The Definite Nature of Indefinite Integrals

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Abstract

The indefinite integral has a dual nature: an **antiderivative** and a **definite integral over variable intervals** $\int_a^x f(t)\,dt$. The latter aspect can be demonstrated effectively in the classroom, using symbolic computation. Using limits of special Riemann sums one can compute certain indefinite integrals $\int_a^x f(t)\,dt$. Examples where this works include t^m (for all real m), $\sin t$, $\cos t$, e^t, $\ln t$. Having gained a concrete understanding of the indefinite integral, students can then use DERIVE's built-in integration `INT(f,x)` to get other indefinite integrals as needed.

This constructive approach allows teaching integrals independently of derivatives.

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1 Introduction

As every calculus student knows (or should know) there are two kinds of integrals of a(n integrable) function f in an interval $[a,b]$:

definite integral, defined as a limit of **Riemann sums**[1]

$$\int_a^b f(x)\,dx := \lim_{\|\mathcal{P}\|\to 0} \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k \;, \tag{1}$$

indefinite integral, for which there are two definitions:

- **antiderivative** (or **primitive**) of f in $[a,b]$, i.e. a function F satisfying

$$F'(x) = f(x) \;, \tag{2}$$

  at all points x in $[a,b]$, with the possible exception of a countable set. F is defined up to a constant, the **constant of integration**.

- **definite integral over a variable interval**,

$$F(x) := \int_a^x f(t)\,dt \;, \tag{3}$$

  with the constant of integration determined by the lower endpoint a.

Definitions (2) and (3) are reconciled by the **Fundamental Theorem of Calculus** (FTC)[2].

Theorem 1 (FTC, First Form; (Howson, 1972, p. 136)) Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, and let $F : [a,b] \to \mathbb{R}$ satisfy the conditions:
(a) the derivative F' exists and $F'(x) = f(x)$ for all $x \in (a,b)$;
(b) the limits $\lim_{x\to a_+} F(x)$ and $\lim_{x\to b_-} F(x)$ exist.
Then

$$\int_a^b f(x)\,dx = \lim_{x\to b_-} F(x) - \lim_{x\to a_+} F(x) \;. \qquad \square \tag{4}$$

Theorem 2 (FTC, Second Form; (Bartle and Sherbert, 1992, p. 252)) Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, and let

$$F(x) := \int_a^x f(t)\,dt \quad \text{for all } x \in [a,b] \;; \tag{5}$$

[1] Notation explained in § 2.

[2] Calculus books do not agree which theorem should be called the FTC, using either Theorem 1 (see e.g. (Shilov, 1973, §9.3, Theorem 9.33)), or Theorem 2 (e.g. (Lang, 1978, Chapter IX, §5, Theorem 5)), or a combined form (e.g. Bartle and Sherbert, 1992, §7.3, p. 253)) emphasizing the inverse nature of differentiation and integration. These are special cases (for one dimension) of Stokes' Theorem relating "surface" and "volume" integrals, see e.g. (Spivak, 1965, Theorem 4-13), which is arguably the "true" FTC.

then F is continuous on $[a, b]$. Moreover, if f is continuous at a point $x \in [a, b]$, then F is differentiable at x and

$$F'(x) = f(x) \ . \qquad \square \tag{2}$$

Theorem 1 is useful for computing definite integrals $\int_a^b f(x)\,dx$ because

- for many familiar functions it is an easy exercise to find, or verify, an antiderivative,
- if F is continuous at the endpoints a and b, then (4) assumes the form

$$\int_a^b f(x)\,dx = F(b) - F(a) \ . \tag{6}$$

Because of (6), several important calculus books[3] delay the definite integral until after indefinite integrals, placing it near the end of one-variable calculus. Such practice is a deviation from the historical order; special definite integrals were evaluated correctly about 2,000 years before calculus[4]. As this chronology suggests, the definite integral is conceptually simpler than differentiation. Moreover, using computers to demonstrate the convergence of Riemann sums, the definite integral can be made concrete and constructive in a way which allows teaching it effectively as early (after limits) as one wishes.

For the definite integral $\int_a^b f(x)\,dx$ to exist, it is only required that f be integrable in $[a, b]$. An integrable function f may have countably many discontinuity points in $[a, b]$, at which points the indefinite integral F may be non-differentiable. However, as long as we use the same indefinite integral F (i.e. the same constant of integration) throughout $[a, b]$, the formula

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{6}$$

is valid. This is illustrated by the following

Example 1 (The sign function) The function

$$\texttt{SIGN}(x) := \begin{cases} -1 & , \ x < 0 \\ \text{undefined} & , \ x = 0 \\ 1 & , \ x > 0 \end{cases} \tag{7}$$

is integrable throughout $\mathbb{R}$,

$$\int_a^x \texttt{SIGN}(t)\,dt = |x| - |a| \ . \tag{8}$$

Therefore $|x|$ is an indefinite integral of $\texttt{SIGN}(x)$. The function $|x|$ is non-differentiable at 0, where $\texttt{SIGN}(x)$ is discontinuous. At all other points, the derivative of $|x|$ is $\texttt{SIGN}(x)$.

[3] E.g. (Bartle and Sherbert, 1992), (Landau, 1951), (Lang, 1978) and (Nikolskii, 1975).

[4] See references to Democritus and Archimedes in any book on the history of mathematics, e.g. (Edwards, 1979, Chapters 1–2).

In applications of the indefinite integral F to compute definite integrals, using (6), the differentiability of F (i.e. its antiderivative aspect) is a secondary issue. The indefinite integral can then be based on

$$F(x) := \int_a^x f(t)\,dt \ , \tag{3}$$

defining it as a definite integral over a variable interval $[a, x]$, independently of differentiability and derivatives.

The purpose of this paper is to demonstrate that this is pedagogically feasible, by illustrating Definition (3) constructively for some familiar functions. Such illustrations require the computation of

- Riemann sums over a variable interval $[a, x]$, and
- limits of such Riemann sums giving $\int_a^x f$ as a definite integral.

This is made possible by symbolic computation, done here with DERIVE.

2 Riemann sums and definite integrals

This section is devoted to computing definite integrals numerically, using limits of Riemann sums. First some notation. Let the function f be defined and bounded on a (closed, finite) interval $[a, b]$. The (**definite**) **Riemann integral** of f from a to b, is defined by

$$\int_a^b f(x)\,dx := \lim_{\|\mathcal{P}\| \to 0} S(f, \mathcal{P}, \xi_1, \ldots, \xi_n) \ , \tag{9}$$

where $S(f, \mathcal{P}, \xi_1, \ldots, \xi_n)$ is a **Riemann sum** of f,

$$S(f, \mathcal{P}, \xi_1, \ldots, \xi_n) := \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k \ , \tag{10}$$

corresponding to:

- a **partition** $\mathcal{P}$ of $[a, b]$, specified by $n + 1$ points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \ , \tag{11}$$

giving n subintervals $I_k := [x_{k-1}, x_k]$ of lengths

$$\Delta x_k := x_k - x_{k-1}, \quad (k = 1, \ldots, n) \ , \tag{12}$$

and,

- n points $\xi_k \in I_k, \quad (k = 1, \ldots, n)$.

The **norm** $\|\mathcal{P}\|$ of the partition $\mathcal{P}$, which features in the limit (9), is defined by

$$\|\mathcal{P}\| := \max_{k=1,\ldots,n} \Delta x_k \ . \tag{13}$$

The function f is **Riemann integrable** on $[a, b]$ if the limit (9) exists, and is the same, for all partitions $\mathcal{P}$ of $[a, b]$, and for all selections of ξ_k in I_k.

Remark 1 (Upper and lower Riemann sums) If the function f, in each subinterval I_k, assumes its maximum value

$$\max_{x_{k-1} \le x \le x_k} f(x) ,$$

and its minimum value

$$\min_{x_{k-1} \le x \le x_k} f(x) ,$$

then substituting these values for $f(\xi_k)$ in (10) gives the **upper** and **lower Riemann sum**, respectively,

$$S_{\text{UPPER}}(f, \mathcal{P}) := \sum_{k=1}^{n} \left(\max_{x_{k-1} \le x \le x_k} f(x) \right) \Delta x_k , \tag{14}$$

$$S_{\text{LOWER}}(f, \mathcal{P}) := \sum_{k=1}^{n} \left(\min_{x_{k-1} \le x \le x_k} f(x) \right) \Delta x_k , \tag{15}$$

which bound all Riemann sums of the same partition,

$$S_{\text{LOWER}}(f, \mathcal{P}) \le S(f, \mathcal{P}, \xi_1, \ldots, \xi_n) \le S_{\text{UPPER}}(f, \mathcal{P}) . \tag{16}$$

We conclude that f is integrable in $[a, b]$ iff

$$\lim_{\|\mathcal{P}\| \to 0} S_{\text{LOWER}}(f, \mathcal{P}) = \lim_{\|\mathcal{P}\| \to 0} S_{\text{UPPER}}(f, \mathcal{P}) , \tag{17}$$

the common value is the integral $\int_a^b f$. However, computing the upper and lower sums is not practical in general, since these sums require the maximum and the minimum of f in each subinterval. An important exception is the class of monotone functions, where the upper and lower sums are quite easy, see Remark 2.

When writing a computer program for Riemann sums, it is convenient to use simple partitions, and uniform selection rules for the points ξ_k. The simplest partition of $[a, b]$ is the **regular partition**,

$$x_k := a + k \, \frac{b-a}{n} \quad \text{and} \quad \Delta x_k = \frac{b-a}{n} \quad \text{for all } k = 0, \ldots, n . \tag{18}$$

With the regular partition, we use two selection rules for the point ξ_k in each subinterval $I_k = [x_{k-1}, x_k]$,

- **left endpoint**: $\xi_k := x_{k-1}$, giving the Riemann sum of f in $[a, b]$ as

$$S_{\text{LEFT}}(f, x, a, b, n) := \left(\frac{b-a}{n} \right) \sum_{k=1}^{n} f \left(a + (k-1) \, \frac{b-a}{n} \right) , \tag{19}$$

- **right endpoint**: $\xi_k := x_k$, giving the Riemann sum

$$S_{\text{RIGHT}}(f, x, a, b, n) := \left(\frac{b-a}{n} \right) \sum_{k=1}^{n} f \left(a + k \, \frac{b-a}{n} \right) . \tag{20}$$

Remark 2 In each subinterval $[x_{k-1}, x_k]$ where the function f is monotone (increasing or decreasing), the maximum and minimum values of f occur at the endpoints x_{k-1}, x_k. For monotone functions, the sums $S_{\text{LEFT}}(f, x, a, b, n)$ and $S_{\text{RIGHT}}(f, x, a, b, n)$ are therefore the upper and lower sums corresponding to the regular partition.

Remark 3 If f is known to be integrable in $[a,b]$, then it suffices to compute the limit of one Riemann sum, say of $S_{\mathrm{LEFT}}(f,x,a,b,n)$ as $n \to \infty$.

Example 2 Consider the function $f(x) = x^2$ in the interval $[0,1]$. The two Riemann sums (19) and (20) then give, omitting arithmetic details,

$$S_{\mathrm{LEFT}}(x^2,x,0,1,n) = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2 = \frac{(n-1)(2n-1)}{6n^2} , \tag{21}$$

$$S_{\mathrm{RIGHT}}(x^2,x,0,1,n) = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{(n+1)(2n+1)}{6n^2} , \tag{22}$$

which converge, as $n \to \infty$, to the same limit, $\frac{1}{3}$. Since the function x^2 is monotone in $[0,1]$, we conclude that

$$\int_0^1 x^2\, dx = \frac{1}{3} .$$

DERIVE **Session 1** DERIVE is used here to illustrate the above concepts, and to calculate few Riemann sums. We omit DERIVE specifics, in particular which command (e.g. `Simplify`, `approX`, `Expand` or `Factor`) is used to transform a given expression. For the instructor, it is, however, necessary to know that DERIVE evaluates a function f at a as a limit `LIM(f,x,a)`. The left Riemann sum (19) of f in $[a,b]$, is then simply

```
S_LEFT(f,x,a,b,n):=(b-a)/n*SUM(LIM(f,x,a+(k-1)*(b-a)/n),k,1,n)
```

Similarly, the Riemann sum (20) is computed by the function

```
S_RIGHT(f,x,a,b,n):=(b-a)/n*SUM(LIM(f,x,a+k*(b-a)/n)),k,1,n)
```

Some teachers may argue that the students will not understand this definition using limits. If so, don't bother the students by showing them the definition, but use `Transfer Load Utility` instead, let the students define an arbitrary function by `F(x):=`, and `Simplify` the left Riemann sum `S_RIGHT(F(x),x,0,n)` to get

3 : $$\frac{b-a}{n} \sum_{k=1}^{n} f\left[a + \frac{k(b-a)}{n}\right] .$$

Therefore, what the students see, is exactly the limit-free definition of the Riemann sum. This is so as DERIVE assumes f to be continuous.

Now, for example, the left Riemann sum of $f(x) = \sin x$ in the interval $[0,\pi]$, with $n = 100$ is computed by

4 : `S_LEFT(SIN(x),x,0,π,100)` , which is approximated, by DERIVE, to 5 : 1.99983 .

Similarly, `S_RIGHT(SIN(x),x,0,π,100)` gives 1.99983. Increasing the number of partition intervals from $n = 100$ to $n = 1000$, the left and right Riemann sums give the same approximate value 2, which is the definite integral of $\sin x$ from 0 to π.

DERIVE being a symbolic algebra package, it is unnecessary to specify the argument n in the above Riemann sums. For example, the expression `S_LEFT(x^2,x,0,1,n)` gives

10 : $$\frac{(n-1)(2n-1)}{6n^2} ,$$ in agreement with (21) .

3 Riemann sums over variable intervals and indefinite integrals

In § 2 we fixed an interval $I = [a, b]$ and computed the definite integral

$$\int_a^b f(t)\, dt \tag{23}$$

a number, with a well-known interpretation as sum of signed areas. In this section we consider the endpoints a and b to be variable. The integral (23) is now a function of the limits of integration a and b, an indefinite integral.

To conform with standard notation, the upper limit of integration is renamed x.

DERIVE **Session 2** We calculate the integral $\int_a^x t^2\, dt$, for general a and x. The fact that the endpoints a and x are arbitrary should not prevent us from using Riemann sums, like those in DERIVE-Session 1. For example, the left Riemann sum `S_LEFT(t^2,t,a,x,n)` gives an expression that depends on a, x and n. We take then the limit, as $n \to \infty$, to get the integral. These two steps can be combined in the statement `LIM(S_LEFT(t^2,t,a,x,n),n,inf)` which is displayed as

1 : $\lim_{n\to\infty}$ `S_LEFT(t^2,t,a,x,n)` and results in 2: $\dfrac{x^3}{3} - \dfrac{a^3}{3}$, the integral $\int_a^x t^2\, dt$.

Next, the left Riemann sum of $\sin t$ from a to x, `S_LEFT(SIN(t),t,a,x,n)` results in a surprisingly nice expression

6 : $$\frac{(a-x)\ \mathtt{COS}\left[a\left[\dfrac{1}{2n}+1\right]-\dfrac{x}{2n}\right]}{2n\ \mathtt{SIN}\left[\dfrac{a}{2n}-\dfrac{x}{2n}\right]} + \frac{(x-a)\ \mathtt{COS}\left[\dfrac{a}{2n}+\dfrac{x(2n-1)}{2n}\right]}{2n\ \mathtt{SIN}\left[\dfrac{a}{2n}-\dfrac{x}{2n}\right]}$$

whose limit[5] as $n \to \infty$,

7 : `LIM(S_LEFT(SIN(t),t,a,x,n),n,inf)` is 8 : $\mathtt{COS}(a) - \mathtt{COS}(x)$.

Similarly, the limit of the left Riemann sum of $\cos t$, from a to x,

11 : `LIM(S_LEFT(COS(t),t,a,x,n),n,inf)` , gives 12 : $\mathtt{SIN}(x) - \mathtt{SIN}(a)$.

The left Riemann sum of e^t, `S_LEFT(EXP(t),t,a,x,n)` gives

13 : $\dfrac{e^{a/n}\,(x-a)\,[e^x - e^a]}{n\left[e^{x/n} - e^{a/n}\right]}$, and the limit, as $n \to \infty$, 14 : $e^x - e^a$.

The left Riemann sum of $\frac{1}{t}$ in $[a, x]$, `S_LEFT(1/t,t,a,x,n)`, gives

[5] This limit is an easy exercise, using

$$\lim_{u\to 0} \frac{\sin u}{u} = 1\ , \quad \text{with} \quad u = \frac{a-x}{2n}\ .$$

15 : $(a-x)\sum_{k=1}^{n}\frac{1}{k\,(a-x)-a(n+1)+x}$

and its limit `LIM(S_LEFT(1/t,t,a,x,n),n,inf)` is 16 : $\text{LN}(x)-\text{LN}(a)$

This approach works for some functions, and fails for others (even for functions with nice antiderivatives). For example, trying to integrate t^m with unspecified m, using `S_LEFT(t^m,t,a,x,n)`, gives an expression whose limit as $n\to\infty$ is too difficult[6]. Another example is $\tan x$. Trying

17 : `LIM(S_LEFT(TAN(t),t,a,x,n),n,inf)` ,

is futile: The left Riemann sum of tan t, `S_LEFT(TAN(t),t,a,x,n)` is,

$$18:\quad \frac{(x-a)\sum_{k=1}^{n}\text{TAN}\left[k\left[\frac{x}{n}-\frac{a}{n}\right]-\frac{x}{n}+a\left[\frac{1}{n}+1\right]\right]}{n}$$

whose limit, as $n\to\infty$, is too difficult. Here we use DERIVE's built in integration,

19 : `INT(TAN(t),t,a,x)` , to get 20 : $\text{LN}(\text{COS}(a))-\text{LN}(\text{COS}(x))$.

We conclude that in symbolic computation of Riemann sums, variable intervals (corresponding to indefinite integrals) pose no greater difficulty than fixed intervals (or definite integrals). The difficult part, as illustrated above for x^m and $\tan x$, is finding a closed expression for the Riemann sum. The next two examples, illustrate how "custom made" Riemann sums are easier than the simple Riemann sums (19) and (20).

Example 3 (A nice trick) We calculate the indefinite integral,

$$\int_a^x \frac{dt}{t^2}\ ,\quad \text{for } 0<a\ .$$

The function $\frac{1}{t^2}$ is continuous in $[a,x]$ for $0<a<x$, and therefore is integrable in $[a,x]$. By Remark 3, it suffices to compute the limit of one Riemann sum.

Let $\{a=x_0<x_1<\ldots<x_{n-1}<x_n=x\}$ be <u>any</u> partition of $[a,x]$, and for each k let the point ξ_k be selected as the geometric mean of the endpoints,

$$\xi_k=\sqrt{x_{k-1}\,x_k}\ ,\quad (k=1,\ldots,n)\ .$$

Then the Riemann sum becomes

$$\sum_{k=1}^{n}\frac{\Delta x_k}{x_{k-1}\,x_k}=\sum_{k=1}^{n}\frac{x_k-x_{k-1}}{x_{k-1}\,x_k}=\sum_{k=1}^{n}\left(\frac{1}{x_{k-1}}-\frac{1}{x_k}\right)=\frac{1}{x_0}-\frac{1}{x_n}=\frac{1}{a}-\frac{1}{x}\ ,$$

giving the correct answer without computing any limit!

[6]We overcome this difficulty in Example 4 by using a tricky Riemann sum.

Example 4 (Another trick) We compute the indefinite integral

$$\int_a^x t^m \, dt \ , \quad \text{where } 0 < a < b \text{ and } m \text{ is real.}$$

It is convenient to use here a "geometric partition", with partition points defined by the geometric sequence,

$$x_k := a \left(\frac{x}{a}\right)^{\frac{k}{n}} \ , \quad (k = 0, 1, \ldots, n) \ , \tag{24}$$

instead of the regular partition. We define a left Riemann sum by taking $\xi_k = x_{k-1}$,

```
S_LEFT_GEOM(f,t,a,x,n):=
     SUM(LIM(f,t,a*(x/a)^((k-1)/n))*(a*(x/a)^(k/n)-a*(x/a)^((k-1)/n)),k,1,n)
```

and use DERIVE to compute[7] the Riemann sum `S_LEFT_GEOM(t^m,t,a,x,n)`, giving

$$\frac{a^{m/n}\left(x^{1/n} - a^{1/n}\right)\left(a^{m+1} - x^{m+1}\right)}{a^{m/n+1/n} - x^{m/n+1/n}}$$

whose limit as $n \to \infty$ is

$$\frac{x^{m+1} - a^{m+1}}{m+1} \ , \tag{25}$$

the indefinite integral of x^m, see Remark 4.

The value $m = -1$ poses no special difficulty. Indeed, `S_LEFT_GEOM(1/t,t,a,x,n)` gives

$$n \left(\frac{x}{a}\right)^{1/n} - n \ ,$$

whose limit, as $n \to \infty$, is $\ln x - \ln a$. This can be obtained alternatively, by taking the limit of (25) as $m \to -1$. Recall that the integrand $1/t$ was handled well also by regular partitions, see Derive Session 2.

Remark 4 Some calculus books give two answers for the integral of x^m,

$$\int x^m \, dx = \begin{cases} \dfrac{x^{m+1}}{m+1} & , \ m \neq -1 \\ \ln x & , \ m = -1 \end{cases} \tag{26}$$

Since an indefinite integral is a definite integral with variable endpoints, it is natural to expect here one answer, which holds for all m. The answer given by DERIVE's built-in integration is pretty:

$$\int x^m \, dx = \frac{x^{m+1} - 1}{m+1} \ , \tag{27}$$

which is (25) with $a = 1$, and so a constant of integration different from (26) is used. The limit of the right hand side, as $m \to -1$, is $\ln x$.

[7] A DERIVE detail: It is required here to `Declare` the variables a and x to be positive.

Remark 5 Note a difficulty in

$$\int \frac{1}{x}\,dx = \ln x + C \tag{28}$$

is that $\ln x$ in the right hand side is defined only for positive x, while the left hand side is defined for all nonzero x. So what is the integral of $1/x$ if x is negative? By changing variables it follows, for $x < 0$,

$$\int \frac{1}{x}\,dx = \ln(-x) + C. \tag{29}$$

Many calculus books write one expression

$$\int \frac{1}{x}\,dx = \ln|x| + C \tag{30}$$

which seems to combine (28) and (29). This is objectionable on two grounds:

- (30) is false in the complex case; indeed

$$\ln(-z) = \ln z + i\,\pi\ ,\ \ (z \in \mathbb{C})\ , \tag{31}$$

showing that (28) is correct for complex z, where the constant of integration is likewise complex.

- (30) is misleading in the real case[8]. The function $1/x$ is not integrable in any interval containing $x = 0$, and therefore (30) cannot unify the integrals (28) and (29) with the same constant of integration C.

It is therefore better to deal with the cases $x > 0$ and $x < 0$ separately, using (28) and (29) respectively[9].

4 Discussion

In the old days of chalk and blackboard, it was difficult to demonstrate the definite integral

$$\text{(1)} \qquad \int_a^b f := \lim_{\|\mathcal{P}\|\to 0} \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k\ ,$$

as limit of Riemann sums, except for few simple examples[10]. The indefinite integral was even farther beyond the means of "numerical illustration".

As shown in § 3, this has already changed. Using symbolic computation we can make the indefinite integral as elementary and palpable as the definite integral. It is even possible to teach indefinite integrals immediately after the definite integral, computing them as definite integrals over variable intervals, for many functions, sufficiently many to get the point across.

We have thus computed here the following indefinite integrals (constants of integration omitted):

$\int x^2\,dx = \frac{x^3}{3}$	$\int \sin x\,dx = -\cos x$	$\int \cos x\,dx = \sin x$
$\int e^x\,dx = e^x$	$\int x^m\,dx = \frac{x^{m+1}-1}{m+1}$ (Remark 4)	$\int \frac{1}{x}\,dx = \ln x$ (Remark 5)

[8]See e.g. (Lang, 1978, pp. 269-270).

[9]This is also how DERIVE gives the indefinite integral of $1/x$.

[10]It can be argued that such numerical "evidence" is not needed for the "true" mathematician, capable of abstraction. However, the majority of students would greatly benefit from such examples, as they try to make sense of Definition (1).

Other indefinite integrals can then be computed using either the built-in integration facility of the symbolic package, or integration techniques, which however require knowledge of differentiation.

Moreover, both definite and indefinite integrals can be taught independently[11] of differentiation, which is no longer a prerequisite for teaching the indefinite integral.

Furthermore, DERIVE may effectively be used for numerical integration with the trapezoid and Simpson rules (Koepf and Ben-Israel, 1993) adding another type of knowledge and experience not available by hand calculations.

It should be noted that some knowledge of complex numbers is required for using symbolic algebra intelligently. As calculus evolves we expect to see, at the high end of the calculus spectrum, complex numbers (even functions of complex variables) covered quite early in the first semester.

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[11] "Independently" means "before", "during" or "after".