A finite sequence of Hahn-type discrete orthogonal polynomials

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ABSTRACT

By considering a specific Sturm-Liouville problem, we introduce a finite sequence of Hahn-type discrete polynomials and prove that they are finitely orthogonal on the real line. We then compute their norm square value by using Dougall's bilateral sum and obtain all moments corresponding to the introduced polynomials.

KEYWORDS

Sturm-Liouville theorem for functions of a discrete variable; Discrete orthogonal polynomials of Hahn-type; Dougall bilateral sum; Norm square values; Moments

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1. Introduction

According to [4, Chapter 2] (see also [8]), the second-order difference equation

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda y(x) = 0, \tag{1}$$

where

$$\Delta y(x) = \nabla y(x+1) = y(x+1) - y(x),$$

and $\sigma(x) = ax^2 + bx + c$, $\tau(x) = dx + e$ with $d \neq 0$ are polynomials of degree at most 2 and 1 can be written as

$$\Delta\Big(\sigma(x)w(x)\nabla y(x)\Big) + \lambda w(x)y(x) = 0,$$

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in which $w(x)$ satisfies the Pearson difference equation

$$\Delta\Big(\sigma(x)w(x)\Big) = \tau(x)w(x). \tag{2}$$

The solutions of equation (1) with

$$\lambda \equiv \lambda_n = -n\Big((n-1)a + d\Big),$$

are polynomials of degree n, say $y(x) = y_n(x)$, and usually called hypergeometric type discrete polynomials. They are orthogonal with respect to the weight function $w(x)$ on the counter set $x = A,\ A+1,\ \ldots,\ B$ [8] as

$$\sum_{x=A}^{B} y_n(x)y_m(x)w(x) = \left(\sum_{x=A}^{B} y_n^2(x)w(x)\right)\delta_{n,m} \quad \text{with} \quad \delta_{n,m} = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases}$$

provided that $w(x) > 0$ for $A \leq x \leq B$ and

$$\sigma(x)w(x)x^k\Big|_{x=A,B+1} = 0,\ \forall k \geq 0. \tag{3}$$

According to [4, Chapter 2], for $y(x) = y_n(x)$, equations (1) and (2) can be written as

$$\sigma_1(x)(\Delta^2 y_n)(x) + \tau_1(x)(\Delta y_n)(x) + \lambda_n\, y_n(x+1) = 0,$$

and

$$\Delta\Big(\sigma_1(x-1)w(x)\Big) = \tau_1(x)w(x+1),$$

where

$$\sigma_1(x) = \sigma(x+1) + \tau(x+1) \quad \text{and} \quad \tau_1(x) = \tau(x+1),$$

or

$$\sigma_2(x)(\nabla^2 y_n)(x) + \tau_2(x)(\nabla y_n)(x) + \lambda_n\, y_n(x-1) = 0,$$

and

$$\Delta\Big(\sigma_2(x+1)w(x)\Big) = \tau_2(x+1)w(x),$$

where

$$\sigma_2(x) = \sigma(x-1) \quad \text{and} \quad \tau_2(x) = \tau(x-1).$$

Hahn's classification [2, 8] states that equation (1) has four orthogonal polynomial solutions, which are known in the literature, respectively, as follows:

- The Charlier polynomials

$$C_n(x;a) = {}_2F_0\left(\begin{array}{c} -n,-x \\ - \end{array} \middle| -\frac{1}{a}\right),$$

orthogonal with respect to the Poisson distribution weight function as

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} C_n(k;a) C_m(k;a) = a^{-n} e^a n!\, \delta_{n,m}\ (a > 0).$$

- The Meixner polynomials

$$M_n(x;\beta,c) = {}_2F_1\left(\begin{array}{c} -n,-x \\ \beta \end{array} \middle| 1-\frac{1}{c}\right),$$

orthogonal with respect to the Pascal distribution weight function as

$$\sum_{k=0}^{\infty} \frac{(\beta)_k c^k}{k!} M_n(k;\beta,c) M_m(k;\beta,c) = \frac{c^{-n} n!}{(\beta)_n (1-c)^{\beta}} \delta_{n,m}\ (\beta > 0,\ 0 < c < 1).$$

- The Kravchuk polynomials

$$K_n(x;p,N) = {}_2F_1\left(\begin{array}{c} -n,-x \\ -N \end{array} \middle| \frac{1}{p}\right), \quad n = 0,1,2,\ldots,N,$$

orthogonal with respect to the Binomial distribution weight function as

$$\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} K_n(k;p,N) K_m(k;p,N) = \frac{(-1)^n n!}{(-N)_n} \left(\frac{1-p}{p}\right)^n \delta_{n,m}\ (0 < p < 1),$$

- and the Hahn polynomials

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{array}{c} -n,-x,n+\alpha+\beta+1 \\ -N,\alpha+1 \end{array} \middle| 1\right), \quad n = 0,1,2,\ldots,N,$$

orthogonal with respect to the Hypergeometric distribution weight function as

$$\begin{aligned}\sum_{k=0}^{N} \binom{\alpha+k}{k} \binom{\beta+N-k}{N-k} & Q_n(k;\alpha,\beta,N) Q_m(k;\alpha,\beta,N) \\ &= \frac{(-1)^n (n+\alpha+\beta+1)_{N+1} (\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n (-N)_n N!} \delta_{n,m}\ (\alpha > -1,\ \beta > -1).\end{aligned}$$

In the above definitions, the functions ${}_2F_0$, ${}_2F_1$ and ${}_3F_2$ are all special cases of the generalized hypergeometric series [2, 3]

$${}_pF_q\left(\begin{array}{c} a_1,a_2,\ldots,a_p \\ b_1,b_2,\ldots,b_q \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!},$$

in which z may be a complex variable and $(a)_k$ is the Pochhammer symbol defined by

$$(a)_0 = 1 \text{ and } (a)_k = a(a+1)(a+2)\cdots(a+k-1).$$

The following table shows the data $\sigma(x)$ and $\tau(x)$ for each of the four above polynomial families [2]:

symbol	$Q_n(x;\alpha,\beta,N)$	$M_n(x;\beta,c)$	$K_n(x;p,N)$	$C_n(x;a)$
$\sigma(x)$	$x(N+\alpha-x)$	x	x	x
$\tau(x)$	$(\beta+1)(N-1)-(\alpha+\beta+2)x$	$(c-1)x+\beta c$	$\frac{Np-x}{1-p}$	$a-x$

As this table shows, all polynomial coefficients σ and τ have real zeros while we can still consider a main case of the real difference equation (1) whose coefficients have complex zeros with four free parameters. According to [4, Chapter 2], if we expand the Pearson equation (2) as follows

$$\frac{w(x+1)}{w(x)} = \frac{\sigma(x)+\tau(x)}{\sigma(x+1)} = \frac{\sigma_1(x-1)}{\sigma_1(x)-\tau_1(x)}, \tag{4}$$

then it is clear in (4) that if $\sigma(x)$ has degree < 2, it leads to the Charlier, Meixner and Krawtchouk polynomials and if $\sigma(x)$ has the exact degree 2, Hahn polynomials are derived if both the numerator $\sigma(x)+\tau(x)$ and the denominator $\sigma(x+1)$ have real factorizations. However, another case in (4) is when the polynomials $\sigma(x)+\tau(x)$ and $\sigma(x+1)$ are real but have complex zeros. In other words, write equation (4) in the expanded form

$$\frac{w(x+1)}{w(x)} = \frac{ax^2+(b+d)x+c+e}{ax^2+(2a+b)x+a+b+c}. \tag{5}$$

Then, two cases can be generally considered for the parameter a in (5), i.e. $a=0$ or $a \neq 0$. If $a=0$, the result is well known in the literature. So, by assuming $a=1\neq 0$ we will reach the simplified equation

$$\frac{w(x+1)}{w(x)} = \frac{x^2+(b+d)x+c+e}{x^2+(b+2)x+b+c+1}. \tag{6}$$

Now, four cases can happen for equation (6):
i) Both numerator and denominator have real zeros, namely

$$\frac{w(x+1)}{w(x)} = \frac{(x+p)(x+q)}{(x+r)(x+s)}, \quad (p,q,r,s\in\mathbb{R}). \tag{7}$$

ii) The numerator has real zeros but the denominator has complex roots, namely

$$\frac{w(x+1)}{w(x)} = \frac{(x+p)(x+q)}{(x+r+is)(x+r-is)} \quad (p,q,r,s\in\mathbb{R}). \tag{8}$$

iii) The numerator has complex zeros but the denominator has real roots, namely

$$\frac{w(x+1)}{w(x)} = \frac{(x+p+iq)(x+p-iq)}{(x+r)(x+s)} \quad (p,q,r,s \in \mathbb{R}). \tag{9}$$

iv) Finally both numerator and denominator have complex zeros, namely

$$\frac{w(x+1)}{w(x)} = \frac{(x+p+iq)(x+p-iq)}{(x+r+is)(x+r-is)} \quad (p,q,r,s \in \mathbb{R}). \tag{10}$$

It is worth mentioning that the solutions of each equations (7), (8), (9) and (10) can be written in terms of the gamma function [13]

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{\prod_{k=0}^{n} (z+k)}, \tag{11}$$

which implies

$$\Gamma(p+iq)\,\Gamma(p-iq) = \Gamma^2(p) \prod_{k=0}^{\infty} \frac{(p+k)^2}{(p+k)^2+q^2}, \tag{12}$$

is a real positive value for any $p > 0$ and $q \in \mathbb{R}$. One of the consequences of (12) is that

$$(p+iq)_n\,(p-iq)_n = \prod_{k=0}^{n-1} \left(q^2+(p+k)^2\right) \qquad (p,q \in \mathbb{R}),$$

is a real positive value, too. Moreover, when $p = m \in \mathbb{N}$, we have [7]

$$\Gamma(m+iq)\,\Gamma(m-iq) = \frac{\pi q}{\sinh(\pi q)} \prod_{k=1}^{m-1} \left(q^2+(m-k)^2\right).$$

Hence, the solutions of equations (7)–(10) can be, respectively, represented as

$$w(x) = w_1(x;p,q,r,s) = \frac{\Gamma(x+p)\Gamma(x+q)}{\Gamma(x+r)\Gamma(x+s)}, \tag{13}$$

$$w(x) = w_2(x;p,q,r,s) = \frac{\Gamma(x+p)\Gamma(x+q)}{\Gamma(x+r+is)\Gamma(x+r-is)}, \tag{14}$$

$$w(x) = w_3(x;p,q,r,s) = \frac{\Gamma(x+p+iq)\Gamma(x+p-iq)}{\Gamma(x+r)\Gamma(x+s)}, \tag{15}$$

$$w(x) = w_4(x;p,q,r,s) = \frac{\Gamma(x+p+iq)\Gamma(x+p-iq)}{\Gamma(x+r+is)\Gamma(x+r-is)}. \tag{16}$$

On the other hand, since a weight function must be always positive, relation (13) implies that $p = q$ and $r = s$, leading to

$$w_1(x;p,p,r,r) = \frac{\Gamma^2(x+p)}{\Gamma^2(x+r)} > 0. \tag{17}$$

Also, by referring to the important relation (12), the function (14) is positive when $p = q$, i.e.

$$w_2(x;p,p,r,s) = \frac{\Gamma^2(x+p)}{\Gamma(x+r+is)\Gamma(x+r-is)} > 0. \tag{18}$$

Similarly, for the case (15) we must have $r = s$, i.e.

$$w_3(x;p,q,r,r) = \frac{\Gamma(x+p+iq)\Gamma(x+p-iq)}{\Gamma^2(x+r)} > 0. \tag{19}$$

Finally, for the case (16) we observe that no restriction is required and $w_4(x;p,q,r,s)$ is always positive. A short look at relations (17), (18) and (19) shows that they are just particular cases of the positive weight function (16). This means that we just deal with a sequence of hypergeometric orthogonal polynomials of a discrete variable with four free parameters which is finitely orthogonal on the real line. In the next section, we study such a sequence and consider three particular cases of it and in section 3, we compute all moments corresponding to the introduced polynomials.

2. A finite sequence of Hahn-type orthogonal polynomials

According to [4, Chapter 2], for $p,q,r,s \in \mathbb{R}$, suppose that

$$\sigma(x) + \tau(x) = (x+p)^2 + q^2,$$

and

$$\sigma(x+1) = (x+r)^2 + s^2,$$

which leads to the real equation

$$\begin{aligned} &\left((x+p+1)^2+q^2\right)(\Delta^2 y_n)(x) + \big(2(p+1-r)x + (p+1)^2 + q^2 - r^2 - s^2\big)(\Delta y_n)(x) \\ &- n(n+1+2p-2r)y_n(x+1) = 0, \end{aligned} \tag{20}$$

or

$$\begin{aligned} &\left((x+r-2)^2+s^2\right)(\nabla^2 y_n)(x) + \big(2(p+1-r)x + p^2 + q^2 - r^2 - s^2 - 2p + 4r - 3\big)(\nabla y_n)(x) \\ &- n(n+1+2p-2r)y_n(x-1) = 0. \end{aligned} \tag{21}$$

By referring to [2, Chap. 5], we look for a polynomial solution of equation (20) in the form

$$y_n(x) = \sum_{k=0}^{n} a_{n,k} \binom{x-1+r+is}{k}, \ a_{n,n} \neq 0,$$

which is a particular case of the general solution

$$y_n(x) = \sum_{k=0}^{\infty} a_{n,k} \binom{x+c}{k}.$$

Since

$$\binom{x+c}{k} = \frac{(-1)^k(-x-c)_k}{k!} = \frac{1}{k!}(x+c)^{\underline{k}},$$

where

$$x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1),$$

and the following relations hold:

$$\Delta(x+c)^{\underline{n}} = n(x+c)^{\underline{n-1}} \ \text{ and } \ x(x+c)^{\underline{n}} = (x+c)^{\underline{n+1}} + (n-c)(x+c)^{\underline{n}}, \tag{22}$$

we can directly reach the following result [4, Chapter 2]:

Proposition 2.1. *The monic polynomial solution of the difference equation (20) can be represented as*

$$\begin{aligned} y_n(x) = \bar{R}_n(x;p,q,r,s) &= \frac{(-r+p+1-i(q+s))_n(-r+p+1+i(q-s))_n}{(n+2p-2r+1)_n} \\ &\quad \times {}_3F_2\left(\left. \begin{matrix} -n, n+1+2p-2r, -x+1-r-is \\ -r+p+1-i(q+s), -r+p+1+i(q-s) \end{matrix} \right| 1 \right) \\ &= (-1)^n \frac{(-r+p+1+i(q+s))_n(-r+p+1+i(q-s))_n}{(n+2p-2r+1)_n} \\ &\quad \times {}_3F_2\left(\left. \begin{matrix} -n, n+1+2p-2r, x+p+iq \\ -r+p+1+i(q+s), -r+p+1+i(q-s) \end{matrix} \right| 1 \right). \end{aligned} \tag{23}$$

Proof. For $c = r+is-1$, if in (20) we substitute

$$\bar{R}_n(x;p,q,r,s) = \sum_{k=0}^{\infty} a_{n,k}(x+c)^{\underline{k}},$$

and use the properties (22), we reach the recurrence equation

$$\begin{aligned}&(k-n)\,(k+n+1+2\,p-2\,r)\,a_{n,k}+(k+1)\left(-s^2+2i\,(p+k-r+1)\,s+p^2\right.\\&\left.+\,(4\,k-2\,n-2\,r+4)\,p+2\,k^2-4\,kr-n^2+2\,nr+q^2+r^2+5\,k-n-4\,r+3\right)a_{n,k+1}\\&-(k+1)\,(k+2)\,(iq+is+k+p-r+2)\,(iq-is-k-p+r-2)\,a_{n,k+2}=0.\end{aligned}$$

Now, by using the Petkovšek-van-Hoeij algorithm [3, 10, 12], we can solve the above recurrence relation and apply Koepf's `Sumtohyper` procedure of the `hsum.mpl` package [3] in order to write the monic solution in terms of hypergeometric series.

The second representation of $\bar{R}_n$ comes from (21) following the same approach and using the basis $(-x+c)^{\underline{k}}$ for $c=-p-iq$ with the property

$$\nabla(-x+c)^{\underline{k}}=-k(-x+c)^{\underline{k-1}}.$$

Using Zeilberger's algorithm [3, 11] implemented in the `hsum.mpl` package by the `sumrecursion` command, one can show that both mentioned representations of $\bar{R}_n$ are solutions of the following three-term recurrence relation

$$\bar{R}_{n+1}(x;p,q,r,s)=(x-c_n)\bar{R}_n(x;p,q,r,s)-d_n\bar{R}_{n-1}(x;p,q,r,s),$$

where

$$c_n=-\frac{(p+r-1)\,n^2+(p+r-1)\,(1+2\,p-2\,r)\,n+(-r+p)\left(p^2+q^2-r^2-s^2+2\,r-1\right)}{2\,(n+p-r)\,(n+1+p-r)},\tag{24}$$

and

$$d_n=\frac{n\,(2\,r-2\,p-n)\left((n+p-r)^2+(q-s)^2\right)\left((n+p-r)^2+(q+s)^2\right)}{4\,(n+p-r)^2\,(2\,n+1+2\,p-2\,r)\,(2\,n-1+2\,p-2\,r)},\tag{25}$$

with the unique initial conditions

$$\bar{R}_0(x;p,q,r,s)=1\quad\text{and}\quad\bar{R}_1(x;p,q,r,s)=x+\frac{p^2+q^2-r^2-s^2+2\,r-1}{2(p+1-r)}.$$

□

Remark 1. Relation (23) shows that there is a direct relationship between $\bar{R}_n(x;p,q,r,s)$ and the continuous Hahn polynomials [3, p. 200] defined by

$$P_n(x;a,b,c,d)=\frac{i^n(a+c)_n(a+d)_n}{n!}\,{}_3F_2\left(\begin{array}{c}-n,n+a+b+c+d-1,a+ix\\a+c,a+d\end{array}\Bigg|1\right),$$

so that we have

$$\begin{aligned}\bar{R}_n(x;p,q,r,s) =& \frac{n!}{i^n(n+2p-2r+1)_n} P_n(ix; 1-r-is, 1-r+is, p-iq, p+iq) \\ =& \frac{n! i^n}{(n+2p-2r+1)_n} P_n(-ix; p+iq, p-iq, 1-r+is, 1-r-is).\end{aligned} \tag{26}$$

We are now in a position to prove that the monic polynomials (23) are finitely orthogonal on the real line. For this purpose, we first reconsider equation (20) and write it in a self-adjoint form to obtain

$$\begin{aligned}&\Big[w_4(x;p,q,r,s)(x+p+iq)(x+p-iq) \\ &\quad \times \big(\bar{R}_m(x;p,q,r,s)\bar{R}_n(x+1;p,q,r,s) - \bar{R}_n(x;p,q,r,s)\bar{R}_m(x+1;p,q,r,s)\big)\Big]_{-\infty}^{\infty} \\ &= (n(n+1+2p-2r) - m(m+1+2p-2r)) \sum_{x=-\infty}^{\infty} w_4(x;p,q,r,s)\bar{R}_m(x;p,q,r,s)\bar{R}_n(x;p,q,r,s).\end{aligned} \tag{27}$$

In order to show that the left hand side of (27) is equal to zero when $m \neq n$, we can use the following limit relations

$$\lim_{x\to\infty} \frac{\Gamma(x+a)}{\Gamma(x)x^a} = 1 \quad \text{and} \quad \lim_{x\to-\infty} \frac{\Gamma(x+a)}{\Gamma(x)x^a} = (-1)^a \quad (\forall a \in \mathbb{C}), \tag{28}$$

which can be proved directly via the limit definition (11). Since

$$\deg\Big(\bar{R}_m(x;p,q,r,s)\bar{R}_n(x+1;p,q,r,s) - \bar{R}_n(x;p,q,r,s)\bar{R}_m(x+1;p,q,r,s)\Big) = n+m-1,$$

the left hand side of (27) is equal to zero for $n \neq m$ if

$$\lim_{x\to\pm\infty} \frac{\Gamma(x+p+iq)\Gamma(x+p-iq)}{\Gamma(x+r+is)\Gamma(x+r-is)} x^{k+2} = 0 \text{ for any } k = 0,1,\ldots,n+m-1. \tag{29}$$

By noting (28), it is now straightforward to verify that (29) is equivalent to

$$\lim_{x\to\infty} x^{k+2-2r+2p} = 0 \text{ and } \lim_{x\to-\infty} (-x)^{k+2-2r+2p} = 0 \quad \text{for any } k = 0,1,\ldots,n+m-1. \tag{30}$$

Finally if in (30) we take $\max\{m,n\} = N$, the left hand side of (27) would be equal to zero if

$$2N+1-2r+2p < 0 \Leftrightarrow N < r-p-\frac{1}{2}. \tag{31}$$

In order to take care of the poles of c_n in (24), we extend the condition (31) to

$$N < r-p-1 < r-p-\frac{1}{2}. \tag{32}$$

The interesting point is that if the latter condition (32) is satisfied, the coefficient d_n in (25) is automatically positive. In fact, the sign of d_n is the sign of

$$\frac{2\,r-2\,p-n}{(2\,n+1+2\,p-2\,r)\,(2\,n-1+2\,p-2\,r)}.$$

Since $n<\max\{m,n\}=N<r-p-\frac{1}{2}$, then $2r-2p-n>n+1>0$ and $2n+1+2p-2r<0$. Moreover, $2n+2p-2r-1<2n+2p-2r+1<0$ and we deduce that

$$\frac{2\,r-2\,p-n}{(2\,n+1+2\,p-2\,r)\,(2\,n-1+2\,p-2\,r)}>0,$$

that is, $d_n>0$ if (32) is valid. Favard's theorem can be therefore applied to conclude that the monic polynomial family $\{\bar{R}_n(x;p,q,r,s)\}$ is orthogonal with respect to the weight function $w_4(x;p,q,r,s)$. By noting these comments, it now remains to compute the norm square value

$$\sum_{x=-\infty}^{\infty} w_4(x;p,q,r,s)\bar{R}_n^2(x;p,q,r,s)=\left(\sum_{x=-\infty}^{\infty} w_4(x;p,q,r,s)\right)\prod_{k=1}^{n} d_k.$$

From (25), first we have

$$\prod_{k=1}^{n} d_k=\Big\{n!\,(2r-2p-n)_n\,(p-r+1+i(q-s))_n\,(p-r+1-i(q-s))_n\,(p-r+1+i(q+s))_n$$
$$(p-r+1-i(q+s))_n\Big\}\Big/\Big\{2^{4n}\,(p-r+1)_n^2\,(p-r+3/2)_n\,(p-r+1/2)_n\Big\}.$$

On the other hand, using Dougall's bilateral sum [2, p. 7]

$$\sum_{n=-\infty}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(d+n)}=\frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)},\quad a,b\notin\mathbb{Z},$$

the moment of order zero can be computed as

$$\sum_{x=-\infty}^{\infty} w_4(x;p,q,r,s)=\sum_{x=-\infty}^{\infty}\frac{\Gamma(x+p+iq)\Gamma(x+p-iq)}{\Gamma(x+r+is)\Gamma(x+r-is)}$$
$$=\frac{\Gamma(p+iq)\Gamma(p-iq)\Gamma(1-p+iq)\Gamma(1-p-iq)\Gamma(2r-2p-1)}{\Gamma(r-p+i(s-q))\Gamma(r-p-i(s-q))\Gamma(r-p-i(s+q))\Gamma(r-p+i(s+q))}>0. \tag{33}$$

Theorem 2.2. *The polynomial set* $\{\bar{R}_n(x;p,q,r,s)\}_{n=0}^{N<r-p-1}$ *is finitely orthogonal*

with respect to the weight function $w_4(x;p,q,r,s)$ on the real line so that we have

$$\sum_{x=-\infty}^{\infty}\frac{\Gamma(x+p+iq)\Gamma(x+p-iq)}{\Gamma(x+r+is)\Gamma(x+r-is)}\bar{R}_n(x;p,q,r,s)\bar{R}_m(x;p,q,r,s)$$
$$=\Big\{n!\,(2r-2p-n)_n\,(p-r+1+i(q-s))_n\,(p-r+1-i(q-s))_n\,(p-r+1+i(q+s))_n$$
$$(p-r+1-i(q+s))_n\,\Gamma(p+iq)\Gamma(p-iq)\Gamma(1-p+iq)\Gamma(1-p-iq)\Gamma(2r-2p-1)\Big\}\,\delta_{m,n}$$
$$\Big/\Big\{2^{4n}\,(p-r+1)_n^2\,(p-r+3/2)_n\,(p-r+1/2)_n\,\Gamma(r-p+i(s-q))\Gamma(r-p-i(s-q))$$
$$\Gamma(r-p-i(s+q))\Gamma(r-p+i(s+q))\Big\}.$$

As we pointed out, there are three particular cases of the weight function $w_4(x;p,q,r,s)$, i. e., relations (17), (18), (19). Hence, three special cases of the main theorem 2.2 can be deduced as follows [4, Chapter 2].

Corollary 2.3. *The polynomial set $\{\bar{R}_n(x;p,0,r,0)\}_{n=0}^{N<r-p-1}$ is finitely orthogonal with respect to the weight function $w_1(x;p,p,r,r)$ on the real line so that we have*

$$\sum_{x=-\infty}^{\infty}\frac{\Gamma^2(x+p)}{\Gamma^2(x+r)}\bar{R}_n(x;p,0,r,0)\bar{R}_m(x;p,0,r,0)$$
$$=\Big\{n!\,(2r-2p-n)_n\,(p-r+1)_n^2\,\Gamma^2(p)\Gamma^2(1-p)\Gamma(2r-2p-1)\Big\}\,\delta_{m,n}$$
$$\Big/\Big\{2^{4n}\,(p-r+3/2)_n\,(p-r+1/2)_n\,\Gamma^4(r-p)\Big\}.$$

Corollary 2.4. *The polynomial set $\{\bar{R}_n(x;p,0,r,s)\}_{n=0}^{N<r-p-1}$ is finitely orthogonal with respect to the weight function $w_2(x;p,p,r,s)$ on the real line so that we have*

$$\sum_{x=-\infty}^{\infty}\frac{\Gamma^2(x+p)}{\Gamma(x+r+is)\Gamma(x+r-is)}\bar{R}_n(x;p,0,r,s)\bar{R}_m(x;p,0,r,s)$$
$$=\Big\{n!\,(2r-2p-n)_n\,(p-r+1-is)_n\,(p-r+1+is)_n\,(p-r+1+is)_n$$
$$(p-r+1-is)_n\,\Gamma^2(p)\Gamma^2(1-p)\Gamma(2r-2p-1)\Big\}\,\delta_{m,n}$$
$$\Big/\Big\{2^{4n}\,(p-r+1)_n^2\,(p-r+3/2)_n\,(p-r+1/2)_n\,\Gamma(r-p+is)\Gamma(r-p-is)$$
$$\Gamma(r-p-is)\Gamma(r-p+is)\Big\}.$$

Corollary 2.5. *The polynomial set $\{\bar{R}_n(x;p,q,r,0)\}_{n=0}^{N<r-p-1}$ is finitely orthogonal*

with respect to the weight function $w_3(x;p,q,r,r)$ on the real line so that we have

$$\sum_{x=-\infty}^{\infty} \frac{\Gamma(x+p+iq)\Gamma(x+p-iq)}{\Gamma^2(x+r)} \bar{R}_n(x;p,q,r,0)\bar{R}_m(x;p,q,r,0)$$
$$= \Big\{ n!\,(2r-2p-n)_n\,(p-r+1+iq)_n\,(p-r+1-iq)_n\,(p-r+1+iq)_n$$
$$(p-r+1-iq)_n\,\Gamma(p+iq)\Gamma(p-iq)\Gamma(1-p+iq)\Gamma(1-p-iq)\Gamma(2r-2p-1) \Big\}\,\delta_{m,n}$$
$$\Big/ \Big\{ 2^{4n}\,(p-r+1)_n^2\,(p-r+3/2)_n\,(p-r+1/2)_n\,\Gamma(r-p-iq)\Gamma(r-p+iq)$$
$$\Gamma(r-p-iq)\Gamma(r-p+iq) \Big\}.$$

3. Moments of the polynomials $\bar{R}_n(x;p,q,r,s)$

As stated in [5, 6], to compute the moments of a continuous or discrete distribution, different bases are usually considered. For example, the canonical basis $\{x^j\}_{j\geq 0}$ is used in the continuous normal distribution, while for the Jacobi weight function $(1-x)^{\alpha}(1+x)^{\beta}$ as the shifted beta distribution [1] on [-1, 1], using one of the two basis $\{(1-x)^j\}_{j\geq 0}$ or $\{(1+x)^j\}_{j\geq 0}$ is appropriate for this purpose. In the negative discrete hypergeometric distribution corresponding to Hahn polynomials, it is more convenient to use the Pochhammer basis $\{(-x)_n\}_{n\geq 0}$, instead of the canonical basis to get

$$\sum_{x=0}^{N-1} \frac{\Gamma(N)\Gamma(\alpha+\beta+2)\Gamma(\alpha+N-x)\Gamma(\beta+x+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)\Gamma(N-x)\Gamma(x+1)}(-x)_n = (-1)^n \frac{(1-N)_n(\beta+1)_n}{(\alpha+\beta+2)_n}.$$

Similarly, for the weight function $w_4(x;p,q,r,s)$, we can use the shifted Pochhammer basis $\{(-x+1-r-is)_n\}_{n\geq 0}$ to compute the moments of the form

$$\mu_n = \sum_{x=-\infty}^{\infty} \frac{\Gamma(x+p+iq)\Gamma(x+p-iq)}{\Gamma(x+r+is)\Gamma(x+r-is)}(-x+1-r-is)_n.$$

Proposition 3.1. *For $n = 0,1,2,\ldots$, the above moments μ_n are solutions of the recurrence equation*

$$\begin{aligned}(n+2p-2r+2)\mu_{n+1} + \Big(&-2\,n^2 + (2\,is - 4\,p + 4\,r - 3)\,n - p^2 \\ &+ (2\,is + 2\,r - 2)\,p - r^2 + (-2\,is + 2)\,r + 2\,is - q^2 + s^2 - 1\Big)\mu_n \\ + n\Big(n^2 - 2\,(is - p + r)\,n + p^2 - 2\,(is + r)\,p + 2irs + q^2 + r^2 - s^2\Big)\mu_{n-1} &= 0. \qquad (34)\end{aligned}$$

Proof. Sum the Pearson equation

$$\Delta(\sigma(x)w_4(x;p,q,r,s))(-x+1-r-is)_n = \tau(x)w_4(x;p,q,r,s)(-x+1-r-is)_n,$$

from $-\infty$ to ∞ and use, respectively, summation by parts, the boundary conditions

(3) and the first equation in (22) to get

$$n\sum_{x=-\infty}^{\infty}\sigma(x+1)w_4(x+1;p,q,r,s)(-x+1-r-is)_{n-1}=\sum_{x=-\infty}^{\infty}\tau(x)w_4(x;p,q,r,s)(-x+1-r-is)_n.$$

From (4), we on the other side get

$$\sigma(x+1)w_4(x+1;p,q,r,s)=(\sigma(x)+\tau(x))w_4(x;p,q,r,s),$$

and the result follows using the second equation in (22). □

To compute the explicit form of the moments, it is now enough to apply the following formula given by N. Sadjang in [9, Theo. 50]

$$\mu_n=I_0(n)\bar{R}_0(x;p,q,r,s)\mu_0,$$

where μ_0 is given in (33) and $I_0(n)$ can be derived from the inversion formula

$$\begin{aligned}(-x+1-r-is)_n&=\sum_{m=0}^{n}I_m(n)\bar{R}_m(x;p,q,r,s)\\&=\sum_{m=0}^{n}\frac{(-n)_m(m+n+2p-2r+2)_{n-m}}{4^{n-m}m!(m+p-r+3/2)_{n-m}(m+p-r+1)_{n-m}}\\&\times(m+p+1-r-i(q+s))_{n-m}(m+p+1-r+i(q-s))_{n-m}\bar{R}_m(x;p,q,r,s).\end{aligned}\tag{35}$$

Equation (35) follows from the inversion formula [14, Prop. 9]

$$(a+ix)_n=\sum_{m=0}^{n}\frac{i^m n!(m+a+c)_{n-m}(m+a+d)_{n-m}}{(n-m)!(m+a+b+c+d-1)_m(2m+a+b+c+d)_{n-m}}P_m(x;a,b,c,d),$$

of the continuous Hahn polynomials and the relationship between $\bar{R}_n(x;p,q,r,s)$ and the continuous Hahn polynomials in (26).

Hence, we obtain

$$\begin{aligned}\mu_n=&\frac{\Gamma(p+iq)\Gamma(p-iq)\Gamma(1-p+iq)\Gamma(1-p-iq)\Gamma(2r-2p-1)}{\Gamma(r-p+i(s-q))\Gamma(r-p-i(s-q))\Gamma(r-p-i(s+q))\Gamma(r-p+i(s+q))}\\&\times\frac{(n+2p-2r+2)_n(-r+p+1-i(q+s))_n(-r+p+1+i(q-s))_n}{4^n(p-r+3/2)_n(p-r+1)_n},\end{aligned}$$

which directly satisfies the recurrence relation (34).

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