/2-\lambda-i)_m(-i-j-2\lambda)_m m!}.\text{[2]} \tag{21} $$

It follows that the linearization coefficients in

$$ P_i^{(\lambda,\lambda)}(x)P_j^{(\lambda,\lambda)}(x) = \sum_{k=0}^{i+j} L_{ij}(i+j-k)P_{i+j-k}^{(2\lambda,2\lambda)}(x), \tag{22} $$

are given by 0 if $k = 2m+1$, which can be also proven directly by the symmetry property of the ultraspherical polynomials $\{P_n^{(\lambda,\lambda)}\}_n$, and

$$ \begin{aligned} L_{ij}(i+j-2m) &= \binom{i+j}{i}\frac{(4\lambda+1)_{i+j-2m}(2\lambda+1)_{i+j}(2(i+j-2m)+4\lambda+1)}{(2\lambda+1)_{i+j-2m}(4\lambda+1)_{2(i+j)-2m+1}} \\ &\quad\times \frac{(2\lambda+1)_{2i}(2\lambda+1)_{2j}}{(2\lambda+1)_i(2\lambda+1)_j}\frac{(-2i-2\lambda)_{2m}}{(-2j-2\lambda)_{2m}} \\ &\quad\times {}_3F_2\left(\begin{matrix}-2m, -4\lambda-1-2(i+j)+2m, -i\\ -2i-2\lambda, -i-j\end{matrix}\,\middle|\,1\right)\frac{(-\lambda-j)_m}{4^m(1/2-\lambda-i)_m(-i-j-2\lambda)_m m!}. \end{aligned} \tag{23} $$

Next, we use the above results to obtain a reduction formula for a finite sum of a terminating hypergeometric function, using for this purpose the well-known connection and linearization formulae for Gegenbauer polynomials.

The Gegenbauer polynomials are Jacobi polynomials with $\alpha = \beta = \mu - \frac{1}{2}$ and another standardization:

$$ C_n^\mu(x) = \frac{(2\mu)_n}{(\mu+\frac{1}{2})_n}P_n^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(x). \tag{24} $$

The connection and linearization formulae are, respectively, given by the formulae ([16, p. 39], compare [17])

$$ C_n^\omega(x) = \sum_{k=0}^{[\frac{n}{2}]}\frac{(\mu+n-2k)(\omega-\mu)_k(\omega)_{n-k}}{k!(\mu)_{n+1-k}}C_{n-2k}^\mu(x), \tag{25} $$

and

$$ C_i^\mu(x)C_j^\mu(x) = \sum_{k=0}^{\min(i,j)}\frac{(i+j+\mu-2k)}{(i+j+\mu-k)}\frac{(\mu)_k(\mu)_{i-k}(\mu)_{j-k}(2\mu)_{i+j-k}}{k!(i-k)!(j-k)!(\mu)_{i+j-k}}\frac{(i+j-2k)!}{(2\mu)_{i+j-2k}}C_{i+j-2k}^\mu(x). \tag{26} $$

That leads, by virtue of (24), to the following connection and linearization formulae for the ultraspherical polynomials

$$ \begin{aligned} P_{i+j-2k}^{(2\lambda,2\lambda)}(x) &= \frac{(2\lambda+1)_{i+j-2k}}{(4\lambda+1)_{i+j-2k}}\sum_{p=0}^{[\frac{i+j}{2}]-k}\frac{(\lambda+i+j-2k-2p+\frac{1}{2})(\lambda)_p(2\lambda+\frac{1}{2})_{i+j-2k-p}}{p!(\lambda+\frac{1}{2})_{i+j-2k-p+1}} \\ &\quad\times\frac{(2\lambda+1)_{i+j-2k-2p}}{(\lambda+1)_{i+j-2k-2p}}P_{i+j-2k-2p}^{(\lambda,\lambda)}, \end{aligned} \tag{27} $$

and

$$ \begin{aligned} P_i^{(\lambda,\lambda)}(x)P_j^{(\lambda,\lambda)}(x) &= \frac{(\lambda+1)_i(\lambda+1)_j}{(2\lambda+1)_i(2\lambda+1)_j}\sum_{k=0}^{\min(i,j)}\frac{(\lambda+i+j-2k+\frac{1}{2})(i+j-2k)!}{(\lambda+i+j-k+\frac{1}{2})k!(i-k)!(j-k)!} \\ &\quad\times\frac{(2\lambda+1)_{i+j-k}(\lambda+\frac{1}{2})_k(\lambda+\frac{1}{2})_{i-k}(\lambda+\frac{1}{2})_{j-k}}{(\lambda+\frac{1}{2})_{i+j-k}(\lambda+1)_{i+j-2k}}P_{i+j-2k}^{(\lambda,\lambda)}(x). \end{aligned} \tag{28} $$

Substituting (27) in (22), using (23) and comparing with (28), we get the following reduction formula, for $0 \le k \le \min(i,j)$,

$$ \begin{aligned} &\sum_{p=0}^{k}\frac{(\lambda)_{k-p}(2\lambda+\frac{1}{2})_{i+j-k-p}}{(4\lambda+1)_{2i+2j-2p+1}(\frac{1}{2}-\lambda-j)_p}\frac{[2(i+j-2p)+(4\lambda+1)]}{p!(k-p)!2^{2p}(\lambda+\frac{1}{2})_{i+j-p-k+1}}\frac{\binom{\lambda+i}{p}}{\binom{2\lambda+i+j}{p}} \\ &\quad\times {}_3F_2\left(\begin{matrix}-2p, -4\lambda-1-2(i+j)+2p, -i\\ -2i-2\lambda, -i-j\end{matrix}\,\middle|\,1\right) \\ &= \frac{\binom{i}{k}\binom{j}{k}}{\binom{i+j}{2k}}\frac{k!}{(2k)!}\frac{(2\lambda+1+i+j-2k)_k(\lambda+1)_i(\lambda+1)_j}{(2\lambda+1)_{i+j}(2\lambda+1)_{2i}(2\lambda+1)_{2j}}\frac{(\lambda+\frac{1}{2})_k(\lambda+\frac{1}{2})_{i-k}(\lambda+\frac{1}{2})_{j-k}}{(\lambda+\frac{1}{2})_{i+j+1-k}}. \end{aligned} \tag{29} $$

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[2] Note that (21) is a variant of the Watson–Whipple formula (see e.g. [10], Table 6.1 on p. 84), hence our deduction gives a simple proof of this formula.

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