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Bounded Nonvanishing Functions and Bateman Functions

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We consider the family $\widetilde{B}$ of bounded nonvanishing analytic functions $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ in the unit disk. The coefficient problem had been extensively investigated (see e.g. [2, 13, 14, 16–18, 20]), and it is known that

$$|a_n| \leq \frac{2}{e}$$

for $n = 1, 2, 3$, and 4. That this inequality may hold for $n \in \mathbb{N}$, is known as the Krzyż conjecture. It turns out that for $f \in \widetilde{B}$ with $a_0 = e^{-t}$

$$f(z) \prec e^{-t((1+z)/(1-z))}$$

so that the superordinate functions $e^{-t((1+z)/(1-z))} = \sum_{k=0}^{\infty} F_k(t) z^k$ are of special interest. The corresponding coefficient functions $F_k(t)$ had been independently considered by Bateman [3] who had introduced them with the aid of the integral representation

$$F_k(t) = (-1)^k \frac{2}{\pi} \int_0^{\pi/2} \cos(t \tan\theta - 2k\theta)\, d\theta.$$

We study the Bateman functions and formulate properties that give insight in the coefficient problem in $\widetilde{B}$.

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1. INTRODUCTION

We consider functions that are analytic in the unit disk

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}.$$

An analytic function f is called subordinate to g, if $f = g \circ \omega$ for some analytic function ω with $\omega(0) = 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$; we write $f \prec g$. The subordination principle states that if g is univalent then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$, see e.g. [15], §23.

Let $\widetilde{B}$ denote the family of bounded nonvanishing analytic functions $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ in $\mathbb{D}$. As f is nonvanishing, we have $\mathrm{Re}\ln f(z) < 0$, and by the subordination principle it turns out that for $a_0 = e^{-t}$

$$-\ln f(z) \prec t\frac{1+z}{1-z},$$

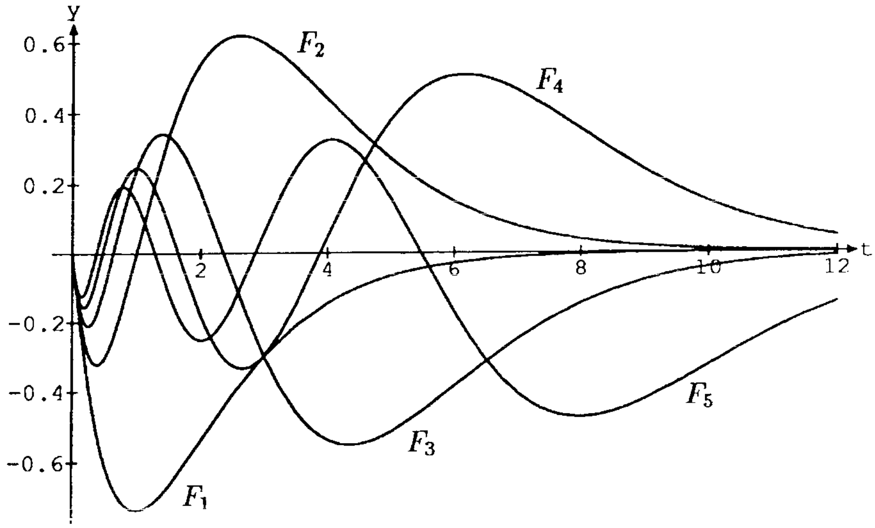

FIGURE 1. Graphs of the functions $F_n(t)$ $(n = 1,\dots,5)$.

and so $f(z) = e^{-t} + a_1 z + a_2 z^2 + \cdots \in \widetilde{B}$ if and only if

$$f(z) \prec e^{-t((1+z)/(1-z))}. \tag{1}$$

Thus the superordinate functions

$$G(t,z) = e^{-t((1+z)/(1-z))} =: \sum_{k=0}^{\infty} F_k(t) z^k \tag{2}$$

are of special interest. Graphs of the functions $F_n(t)$ $(n = 1,\dots,5)$ are given in Figure 1. The following is a list of the first functions F_n:

$$F_0(t) = e^{-t}$$

$$F_1(t) = -2te^{-t},$$

$$F_2(t) = 2e^{-t}(-1+t)t,$$

$$F_3(t) = \frac{2te^{-t}(-3+6t-2t^2)}{3},$$

$$F_4(t) = \frac{2te^{-t}(-3+9t-6t^2+t^3)}{3},$$

$$F_5(t) = \frac{2te^{-t}(-15+60t-60t^2+20t^3-2t^4)}{15}.$$

We consider the coefficient problem ($n \in \mathbb{N}$) to find

$$A_n := \max_{f \in \widetilde{B}} |a_n(f)|.$$

That the maximum exists for all $n \in \mathbb{N}$ follows from the fact that the union of $\widetilde{B}$ with the constant functions $c \in \overline{\mathbb{D}}$ forms a compact family of analytic functions. For the coefficient problem it is no loss of generality to assume that $a_0 > 0$ so that we can assume that (1) holds for some $t > 0$. For small n it is then easy to solve the coefficient problem using subordination techniques.

As $f \prec g$ implies that $|a_1(f)| \leq |a_1(g)|$ (see e.g. [15], Theorem 212), we have

$$A_1 = \max_{t \geq 0} |a_1(e^{-t((1+z)/(1-z))})| = \max_{t \geq 0} |F_1(t)| = \max_{t \geq 0} 2te^{-t} = \frac{2}{e}$$

with equality iff $t = 1$, and $f(z) = \eta e^{-((1+\xi z)/(1-\xi z))}(|\xi| = |\eta| = 1)$.

By the composition with a Möbius transform, this leads to the inequality (see [16])

$$(1 - |z|^2)|f'(z)| \leq \frac{2}{e} \qquad (z \in \mathbb{D}) \tag{3}$$

from which we may deduce by a standard technique (see e.g. [7], p. 72, Exercise 17) that

$$n|a_n(f)| = |a_{n-1}(f')| = \left| \frac{1}{2\pi i} \int_{\partial \mathbb{D}_r} \frac{f'(\zeta)}{\zeta^n} d\zeta \right| \leq \frac{1}{r^{n-1}} \left(\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})| \, d\theta \right)$$

$$\leq \frac{1}{r^{n-1}(1 - r^2)} \frac{2}{e} \leq \frac{2}{e} \frac{n+1}{2} \left(1 + \frac{2}{n-1}\right)^{(n-1)/2} < n$$

where we used (3) and chose $r^2 = (n-1)/(n+1)$. Unfortunately this estimate is just too weak to be of value: The bound 1 for A_n is very elementary, and holds even for all functions bounded by 1. Each global bound less than 1 would be new, however.

It is similarly easy to solve the coefficient problem for $n = 2$ using subordination techniques, ([15], Theorem 212, see e.g. [10]).

Using several methods it was shown that

$$A_n = \frac{2}{e} \tag{4}$$

for $n = 1, 2, 3$, and 4. Obviously $G(1, z^n)$ has nth coefficient equal to $2/e$, which makes these results sharp. That (4) may hold for $n \in \mathbb{N}$, is known as the Krzyż conjecture.

If the subordinate function has very regular coefficient behavior, then global coefficient results are available: If

$$\sum_{k=0}^{\infty} a_k z^k \prec \sum_{k=0}^{\infty} b_k z^k, \tag{5}$$

and if the coefficient sequence b_n is nonnegative, nonincreasing, and convex, then $|a_n| \le b_0$ for all $n \in \mathbb{N}_0$, and if the coefficient sequence b_n is nonnegative, nondecreasing, and convex, then $|a_n| \le b_n$ all $n \in \mathbb{N}_0$ (see e.g. [15], Theorem 216). On the other hand, the coefficient sequences of our subordinate functions $G(t, z)$ are highly irregular for all $t > 0$.

Another important result, however, can be obtained by subordination techniques, as well. It is well known that if (5) holds, then

$$\sum_{k=0}^{\infty} |a_k|^2 \le \sum_{k=0}^{\infty} |b_k|^2.$$

Especially: If an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ of the unit disk is bounded by 1, then $f \prec z$, and the relation

$$\sum_{k=0}^{\infty} |a_k|^2 \le 1$$

(following also directly from Parseval's identity) is obtained (for sharper versions see also [7], Theorem 6.1 and Theorem 6.2). Equality holds if and only if (see [19], Theorem 3) f is an inner function, i.e. if the radial limit $f(e^{i\theta}) := \lim_{r\to 1} f(re^{i\theta}) = 1$ for almost all $e^{i\theta}$ on the unit circle $\partial\mathbb{D}$. Nonvanishing inner functions with positive $f(0)$ have the representation (see e.g. [9], second theorem on p. 66)

$$f(z) = \exp\left(-\int_{\partial\mathbb{D}} \frac{1+e^{i\theta}z}{1-e^{i\theta}z}\, d\mu(\theta)\right),$$

where μ is a singular positive measure on the unit circle $\partial\mathbb{D}$. If we choose a point measure μ concentrating its full measure t at the point $\theta = 1$, we get the function $G(t, z) = e^{-t((1+z)/(1-z))}$ of Equation (2) so that we are lead to the identity

$$\sum_{k=0}^{\infty} F_k^2(t) = 1.$$

For each individual coefficient of $G(t, z)$ we thus have the (weak) inequality

$$|F_n(t)| \le \sqrt{1 - F_0^2(t)} = \sqrt{1 - e^{-2t}} \qquad (n \in \mathbb{N}). \tag{6}$$

It is the purpose of this paper to develop further properties, especially inequalities, for the functions F_n $(n \in \mathbb{N})$, giving more insight in the coefficient problem for $\widetilde{B}$.

2. A COLLECTION OF PROPERTIES OF THE BATEMAN FUNCTIONS

In [3] (see also [1], §13.6) Bateman introduced the functions $(x \ge 0)$

$$k_n(x) := \frac{2}{\pi} \int_0^{\pi/2} \cos(x \tan\theta - n\theta)\, d\theta,$$

and he verified that ([3], formula (2.7))

$$k_{2m}(x) = (-1)^m e^{-x} (L_m(2x) - L_{m-1}(2x)) \tag{7}$$

where $L_m(t)$ denotes the mth Laguerre polynomial. On the other hand if one defines the functions F_n ($n \in \mathbb{N}_0$) with the aid of the generating function

$$e^{-t((1+z)/(1-z))} =: \sum_{k=0}^{\infty} F_k(t) z^k,$$

one gets immediately (see [10], formula (14), and p. 178)

$$F_n(t) = e^{-t}(L_n(2t) - L_{n-1}(2t)) \tag{8}$$

and a comparison of (7) and (8) yields the relation

$$F_n(t) = (-1)^n k_{2n}(t)$$

so that we get the Bateman representation

$$F_n(t) = (-1)^n \frac{2}{\pi} \int_0^{\pi/2} \cos(t \tan\theta - 2n\theta)\, d\theta \tag{9}$$

for our functions F_n. By Bateman's work we are prepared to state many further properties: For $n \in \mathbb{N}$ the function F_n satisfies the differential equation (see [3], formula (5.1))

$$tF_n''(t) = (t - 2n)F_n(t) \tag{10}$$

with the initial values

$$F_n(0) = 0 \qquad \text{and} \qquad F_n'(0) = -2, \tag{11}$$

and the Rodrigues type formula (see [3], formula (31))

$$F_n(t) = \frac{te^t}{n!} \frac{d^n}{dt^n}(e^{-2t} t^{n-1}).$$

The differential equation can also be obtained completely algorithmically (see [11]–[12]).

Further we get the following connection with the generalized Laguerre polynomials (see [23], p. 216, formula (1.15))

$$F_n(t) = e^{-t} L_n^{(-1)}(2t), \tag{12}$$

and (see [22], formula (5.2.1))

$$F_n(t) = -e^{-t} \frac{2t}{n} L_{n-1}^{(1)}(2t), \tag{13}$$

from which one may deduce the hypergeometric representation

$$F_n(t) = -2te^{-t}\, {}_1F_1\left(\left. \begin{matrix} 1-n \\ 2 \end{matrix} \right| 2t \right),$$

and the explicit representation

$$F_n(t) = \frac{e^{-t}}{n} \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!} \binom{n}{k} (2t)^k.$$

Bateman obtained further relations: a difference equation ([3], formula (4.1))

$$(n-1)(F_n(t)-F_{n-1}(t))+(n+1)(F_n(t)-F_{n+1}(t))=2tF_n(t) \tag{14}$$

that is also an easy consequence of the defining equation using the generating function, he obtained a difference differential equation ([3], formula (4.2))

$$(n+1)F_{n+1}(t)-(n-1)F_{n-1}(t)=2tF_n'(t),$$

and a system of differential equations ([3], formula (4.3))

$$F_n'(t)-F_{n+1}'(t)=F_n(t)+F_{n+1}(t), \tag{15}$$

from which he is led to the inequalities for F_n ([3], formula (4.4))

$$|F_n(t)|\le\frac{2n}{t}\qquad(n>2), \tag{16}$$

and for F_n' ([3], formula (4.5))

$$|F_n'(t)|\le\frac{n}{t}\qquad(n>2). \tag{17}$$

For large t the first inequality is a refinement of the trivial estimate

$$|F_n(t)|\le 1 \tag{18}$$

that follows from (6) or from the Bateman representation (9).

Finally Bateman obtained the following statements about integrals of products F_nF_m $(n,m\in\mathbb{N})$ (see [3], formula (2.91))

$$\int_0^\infty F_n^2(t)\,dt=1\qquad\text{and}\qquad\int_0^\infty F_n(t)F_m(t)\,dt=\begin{cases}0 & \text{if}\quad |n-m|\neq 1\\ \frac{1}{2} & \text{otherwise}\end{cases} \tag{19}$$

We state further properties: The functions F_n $(n\in\mathbb{N})$ have a zero at the origin and $n-1$ further positive real zeros (see e.g. [23], *Nullstellensatz*, p. 123) (indeed, by (13), $F_n(t)$ has the same zeros as $L_{n-1}^{(1)}(2t)$).

From the differential equation (10) we moreover see that at $t=2n$ there is a point of inflection, and as $F_n(t)\to 0$ for $t\to\infty$, and all other points of inflection lie at the zeros of F_n one easily deduces that $t=2n$ must be the largest point of inflection of F_n implying that all the zeros of F_n lie in the interval $[0,2n)$. The successive relative maxima of $|F_n|$ lying between the zeros of F_n form an increasing sequence (see [22], Theorem 7.6.2, $\alpha=-1$), so that the largest value attained by $|F_n(t)|$ is attained at the last zero of F_n' which is seen to lie between the last zero T_n of F_n and the point $t=2n$. For small n the mentioned qualitative properties of F_n can be recognized in Figure 1.

By a result of Hahn ([8], formula (17)) the last zero T_n (being the last zero of $L_{n-1}^{(1)}(2t)$) satisfies the relation

$$4n-2-C_1\sqrt[3]{4n-2} < 2T_n < 4n-2-C_2\sqrt[3]{4n-2} \tag{20}$$

with two positive constants $C_1, C_2 \in \mathbb{R}^+$ that are independent of n, in particular

$$\lim_{n\to\infty} \frac{T_n}{n} = 2. \tag{21}$$

The right hand side of (20) leads to the sharpened inequality

$$T_n < 2n - 1,$$

and Puiseux series expansion of (20) yields the refinement of (21)

$$\frac{T_n}{n} = 2 - O\left(\left(\frac{1}{n}\right)^{2/3}\right).$$

3. REPRESENTATION BY RESIDUES

To the system of differential equations given by (15) together with the initial conditions $F_n(0) = 0$ $(n \in \mathbb{N})$ the technique of Laplace transformation

$$\mathcal{L}(f)(z) := \int_0^\infty e^{-zt} f(t)\, dt$$

can be applied to deduce a representation by residues for F_n. It is well-known that $\mathcal{L}(f') = z\mathcal{L}(f) - f(0)$ (see e.g. [6], Satz 9.1) so that we obtain $(n \in \mathbb{N})$

$$(z+1)\mathcal{L}(F_{n+1}) = (z-1)\mathcal{L}(F_n)$$

or

$$\mathcal{L}(F_{n+1}) = \frac{z-1}{z+1}\mathcal{L}(F_n).$$

Induction shows then that for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$

$$\mathcal{L}(F_{n+k}) = \left(\frac{z-1}{z+1}\right)^k \mathcal{L}(F_n). \tag{22}$$

To obtain the initial function $\mathcal{L}(F_1)$, we use $F_0(t) = e^{-t}$ to get first

$$\mathcal{L}(F_0)(z) = \int_0^\infty e^{(z-1)t}\, dt = \frac{1}{1+z}.$$

Further from (15) with $n = 0$ we are led to

$$(z+1)\mathcal{L}(F_1) = (z-1)\mathcal{L}(F_0) - 1$$

or

$$\mathcal{L}(F_1)(z) = -\frac{2}{(1+z)^2}.$$

Thus by an application of (22) with $n = 1$ we have finally

$$\mathcal{L}(F_k)(z) = -\frac{2}{(1+z)^2}\left(\frac{z-1}{z+1}\right)^{k-1}.$$

If we use now the inverse Laplace transform (see e.g. [6], p. 170, formula (15)), we get

$$F_k(t) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma_R} e^{tz}\mathcal{L}(F_k)(z)\,dz,$$

where $\gamma_R : [-R,R] \to \mathbb{C}$ is given by $\gamma_R(\tau) = i\tau$, and therefore we have the integral representation

$$F_k(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{it\tau}\frac{(\tau+i)^{k-1}}{(\tau-i)^{k+1}}\,d\tau.$$

By a standard procedure this can be identified as the residue (see e.g. [4], p. 217, formula (12))

$$\int_{-\infty}^{\infty} e^{it\tau}\frac{(\tau+i)^{k-1}}{(\tau-i)^{k+1}}\,d\tau = 2\pi i\,\mathrm{Res}\left(e^{itz}\frac{(z+i)^{k-1}}{(z-i)^{k+1}}\right).$$

and therefore we have the representation ($k \in \mathbb{N}$)

$$F_k(t) = 2i\,\mathrm{Res}\left(e^{itz}\frac{(z+i)^{k-1}}{(z-i)^{k+1}}\right).$$

These results are collected in

THEOREM 1 *The Bateman functions F_k ($k \in \mathbb{N}$) satisfy the integral representation*

$$F_k(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{it\tau}\frac{(\tau+i)^{k-1}}{(\tau-i)^{k+1}}\,d\tau$$

and therefore the residual representation

$$F_k(t) = 2i\,\mathrm{Res}\left(e^{itz}\frac{(z+i)^{k-1}}{(z-i)^{k+1}}\right).$$

4. RESULTS DEDUCED FROM THE DIFFERENTIAL EQUATION

In this section we deduce another statement about an integral involving the Bateman functions and get an estimate for $|F_n|$ using its differential equation (10). Multiplying (10) by $2F_n'(t)/t$, we have

$$2F_n'(t)F_n''(t) = 2F_n(t)F_n'(t) - \frac{2n}{t}2F_n(t)F_n'(t).$$

We integrate from 0 to t, and get for $n \in \mathbb{N}$ using the initial values (11)

$$(F_n')^2(t) - 4 = F_n^2(t) - \int_0^t \frac{2n}{\tau}2F_n(\tau)F_n'(\tau)\,d\tau. \tag{23}$$

For the last integral we get integrating by parts

$$\int_0^t \frac{2n}{\tau} 2F_n(\tau)F_n'(\tau)\,d\tau = \frac{2n}{\tau}F_n^2(\tau)\Big|_0^t + \int_0^t \frac{2n}{\tau^2}F_n^2(\tau)\,d\tau$$
$$= \frac{2n}{t}F_n^2(t) - 2nF_n(0)F_n'(0) + \int_0^t 2n\left(\frac{F_n(\tau)}{\tau}\right)^2 d\tau$$
$$= \frac{2n}{t}F_n^2(t) + 2n\int_0^t \left(\frac{F_n(\tau)}{\tau}\right)^2 d\tau.$$

So we have the identity

$$(F_n')^2(t) - 4 = F_n^2(t) - \frac{2n}{t}F_n^2(t) - 2n\int_0^t \left(\frac{F_n(\tau)}{\tau}\right)^2 d\tau. \qquad (24)$$

From this identity by letting $t \to \infty$ we are led to the statement

$$\int_0^\infty \left(\frac{F_n(\tau)}{\tau}\right)^2 d\tau = \frac{2}{n}$$

as $\lim_{t\to\infty} F_n(t) = \lim_{t\to\infty} F_n'(t) = 0$. Therefore in particular ($t \geq 0$)

$$\int_0^t \left(\frac{F_n(\tau)}{\tau}\right)^2 d\tau < \frac{2}{n}. \qquad (25)$$

At this point we like to mention that from (24) it is now very easy to deduce the inequality $t < 2n$ for a local extremum of F_n, again (compare §2), as an application of (25) yields

$$F_n^2(t) - \frac{2n}{t}F_n^2(t) = (F_n')^2(t) - 4 + 2n\int_0^t \left(\frac{F_n(\tau)}{\tau}\right)^2 d\tau < (F_n')^2(t),$$

and therefore for any point with $F_n'(t) = 0$ we get $t < 2n$.

To deduce an estimate for $|F_n|$ we regroup (24) and get

$$\frac{2n-t}{t}F_n^2(t) = 4 - (F_n')^2(t) - 2n\int_0^t \left(\frac{F_n(\tau)}{\tau}\right)^2 d\tau < 4,$$

and for $t < 2n$ (which is the critical region) finally

$$|F_n^2(t)| < \frac{4t}{2n-t}. \qquad (26)$$

We note that, however, this improves (6) for small t only, see Figure 2.

In the next section we will give a further improvement of (26).

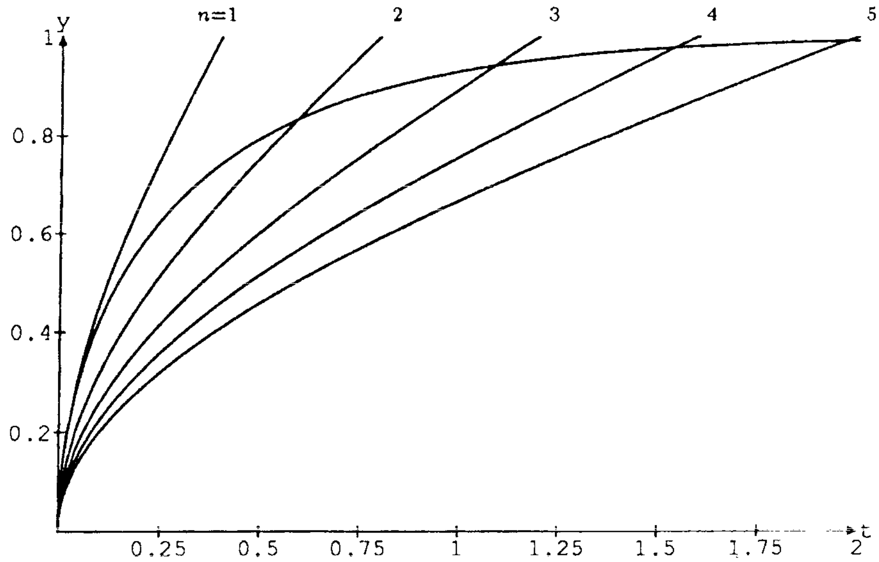

FIGURE 2. The estimates (6) and (26) for $n = 1, \ldots, 5$.

5. ESTIMATES BY THE SZEGÖ METHOD

We consider the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ ($\alpha \in \mathbb{R}$) given by their Rodrigues formula

$$x^\alpha e^{-x} L_n^{(\alpha)}(x) = \frac{1}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}) \tag{27}$$

(see [23], p. 213, formula (1.3)). Szegö ([21], see [22], p. 159, formula (7.21.3)) considered the case $\alpha = 0$, and was led to the inequality

$$e^{-x/2}|L_n^{(0)}(x)| < 1 \qquad (x > 0). \tag{28}$$

Using a similar method we get the following development. For $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{Z}$ the function $f_{n\alpha}(x) := e^{-x}x^{n+\alpha}$ is analytic in $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. By (27) we have

$$x^\alpha e^{-x} L_n^{(\alpha)}(x) = \frac{1}{n!} f_{n\alpha}^{(n)}(x).$$

If $x \in \mathbb{R}^+$, then for $z = x + re^{i\theta}$ ($r \in (0, x)$) we have by Taylor's formula

$$f_{n\alpha}(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f_{n\alpha}^{(k)}(x)(z-x)^k,$$

and Cauchy's integral formula gives the estimate

$$\frac{1}{k!}|f_{n\alpha}^{(k)}(x)| < \frac{1}{r^k} \max_{0 \le \theta \le 2\pi} |f_{n\alpha}(x + re^{i\theta})|.$$

Especially for $k = n$ this yields

$$x^\alpha e^{-x} |L_n^{(\alpha)}(x)| r^n < \max_{0 \le \theta \le 2\pi} |f_{n\alpha}(x + re^{i\theta})|. \tag{29}$$

If furthermore $n + \alpha > 0$, then $f_{n\alpha}$ is analytic in all of $\mathbb{C}$, and (29) holds for all $r \in \mathbb{R}^+$. This case will be studied now.

To give an estimate of $\max_{0 \le \theta \le 2\pi} |f_{n\alpha}(x + re^{i\theta})|$, we expand

$$f_{n\alpha}(x + re^{i\theta}) = e^{-(x + re^{i\theta})}(x + re^{i\theta})^{n+\alpha}$$

to get

$$\max_{0 \le \theta \le 2\pi} |f_{n\alpha}(x + re^{i\theta})| = e^{-x} \max_{0 \le \theta \le 2\pi} e^{-r\cos\theta}(x^2 + r^2 + 2rx\cos\theta)^{(n+\alpha)/2}.$$

Together with (29) we have therefore

$$x^\alpha |L_n^{(\alpha)}(x)| r^n < \max_{0 \le \theta \le 2\pi} e^{-r\cos\theta}(x^2 + r^2 + 2rx\cos\theta)^{(n+\alpha)/2}.$$

We set now $\lambda := \cos\theta \in [-1, 1]$,

$$p(\lambda) := x^2 + r^2 + 2rx\lambda, \qquad \text{and} \qquad q(\lambda) := e^{-r\lambda} p(\lambda)^{(n+\alpha)/2},$$

and have therefore

$$x^\alpha |L_n^{(\alpha)}(x)| r^n < \max_{-1 \le \lambda \le 1} q(\lambda). \tag{30}$$

As

$$q'(\lambda) = re^{-r\lambda} p(\lambda)^{(n+\alpha-2)/2}(x(n+\alpha) - p(\lambda))$$

we get for a possible critical point λ_0 of q the relation

$$\lambda_0 = \frac{x(n+\alpha) - (x^2 + r^2)}{2rx}.$$

At the point $\lambda = \lambda_0$ we have furthermore

$$p(\lambda_0) = x^2 + r^2 + 2rx\lambda_0 = x(n+\alpha) > 0,$$

hence

$$q''(\lambda_0) = -2r^2 x e^{-r\lambda_0} p(\lambda_0)^{(n+\alpha-2)/2} < 0,$$

and λ_0 maximizes q. Therefore, from (30) we get

$$x^\alpha |L_n^{(\alpha)}(x)| r^n < q(\lambda_0) \tag{31}$$

if $-1 \le \lambda_0 \le 1$.

We consider now the case $x \in (0, 4(n+\alpha)]$ (which with respect to the representation (13) corresponds to the critical region $(0, 2n)$ for $t = x/2$), and choose $r := \sqrt{x(n+\alpha)}$. In this case we have $\lambda_0 = \frac{1}{2}\sqrt{x/(n+\alpha)} \in [-1, 0)$. Hence, (31) implies

$$x^\alpha |L_n^{(\alpha)}(x)| (x(n+\alpha))^{n/2} < e^{x/2}(x(n+\alpha))^{(n+\alpha)/2},$$

and therefore finally

THEOREM 2 *For the generalized Laguerre polynomials* $L_n^{(\alpha)}$ $(\alpha \in \mathbb{Z})$ *the estimate*

$$e^{-x/2}|L_n^{(\alpha)}(x)| < \left(\frac{n+\alpha}{x}\right)^{\alpha/2} \tag{32}$$

holds for $x \in (0, 4(n+\alpha)]$ *if* $n + \alpha > 0$.

If we define the functions $(\alpha \in \mathbb{R})$

$$F_n^{(\alpha)}(t) := e^{-t} L_n^{(\alpha)}(2t) \tag{33}$$

then (32) reads $(x = 2t)$

$$|F_n^{(\alpha)}(t)| < \left(\frac{n+\alpha}{2t}\right)^{\alpha/2} \qquad (t \in (0, 2(n+\alpha)]).$$

For $\alpha = 0$, we have Szegö's result (28) in this interval, and for $\alpha = 1$ we get in view of representation (13)

$$|F_n(t)| < \sqrt{\frac{2t}{n}} \qquad (t \in (0, 2n)). \tag{34}$$

This inequality improves (26) as Puiseux expansion yields

$$\sqrt{\frac{4t}{2n-t}} = \sqrt{\frac{2t}{n}} + \frac{1}{2\sqrt{2}}\left(\frac{t}{n}\right)^{3/2} + P\left(\frac{t}{n}\right)$$

with some positive function P.

Note that the special choice $\alpha = -1$ (and *not* the value $\alpha = 0$) generates the Bateman functions $F_n(t) = F_n^{(-1)}(t)$.

6. ASYMPTOTIC ESTIMATES

We consider the functions $(\alpha \in \mathbb{R})$

$$F_n^{(\alpha)}(t) = e^{-t} L_n^{(\alpha)}(2t) = e^{-t}(L_n^{(\alpha+1)}(2t) - L_{n-1}^{(\alpha+1)}(2t)) = F_n^{(\alpha+1)}(t) - F_{n-1}^{(\alpha+1)}(t) \tag{35}$$

of (33) (see [23], p. 216, formula (1.15)) now in more detail. Taking derivative yields

$$\begin{aligned}(F_n^{(\alpha)})'(t) &= -F_n^{(\alpha)}(t) + 2e^{-t}(L_n^{(\alpha)})'(2t) \\ &\overset{(35)}{=} -e^{-t}(L_n^{(\alpha+1)}(2t) - L_{n-1}^{(\alpha+1)}(2t)) + 2e^{-t}(L_n^{(\alpha)})'(2t) \\ &= -e^{-t}(L_n^{(\alpha+1)}(2t) + L_{n-1}^{(\alpha+1)}(2t)) \\ &= -(F_n^{(\alpha+1)}(t) + F_{n-1}^{(\alpha+1)}(t)),\end{aligned} \tag{36}$$

where the relation about $(L_n^{(\alpha)})'$ corresponds to ([23], p. 215, formula (1.12)).

Moreover the program [12] generates the differential equation

$$(1 + \alpha + 2n - t)F_n^{(\alpha)}(t) + (1 + \alpha)(F_n^{(\alpha)})'(t) + t(F_n^{(\alpha)})''(t) = 0$$

for the functions $F_n^{(\alpha)}$ with respect to the variable t, and the recurrence equation

$$(-1 + \alpha + n)F_{n-2}^{(\alpha)} + (1 - \alpha - 2n + 2t)F_{n-1}^{(\alpha)} + nF_n^{(\alpha)} = 0$$

with respect to the variable n.

Assume now, $t_1 < t_2$. Then we get by an integration

$$(F_n^{(\alpha)})^2(t_2) - (F_n^{(\alpha)})^2(t_1) = 2\int_{t_1}^{t_2} F_n^{(\alpha)}(t)(F_n^{(\alpha)})'(t)\,dt$$

$$\overset{(35),(36)}{=} -2\int_{t_1}^{t_2} ((F_n^{(\alpha+1)}(t))^2 - (F_{n-1}^{(\alpha+1)}(t))^2)\,dt.$$

If we choose $\alpha = 0$, we get in particular

$$(F_n^{(0)})^2(t_2) - (F_n^{(0)})^2(t_1) = -2\int_{t_1}^{t_2} ((F_n^{(1)}(t))^2 - (F_{n-1}^{(1)}(t))^2)\,dt. \tag{37}$$

Together with the relation ($n \in \mathbb{N}$)

$$F_n(t) = F_n^{(-1)}(t) = -\frac{2t}{n}F_{n-1}^{(1)}(t)$$

(see (12) and (13), or [22], p. 98, formula (5.2.1)) it follows from (37) that

$$(F_n^{(0)})^2(t_2) - (F_n^{(0)})^2(t_1) = \int_{t_1}^{t_2} F_{n+1}^2(t)\frac{(n+1)^2}{2t^2}\,dt - \int_{t_1}^{t_2} F_n^2(t)\frac{n^2}{2t^2}\,dt.$$

We now let $t_2 \to \infty$. Then $F_n^{(0)}(t_2) \to 0$, and as

$$\int_0^\infty F_n^2(t)\,dt = 1 \tag{38}$$

(see (19), compare [3], formulae (2.7) and (2.91)), we get further

$$(F_n^{(0)})^2(t_1) = \int_{t_1}^\infty F_{n+1}^2(t)\frac{(n+1)^2}{2t^2}\,dt - \int_{t_1}^\infty F_n^2(t)\frac{n^2}{2t^2}\,dt$$

$$\le \int_{t_1}^\infty F_{n+1}^2(t)\frac{(n+1)^2}{2t^2}\,dt \le \frac{(n+1)^2}{2t_1^2}\left(1 - \int_0^{t_1} F_{n+1}^2(t)\,dt\right)$$

where the last inequality is deduced with (38). So we have finally the inequality ($n \in \mathbb{N}$, $t > 0$)

$$|F_n^{(0)}(t)| \le \frac{n+1}{\sqrt{2}t}.$$

This sharpens the result of Szegö (28) for large t. From (35) and (36) it follows that

$$F_n^{(\alpha)}(t) + (F_n^{(\alpha)})'(t) = -2F_{n-1}^{(\alpha+1)}(t).$$

We deduce now for a critical point t_k of F_n with $F_n'(t_k) = 0$ the relation

$$|F_n(t_k)| = 2|F_{n-1}^{(0)}(t_k)| \leq \sqrt{2}\frac{n}{t_k}. \tag{39}$$

Especially is this $\leq 2/e$ for

$$\frac{t_k}{n} \geq \frac{e}{\sqrt{2}} \approx 1.92211551407955841\ldots. \tag{40}$$

It is now remarkable that by the result of Hahn (21) for $n \to \infty$ the most important critical point T of F_n which produces the maximal value of F_n has the property $T/n \to 2$ as $T_n < T < 2$. This gives finally the following

THEOREM 3 *The Krzyż conjecture is asymptotically true for the superordinate functions* $e^{-t((1+z)/(1-z))}$, *i.e. we have* $|a_n(e^{-t((1+z)/(1-z))})| \leq 2/e$ *for* $n \geq N$.

We will now strengthen this result.

Therefore let an arbitrary positive zero t_n of F_n be given. Then t_n is also a zero of $h_n := F_n^2$, and as

$$h_n''(t) = 2(F_n'(t))^2 + 2F_n(t)F_n''(t),$$

by the differential equation for F_n we get

$$h_n''(t) = 2(F_n'(t))^2 + 2\frac{t-2n}{t}F_n^2(t). \tag{41}$$

From this we may deduce that $h_n''(t) > 0$ for $t \geq 2n$. Now, however, we consider the interval between t_n and the smallest relative extremum $t_n^* > t_n$ of F_n, i.e. the smallest zero t_n^* of F_n' after t_n. Then obviously h_n is strictly increasing in $[t_n, t_n^*]$, further $h_n'(t_n) = h_n'(t_n^*) = 0$, and therefore $h_n'|_{[t_n,t_n^*]}$ assumes an absolute maximum at some interior point $t_n^{**} \in (t_n, t_n^*)$, where $h_n''(t_n^{**}) = 0$.

From (41) we deduce

$$(F_n'(t_n^{**}))^2 = \frac{2n - t_n^{**}}{t_n^{**}}F_n^2(t_n^{**}),$$

and therefore by (18)

$$|F_n'(t_n^{**})| = \sqrt{\frac{2n - t_n^{**}}{t_n^{**}}}|F_n(t_n^{**})| < \sqrt{\frac{2n - t_n^{**}}{t_n^{**}}}.$$

As $\sqrt{(2n-t)/t}$ is strictly decreasing for $t \in (0, 2n)$, it follows furthermore that

$$|F_n'(t_n^{**})| < \sqrt{\frac{2n - t_n}{t_n}},$$

and finally (h_n' is positive)

$$h_n'(t_n^{**}) = 2|F_n(t_n^{**})|\,|F_n'(t_n^{**})| < 2\sqrt{\frac{2n - t_n}{t_n}}$$

using (18) again. As t_n^{**} is the global maximum of h'_n in $[t_n, t_n^*]$, we therefore are led to the inequalities

$$0 < h'_n(t) < 2\sqrt{\frac{2n - t_n}{t_n}} \tag{42}$$

for all $t \in (t_n, t_n^*)$.

We are interested in $h_n(t_n^*)$, the value of h_n at its maximum t_n^*. Therefore let $p > 0$ be given such that $h_n(t_n^*) > 1/p$. As $h_n(t_n) = 0$, and h_n is strictly increasing, there is some $\tilde{t}_n \in (t_n, t_n^*)$ with $h_n(\tilde{t}_n) = 1/p$. The mean value theorem then shows the existence of $\xi_n \in (\tilde{t}_n, t_n^*)$ with

$$\frac{h_n(t_n^*) - h_n(\tilde{t}_n)}{t_n^* - \tilde{t}_n} = h'_n(\xi_n),$$

and therefore by (42)

$$\frac{h_n(t_n^*) - h_n(\tilde{t}_n)}{t_n^* - \tilde{t}_n} < 2\sqrt{\frac{2n - t_n}{t_n}}$$

or

$$h_n(t_n^*) < h_n(\tilde{t}_n) + 2\sqrt{\frac{2n - t_n}{t_n}}(t_n^* - \tilde{t}_n) = \frac{1}{p} + 2\sqrt{\frac{2n - t_n}{t_n}}(t_n^* - \tilde{t}_n).$$

By (19) we have

$$\int_0^\infty h_n(\tau)\,d\tau = 1,$$

and thus by the integral mean value theorem (h_n is nonnegative)

$$1 > \int_{\tilde{t}_n}^{t_n^*} h_n(\tau)\,d\tau = h_n(\eta_n)(t_n^* - \tilde{t}_n)$$

for some $\eta_n \in (\tilde{t}_n, t_n^*)$. As h_n is strictly increasing, we therefore get $h_n(\eta_n) > h_n(\tilde{t}_n) = 1/p$ implying

$$1 > \frac{1}{p}(t_n^* - \tilde{t}_n) \qquad \text{or} \qquad t_n^* - \tilde{t}_n < p.$$

Finally we have

$$h_n(t_n^*) < \frac{1}{p} + 2p\sqrt{\frac{2n - t_n}{t_n}}.$$

We were led to this inequality under the assumption that $h_n(t_n^*) > 1/p$. If $h_n(t_n^*) \le 1/p$, however, then the same conclusion follows trivially, so that the above calculations can be summarized by the following

LEMMA 1 *Let $h_n(t) = F_n^2(t)$, let t_n be a positive zero of F_n, let t_n^* the lowest zero of F'_n that is larger than t_n, and let $p > 0$. Then*

$$h_n(t_n^*) < \frac{1}{p} + 2p\sqrt{\frac{2n - t_n}{t_n}}.$$

We now emphasize on the largest zero $t_n = T_n$ of F_n. By the results of §2 the global maximum of F_n is attained at the last zero T_n^* of F_n' which lies in the interval $(T_n, 2n)$, and is therefore the smallest zero of F_n' after T_n. So Lemma 1 applies.

By a result of Bottema and Hahn (see [5] and [8], p. 228, last formula), the inequality

$$T_n > 2n - \tfrac{3}{2} - 8\sqrt{2}\sqrt{n-1} =: \tau_n \tag{43}$$

($n \geq 33$) holds for the last zero of F_n (or $L_{n-1}^{(1)}$). As $\sqrt{(2n-t)/t}$ is strictly decreasing for $t \in (0, 2n)$, we have the inequality

$$\sqrt{\frac{2n - T_n}{T_n}} < \sqrt{\frac{2n - \tau_n}{\tau_n}}.$$

Puiseux expansion yields the asymptotic expression ($n \to \infty$)

$$\sqrt{\frac{2n-\tau_n}{\tau_n}} = 2\sqrt[4]{2}\,\frac{1}{n^{1/4}} + \frac{131}{16\sqrt[4]{2}}\,\frac{1}{n^{3/4}} + O\left(\frac{1}{n^{5/4}}\right),$$

especially is

$$\sqrt{\frac{2n-\tau_n}{\tau_n}} \sim \frac{1}{n^{1/4}}.$$

In our calculations the value p was arbitrary, so we have the freedom to choose it properly. The asymptotics suggest the choice $p \sim n^{1/8}$. For any $a > 0$ we get therefore

$$h_n(T_n^*) < \frac{a}{n^{1/8}} + 2\frac{n^{1/8}}{a}\sqrt{\frac{2n - T_n}{T_n}} < \frac{a}{n^{1/8}} + 2\frac{n^{1/8}}{a}\sqrt{\frac{2n-\tau_n}{\tau_n}} \sim \frac{1}{n^{1/8}} < \frac{b}{n^{1/8}}$$

for some $b > 0$.

We choose the value $a = 2\sqrt[8]{2}$ (minimizing the leading term in the corresponding Puiseux expansion) and get the global estimate

$$h_n(T_n^*) < 2\sqrt[8]{2}\,\frac{1}{n^{1/8}} + \frac{\sqrt{\frac{3}{2} + 8\sqrt{2}\sqrt{n-1}}}{\sqrt[8]{2}\sqrt{2n - \frac{3}{2} - 8\sqrt{2}\sqrt{n-1}}}\,n^{1/8}.$$

Now we remember that F_n takes its global maximum over $\mathbb{R}^+$ at the point T_n^*, and so does h_n. We therefore have for all $a > 0$, $n \in \mathbb{N}$ and $t > 0$ the inequality

$$|F_n(t)| < \sqrt{2\sqrt[8]{2}\,\frac{1}{n^{1/8}} + \frac{\sqrt{\frac{3}{2} + 8\sqrt{2}\sqrt{n-1}}}{\sqrt[8]{2}\sqrt{2n - \frac{3}{2} - 8\sqrt{2}\sqrt{n-1}}}\,n^{1/8}}$$

$$= \frac{2\sqrt[16]{2}}{n^{1/16}} + \frac{131}{64\; 2^{7/16} n^{9/16}} + O\left(\frac{1}{n^{17/16}}\right). \tag{44}$$

We mention that we get a better asymptotic estimate if we use the sharper left hand inequality (20) instead of (43), set $\tau_n^* := -1 + 2n - C(2n-1)$ (C constant) leading to the asymptotic result

$$\sqrt{\frac{2n-\tau_n^*}{\tau_n^*}} \sim \frac{1}{n^{1/3}},$$

and therefore by the choice $p \sim n^{1/6}$ and the same procedure as above to the

THEOREM 4 *For all $t \in \mathbb{R}^+$ we have the asymptotic inequality* ($n \geq N$)

$$|F_n(t)| < \frac{c}{n^{1/12}}$$

for some $c > 0$, and in particular the limiting value

$$\lim_{n\to\infty} |F_n(t)| = 0.$$

Obviously this theorem strengthens Theorem 3.

In principle (44) enables one to prove the statement

$$|F_n(t)| \leq \frac{2}{e}$$

for all $n \in \mathbb{N}$. Therefore one shows that the estimation function

$$E(n) = \sqrt{2\sqrt[8]{2}\frac{1}{n^{1/8}} + \frac{\sqrt{\frac{3}{2} + 8\sqrt{2}\sqrt{n-1}}}{\sqrt[8]{2}\sqrt{2n - \frac{3}{2} - 8\sqrt{2}\sqrt{n-1}}}n^{1/8}}$$

of (44) is decreasing, and as $E(17821075) > 2/e$ and $E(17821076) < 2/e$, it remains to prove the result for only a finite number of initial values.

The number of initial values, however, can be decisively lowered using that by (40) $|F_n(t)| \leq 2/e$ whenever $T_n^*/n \geq e/\sqrt{2}$, especially if $T_n/n \geq e/\sqrt{2}$. From the Bottema–Hahn bound

$$\frac{T_n}{n} > \frac{\tau_n}{n} = 2 - \frac{3}{2n} - 8\sqrt{2}\frac{\sqrt{n-1}}{n} =: e(n)$$

we obtain first by the calculation

$$e'(n) = \frac{-16\sqrt{2} + 3\sqrt{n-1} + 8\sqrt{2}n}{2n^2\sqrt{n-1}}$$

that $e(n)$ is increasing for $n > 2$, and as $\lim_{n\to\infty} e(n) = 2$ there is exactly one solution $n_0 \geq 2$ of the equation $e(n) = e/\sqrt{2}$, and $T_n/n > e(n) \geq e(n_0) = e/\sqrt{2}$ for $n > n_0$. A numerical calculation shows that $n_0 \approx 21138.7$ so that we are led to the

THEOREM 5 *The inequality* $|F_n(t)| \leq 2/e$ *is true for all* $n \in \mathbb{N}$ *and all* $t > 0$ *if it is true for* $n \leq 21138$.

7. ESTIMATES FOR THE DERIVATIVE

By (23) it follows that at the zeros of F_n the derivative F_n' satisfies the relation $|F_n'(t)| \leq 2$. This result holds for all $t \geq 0$ which can be seen as follows: Using (36) with $\alpha = -1$ we have

$$F_n'(t) = -e^{-t}(L_n(2t) + L_{n-1}(2t)), \tag{45}$$

and by an application of the Szegö result (28) it follows for $t > 0$

$$|F_n'(t)| < 2. \tag{46}$$

This shows that for all $n \in \mathbb{N}$ the derivatives $|F_n'|$ have their maximal value at the origin where $|F_n'(0)| = 2$ (see (11)). We note moreover, even though the derivative $F := F_n'$ has a representation (45) similar to that of F_n itself, it satisfies the much more complicated differential equation

$$(-4n^2 + 4nt - t^2)F(t) - 2nF'(t) + (-2nt + t^2)F''(t) = 0.$$

8. THE FUNCTIONS H_n

In this section we collect the explicit inequalities that we deduced, and formulate a conjecture concerning the Bateman functions.

As the last point of inflection of the functions F_n is at the point $t = 2n$ which increases with increasing n, it is reasonable to introduce the functions

$$H_n(t) := (-1)^n F_n(nt)$$

that have common absolute values with F_n which, however, are attained at different points. The scale on the t-axis is here such that the last point of inflection lies at $t = 2$ for all functions H_n ($n \in \mathbb{N}$), and H_n is positive for $t \geq 2$. It is easy to deduce the differential equation

$$tH_n''(t) = n^2(t-2)H_n(t) \tag{47}$$

satisfied by H_n. The inequalities that we deduced for F_n read as follows for H_n: The trivial estimate (18) gives

$$|H_n(t)| \leq 1,$$

(16) yields ($n > 2$)

$$|H_n(t)| \leq \frac{2}{t},$$

the refinement (39) gives

$$|H_n(t_k)| \leq \frac{\sqrt{2}}{t_k}$$

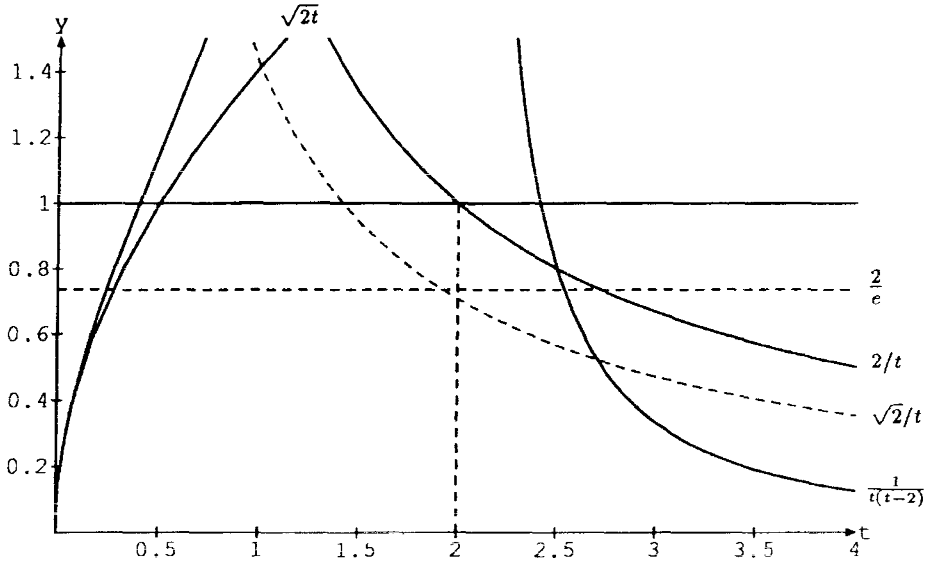

FIGURE 3. Estimates for the functions H_n

for a critical point t_k of H_n, (26) implies

$$|H_n(t)| < \sqrt{\frac{4t}{2-t}},$$

and finally (34) yields

$$|H_n(t)| < \sqrt{2t}.$$

These estimates commonly do not depend on n. One more estimate will be added in the next section. Figure 3 shows them graphically.

Figures 4 and 5 show the graphs of the functions H_n ($n = 1,\ldots,20$). We conjecture that H_n is strictly decreasing for increasing n at the point $t = 2$. Note that by the result of Hahn (21) this is not true for any $t < 2$. In the next section we will show, however, that $\lim_{n\to\infty} H_n(t) = 0$ for each $t > 2$.

9. ESTIMATES FOR LARGE t

In this section we show that for all $t > 2$ the values $H_n(t)$ tend to 0 for $n \to \infty$.

The inequalities (46), and (17) correspond to the inequalities

$$|H_n'(t)| \le 2n,$$

and

$$|H_n'(t)| \le \frac{n}{t} \tag{48}$$

for the derivative of H_n, respectively.

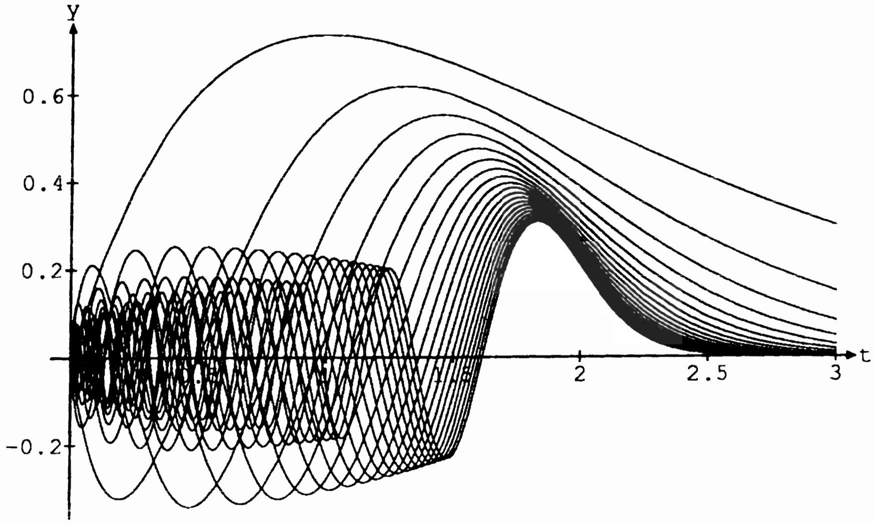

FIGURE 4. The functions H_n ($n = 1,\ldots,20$).

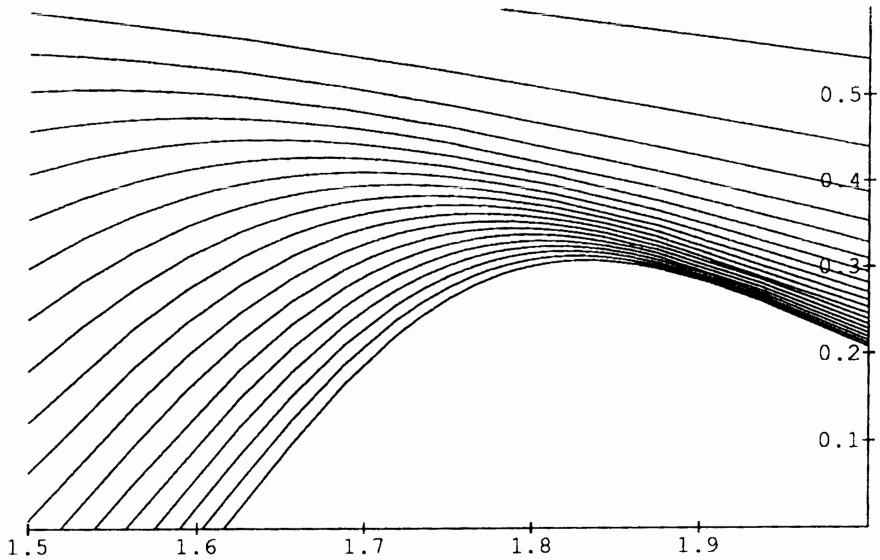

FIGURE 5. The functions H_n ($n = 1,\ldots,20$) in the interval [1.5,2].

By the differential equation (47) for H_n we have

$$H_n(t) = \frac{t}{n^2(t-2)} H_n''(t). \tag{49}$$

Let now $2 < t_1 < t_2$ be given. By definition H_n is strictly positive in $[t_1,t_2]$, and by (49) H_n'' is strictly positive in $[t_1,t_2]$. Therefore

$$\int_{t_1}^{t_2} H_n(t)\,dt = \frac{1}{n^2}\int_{t_1}^{t_2} \frac{t}{t-2}H_n''(t)\,dt.$$

As the function $t/(t-2)$ is strictly decreasing, we have

$$\max_{t\in[t_1,t_2]} \frac{t}{t-2} = \frac{t_1}{t_1-2},$$

and therefore

$$\frac{1}{n^2}\int_{t_1}^{t_2} \frac{t}{t-2}H_n''(t)\,dt \le \frac{1}{n^2}\frac{t_1}{t_1-2}\int_{t_1}^{t_2} H_n''(t)\,dt.$$

As H_n'' is positive in $[t_1,t_2]$, and as $\lim_{t\to\infty} H_n'(t) = 0$, it follows that H_n' is negative and increasing, and therefore

$$\int_{t_1}^{t_2} H_n(t)\,dt \le \frac{1}{n^2}\frac{t_1}{t_1-2}|H_n'(t_1) - H_n'(t_2)| \le \frac{1}{n^2}\frac{t_1}{t_1-2}|H_n'(t_1)|$$

$$\le \frac{1}{n^2}\frac{t_1}{t_1-2}\frac{n}{t_1} \le \frac{1}{n(t_1-2)}$$

where we used (48). For fixed $t_2 > 2$ we set now $t_1 := (2+t_2)/2$, and with the integral mean value theorem we find $\tau \in [t_1,t_2]$ with

$$H(t_2) \le H(\tau) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} H_n(t)\,dt \le \frac{1}{(t_2-t_1)(t_1-2)}\frac{1}{n} = \frac{4}{(t_2-2)^2}\frac{1}{n}.$$

Another estimate for large t which is independent of n, will be established now. Let again $2 < t_1 < t_2$. Then by (49)

$$|H_n^2(t_2) - H_n^2(t_1)| = \left|\int_{t_1}^{t_2} (H_n^2(t))'\,dt\right| = \left|\int_{t_1}^{t_2} 2H_n'(t)H_n(t)\,dt\right|$$

$$= \left|\int_{t_1}^{t_2} \frac{t}{n^2(t-2)}H_n'(t)H_n''(t)\,dt\right|$$

$$\le \frac{t_1}{n^2(t_1-2)}\int_{t_1}^{t_2} |2H_n'(t)H_n''(t)|\,dt$$

$$= \frac{t_1}{n^2(t_1-2)}|(H_n')^2(t_2) - (H_n')^2(t_1)|.$$

We let now $t_2 \to \infty$, and get $(t := t_1)$ using (48)

$$|H_n(t)|^2 \le \frac{t}{n^2(t-2)}|(H_n')^2(t)| \le \frac{1}{t(t-2)}.$$

This estimate improves the earlier ones for large t, see Figure 3.

Finally we show that uniformly with respect to n the functions H_n decrease faster than each negative power.

THEOREM 6 *For each $k \in \mathbb{N}_0$ there is a constant $C_k > 0$ such that*

$$|H_n(t)| \le \frac{C_k}{t^k} \qquad (t > 0,\ n \in \mathbb{N})$$

that is independent of n.

Proof We prove the result by induction with respect to k. For $k = 0$ the statement is trivially true, see (18). Assume now the statement holds for some $k \in \mathbb{N}_0$, i.e.

$$|F_n(t)| \le C_k \left(\frac{n}{t}\right)^k$$

Then we get using (14)

$$\begin{aligned} |F_n(t)| &= \frac{1}{2t}|(n-1)(F_n(t) - F_{n-1}(t)) + (n+1)(F_n(t) - F_{n+1}(t))| \\ &\le \frac{2(n-1)C_k + (n+1)C_k + (n+1)C_k D_k}{2t}\left(\frac{n}{t}\right)^k \\ &\le \frac{2(n-1)C_k D_k + (n+1)C_k D_k + (n+1)C_k D_k}{2t}\left(\frac{n}{t}\right)^k \\ &= 2C_k D_k \left(\frac{n}{t}\right)^{k+1}, \end{aligned}$$

where we chose $D_k \ge 1$ such that $(n+1)^k \le D_k b^k$ (the choice $D_k = 2^k$ does the job required as $(n+1)^k \le (2n)^k \le 2^k n^k$). This yields the result. ■

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