***On the interplay between geometrical and analytical properties of univalent functions***

# HABILITATIONSSCHRIFT


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# 1. Introduction

## 1.1 Univalent functions

We consider functions that are analytic in the unit disk

$$\mathbb{D} := \{z \in \mathbb{C} |\; |z| < 1\} .$$

A function is called *univalent* (or *schlicht*) if it is one-to-one. The Riemann mapping theorem guarantees the existence of a univalent map $f : \mathbb{D} \to G$ for each simply connected domain $G \subsetneqq \mathbb{C}$. Moreover $f$ is uniquely determined except of the composition with rotations $z \mapsto e^{i\alpha}z$ of $\mathbb{D}$.

If $(G_n)$ is a sequence of simply connected domains with $a \in G_n, n \in \mathbb{N}$, then the largest domain $G$ containing $a$ and having the property that each compact subset of $G$ lies in all but a finite number of the domains $G_n$ is called the *kernel* of $(G_n)$. If no such domain exists then the kernel is $\{a\}$. A sequence $(G_n)$ is said to converge to $G$, if each subsequence has the kernel $G$. We write $G_n \to G$. The *Carathéodory kernel theorem* states that a sequence $(f_n)$ of univalent functions with $f_n(0) = a$ and $f_n'(0) > 0$ converges locally uniformly to $f$, if and only if $f_n(\mathbb{D})$ converges to $f(\mathbb{D})$.

If we speak about convergence of a sequence $(f_n)$ of analytic functions, we mean locally uniform convergence and write $f_n \to f$. The family $A$ of *analytic functions of* $\mathbb{D}$ together with this topology is a Fréchet space, i.e. a locally convex complete metrizable linear space.

A sequence of univalent functions not converging locally uniformly to $\infty$ is normal, and there is a convergent subsequence. The limit function is univalent or constant. When considering sequences of univalent functions, we often assume without loss of generality that they converge instead of choosing a subsequence.

The family $S$ of univalent functions that are normalized by $f(0) = 0$, $f'(0) = 1$, i.e.

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1.1}$$

is a compact subset of $A$.

A function $f \in A$ is called $m$-fold symmetric if it has the special form ($m \in \mathbb{N}$)

$$f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \cdots \ , \tag{1.2}$$

which is equivalent to the fact that the Riemann image surface $F$ is $m$-fold symmetric with respect to the origin, i.e. for all $w \in F$, $k = 1, \ldots, m$ also the points $e^{2\pi i k/m}w \in F$.

(References: [13], [17], [49].)

## 1.2 Functions with positive real part

Let $P$ denote the subset of $A$ of functions $p$ with positive real part that are normalized by $p(0) = 1$.

A function of the form

$$p(z) = \int_{\partial\mathbb{D}} \frac{1 + xz}{1 - xz} d\mu(x) \ , \tag{1.3}$$

where $\mu$ denotes a Borel probability measure on $\partial\mathbb{D}$, clearly has positive real part, because the kernel functions have this property. The famous *Herglotz representation theorem* states that the converse is also true. This is equivalent to the fact that the *extreme points* of $P$ (i.e. the points which have no proper convex representation within the convex set $P$) are the kernel functions of representation (1.3), which map $\mathbb{D}$ univalently onto the right halfplane $\{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}$ (see e.g. [53], [20]); we write $\mathrm{E}\,(P) = \{\frac{1+xz}{1-xz} | \ x \in \partial\mathbb{D}\}$. By the Krein-Milman theorem their *closed convex hull* $\overline{\mathrm{co}}\,(\mathrm{E}\,P)$ is all of $P$ and so their *convex hull* $\mathrm{co}\,(\mathrm{E}\,P)$ lies dense in $P$ with respect to the topology of locally uniform convergence (which makes $P$ compact), so that each function $p \in P$ can be locally uniformly approximated by functions $p_n$ of the form

$$p_n(z) = \sum_{k=1}^{n} \mu_k \frac{1 + x_k z}{1 - x_k z}, \quad |x_k| = 1, \ \mu_k > 0 \quad (k = 1, \ldots, n),$$

$$\sum_{k=1}^{n} \mu_k = 1, \quad n \in \mathbb{N}. \tag{1.4}$$

The functions of the form (1.4) give the so-called *Carathéodory boundary* of $P$.

A function $f$ is called *subordinate* to $g$, if $f = g \circ \omega$ for some function $\omega \in A$ with $\omega(0) = 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$; we write $f \prec g$. The *subordination principle* states that if $g$ is univalent then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$, and so $p \in P$ iff $p \prec \frac{1+z}{1-z}$. If $f \prec g$ then by Schwarz's Lemma $f(\mathbb{D}_r) \subset g(\mathbb{D}_r)$ for all $r \in ]0,1[$ where $\mathbb{D}_r := \{z \in \mathbb{C} \mid |z| < r\}$.

By $B$ we denote the family of functions $\omega \in A$ with $\omega(0) = 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$, and by $\operatorname{Sub} F$ the family of functions which are subordinate to some $f \in F$.

A compact family which is similar to $P$ is the class $\widetilde{P}$ of functions $p$ normalized by $p(0) = 1$ for which there is some $\alpha \in \mathbb{R}$ such that the real part of $e^{i\alpha}p$ is positive. One sees that $p \in \widetilde{P}$ iff $p \prec \frac{1+yz}{1-z}$, where $y = e^{-2i\alpha}$. A slight modification of Herglotz's theorem gives that each function $p \in \widetilde{P}$ can be approximated by functions of the form

$$p_n(z) = \sum_{k=1}^{n} \mu_k \frac{1 + yx_k z}{1 - x_k z}, \quad |y| = |x_k| = 1,\ \mu_k > 0 \quad (k = 1, \ldots, n),$$

$$\sum_{k=1}^{n} \mu_k = 1, \quad n \in \mathbb{N}\,, \tag{1.5}$$

in other words

**Lemma 1.1** *The functions of the form* (1.5) *form a dense subset of* $\widetilde{P}$.

(For details see e.g. [20], chapter 3, and [24].)

**Lemma 1.2** *Each function of the form* (1.5) *has a representation*

$$p_n(z) = \prod_{k=1}^{n} \frac{1 - y_k z}{1 - x_k z}\,, \tag{1.6}$$

*where*

$$|x_k| = |y_k| = 1 \quad (k = 1, \ldots, n) \tag{1.7}$$

*and*

$$\arg x_1 < \arg y_1 < \arg x_2 < \arg y_2 < \cdots < \arg x_n < \arg y_n < \arg x_1 + 2\pi. \tag{1.8}$$

*Proof:* The function $p_n$ given by (1.5) is rational in $\widehat{\mathbb{C}}$ of degree $n$ with exactly $n$ poles at the points $\overline{x_k}$, and $p_n(0) = 1$, so that (1.6) holds. As a convex combination of functions subordinate to $\frac{1+yz}{1-z}$ also $p_n \prec \frac{1+yz}{1-z}$, and so $p_n(\mathbb{D})$ lies in some halfplane $H$ whose boundary contains the origin, and in particular $p_n$ is nonvanishing in $\mathbb{D}$. From this it follows that $|y_k| \leqq 1$ $(k = 1, \dots, n)$. On the other hand

$$p_n(\infty) = -y \sum_{k=1}^{n} \mu_k = -y = \prod_{k=1}^{n} \frac{y_k}{x_k},$$

so that $\prod_{k=1}^{n} |y_k| = 1$, which leads to (1.7). From (1.6) it follows with the aid of the identity

$$\arg(1+x) = \frac{1}{2} \arg x, \quad |x| = 1,\ x \neq -1, \tag{1.9}$$

that for $e^{i\theta} \neq \overline{x_k}, \overline{y_k} \quad (k = 1, \dots, n)$

$$\arg\Big(p_n(e^{i\theta})\Big) = \frac{1}{2} \arg \left( \prod_{k=1}^{n} \frac{y_k}{x_k} \right) \qquad (\mathrm{mod}\ \pi), \tag{1.10}$$

so that the curve $\{p_n(e^{i\theta})\}$ lies on a line $\ell$ through the origin, and $p_n(\mathbb{D}) \subset H$ then implies that $p_n(\mathbb{D}) = H$ where $H$ denotes the halfplane with $\ell = \partial H$ and $1 \in H$. In particular, $p_n(e^{i\theta})$ does not contain a turning point $\theta_0$ where $p_n'(e^{i\theta_0}) = 0$. Suppose now that (1.8) does not hold. Then there exist two zeros of $p_n(e^{i\theta})$, $\theta_1$ and $\theta_2$, say, without pole between them (on $\partial\mathbb{D}$), so that $p_n(e^{i\theta})$ must change its direction on $\ell$ for some $\theta_0 \in ]\theta_1, \theta_2[$. Here $p_n'(e^{i\theta_0}) = 0$, and we have a contradiction. □

On the other hand, functions of the form (1.6) - (1.8) are elements of $\widetilde{P}$ as the following lemma shows.

**Lemma 1.3** *The functions of the form* (1.6) - (1.8) *form a dense subset of* $\widetilde{P}$.

*Proof:* By Lemma 1.1 the functions of form (1.5) are dense in $\widetilde{P}$, and by Lemma 1.2 they have a representation of the form (1.6) - (1.8). Now we show that functions of the form (1.6) - (1.8) lie in $\widetilde{P}$, which gives the result.

As above we get (1.10), and the curve $\{p_n(e^{i\theta})\}$ lies on a line $\ell$ through the origin. Next we shall show that $p_n'(z) \neq 0$ for $z \in \partial\mathbb{D}$, and from this it follows that $p_n(\mathbb{D})$ must lie on one side of $\ell$, because $p_n(e^{i\theta})$ does not change its direction by moving on $\ell$ while $\theta$ varies from 0 to $2\pi$. Hence $p \in \widetilde{P}$.

The zeros $\overline{y_k}$ and the poles $\overline{x_k}$ of $p_n$ are pairwise different by (1.8), so that they have order one and $p_n'(\overline{x_k}), p_n'(\overline{y_k}) \neq 0 \ \ (k = 1, \ldots, n)$. It remains to show that

$$z\frac{p_n'}{p_n}(z) \neq 0 \text{ for } z \in \partial\mathbb{D}, \quad z \neq \overline{x_k}, \overline{y_k} \quad (k = 1, \ldots, n) .$$

From representation (1.6) it follows for $z \in \partial\mathbb{D}$ that

$$z\frac{p_n'}{p_n}(z) = \sum_{k=1}^{n} \left( \frac{1}{1 - \overline{y_k}\, \overline{z}} - \frac{1}{1 - \overline{x_k}\, \overline{z}} \right) .$$

The real part of this sum equals zero because the same is true for each summand. On the other hand we get for $z = e^{i\theta}$

$$\operatorname{Im} \left( z\frac{p_n'}{p_n}(z) \right) = \frac{1}{2} \sum_{k=1}^{n} \left( \cot \frac{\varphi_k - \theta}{2} - \cot \frac{\psi_k - \theta}{2} \right) ,$$

if we write

$$\varphi_k := \arg \overline{x}_k \ ; \quad \psi_k := \arg \overline{y}_k \ .$$

Now let $\theta \in [0, 2\pi]$ be given. Then rearrange the values of $\varphi_k$ and $\psi_k$ modulo $2\pi$, such that

$$\theta < \varphi_N < \psi_N < \varphi_{N+1} < \psi_{N+1} < \cdots < \varphi_n < \psi_n < \varphi_1 < \psi_1 < \cdots < \varphi_{N-1} < \psi_{N-1} < \theta + 2\pi \tag{1.11}$$

or

$$\theta < \psi_N < \varphi_N < \psi_{N+1} < \varphi_{N+1} < \cdots < \psi_n < \varphi_n < \psi_1 < \varphi_1 < \cdots < \psi_{N-1} < \varphi_{N-1} < \theta + 2\pi \tag{1.12}$$

holds which is possible by (1.8) if $\theta \neq \varphi_k, \psi_k \ \ (k = 1, \ldots, n)$. Write

$$a_k := \cot \frac{\psi_k - \theta}{2} \ ; \quad b_k := \cot \frac{\varphi_k - \theta}{2} \ ,$$

then

$$\operatorname{Im} \left( z\frac{p_n'}{p_n}(z) \right) = \frac{1}{2} \sum_{k=1}^{n} (b_k - a_k) \ .$$

Suppose now, (1.11) holds, then $b_k - a_k > 0 \ \ (k = 1, \ldots, n)$, because the function cot is strictly decreasing in $]0, \pi[$. Thus $\operatorname{Im}(zp_n'/p_n(z)) > 0$. If (1.12) holds, $\operatorname{Im}(zp_n'/p_n) < 0$ follows similarly. This finishes the proof that $zp_n'/p_n$ has no zero on $\partial\mathbb{D}$. □

## 1.3 Polygons and Schwarz-Christoffel mappings

Let $f \in A$ be continuous in $\overline{\mathbb{D}}$ and have a Riemann surface $F$ as image domain whose boundary consists of a finite number of linear arcs, such that the boundary correspondence $\partial\mathbb{D} \to \partial F$ is one-to-one. Then $F$ is called a *polygon*. Let $F$ have $n$ vertices of inner angles $\alpha_k\pi$ $(k = 1, \ldots, n)$. We do not suppose $f$ to be univalent, so that $\alpha_k > 2$ is possible, whereas for univalent polygons

$$\alpha_k \leqq 2 \quad (k = 1, \ldots, n)\,. \tag{1.13}$$

If we have a bounded vertex then

$$\alpha_k > 0\,. \tag{1.14}$$

If a vertex lies at infinity we measure the angle on the Riemann sphere and have

$$\alpha_k \geqq 0\,, \tag{1.15}$$

where $\alpha_k = 0$ is a zero angle which corresponds to two parallel rays of $\partial F$.

Let now $x_k$ be the *prevertices*, i.e. the preimages under $f$ of the vertices $f(x_k)$. Then the *Schwarz-Christoffel formula* is the representation

$$\frac{f''}{f'}(z) = -2\sum_{k=1}^{n} \frac{\mu_k}{z - x_k}\,, \tag{1.16}$$

where

$$2\mu_k\pi := \begin{cases} (1 - \alpha_k)\pi & \text{if } f(x_k) \text{ is bounded} \\ (1 + \alpha_k)\pi & \text{if } f(x_k) \text{ is unbounded} \end{cases} \tag{1.17}$$

denote the *outer angles*. The formula

$$\sum_{k=1}^{n} \mu_k = 1 \tag{1.18}$$

corresponds in the bounded (univalent) case both to the rule for the sum of angles in an $n$-gon and to the fact that the increment of the tangent direction is exactly $2\pi$ when surrounding the polygon on $\partial F$ one time.

On the other hand, if $f$ fulfills (1.16) and (1.18) with $x_k \in \partial\mathbb{D}$ for $k = 1, \ldots, n$, then the Riemann image surface $f(\mathbb{D})$ is a polygon.

If $f(x_k)$ is bounded then relation (1.14) yields

$$\mu_k < \frac{1}{2}\,, \tag{1.19}$$

whereas for unbounded $f(x_k)$ relations (1.15) and (1.17) give

$$\mu_k \geqq \frac{1}{2} \,. \tag{1.20}$$

If $f$ is univalent, then (1.13) leads to

$$\mu_k \geqq -\frac{1}{2} \quad (k = 1, \ldots, n) \,. \tag{1.21}$$

(References: [31], [55].)

## 1.4 Convex and starlike functions

A function $f \in A$ is called *convex* if it maps $\mathbb{D}$ univalently onto a convex domain, and it is called *starlike* if it maps $\mathbb{D}$ univalently onto a domain which is starlike with respect to $f(0) = 0$.

Clearly a polygon is convex if $\alpha_k < 1$ $(k = 1, \ldots, n)$ or equivalently if $\mu_k > 0$ $(k = 1, \ldots, n)$. So by (1.16) it follows that

$$1 + z\frac{f''}{f'}(z) = \sum_{k=1}^{n} \mu_k \frac{1 + \overline{x_k}z}{1 - \overline{x_k}z} \,, \tag{1.22}$$

if one uses (1.18). Thus

$$1 + z\frac{f''}{f'} \in P \,. \tag{1.23}$$

On the other hand, if (1.23) holds, then by (1.4) $f$ can be approximated by convex Schwarz-Christoffel mappings, and the Carathéodory kernel theorem shows that $f(\mathbb{D})$ is convex. So (1.23) is a necessary and sufficient condition for $f$ to be convex.

Let $K$ denote the family of convex functions that are normalized by (1.1).

It is well-known that a function $f$ is starlike if and only if

$$z\frac{f'}{f} \in P \tag{1.24}$$

(see e.g. [49]).

By (1.4) and Lemmas 1.1 and 1.2 the function $zf'/f$ can be approximated by functions of the form

$$z\frac{f_n'}{f_n}(z) = p_n(z) = \sum_{k=1}^{n} \mu_k \frac{1 + x_k z}{1 - x_k z} = \prod_{k=1}^{n} \frac{1 - y_k z}{1 - x_k z} \,,$$

$$|x_k| = |y_k| = 1,\ \mu_k > 0 \quad (k = 1, \ldots, n),\ \sum_{k=1}^{n} \mu_k = 1,\ \ n \in \mathbb{N}\ ,$$

with the property (1.8), so that

$$\frac{f_n''}{f_n'} = \frac{p_n(z) - 1}{z} + \frac{p_n'}{p_n}(z) = -2\sum_{k=1}^{n} \frac{-1/2}{z - \overline{y_k}} - 2\sum_{k=1}^{n} \frac{\mu_k + 1/2}{z - \overline{x_k}}\ ,$$

from which we can see that $f_n$ is a Schwarz-Christoffel mapping with $n$ finite vertices of inner angle $2\pi$, and alternating $n$ vertices at $\infty$. This is a special case of linearly accessibility which will be considered later.

Let $St$ denote the family of starlike functions that are normalized by (1.1).

## 1.5 Functions of bounded boundary rotation

The *boundary rotation* of a polygon $F$ is the total change of the tangent direction when surrounding the boundary of the polygon one time and can be calculated as the sum of the absolute value of the outer angles

$$\operatorname{br}(F) = \sum_{k=1}^{n} 2|\mu_k|\pi\ . \tag{1.25}$$

The boundary rotation of the corresponding Schwarz-Christoffel mapping is defined to be the boundary rotation of its image polygon. A function $f$ has boundary rotation $K\pi$, if it can be approximated by Schwarz-Christoffel mappings with respect to locally uniform convergence, i.e. if

$$\frac{f''}{f'}(z) = -2 \int\limits_{\partial \mathbb{D}} \frac{d\mu(x)}{z - x} \tag{1.26}$$

where $\mu$ is a signed measure on $\partial\mathbb{D}$ with the properties

$$\int\limits_{\partial \mathbb{D}} d\mu(x) = 1 \tag{1.27}$$

and

$$\operatorname{br}(f) = 2\pi \int\limits_{\partial \mathbb{D}} |d\mu(x)| = K\pi\ . \tag{1.28}$$

Representation (1.26) is called the *Paatero representation* of $f$. By the Herglotz formula (1.3) and the representation of $\mu$ as the difference of two positive measures the Paatero representation (1.26) is equivalent to the existence of two functions $p_1, p_2 \in P$ such that

$$1 + z\frac{f''}{f'}(z) = \left(\frac{K}{4} + \frac{1}{2}\right) \cdot p_1 - \left(\frac{K}{4} - \frac{1}{2}\right) \cdot p_2 \,. \tag{1.29}$$

Let $V(K)$ denote the family of functions of bounded boundary rotation at most $K\pi$ that are normalized by (1.1). So $V(K)$ is the locally uniform closure of the corresponding family of normalized Schwarz-Christoffel mappings of bounded boundary rotation at most $K\pi$.

Generalized polygons with an infinite number of vertices $w_k$ $(k \in \mathbb{N})$ of outer angle $2\mu_k\pi$ with $\sum_{k=1}^{\infty} |\mu_k| < \infty$ are examples of functions of bounded boundary rotation.

(References: [43], [55], [21], see also chapter 2.)

## 1.6 Linearly accessible domains and close-to-convex functions

A domain $F$ is called *(angularly) accessible* of order $\beta$ $(\beta \in [0,1])$, if it is the complement of the union of rays that are pairwise disjoint except that the origin of one ray may lie on another one of the rays, such that every ray is the bisector of a sector of angle $(1-\beta)\pi$ which wholly lies in the complement of $F$. If $\beta = 1$ then $F$ is called *(strictly) linearly accessible* (see [5], [54], [48]). A function $f$ is called *close-to-convex* of order $\beta$ $(\beta \in [0,1])$, (for reasons which shall be seen later) if $f(\mathbb{D})$ is accessible of order $\beta$. We shall give an analytical characterization for $f$ to be close-to-convex of order $\beta$, which is for $\beta = 1$ originally due to Lewandowski [37] - [38] and for $\beta < 1$ to Pommerenke [48] (who did not give a proof for his statement) and has been the original definition of close-to-convexity given by Kaplan [22]. Therefore we use Lemma 1.3.

**Theorem 1.1** *Let $f$ be univalent and $f(\mathbb{D})$ accessible of order $\beta$. Then there exist a convex function $g$ and a function $p \in \widetilde{P}$ such that the representation*

$$f' = g' \cdot p^\beta \tag{1.30}$$

*holds.*

*Proof:* Suppose firstly, $\beta = 1$. Then by the geometrical definition we have $f(\mathbb{D}) = \mathbb{C} \setminus \bigcup_{t \in T} \gamma_t$, where $\gamma_t$ are rays that are pairwise disjoint except that the origin of one ray may lie on another one of the rays, and $T$ is a suitably chosen parameter set, which is separable (e.g. $T \subset \mathbb{R}^3$). Choose a dense subset $\{t_n \in T \mid n \in \mathbb{N}\}$ of $T$ and define $f_n$ by

$$f_n(\mathbb{D}) := \mathbb{C} \setminus \bigcup_{k=1}^{n} \gamma_{t_k}, \quad \arg f_n(0) := \arg f(0)\,. \tag{1.31}$$

There is no loss of generality to assume that $(\gamma_{t_k})$ $(k \in \mathbb{N})$ are pairwise disjoint, because if some of the chosen rays has its origin lying on another ray, we shorten it by $1/n$ and get the same conclusion. Obviously $f_n \to f$, because $f_n(\mathbb{D}) \to f(\mathbb{D})$ in the sense of Carathéodory kernel convergence. This shows that it suffices to show the conclusion for functions $f_n$ satisfying (1.31), because $\{f_n\}$ is a normal family and the functions $f$ with representation (1.30) form a closed subset of $A$.

Observe that $f_n$ is a certain Schwarz-Christoffel mapping with $n$ finite vertices at the points $w_k =: f_n(\overline{y_k})$, say. The inner angle at each of those hairpin vertices is $2\pi$. The other $n$ vertices alternate with $w_k$ and lie at $\infty =: f_n(\overline{x_k})$, say. The inner angles $\alpha_k\pi$ at those vertices satisfy $\alpha_k \geqq 0$, and their sum fulfills $\sum\limits_{k=1}^{n} \alpha_k\pi = 2\pi$, because $f_n$ is univalent (in other words: the rays are traversed at $\infty$ systematically with increasing argument when surrounding the polygon), so that by (1.16) and (1.17)

$$\begin{aligned}\frac{f_n''}{f_n'}(z) &= -2\sum_{k=1}^{n} \frac{-1/2}{z-\overline{y_k}} - 2\sum_{k=1}^{n} \frac{(1+\alpha_k)/2}{z-\overline{x_k}} = \\ &\sum_{k=1}^{n}\left(\frac{1}{z-\overline{y_k}} - \frac{1}{z-\overline{x_k}}\right) - 2\sum_{k=1}^{n} \frac{\alpha_k/2}{z-\overline{x_k}}\ .\end{aligned} \tag{1.32}$$

The choice (1.6) gives a function $p_n \in \widetilde{P}$ as Lemma 1.3 shows because (1.7) and (1.8) are fulfilled, and

$$\frac{g_n''}{g_n'}(z) := -2\sum_{k=1}^{n} \frac{\alpha_k/2}{z-\overline{x_k}}\,, \quad g_n'(0) := f_n'(0)$$

gives a convex polygon. Then from (1.32) it follows that

$$\frac{f_n''}{f_n'} = \frac{p_n'}{p_n} + \frac{g_n''}{g_n'}\,, \quad f_n'(0) = g_n'(0)$$

which is equivalent to $f_n' = g_n' \cdot p_n$ .

Now suppose $0 < \beta < 1$. Then for each $\gamma_t$, $(t \in T)$, the sector $S_t$ of angle $(1-\beta)\pi$ which is symmetric with respect to $\gamma_t$ lies in $\mathbb{C} \setminus f(\mathbb{D})$. Define here

$$f_n(\mathbb{D}) := \mathbb{C} \setminus \bigcup_{k=1}^{n} S_{t_k}, \quad \arg f_n(0) := \arg f(0) . \tag{1.33}$$

for a certain dense subset $\{t_n \in T \mid n \in \mathbb{N}\}$ of $T$. Then $f_n \to f$, and it suffices to show the conclusion for functions $f_n$ satisfying (1.33).

Observe that $f_n(\mathbb{D})$ is a polygon with $2n$ vertices, $n$ of them of inner angle $(1+\beta)\pi$ at the origins of $S_{t_k}$ $(k = 1, \ldots, n)$. Let the sectors $S_{t_k}$ be ordered in the same way as their origins – which are vertices of $f_n(\mathbb{D})$ – when traversing $\partial\mathbb{D}$ in positive sense. Now the polygon $f_n(\mathbb{D})$ has a finite vertex between the origins $S_{t_k}$ and $S_{t_{k+1}}$ when surrounding $f_n(\mathbb{D})$ if they intersect, and has a vertex at $\infty$ if they do not. Let $\alpha_k\pi$ be the angle between the directions of $\gamma_{t_k}$ and $\gamma_{t_{k+1}}$. Then in either case the outer angle is seen to be $2\mu_k\pi = (\alpha_k + \beta)\pi$, so that

$$\frac{f_n''}{f_n'}(z) = \beta \sum_{k=1}^{n} \left( \frac{1}{z - \overline{y_k}} - \frac{1}{z - \overline{x_k}} \right) - 2 \sum_{k=1}^{n} \frac{\alpha_k/2}{z - \overline{x_k}} .$$

Because $\sum\limits_{k=1}^{n} \alpha_k\pi = 2\pi$, this gives the result as above. □

It is decisive that the converse is also true. For this reason the functions are called close-to-convex.

**Theorem 1.2** *Let $\beta \in\ ]0,1]$ and let $f$ have a representation of the form* (1.30) *for some convex function $g$ and some $p \in \widetilde{P}$. Then $f$ is univalent and $f(\mathbb{D})$ is accessible of order $\beta$.*

*Proof:* The function $h = f \circ g^{-1}$ is defined in the convex domain $g(\mathbb{D})$ and fulfills for $z_1, z_2 \in g(\mathbb{D})$

$$h(z_2) - h(z_1) = \int_{z_1}^{z_2} h'(z)dz = (z_2 - z_1) \int_0^1 h'(tz_2 + (1-t)z_1)dt \neq 0 ,$$

since $\mathrm{Re}\,(e^{i\alpha}h') = \mathrm{Re}\,(e^{i\alpha}f'/g') > 0$ for some $\alpha \in \mathbb{R}$, so that $h$ and therefore $f$ is univalent.

We prove the rest of the result also by an approximation argument. Therefore we need to know that the family of domains that are accessible of order $\beta$ is closed with respect to Carathéodory kernel convergence, i.e. a convergent sequence of domains which are accessible of order $\beta$ that does not converge to a singleton converges to a domain accessible of order $\beta$.

Suppose $G_n$ are accessible of order $\beta$ and $G_n \to G$. Each boundary point $w \in \partial G$ is the limit point of a sequence $w_n$ of boundary points of $G_n$. Each $w_n$ is the vertex of a sector $S_n$ which lies in $\mathbb{C} \backslash G_n$. Let $\gamma_n$ denote the bisector of $S_n$. Then one chooses a subsequence such that there is a limit direction of the directions of $\gamma_n$ and thus a limit ray $\gamma$. Let $S$ be the corresponding symmetric sector of angle $(1-\beta)\pi$. Carathéodory kernel convergence shows that $S \subset \mathbb{C} \setminus G$. Furthermore a simple argument also shows that the rays which correspond to different boundary points of $G$ are pairwise disjoint. For the details see [5], Lemma 3.

Suppose now, $f$ has a representation (1.30). Then

$$\frac{f''}{f'} = \beta\frac{p'}{p} + \frac{g''}{g'} \,. \tag{1.34}$$

Each function of this form can be approximated by functions $f_n$ of the same form where $g$ is a convex Schwarz-Christoffel mapping and $p$ has a representation (1.6) - (1.8) as Lemma 1.3 shows. So we get for the approximants

$$\frac{f_n''}{f_n'}(z) = -2\sum_{k=1}^{n} \frac{\beta/2}{z-\overline{x_k}} - 2\sum_{k=1}^{n} \frac{-\beta/2}{z-\overline{y_k}} - 2\sum_{k=1}^{m} \frac{\mu_k}{z-w_k} \tag{1.35}$$

where the numbers $x_k, y_k$ alternate with each other on $\partial\mathbb{D}$, $\mu_k > 0, |w_k| = 1$ $(k = 1, \ldots, m)$, $\sum\limits_{k=1}^{n} \mu_k = 1$, and $n, m \in \mathbb{N}$. Without loss of generality we can assume that $g$ is bounded (i.e. $\mu_k < 1/2$ $(k = 1, \ldots, m)$) because otherwise we approximate $g$ by bounded convex polygons. On similar reasons we suppose that the numbers $w_k$ are pairwise different from $\overline{x_k}$ and $\overline{y_k}$.

From (1.35) one sees that $f_n(\mathbb{D})$ is a polygon, and because it has the form (1.34) it is *a priori* close-to-convex and hence univalent.

Now suppose first, $\beta = 1$. Then there are $n$ vertices at $\infty$ of angle zero, and alternately $n$ finite hairpin vertices of angle $2\pi$. Furthermore there are $m$ finite convex vertices.

At first we prove that the complement $E$ of $F := f_n(\mathbb{D})$ contains the $n$ rays $\gamma_k$ $(k = 1, \ldots, n)$, which come from the hairpin vertices $O_k$. Clearly a segment $\sigma$ of $\gamma_k$ containing $O_k$ lies in $E$. Suppose now that there is a point $Q \in \gamma_k$ which lies in $F$. Then there is a curve $\Gamma$ which connects $O_k$ with $Q$ within $F$ because $O_k$ is an accessible boundary point. The segment of $\gamma_k$

from $O_k$ to $Q$ and $\Gamma$ encloses a bounded region. It contains in its interior some point $P$ of $\partial F$ as from $Q \in F$ it follows that there is a convex vertex before and after $O_k$. That part of $\partial F$ from $O_k$ to the next or last vertex at infinity which contains $P$ is called $\delta$. Now because $\delta$ is unbounded it must cross $\gamma_k$ between $\sigma$ and $Q$. But this contradicts the fact that all vertices of $\delta$ are convex. Thus $\gamma_k \subset E$.

The rays $\gamma_k$ $(k = 1, \ldots, n)$ are pairwise disjoint because of the univalence. Let them be ordered in the same way as their origins $O_k$ when traversing $\partial\mathbb{D}$ in positive sense.

When traversing from $O_k$ to $O_{k+1}$ along $\partial F$ there is exactly one vertex at $\infty$ (of angle zero) between $O_k$ and $O_{k+1}$, because the numbers $x_k, y_k$ are alternating on $\partial\mathbb{D}$. So the rays $\gamma_k$ are separated by half parallel strips and lie in components $G_k$ of $E$ which are pairwise disjoint.

Furthermore $E = \bigcup\limits_{k=1}^{n} G_k$, because in a neighborhood of infinity $E$ has exactly $n$ components ($f$ has exactly $n$ poles on $\partial\mathbb{D}$), so that an additional component would be bounded contradicting the simply connectivity of $F$.

So, for to fill $E$ with rays that are pairwise disjoint, it is enough to do this for the components $G_k$. But this is easily done.

Take the parallels of $\gamma_k$ from $O_k$ to the next vertex $P_1$ with origins on $\partial F$. Because all vertices before the next vertex at $\infty$ are convex, we may choose from $P_1$ on as direction of a new family of parallel rays the boundary direction of $F$ before $P_1$, and fill the remaining sector arbitrarily. Note that in this case the origin of some ray lies on another one of the rays. Continue the procedure from $P_1$ to the next vertex $P_2$ and so on until $P_j = \infty$. Finally apply the same process from $O_k$ to the last vertex at $\infty$ before $O_k$. This gives a suitable representation of $G_k$ as union of rays that are pairwise disjoint and finishes the proof for $\beta = 1$.

Now suppose, $0 < \beta < 1$. Because $\mu_k < 1/2$ $(k = 1, \ldots, m)$, $f$ is bounded, i.e. all vertices are finite. There are exactly $n$ vertices of angle $(1 - \beta)\pi$, alternately $n$ vertices of angle $(1 + \beta)\pi$, and finally $m$ convex vertices. The vertices $O_k$ $(k = 1, \ldots, n)$, of inner angle $(1 + \beta)\pi$ define sectors $S_k$ $(k = 1, \ldots, n)$, of angle $(1 - \beta)\pi$ which lie in $E := \mathbb{C} \setminus f(\mathbb{D})$. Let $\gamma_k$ denote the bisector of $S_k$ $(k = 1, \ldots, n)$. Because all other $n + m$ vertices are bounded and convex, $E$ can be filled with rays $\gamma_t$ $(t \in T)$ that are pairwise disjoint such that for each $\gamma_t$ the symmetric sector $S_t$ of angle $(1 - \beta)\pi$ lies in $E$, if we choose $\gamma_t$ to be parallel to $\gamma_k$ in a neighborhood of $O_k$ $(k = 1, \ldots, n)$ and proceed as above. $\square$

As usual, we call a function close-to-convex of order $\beta$ if it has a representation (1.30) also if $\beta > 1$. Of course those functions must not be univalent. For $\beta \geqq 0$ let $C(\beta)$ denote the family of close-to-convex functions of order $\beta$ that are normalized by (1.1).

By the analytic definitions of convex and starlike functions (1.23) and (1.24), the convex auxiliary function $g$ can be replaced by some starlike auxiliary function $h$ such that there is a representation

$$f'(z) = \frac{h(z)}{z} \cdot p^{\beta} \tag{1.36}$$

for $f \in C(\beta)$.
(Reference: [30].)

## 1.7 Invariants under similarities and the Nehari criterion

If $f \in S$, i.e. $f$ is a univalent function that is normalized by (1.1), then the renormalized composition $g$ of $f$ with a univalent automorphism $\omega : \mathbb{D} \to \mathbb{D}$ of the unit disk

$$\omega(z) := x\frac{z+a}{1+\overline{a}z}, \quad a \in \mathbb{D}, \quad |x| = 1\,, \tag{1.37}$$

given by

$$g := \frac{f \circ \omega - f \circ \omega(0)}{(f \circ \omega)'(0)}\,, \tag{1.38}$$

lies in $S$. Pommerenke [46] - [47] called families with this property linearly invariant, and showed that many results about univalent functions are effected by this property. The function $g$ is called *Koebe transform* of $f$, it has (in the univalent case) a range $G$ which is *similar* to the range $F$ of $f$, i.e. $G = aF + b$ $(a, b \in \mathbb{C})$, and all normalized functions with a similar domain have this form. The second coefficient of $g$ has for $x = 1$ absolute value

$$\varkappa(f;a) := |a_2(g)| = \left| -\overline{a} + \frac{1}{2}(1-|a|^2)\frac{f''}{f'}(a) \right| \,. \tag{1.39}$$

We call $\varkappa$ the *Koebe expression* of $f$.

For a locally univalent function $f$ we define the *order* of $f$ by

$$\operatorname{ord}(f) := \sup_{a \in \mathbb{D}} \varkappa(f;a)\,.$$

It represents the order of the linearly invariant family $\mathrm{Lin}(f)$ generated by $f$, and it is bounded if and only if $\mathrm{Lin}(f)$ is normal (see [46], Folgerung 1.1).

For an analytical expression to have a geometrical meaning the expression must have a certain invariance property with respect to the composition with automorphisms of $\mathbb{D}$, because the range is invariant under this composition.

We have for the Koebe expression

**Lemma 1.4** *If $f \in A$ is locally univalent and $\omega$ is defined by* (1.37), *then for $g = f \circ \omega$ holds*

$$\varkappa(g; z) = \varkappa(f; \omega(z)) \quad (z \in \mathbb{D})\,. \tag{1.40}$$

*Proof:* The relations

$$\frac{g''}{g'} = \omega' \frac{f''}{f'} + \frac{\omega''}{\omega'}$$

and

$$1 - |\omega(z)|^2 = 1 - \frac{z+a}{1+\overline{a}z}\frac{\overline{z}+\overline{a}}{1+a\overline{z}} = \frac{(1-|z|^2)(1-|a|^2)}{(1+\overline{a}z)(1+a\overline{z})}$$

imply

$$\begin{aligned}
\varkappa(g;z)^2 &= \left|-\overline{z} + \frac{1}{2}(1-|z|^2)\left(x\frac{1-|a|^2}{(1+\overline{a}z)^2}\frac{f''}{f'}(\omega) - \frac{2\overline{a}}{1+\overline{a}z}\right)\right|^2 \\
&= \left(-\frac{\overline{z}+\overline{a}}{1+\overline{a}z} + \frac{1}{2}\frac{(1-|z|^2)(1-|a|^2)}{(1+\overline{a}z)^2} x \frac{f''}{f'}(\omega)\right) \\
&\quad \cdot \left(-\frac{z+a}{1+a\overline{z}} + \frac{1}{2}\frac{(1-|z|^2)(1-|a|^2)}{(1+a\overline{z})^2}\overline{x}\overline{\left(\frac{f''}{f'}(\omega)\right)}\right) \\
&= |\omega|^2 - \frac{1}{2}\frac{(1-|z|^2)(1-|a|^2)}{(1+\overline{a}z)(1+a\overline{z})}\left(\frac{\overline{z}+\overline{a}}{1+a\overline{z}}\overline{x}\overline{\left(\frac{f''}{f'}(\omega)\right)} + \frac{z+a}{1+\overline{a}z}x\frac{f''}{f'}(\omega)\right) \\
&\quad + \frac{1}{4}\frac{(1-|z|^2)^2(1-|a|^2)^2}{(1+\overline{a}z)^2(1+a\overline{z})^2}\left|\frac{f''}{f'}(\omega)\right|^2 \;=\; \varkappa(f;\omega(z))^2\,.
\end{aligned}$$

□

Moreover, $\varkappa$ as a function of $f$ does only depend on $f''/f'$, so that it is also invariant under similarities of the range. From this it follows that (for univalent $f$) the expressions

$$\mathrm{ord}\,(f) \qquad \text{and} \qquad \inf_{a\in\mathbb{D}} \varkappa(f;a)$$

as well as

$$\limsup_{a\to\partial\mathbb{D}} \varkappa(f;a) \qquad \text{and} \qquad \liminf_{a\to\partial\mathbb{D}} \varkappa(f;a)$$

represent geometric properties which are invariant under similarities.

The same is true for the expression $\sigma$, which is defined with the aid of the Schwarzian derivative $S_f$ of $f$, i.e.

$$S_f := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 , \tag{1.41}$$

namely ($a \in \mathbb{D}$)

$$\sigma(f;a) := (1-|a|^2)^2 |S_f(a)| . \tag{1.42}$$

**Lemma 1.5** *If $f \in A$ is locally univalent and $\omega$ is defined by* (1.37), *then for $g = f \circ \omega$ holds*

$$\sigma(g;z) = \sigma(f;\omega(z)) \quad (z \in \mathbb{D}) . \tag{1.43}$$

*Proof:* The well-known invariance property of the Schwarzian derivative

$$S_g = S_f(\omega) \cdot (\omega')^2$$

implies the result similarly as in the above case. □

We call $\sigma$ the *Nehari expression* of $f$, because Nehari has shown that $\sigma(f;z) \leqq 2$ implies univalence, and on the other hand univalent functions satisfy $\sigma(f;z) \leqq 6$. Moreover, convex functions fulfill $\sigma(f;z) \leqq 2$ (see [41], [42] and [32]).

## 1.8 Logarithmic derivative and the Becker criterion

Another important univalence criterion involves the logarithmic derivative and is due to Becker. We call ($a \in \mathbb{D}$)

$$\lambda(f;a) := (1-|a|^2)\left|\frac{f''}{f'}(a)\right| \tag{1.44}$$

the *Becker expression* of $f$. Beckers criterion states that $\lambda(f;z) \leqq 1$ implies the univalence of $f$. On the other hand univalent functions satisfy $\lambda(f;z) \leqq 6$ (see [3]).

Let us note the following correspondence between the Nehari and Becker conditions.

**Lemma 1.6** *Let $f \in A$ be locally univalent. Then*

(a): $$\sup_{z\in\mathbb{D}} \lambda(f;z) \leqq \lambda \quad \Longrightarrow \quad \sup_{z\in\mathbb{D}} \sigma(f;z) \leqq 4\lambda + \tfrac{1}{2}\lambda^2 ,$$

(b): $$\sup_{z\in\mathbb{D}} \sigma(f;z) \leqq \sigma \quad \Longrightarrow \quad \sup_{z\in\mathbb{D}} \lambda(f;z) \leqq 2\,(1+\tfrac{\sigma}{2})^{1/2} + 2 .$$

*Proof:* Statement (a) is proved in [14]. (A sharper version of it is given in [57]). For to prove (b) observe that the functions $f$ satisfying $\sigma(f;z) \leqq \sigma$ ($z \in \mathbb{D}$) form a linearly invariant family of order $(1+\sigma/2)^{1/2}$ (see [46], Folgerung 2.3). Therefore $\varkappa(f;z) \leqq (1+\sigma/2)^{1/2}$ (see [46], Lemma 1.2) which implies the result. □

# 2. Geometrical interpretation of the Koebe, Nehari and Becker expressions

## 2.1 Polygons

Let $F = f(\mathbb{D})$ be a polygon with inner angles $\alpha_k\pi$ $(k = 1, \ldots, n)$, so that $f$ has a Schwarz-Christoffel representation

$$\frac{f''}{f'}(z) = -2\sum_{k=1}^{n} \frac{\mu_k}{z - x_k}, \quad |x_k| = 1 \quad (k = 1, \ldots, n), \quad \sum_{k=1}^{n} \mu_k = 1, \tag{2.1}$$

where $2\mu_k\pi$ $(k = 1, \ldots, n)$ are the outer angles (1.17) and $x_k$ $(k = 1, \ldots, n)$ are the prevertices.

We write $z = re^{i\theta}$ and define

$$b_k(z) := \frac{1 - \overline{z}x_k}{z - x_k}. \tag{2.2}$$

Obviously $|b_k| = 1$ $(k = 1, \ldots, n)$ for all $z \in \mathbb{D}$. We get then for the Koebe, Becker and Nehari expressions

$$\begin{aligned} \varkappa(f; z) &= \left| -\overline{z} + \frac{1}{2}(1 - |z|^2)\frac{f''}{f'}(z) \right| \\ &= \left| \overline{z} + (1 + r)\sum_{k=1}^{n} \frac{1 - r}{z - x_k}\mu_k \right| = \left| \sum_{k=1}^{n} \mu_k b_k(z) \right|, \end{aligned} \tag{2.3}$$

$$\lambda(f; z) = (1 - |z|^2)\left| \frac{f''}{f'}(z) \right| = 2(1 + r)\left| \sum_{k=1}^{n} \frac{1 - r}{z - x_k}\mu_k \right|, \tag{2.4}$$

and, since for the Schwarzian derivative one has

$$\begin{aligned} S_f(z) &= 2\sum_{k=1}^{n} \frac{\mu_k}{(z - x_k)^2} - 2\left( \sum_{k=1}^{n} \frac{\mu_k}{z - x_k} \right)^2 \\ &= \sum_{j,k=1}^{n} \mu_j\mu_k \left( \frac{1}{z - x_j} - \frac{1}{z - x_k} \right)^2, \end{aligned} \tag{2.5}$$

finally

$$\begin{aligned}
\sigma(f;z) &= (1-|z|^2)^2|S_f(z)| \\
&= 2(1+r)^2\left|\sum_{k=1}^{n}\left(\frac{1-r}{z-x_k}\right)^2\mu_k - \left(\sum_{k=1}^{n}\frac{1-r}{z-x_k}\mu_k\right)^2\right| \\
&= \left|\sum_{j,k=1}^{n}\mu_j\mu_k\Big(b_j(z)-b_k(z)\Big)^2\right| . \qquad (2.6)
\end{aligned}$$

The following lemma will be used to examine the boundary behaviour of these expressions.

**Lemma 2.1** *Let* $|x_k| = 1$*, then*

$$\lim_{r\to 1}\frac{1-r}{re^{i\theta}-x_k} = \begin{cases} -\overline{x_k} & \text{if } \theta = \arg x_k \\ 0 & \text{otherwise} \end{cases} .$$

*Proof:* We have

$$\frac{1-r}{re^{i\theta}-x_k} = -\overline{x_k}(1-r)\frac{1}{1-re^{i\theta}\overline{x_k}} .$$

If now $e^{i\theta} \neq x_k$, then the last fraction is bounded, so that the right hand side tends to zero, while for $e^{i\theta} = x_k$ we have $(1-r)/(1-re^{i\theta}\overline{x_k}) \equiv 1$. □

Therefore we get from (2.3) - (2.6):

**Lemma 2.2** *If $f$ is a Schwarz-Christoffel mapping* (2.1)*, then*

(a) $$\lim_{r\to 1}\varkappa(f;re^{i\theta}) = \begin{cases} |1-2\mu_k| = \alpha_k & \text{if } \theta = \arg x_k \\ 1 & \text{otherwise} \end{cases} ,$$

(b) $$\lim_{r\to 1}\lambda(f;re^{i\theta}) = \begin{cases} 4|\mu_k| & \text{if } \theta = \arg x_k \\ 0 & \text{otherwise} \end{cases} ,$$

(c) $$\lim_{r\to 1}\sigma(f;re^{i\theta}) = \begin{cases} 8|\mu_k(1-\mu_k)| = 2|1-\alpha_k^2| & \text{if } \theta = \arg x_k \\ 0 & \text{otherwise} \end{cases} .$$

Now it follows

**Theorem 2.1** *If $f$ is a Schwarz-Christoffel mapping* (2.1), *then*

(a1) $\limsup_{z\to\partial\mathbb{D}} \varkappa(f;z) = \max_{0\leqq k\leqq n} |1-2\mu_k| = \max_{0\leqq k\leqq n} \alpha_k$ ,

(a2) $\liminf_{z\to\partial\mathbb{D}} \varkappa(f;z) = \min_{0\leqq k\leqq n} |1-2\mu_k| = \min_{0\leqq k\leqq n} \alpha_k$ ,

(b) $\limsup_{z\to\partial\mathbb{D}} \lambda(f;z) = 4 \max_{1\leqq k\leqq n} |\mu_k|$ ,

(c) $\limsup_{z\to\partial\mathbb{D}} \sigma(f;z) = 8 \max_{1\leqq k\leqq n} |\mu_k(1-\mu_k)| = 2 \max_{1\leqq k\leqq n} |1-\alpha_k^2|$ ,

*where* $\mu_0 := 0$ *and* $\alpha_0 := 1$.

We remark that (a) can be interpreted in the following way: the limsup of the Koebe expression measures the largest inner angle divided by $\pi$, where we have to take into consideration the angle $\pi$ of each smooth boundary point, whereas the liminf of the Koebe expression measures the smallest inner angle divided by $\pi$. It is a special property of polygons that every boundary point is either smooth or a vertex. We shall see later that these considerations can be generalized to a larger class of functions whose images have this property, namely to functions with bounded boundary rotation.

On the other hand, by reason of (1.19) - (1.20) the limsup of the Becker expression measures whether the polygon is bounded:

**Corollary 2.1** *If $f$ is a Schwarz-Christoffel mapping* (2.1), *and if $\alpha_k \leqq 2$ ($k = 1, \ldots, n$) (in particular, if $f$ is univalent), then*

$$\limsup_{z\to\partial\mathbb{D}} \lambda(f;z) \geqq 2 \quad \Longleftrightarrow \quad f \text{ is unbounded} .$$

## 2.2 Domains with the angle property

Let $F$ be a simply connected plain domain or Riemann surface. Then we say that $F$ has the *angle property*, if each boundary point is either smooth, i.e. there is a tangent there, or it is a vertex, i.e. there exist two halftangents corresponding to the left and right derivatives of some parametric representation of the boundary curve. An analytic function $f : \mathbb{D} \to F$ which extends continuously to the boundary of $\mathbb{D}$ has the angle property if its Riemann image surface $F$ has it.

If $F$ has the angle property, then at each boundary point we define the *inner angle* to be the angle between the halftangents measured from the interior of $F$. The inner angle always exists and equals $\pi$ at each smooth boundary point. With $\alpha_{\max}\pi$ and $\alpha_{\min}\pi$ we denote the supremum and the infimum of the inner angles of $F$ and we speak about the *largest* and the *smallest* inner angle of $F$.

The definitions also apply if $F$ is unbounded considering tangents and halftangents on the Riemann sphere. An unbounded $F$ with the angle property must have an inner angle also at each point on $\partial F$ which is unbounded.

The outer angle at some vertex is defined as in the case of polygons by (1.17), and its absolute value measures the change of the tangent direction at the vertex discarding the direction of the change. The outer angle at some smooth boundary point equals zero. By $2\mu_{\max}\pi$ and $2\mu_{\min}\pi$ we denote the supremum and the infimum of the absolute value of the outer angles of $F$. Remark that in the unbounded case the outer angle has not the same geometrical meaning as in the bounded case, in particular if $\infty$ is a smooth boundary point, then the corresponding outer angle $2\mu_k\pi$ does not equal zero but equals $2\pi$.

## 2.3 Functions of bounded boundary rotation

In this section we generalize some of the results for Schwarz-Christoffel mappings to functions of bounded boundary rotation. It is a result essentially due to Paatero that functions of bounded boundary rotation have the angle property (see [43]), so that there exist the largest and the smallest inner and outer angles $\alpha_{\max}\pi$, $\alpha_{\min}\pi$, $2\mu_{\max}\pi$ and $2\mu_{\min}\pi$. This result is contained in the following

**Theorem 2.2** *Let $f \in V(K)$ have boundary rotation $K\pi$. Then $f$ has the Paatero representation*

$$\frac{f''}{f'}(z) = -2 \int\limits_{\partial\mathbb{D}} \frac{d\mu(x)}{z-x}, \quad \int\limits_{\partial\mathbb{D}} d\mu(x) = 1, \quad \int\limits_{\partial\mathbb{D}} |d\mu(x)| = \frac{K}{2} \tag{2.7}$$

*for some signed measure $\mu$ on $\partial\mathbb{D}$, and it has a spherically continuous extension $f : \overline{\mathbb{D}} \to \widehat{\mathbb{C}}$. Each boundary point $f(x)$ $(x = e^{i\theta})$ has either*

(a) *a local tangent of direction*

$$T(\theta) = \lim_{r\to 1} \arg\left(e^{i\theta} f'(re^{i\theta})\right) + \frac{\pi}{2}, \tag{2.8}$$

*which corresponds to the fact that $\mu(\{x\}) = 0$,* *or*

(b) *two local halftangents, so that $f(x)$ is a vertex of $\partial f(\mathbb{D})$ of an outer angle $2\mu(\{x\})\pi$, which corresponds to the fact that $\mu(\{x\}) \neq 0$.*

*In particular: $f$ has the angle property.*
*Moreover the images of the radial rays $f_\theta(r) := f(re^{i\theta})$ $(r \in [0,1])$ divide the inner angle of $\partial f(\mathbb{D})$ at $f(x)$ in two equal parts.*

*Proof:* Let $f$ have boundary rotation $K\pi$. Then there is a Paatero representation (2.7). In this context it is more convenient to write (2.7) as a Stieltjes integral representation with the distribution function $m : [0, 2\pi] \to \mathbb{R}$ of $\mu$ defined by

$$m(t) := \frac{1}{2}\left(\mu([0, e^{it}[) + \mu([0, e^{it}])\right) + C\,, \tag{2.9}$$

where $C \in \mathbb{R}$ is such that

$$\int_0^{2\pi} \left(m(t) - \frac{t}{2\pi}\right) dt = 0\,. \tag{2.10}$$

The Paatero representation (2.7) then reads

$$\frac{f''}{f'}(z) = -2\int_0^{2\pi} \frac{dm(t)}{z - e^{it}},\ \int_0^{2\pi} dm(t) = 1,\ \int_0^{2\pi} |dm(t)| = \frac{K}{2}\,. \tag{2.11}$$

Therefore it follows by integration (using the normalization $f'(0) = 1$) that

$$\begin{aligned} \ln f'(z) &= \int_0^z \frac{f''}{f'}(\zeta)d\zeta = -2\int_0^{2\pi} dm(t)\left(\int_0^z \frac{d\zeta}{\zeta - e^{it}}\right) \\ &= -2\int_0^{2\pi} \ln\left(1 - e^{-it}z\right) d\left(m(t) - \frac{t}{2\pi}\right)\,, \end{aligned} \tag{2.12}$$

as

$$\int_0^{2\pi} \ln\left(1 - e^{-it}z\right) dt = 0 \qquad (z \in \mathbb{D})\,.$$

Observe that $m(t) - t/(2\pi)$ is periodic with period $2\pi$ by (2.11), so that an integration by parts gives with the aid of (2.10) that

$$\begin{aligned}
\ln f'(z) &= 2i \int_0^{2\pi} \frac{e^{-it}z}{1-e^{-it}z}\left(m(t) - \frac{t}{2\pi}\right) dt \\
&= i \int_0^{2\pi} \frac{1+e^{-it}z}{1-e^{-it}z}\left(m(t) - \frac{t}{2\pi}\right) dt \qquad (2.13)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\arg f'(z) &= \operatorname{Re} \int_0^{2\pi} \frac{1+e^{-it}z}{1-e^{-it}z}\left(m(t) - \frac{t}{2\pi}\right) dt \\
&= \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2}\left(m(t) - \frac{t}{2\pi}\right) dt\,. \qquad (2.14)
\end{aligned}$$

By the definition (2.9) of $m$ it follows (see e.g. [17], p. 336) that ($z = re^{i\theta}$)

$$\lim_{r\to 1} \arg f'(z) = 2\pi\left(m(\theta) - \frac{\theta}{2\pi}\right)\,,$$

so that

$$\lim_{r\to 1} \arg zf'(z) = 2m(\theta)\pi\,. \qquad (2.15)$$

This implies that

$$N(\theta) := \lim_{r\to 1} \arg zf'(z) \qquad (2.16)$$

exists for each $\theta \in [0, 2\pi]$ and is a function of bounded variation with

$$N(\theta) = 2m(\theta)\pi\,, \quad \int_0^{2\pi} |dN(\theta)| = \operatorname{br}(f) = K\pi\,. \qquad (2.17)$$

To get (a) and (b) we now use Paatero's result that $f$ has a continuous extension to $\overline{\mathbb{D}}$, and that at each finite boundary point $f(e^{i\theta})$ there is either a tangent to $\partial f(\mathbb{D})$ of direction $T(\theta)$ if $m$ is continuous at $\theta$, or two half-tangents of direction $T(\theta - 0)$ and $T(\theta + 0)$ such that $\partial f(\mathbb{D})$ has a vertex at $f(e^{i\theta})$ whose outer angle equals the total jump of $m$ at $\theta$ (see [43], §7). An inspection of Paatero's proof shows that the same conclusion follows if

$f(x) = \infty$, replacing the euclidean by the spherical distance and measuring angles and directions spherically.

Finally observe that $\arg zf'(z)$ gives the normal direction of the level curve $f_r(\theta) := f(re^{i\theta})$ $(\theta \in [0, 2\pi])$ at the point $z = re^{i\theta}$, so that $\arg zf'(z) + \pi/2$ is the direction of the tangent. On the other hand the image of the radial ray $f_\theta$ cuts $f_r$ perpendicularly for all $r \in ]0,1[$ as $f$ is locally conformal. By (a) this remains true for $r = 1$, if at $f(x)$ there exists a tangent, implying that $f_\theta$ divides the inner angle (namely $\pi$) in two equal parts. If $f(x)$ is a vertex of $\partial f(\mathbb{D})$, then by (2.9) the same conclusion follows. □

Each signed measure $\mu$ on $\partial\mathbb{D}$ has a *Lebesgue decomposition* as the sum of some *discrete*, some *continuously singular* and some *absolutely continuous part* with respect to Lebesgue measure $\lambda_1$, i.e.

$$\mu = \mu_{\text{disc}} + \mu_{\text{sing}} + \mu_{\text{abs}} \ ,$$

(see e.g. [52], p. 240) where

$$\mu_{\text{disc}} = \sum_{k=1}^{\infty} \mu_k \delta_{x_k} \tag{2.18}$$

($\delta_x$ is the *Dirac measure* at $x$). We write $\mu_{\text{cont}} := \mu_{\text{sing}} + \mu_{\text{abs}}$ for the *continuous part of* $\mu$.

The theorem has the consequence that

**Corollary 2.2** *Let $f \in V(K)$ such that the corresponding signed measure $\mu$ has a decomposition $\mu = \mu_{\text{disc}} + \mu_{\text{cont}}$. Then $\partial f(\mathbb{D})$ is smooth up to a countable number of vertices $f(x_k)$ $(k \in \mathbb{N})$ of outer angles $2\mu_k\pi$, say, and there is a one-to-one correspondence between those boundary points and $\mu_{\text{disc}}$ such that* (2.18) *holds.*
*For the largest and the smallest inner and outer angles $\alpha_{\max}\pi$, $\alpha_{\min}\pi$, $2\mu_{\max}\pi$ and $2\mu_{\min}\pi$ it follows*

(a1) $\alpha_{\max} = \max\limits_{k \in \mathbb{N}_0} |1 - 2\mu_k| \ ,$

(a2) $\alpha_{\min} = \min\limits_{k \in \mathbb{N}_0} |1 - 2\mu_k| \ ,$

(b1) $2\mu_{\max} = 2 \max\limits_{k \in \mathbb{N}} |\mu_k| \ ,$

(b2) $2\mu_{\min} = 0 \ ,$

*where* $\alpha_0 := 1$.

*Proof:* We have only to prove that the desired maxima and minima exist. But this follows easily as (a): $\mu_k \to 0$ for $k \to \infty$, and so 0 is the only cluster point of $\{\mu_k \mid k \in \mathbb{N}\}$, and (b): $\{\mu_k \mid k \in \mathbb{N}\}$ is bounded. $\square$

Part (b2) of the corollary is obviously equivalent to the existence of some smooth boundary point. The existence of the maxima and minima considered shows that $\alpha_{\max}\pi$, $\alpha_{\min}\pi$, $2\mu_{\max}\pi$ and $2\mu_{\min}\pi$ in fact represent the maximum and minimum of the inner and outer angles.

Now we are ready to generalize Theorem 2.1 to functions of bounded boundary rotation. Therefore we deduce the following formulas for the Koebe, Becker and Nehari expressions for functions of bounded boundary rotation with a representation (2.7) similar to (2.3) - (2.6): let

$$b(z;x) := \frac{1-\overline{z}x}{z-x}\,, \tag{2.19}$$

then

$$\varkappa(f;z) = \left|\overline{z} + (1+r)\int_{\partial\mathbb{D}} \frac{1-r}{z-x}d\mu(x)\right| = \left|\int_{\partial\mathbb{D}} b(z;x)d\mu(x)\right|\,, \tag{2.20}$$

$$\lambda(f;z) = 2(1+r)\left|\int_{\partial\mathbb{D}} \frac{1-r}{z-x}d\mu(x)\right|\,, \tag{2.21}$$

and, since

$$\begin{aligned} S_f(z) &= 2\int_{\partial\mathbb{D}} \frac{d\mu(x)}{(z-x)^2} - 2\left(\int_{\partial\mathbb{D}} \frac{d\mu(x)}{z-x}\right)^2 \\ &= \int_{(\partial\mathbb{D})^2} d\mu(x)d\mu(y)\left(\frac{1}{z-x} - \frac{1}{z-y}\right)^2\,, \end{aligned} \tag{2.22}$$

we get

$$\begin{aligned} \sigma(f;z) &= 2(1+r)^2\left|\int_{\partial\mathbb{D}} \left(\frac{1-r}{z-x}\right)^2 d\mu(x) - \left(\int_{\partial\mathbb{D}} \frac{1-r}{z-x}d\mu(x)\right)^2\right| \\ &= \left|\int_{(\partial\mathbb{D})^2} d\mu(x)d\mu(y)\big(b(z;x) - b(z;y)\big)^2\right|\,. \end{aligned} \tag{2.23}$$

**Theorem 2.3** *Let $f \in V(K)$ such that the corresponding signed measure $\mu$ has discrete part $\mu_{\text{disc}}$ of form* (2.18). *Then*

(a1) $\limsup\limits_{z\to\partial\mathbb{D}} \varkappa(f;z) = \alpha_{\max}$ ,

(a2) $\liminf\limits_{z\to\partial\mathbb{D}} \varkappa(f;z) = \alpha_{\min}$ ,

(b) $\limsup\limits_{z\to\partial\mathbb{D}} \lambda(f;z) = 4\mu_{\max}$ ,

(c) $\limsup\limits_{z\to\partial\mathbb{D}} \sigma(f;z) = 8\max\limits_{k\in\mathbb{N}} |\mu_k(1-\mu_k)|$ .

*Proof:* Let $f \in V(K)$ with corresponding signed measure $\mu$. As usual we write $\mu = \mu_{\text{disc}} + \mu_{\text{cont}}$ such that (2.18) holds. Then $\sum\limits_{k=1}^{\infty} |\mu_k| \leqq K/2$. Let $\varepsilon > 0$ be given. Now choose $n \in \mathbb{N}$ large enough that

$$\sum_{k=n+1}^{\infty} |\mu_k| \leqq \varepsilon \tag{2.24}$$

and that the maximal value $\max\limits_{k\in\mathbb{N}} |\mu_k| = |\mu_{k_0}|$ is attained for $k_0 \leqq n$.

Let us first consider (b). The integral on the right hand side of (2.21) can be decomposed in three terms ($z = re^{i\theta}$)

$$\begin{aligned} I(z) &= \int\limits_{\partial\mathbb{D}} \frac{1-r}{z-x}\,d\mu(x) \\ &= \sum_{k=1}^{n} \mu_k \frac{1-r}{z-x_k} + \sum_{k=n+1}^{\infty} \mu_k \frac{1-r}{z-x_k} + \int\limits_{\partial\mathbb{D}} \frac{1-r}{z-x}\,d\mu_{\text{cont}}(x) \\ &= \qquad I_1(z) \qquad + \qquad I_2(z) \qquad + \qquad I_3(z)\,. \end{aligned} \tag{2.25}$$

For $I_1$ we get by Lemma 2.1

$$\lim_{r\to 1} I_1(re^{i\theta}) = \begin{cases} -\mu_k \overline{x_k} & \text{if } \theta = \arg x_k \quad (k = 1,\ldots,n) \\ 0 & \text{otherwise} \end{cases} ,$$

so that

$$\limsup_{z\to\partial\mathbb{D}} |I_1(z)| = \max_{1\leqq k\leqq n} |\mu_k| = \max_{k\in\mathbb{N}} |\mu_k| = \mu_{\max}$$

by the choice of $n$ and by Corollary 2.2. Thus it remains to show that $I_2$ and $I_3$ tend to zero as $r$ tends to 1. This follows for $I_2$ from (2.24) and for

$I_3$ from the continuity of $\mu_{\text{cont}}$ (see e.g. [18]), which finishes the proof for (b).
(a): As above we have a decomposition (see (2.20))

$$\begin{aligned}
\varkappa(f;z) &= \left| \int\limits_{\partial\mathbb{D}} \frac{1-\overline{z}x}{z-x} d\mu(x) \right| \\
&= \left| \sum_{k=1}^{n} \mu_k \frac{1-\overline{z}x_k}{z-x_k} + \sum_{k=n+1}^{\infty} \mu_k \frac{1-\overline{z}x_k}{z-x_k} + \int\limits_{\partial\mathbb{D}} \frac{1-\overline{z}x}{z-x} d\mu_{\text{cont}}(x) \right| \\
&= \left| \quad I_1(z) \quad + \quad I_2(z) \quad + \quad I_3(z) \quad \right| ,
\end{aligned}$$

for which we conclude

$$\lim_{r\to 1} I_1(re^{i\theta}) = \begin{cases} \overline{x_k}(1-2\mu_k) & \text{if } \theta = \arg x_k \quad (k=1,\ldots,n) \\ 1 & \text{otherwise} \end{cases} ,$$

and $\lim\limits_{r\to 1} I_2(re^{i\theta}) = \lim\limits_{r\to 1} I_3(re^{i\theta}) = 0$, as $|b(z;x_k)|$ is bounded by 1 for $z \in \mathbb{D}$ and $x_k \in \partial\mathbb{D}$.
(c): The same procedure shows that for $\lim\limits_{r\to 1} \sigma(f;re^{i\theta})$ also the discrete part of $\mu$ is decisive. □

As consequence we have

**Corollary 2.3** *Let $f \in V(K)$. Then*

$$\limsup_{z\to\partial\mathbb{D}} \lambda(f;z) = 0 \quad \Longleftrightarrow \quad f \text{ is bounded and } \partial f(\mathbb{D}) \text{ is smooth} .$$

*Proof:* By Theorem 2.3 the left hand side is equivalent to $\mu_{\max} = 0$, and this obviously is equivalent to the fact that $\mu_{\text{disc}} = 0$, which by Corollary 2.2 is equivalent to the smoothness and boundedness of $\partial f(\mathbb{D})$. □

Moreover

**Corollary 2.4** *Let $f \in V(K)$ such that the corresponding signed measure $\mu$ has discrete part $\mu_{\text{disc}}$ of form* (2.18). *If further $\mu_k \geqq -1/2$ $(k \in \mathbb{N})$ (in particular, if $f$ is univalent), then*

$$\limsup_{z\to\partial\mathbb{D}} \lambda(f;z) \left. \begin{matrix} < 2 \\ > 2 \\ = 2 \end{matrix} \right\} \quad \Longrightarrow \quad \left\{ \begin{matrix} f \text{ is bounded} \\ f \text{ is unbounded} \\ f(\mathbb{D}) \text{ has a vertex of inner angle zero} \end{matrix} \right. .$$

*Proof:* By Theorem 2.3 the expression $\limsup_{z \to \partial \mathbb{D}} \lambda(f;z)$ equals $4\mu_{\max}$. Let now first this term be less or greater than 2. Then by Theorem 2.2 $f(\mathbb{D})$ has vertices of outer angles $2\mu_k\pi$ $(k \in \mathbb{N})$, and so is bounded and unbounded respectively by the definition of a vertex at $\infty$. On the other hand, if it equals 2, then necessarily there is a vertex which corresponds to $\mu_{\max} = 1/2$ of outer angle $\pi$, which gives the result. □

Becker ([4], p. 414) conjectured that for $f \in S$ with Jordan domain $f(\mathbb{D})$ the condition

$$\limsup_{z \to \partial \mathbb{D}} \lambda(f;z) < 2 \tag{2.26}$$

implies that $f$ has a quasiconformal extension to $\mathbb{C}$. This conjecture is true for functions of bounded boundary rotation.

**Corollary 2.5** *Let $f \in S$ have bounded boundary rotation. Then* (2.26) *implies that $f$ has a quasiconformal extension to $\mathbb{C}$.*

*Proof:* Suppose $f \in V(K)$. As $f$ is univalent, by Corollary 2.4 condition (2.26) implies that $f$ is bounded. By Theorem 2.3 it follows moreover that $\mu_{\max} =: \frac{1}{2}(1-\varepsilon)$ for some $\varepsilon > 0$. So for all vertices the relation $|1-\alpha_k| \leqq 2\mu_{\max} = 1-\varepsilon$ holds, and therefore $\varepsilon \leqq \alpha_k \leqq 2-\varepsilon$ $(k \in \mathbb{N})$, so that there is a vertex of smallest angle $\alpha_{\min}\pi \geqq \varepsilon\pi$ and a vertex of largest angle $\alpha_{\max}\pi \leqq (2-\varepsilon)\pi$. Because $\mu_k \to 0$ as $k \to \infty$ there are only a finite number of vertices with an outer angle near $\pm\pi$ (i.e. an inner angle $\alpha_k\pi \approx 0$ or $\alpha_k\pi \approx 2\pi$), so that the local characterization of quasicircles due to Ahlfors ([2], see [34], chapter II, §8) shows that $\partial f(\mathbb{D})$ is a quasicircle. □

Corollary 2.2 gives a one-to-one correspondence between the discrete part of the signed measure $\mu$ associated with $f$ and the vertices of $\partial f(\mathbb{D})$. Therefore it is of some interest to decide what kinds of boundary smoothness are typical for the parts of $\mu$ absolutely continuous and continuously singular with respect to Lebesgue measure. Here we get a partial result.

**Lemma 2.3** *Let $f \in V(K)$ with $f(\mathbb{D}) = F$ and $z_0 = e^{i\theta_0}$ such that $f(z_0)$ is a point where the boundary curve $f(e^{i\theta})$ is analytic. Then the function $m$ associated with $f$ by* (2.11) *is a $C^\infty$-function in a neighborhood of $\theta_0$.*

*Proof:* As $\partial F$ is analytic at $f(z_0)$ the Schwarz reflection principle shows that $f$ has an analytic extension at $z_0$. So in particular $f$ is analytic in a certain neighborhood $U$ of $z_0$ on the boundary of $\mathbb{D}$, and so is $f'$. We deduce that moreover $f'(z_1) \neq 0$ for $z_1 = e^{i\theta_1} \in U$. Suppose the contrary, then $f'$ has an expansion ($\alpha_1 \neq 0$)

$$f'(z) = \alpha_1(z - z_1)^k + \alpha_2(z - z_1)^{k+1} + \dots$$

for some $k \in \mathbb{N}$, which leads to

$$\frac{f''}{f'}(z) = \frac{k}{z - z_1} + H(z) \tag{2.27}$$

with $H$ analytic in $U$. By the identity theorem for analytic functions (2.27) holds also in $\mathbb{D}$ so that by Theorem 2.2 $\partial f(\mathbb{D})$ has a vertex at $f(z_1)$ (of outer angle $-k\pi$), in contrast to the analycity. Therefore $f'(z_1) \neq 0$, and so $k = 0$ in (2.27), i.e. $f''/f'$ is analytic at $z_1$, and so is $\ln(f')$. In particular $\arg f'(e^{i\theta})$ is in $C^\infty$ at $\theta_1$ and so in $U$. By (2.15) the conclusion follows. □

From this we get

**Theorem 2.4** *Let $f \in V(K)$ with $f(\mathbb{D}) = F$ such that $\partial F$ is analytic except at a countable number of points where $\partial F$ has a tangent. Then the signed measure $\mu$ associated with $f$ by* (2.7) *is absolutely continuous with respect to $\lambda^1$.*

*Proof:* If $\partial F$ is analytic everywhere, then by the Lemma $m$ is in $C^\infty([0, 2\pi])$ and $dm = m'(\theta)d\theta$, where $m'$ in particular is integrable and its integral gives $m$, so $m$ and thus $\mu$ is absolutely continuous. If there is at most a countable number of points of nonanalycity on $\partial f(\mathbb{D})$, then – as there is no vertex – $m$ is the sum of the above constructed absolutely continuous part and some continuously singular part $m_{\text{sing}}$ with $m'_{\text{sing}} = 0$ a.e.. Moreover $m'_{\text{sing}}$ is continuous in $[0, 2\pi]$ except of some countable set $\Omega$ by the Lemma. So it has a unique continuous extension to $[0, 2\pi] \setminus \Omega$ which vanishes. Finally $\mu_{\text{sing}}$ must vanish as it is continuous in $[0, 2\pi]$ and its support $\Omega$ is countable. □

Similarly one gets if there are vertices

**Theorem 2.5** *Let $f \in V(K)$ with $f(\mathbb{D}) = F$ such that $\partial F$ is analytic except of at most a countable number of points where $\partial F$ has a tangent and a countable number of vertices $w_k = f(x_k)$ of outer angle $2\mu_k\pi$ ($k \in \mathbb{N}$). Then the signed measure $\mu$ associated with $f$ fulfills $\mu = \mu_{\text{disc}} + \mu_{\text{abs}}$ such that* (2.18) *holds.*

## 2.4 Convex functions

The results of the last section apply to convex functions. In this section we shall show that in the special case of convex functions also corresponding results for the terms $\sup_{z\in\mathbb{D}} \varkappa(z)$, $\inf_{z\in\mathbb{D}} \varkappa(z)$ and $\sup_{z\in\mathbb{D}} \sigma(z)$ are available. On the other hand our results give analytic representations for $\alpha_{\max}$, $\alpha_{\min}$ and $2\mu_{\max}$. We remark that Pommerenke gave the following representation for the maximal outer angle

$$2\mu_{\max} = \lim_{r\to 1} \frac{\ln\left(\max\limits_{|z|=r} |f'(z)|\right)}{\ln \frac{1}{1-r}} .$$

(see [44], Theorem 1).

**Theorem 2.6** *Let $f \in K$, then*

(a1) $\quad \sup\limits_{z\in\mathbb{D}} \varkappa(f;z) = \alpha_{\max} = 1 ,$

*in fact,* (a1) *is equivalent to the convexity of $f$.*

(b) $\quad \limsup\limits_{z\to\partial\mathbb{D}} \lambda(f;z) = 4\mu_{\max} = \begin{cases} 2(1-\alpha_{\min}) & \text{if } f \text{ is bounded} \\ 2(1+\alpha_{\min}) & \text{if } f \text{ is unbounded} \end{cases} ,$

(c1) $\quad \limsup\limits_{z\to\partial\mathbb{D}} \sigma(f;z) = 8\mu_{\max}(1-\mu_{\max}) = 2(1-\alpha_{\min}^2) .$

*If furthermore $f$ is unbounded, then*

(a2) $\quad \inf\limits_{z\in\mathbb{D}} \varkappa(f;z) = \alpha_{\min} ,$

(c2) $\quad \sup\limits_{z\in\mathbb{D}} \sigma(f;z) = 2(1-\alpha_{\min}^2) ,$

*and $\alpha_{\min}\pi$ is the angle of $\partial f(\mathbb{D})$ at $\infty$.*

*Proof:* (a1): That this is equivalent to the convexity of $f$ follows from the fact that the universal linearly invariant family of order 1 is the family of convex functions (see [46], Folgerung 1.1 and Folgerung 2.4). On the other hand by Theorem 2.3 this is equivalent to the geometrical fact that all nonsmooth boundary points of $\partial f(\mathbb{D})$ have interior angles less than $\pi$ and the existence of some smooth boundary point.

(b): For convex functions and all $k \in \mathbb{N}$ we have $\mu_k \in ]0,1]$, so that because of the relation $\alpha_k = |1 - 2\mu_k|$ the value $\alpha_{k_0} = \alpha_{\min}$ is attained if the distance of $\mu_{k_0}$ and $1/2$ is minimal. If $f$ is unbounded, then $\mu_{\max} > 1/2$, and this value is easily seen to minimize the distance to $1/2$. Otherwise also the largest value $\mu_{\max} < 1/2$ minimizes this distance, so that finally

$$\alpha_{\min} = |1 - 2\mu_{\max}| \ , \tag{2.28}$$

which leads to the result by Theorem 2.3.
(c1): By Theorem 2.3 it follows that

$$\limsup_{z \to \partial\mathbb{D}} \sigma(f; z) = 8 \max_{k \in \mathbb{N}} |\mu_k(1 - \mu_k)| \, . \tag{2.29}$$

As $\mu_k > 0 \ \ (k \in \mathbb{N})$ and because that value of $\{\mu_k\}$ nearest $1/2$ is $\mu_{\max}$ we see that this value maximizes the right hand side of (2.29) implying the result.
(a2): If $f$ is unbounded, then $\partial f(\mathbb{D})$ has a vertex at $\infty$ of angle $\alpha_1\pi = \alpha_{\min}\pi$ with corresponding outer angle $2\mu_1\pi = 2\mu_{\max}\pi$.

Because $f(\mathbb{D})$ can be approximated by unbounded convex polygonal domains with fixed angle $\alpha_1\pi$ at $\infty$, it is sufficient to consider those Schwarz-Christoffel mappings with $\mu_1 = \frac{1+\alpha_1}{2}$ and $\sum\limits_{k=2}^{n} \mu_k = \frac{1-\alpha_1}{2}$. Therefore we get with (2.3) as $|b_k| = 1 \ \ (k = 1, \ldots, n)$

$$\begin{aligned} \varkappa(f; z) &= \left| \sum_{k=1}^{n} \mu_k b_k \right| = \left| \mu_1 + \sum_{k=2}^{n} \mu_k b_k \overline{b_1} \right| \\ &\geqq \mu_1 - \sum_{k=2}^{n} \mu_k = \frac{1+\alpha_1}{2} - \frac{1-\alpha_1}{2} = \alpha_1 \, . \end{aligned}$$

Theorem 2.3 shows that $\liminf\limits_{z \to \partial\mathbb{D}} \varkappa(f; z) = \alpha_1$, which gives the result.
(c2): Without loss of generality consider the same unbounded convex polygons with fixed angle $\alpha_1\pi$ at $\infty$. Then by (2.6) and (2.3) we get

$$\begin{aligned} \sigma(f; z) &= \left| \sum_{j,k=1}^{n} \mu_j \mu_k \Big(b_j(z) - b_k(z)\Big)^2 \right| \leqq \sum_{j,k=1}^{n} \mu_j \mu_k \Big|b_j(z) - b_k(z)\Big|^2 \\ &\leqq 2 \left( 1 - \left| \sum_{k=1}^{n} \mu_k b_k(z) \right|^2 \right) = 2\Big(1 - \varkappa(f; z)^2\Big) \leqq 2\Big(1 - \alpha_{\min}^2\Big) \, . \end{aligned} \tag{2.30}$$

On the other hand by (c1) $\limsup\limits_{z\to\partial\mathbb{D}} \sigma(f;z) = 2(1-\alpha_{\min}^2)$, which finishes the proof. □

We remark that (c2) for unbounded convex functions is much stronger than the result given in [27], Theorem 3, where the question was solved, which convex functions attain the maximal value 2 for the supremum of the Nehari expression.

We conjecture that the statement (c2) remains true if $f$ is bounded, because it seems to be true numerically. Moreover we conjecture that for bounded convex functions $\inf\limits_{z\in\mathbb{D}} \varkappa(f;z) = 0$.

The statement (a1) shows in particular that for the Koebe expression the sup and the limsup coincide. We shall show in the sequel that for convex functions the Koebe expression satisfies moreover a certain maximum principle. Therefore we need the

**Lemma 2.4** *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be locally univalent. If the Koebe expression $\varkappa(f;z)$ has a local maximum at $z_0 = 0$, then*

$$a_3 = \frac{1}{3}\frac{a_2^2}{|a_2|^2}\left(1 + 2\,|a_2|^2\right),$$

*in particular*

$$\left|3a_3 - 2a_2^2\right| = 1\,. \tag{2.31}$$

*Proof:* Let

$$F(r,\theta) := re^{-i\theta} - \frac{1}{2}(1-r^2)\frac{f''}{f'}(re^{i\theta})$$

and

$$G(r,\theta) := F(r,\theta)\cdot\overline{F(r,\theta)} = \varkappa(f;re^{i\theta})^2\,,$$

then for a local maximum of $\varkappa(f;z)$ at the origin obviously

$$\left.\frac{\partial G}{\partial r}\right|_{r=0} = 0 \tag{2.32}$$

holds for all $\theta \in \mathbb{R}$. From

$$\frac{\partial G}{\partial r} = 2\cdot\mathrm{Re}\left(\frac{\partial F}{\partial r}\cdot\overline{F}\right)$$

and

$$\frac{\partial F}{\partial r}(r,\theta) = e^{-i\theta} - \frac{1}{2}(1-r^2)e^{i\theta}\left(\frac{f''}{f'}\right)'(re^{i\theta}) + r\frac{f''}{f'}(re^{i\theta})$$

we get therefore for all $\theta \in \mathbb{R}$ the relation

$$\operatorname{Re}\left\{\left(e^{-i\theta} - \frac{1}{2}e^{i\theta}\left(\frac{f''}{f'}\right)'(0)\right)\overline{\left(\frac{f''}{f'}(0)\right)}\right\} = 0\,. \tag{2.33}$$

This implies either $a_2 = 0$ – which leads to a local minimum of $\varkappa(f;z)$ at the origin – or, using the notations $(\rho > 0)$

$$2\overline{a_2} = \overline{\left(\frac{f''}{f'}(0)\right)} =: \rho e^{i\varphi}$$

and

$$3a_3 - 2a_2^2 = \frac{1}{2}\left(\frac{f''}{f'}\right)'(0) =: x + iy = b\,,$$

we get for all $\theta \in \mathbb{R}$

$$\operatorname{Re}\Big\{\Big(\cos\theta - i\sin\theta - (\cos\theta + i\sin\theta)(x+iy)\Big)\rho(\cos\varphi + i\sin\varphi)\Big\} =$$

$$\rho\cdot\operatorname{Re}\Big\{\Big((1-x)\cos\theta + y\sin\theta - i\left((1+x)\sin\theta + y\cos\theta\right)\Big)(\cos\varphi + i\sin\varphi)\Big\} =$$

$$\rho\left(\cos\theta\Big[(1-x)\cos\varphi + y\sin\varphi\Big] + \sin\theta\Big[y\cos\varphi + (1+x)\sin\varphi\Big]\right) = 0\,,$$

so that the coefficients of the terms $\cos\theta$ and $\sin\theta$ must vanish. This implies the relations

$$(1-x)\cos\varphi = -y\sin\varphi\,, \tag{2.34}$$

$$(1+x)\sin\varphi = -y\cos\varphi\,, \tag{2.35}$$

from which we deduce by multiplication that

$$|b|^2 = x^2 + y^2 = 1\,,$$

and so (2.31). Now we substitute $b = x + iy =: e^{i\beta}$ into (2.34) and (2.35) and a short calculation gives the two equations

$$\cos(\varphi + \beta) = \cos\varphi\,, \tag{2.36}$$

$$\sin(\varphi + \beta) = -\sin\varphi\,, \tag{2.37}$$

which finally lead to the unique solution $\beta = -2\varphi$ implying the result. □

The next lemma shows that only very special convex functions satisfy (2.31).

**Lemma 2.5** *Let $f \in K$. Then relation* (2.31) *implies that*

$$\frac{f''}{f'}(z) = -2\frac{t}{z-x} - 2\frac{1-t}{z+x} \tag{2.38}$$

*for some $t \in [0,1]$ and some $x \in \partial\mathbb{D}$, in particular: $f(\mathbb{D})$ is either a half-plane, a sector or a parallel strip.*

*Proof:* If $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in K$, then $p(z) = 1 + z\frac{f''}{f'}(z) = 1 + p_1z + p_2z^2 + \cdots \in P$. So

$$|p_2| = \left|6a_3 - 4a_2^2\right| \leqq 2$$

with equality if and only if

$$p(z) = t\left(\frac{1+\overline{x}z}{1-\overline{x}z}\right) + (1-t)\left(\frac{1-\overline{x}z}{1+\overline{x}z}\right)$$

for some $t \in [0,1]$ and some $x \in \partial\mathbb{D}$ (see e.g. [49], Corollary 2.3). This gives the result. □

Now we have

**Theorem 2.7** *Let $f \in K$. Then the Koebe expression $\varkappa(f;z)$ satisfies a maximum principle, i.e. it takes its maximum over each domain $D$ which is properly contained in $\mathbb{D}$ (such that its closure lies in $\mathbb{D}$, too) at the boundary of $D$. In particular: the function*

$$K(r) := \sup_{|z|=r}\left|-\overline{z} + \frac{1}{2}(1-|z|^2)\frac{f''}{f'}(z)\right|$$

*is monotonically increasing for $r \in [0,1[$.*

*Proof:* We shall prove that for $f \in K$ a local maximum of the expression $\varkappa(f;z)$ can only occur at a point $z_0 \in \mathbb{D}$ if $f(\mathbb{D})$ is either a halfplane or a sector, and in those cases the extremal value is attained at a curve joining $z_0$ with the boundary, namely at a Steiner circle, i.e. the image of the segment $]-1,1[$ under an automorphism of $\mathbb{D}$, which gives the result.

Suppose first that $\varkappa(f;z)$ has a local maximum at 0. Then by Lemma 2.4 (2.31) holds and by Lemma 2.5 $f$ is of form (2.38). From this representation one deduces that

$$\overline{z} - \frac{1}{2}(1-r^2)\frac{f''}{f'}(z) = \overline{z} + \frac{1-r^2}{z^2-x^2}\Big(t(z+x) + (1-t)(z-x)\Big)\,,$$

and especially for $z := rx$ $(r \in\,]-1,1[)$ it follows that

$$\overline{z} - \frac{1}{2}(1-r^2)\frac{f''}{f'}(z) = (1-2t)\overline{x} \, .$$

So $\varkappa(f;z)$ is constant on some diameter of $\mathbb{D}$, which was to prove. In the case of a parallel strip $(t = 1/2)$ the extremal value of $\varkappa(f;z)$ obviously is a minimum, so that this case must not be considered.

On the other hand, if $\varkappa(f;z)$ has a local maximum at a point $z_0 \neq 0$, then by Lemma 1.4 the information which we deduced at the origin can be transformed by an automorphism $\omega$ of $\mathbb{D}$, as the family $K$ of convex functions is linearly invariant. This gives the result. □

## 2.5 Convex functions with vanishing second coefficient

Suppose, $f_m$ has the special form $(m \in \mathbb{N})$

$$f_m(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots \, , \tag{2.39}$$

then $f_m \to z$ as $m \to \infty$. Hence $f_m(\mathbb{D})$ tends to a disk in the sense of Carathéodory kernel convergence (if $f_m$ are univalent). So it seems to be plausible that the geometry of $f_m(\mathbb{D})$ will be restricted in some sense in connection with the restriction of some analytic properties.

The next theorem gives a sharp version of these considerations in the case of convex functions. Therefore we need the

**Lemma 2.6** *Let $f \prec g$ and $r \in\,]0,1]$. Then*

$$\sup_{z\in\mathbb{D}_r}(1-|z|^2)\,|f'(z)| \leqq \sup_{z\in\mathbb{D}_r}(1-|z|^2)\,|g'(z)|$$

(see e.g. [49], p. 35, formula (4)).

**Theorem 2.8** (see [23]) *Let $m \in \mathbb{N}$ and $f_m \in K$ of form (2.39). Then $\lambda(f_m;z) \leqq \frac{4}{m}$, and this result is sharp for the function $G_m$ with*

$$G_m'(z) = \frac{1}{(1-z^m)^{2/m}} \, , \quad G(0) = 0 \, . \tag{2.40}$$

*By Theorem 2.6 this has the geometric consequence that for $f_m(\mathbb{D})$ hold*

(a) $\quad 2\mu_{\max}\pi \leqq \frac{1}{m}2\pi$ , *for $m \geqq 2$ moreover*

(b) $\quad \alpha_{\min}\pi \geqq \left(1-\frac{2}{m}\right)\pi$ ,

*and for $m \geqq 3$*

(c) $\quad f_m$ *is bounded.*

*Proof:* For a convex function of the given form it is well-known that

$$f'_m \prec \frac{1}{(1-z)^{2/m}} \tag{2.41}$$

(see e.g. [16]). This statement is equivalent to $\ln f'_m \prec \ln h' := -\frac{2}{m}\ln(1-z)$, so that by the lemma we only have to observe that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{h''}{h'}(z)\right| = \sup_{z\in\mathbb{D}}\frac{2}{m}(1-|z|^2)\frac{1}{1-|z|} = \frac{4}{m}\,.$$

For the function $G_m$, defined by (2.40), one gets, choosing $z=r>0$, that

$$(1-|z|^2)\left|\frac{G''_m}{G'_m}(z)\right| = (1-r^2)\frac{2\cdot r^{m-1}}{1-r^m} = (1+r)\frac{2\cdot r^{m-1}}{1+r+\cdots+r^{m-1}} \xrightarrow{(r\to 1)} \frac{4}{m},$$

which establishes the statement about equality. □

We remark that the statements (b) and (c) are obvious geometrical facts for $m$-fold symmetric convex functions, and the theorem generalizes these facts.

For convex functions with vanishing second coefficient we have as a

**Corollary 2.6** *Let $f \in K$ with $a_2(f)=0$. Then either $f$ is bounded or $f$ is unbounded and has a zero angle at $\infty$.*

*Proof:* Applying the theorem for $m=2$ we get $\mu_{\max} \leqq 1/2$. By the geometrical interpretation as outer angle the result follows. □

Finally we have the

**Corollary 2.7** (see [23]) *Let $f(z)=z+a_2z^2+a_3z^3+\cdots \in K$ with $a_2=a_3=a_4=0$. Then $f$ fulfills the Becker univalence criterion.*

## 2.6 Convex functions with angle $\alpha\pi$ at $\infty$

In Corollary 2.6 geometrical conditions had been given for $f \in K$ with $a_2(f) = 0$: either $f$ is bounded or $f$ is unbounded and $f(\mathbb{D})$ has a zero angle at $\infty$.

In this section we consider unbounded convex functions with given angle at $\infty$ and get results in the opposite direction.

For $\alpha \in [0,1]$ let $K(\alpha) \subset K$ denote the family of unbounded convex functions with inner angle $\alpha\pi$ at $\infty$. Obviously $K(1)$ consists only of halfplane mappings, so

$$K(1) = \left\{ f \in K \;\middle|\; f(z) = \frac{z}{1 - xz},\ x \in \partial\mathbb{D} \right\} .$$

The family $K(\alpha)$ is a linearly invariant family of order 1.

The compactness of $K$ shows that if $\alpha \to 1$ then $f_\alpha \in K(\alpha)$ implies that $f_\alpha \to f \in K(1)$, and so $|a_n(f_\alpha)| \to 1$ for all $n \in \mathbb{N}$. The following theorem gives more detailed information for the second and third coefficients.

**Theorem 2.9** *Let $\alpha \in [0,1]$ and $f \in K(\alpha)$. Then*

(a) $$|a_2(f)| \geqq \inf_{z\in\mathbb{D}} \varkappa(f;z) = \alpha ,$$

(b) $$\limsup_{z\to\partial\mathbb{D}} \lambda(f;z) = 2(1+\alpha) ,$$

(c) $$|a_3(f) - a_2^2(f)| \leqq \sup_{z\in\mathbb{D}} \sigma(f;z) = 2(1-\alpha^2) .$$

*If* $\tau(f;a) := \left|a_3\left(\frac{f\circ\omega - f\circ\omega(0)}{(f\circ\omega)'(0)}\right)\right|$, $\omega(z) = \frac{z+a}{1+\bar{a}z}$ *and* $\mu = \frac{1+\alpha}{2}$, *then for* $\alpha > \frac{1}{2}(\sqrt{13}-3) = 0.3027...$ *furthermore*

(d) $$|a_3(f)| \geqq \inf_{z\in\mathbb{D}} \tau(f;z) \geqq \tfrac{1}{3}(4\mu^2 + 2\mu - 3) = \tfrac{1}{3}(\alpha^2 + 3\alpha - 1) .$$

*For all $f \in K$ holds*

(e) $$\liminf_{z\to\partial\mathbb{D}} \tau(f;z) = \tfrac{1}{3}(3 - 8\mu_{\max} + 8\mu_{\max}^2) ,$$

*in particular for $f \in K(\alpha)$*

(f) $$\liminf_{z\to\partial\mathbb{D}} \tau(f;z) = \tfrac{1}{3}(3 - 8\mu + 8\mu^2) = \tfrac{1}{3}(1 + 2\alpha^2) .$$

*Proof:* The statements (a), (b) and (c) are obvious consequences of Theorem 2.6. Let us now consider the absolute value $\tau(f;a)$ of the third coefficient of the Koebe transform $h := \frac{f\circ\omega - f\circ\omega(0)}{(f\circ\omega)'(0)}$. If $f$ is a polygonal function with an angle $\alpha\pi$ at $\infty$ then without loss of generality $\mu_1 = \mu$ and so $\sum_{k=2}^{n} \mu_k = 1-\mu$. By (2.2) - (2.6), and as $|b_k(a)| \leqq 1$ $(k = 1, \ldots, n)$, we have

$$\begin{aligned}
\tau(f;a) &= \frac{|h'''(0)|}{6} \\
&= \frac{1}{6}\left|\frac{(1-|a|^2)^3 f'''(a) - 6\overline{a}(1-|a|^2)^2 f''(a) + 6\overline{a}^2(1-|a|^2)f'(a)}{(1-|a|^2)f'(a)}\right| \\
&= \left|\frac{1}{6}(1-|a|^2)^2 S_f(a) + \left(-\overline{a} + \frac{1}{2}(1-|a|^2)\frac{f''}{f'}(a)\right)^2\right| \\
&= \left|\frac{1}{6}\sum_{j,k=1}^{n} \mu_j\mu_k\left(b_j(a) - b_k(a)\right)^2 + \left(\sum_{k=1}^{n}\mu_k b_k(a)\right)^2\right| \\
&= \frac{1}{3}\left|\sum_{k=1}^{n}\mu_k b_k(a)^2 + 2\left(\sum_{k=1}^{n}\mu_k b_k(a)\right)^2\right| \\
&= \frac{1}{3}\left|\mu + \sum_{k=2}^{n}\mu_k b_k^2\overline{b_1}^2 + 2\left(\mu^2 + 2\mu\sum_{k=2}^{n}\mu_k b_k\overline{b_1} + \left(\sum_{k=2}^{n}\mu_k b_k\overline{b_1}\right)^2\right)\right| \\
&\geqq \frac{1}{3}\left(\mu + 2\mu^2 - (1-\mu) - 4\mu(1-\mu) - 2(1-\mu)^2\right) \\
&= \frac{1}{3}(4\mu^2 + 2\mu - 3) = \frac{1}{3}(\alpha^2 + 3\alpha - 1)\,,
\end{aligned}$$

which gives the result by approximation.
(e),(f): This is proved in a way similar to the proof of Theorem 2.3. □

We remark that the right hand side of inequality (d) tends to 1 as $\alpha \to 1$, and so gives a rather sharp estimate for values of $\alpha$ near 1. The statement (a) shows that $K(\alpha)$ is an example of a linearly invariant family for which $\inf_{f\in K(\alpha)} |a_2(f)|$ is bounded from below.

## 2.7 Close-to-convex functions

By $K_m, St_m, C_m(\beta)$ and $V_m(K)$ we denote the families of $m$-fold symmetric convex, starlike, close-to-convex functions of order $\beta$ and functions of bounded boundary rotation at most $K\pi$, respectively. It is easy to see - either geometrically with the aid of the content of the introduction or analytically using the original developments (see [43] and [22]) - that in all cases the corresponding functions from $P$ and $\widetilde{P}$ are of the special form

$$p(z) = 1 + c_m z^m + c_{2m} z^{2m} + \cdots . \tag{2.42}$$

Now we consider $m$-fold symmetric close-to-convex functions of order $\beta$. Therefore we need the following

**Lemma 2.7** *Let $x \in \partial\mathbb{D}$, $\lambda \in \mathbb{R}^+$ and $h' \prec \left(\frac{1+xz}{1-z}\right)^{\lambda/2}$. Then $\lambda(h;z) \leqq \lambda$.*

*Proof:* As we have $\ln h' \prec \frac{\lambda}{2} \ln \frac{1+xz}{1-z}$, Lemma 2.6 implies that

$$\sup_{z\in\mathbb{D}_r} \left(1-|z|^2\right) \left|\frac{h''}{h'}(z)\right| \leqq \frac{|1+x|}{2}\lambda ,$$

and so the result follows. □

Now we have

**Theorem 2.10** (see [23]) *Let $m \in \mathbb{N}$, $\beta > 0$ and $f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \cdots$ an $m$-fold symmetric close-to-convex function of order $\beta$. Then $\lambda(f;z) \leqq \frac{4}{m} + 2\beta$, and this is sharp for the function $F_m$ given by*

$$F_m'(z) = \frac{(1+z^m)^\beta}{(1-z^m)^{\beta+2/m}} , \quad f(0) = 0 . \tag{2.43}$$

*Proof:* Let $f$ have the properties considered. Then there is an $m$-fold symmetric convex function $g$, a complex number $x \in \partial\mathbb{D}$ and a function $p \prec \frac{1+xz}{1-z}$ such that $f' = g' \cdot p^\beta$. Thus we have by Theorem 2.8 and Lemma 2.7 with $p := h'$

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{f''}{f'}(z)\right| &\leqq \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{g''}{g'}(z)\right| + \beta \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{p'}{p}(z)\right| \\ &\leqq \frac{4}{m} + 2\beta .\end{aligned}$$

For the function $F_m$, defined by (2.43), one gets, choosing $z = r > 0$, that

$$(1-|z|^2)\left|\frac{F_m''}{F_m'}(z)\right| = (1-r^2)\frac{2\cdot r^{m-1}}{1-r^m} + \beta(1-r^2)\frac{2m\cdot r^{m-1}}{1-r^{2m}} \overset{(\mathrm{r}\to 1)}{\longrightarrow} \frac{4}{m} + 2\beta ,$$

which establishes the statement about equality. $\square$

We remark that for $m = 1$ the statement is an immediate consequence of the linearly invariance of $C(\beta)$ because for $f \in C(\beta)$ one has $\varkappa(f;z) \leqq 1+\beta$ (see e.g. [46], Lemma 1.2), implying that

$$(1-|z|^2)\left|\frac{f''}{f'}(z)\right| \leqq 2\left|-\bar{z}+\frac{1}{2}(1-|z|^2)\frac{f''}{f'}(z)\right| + 2\,|z| \leqq 4+2\beta\,.$$

The theorem gives

**Corollary 2.8** (see [23]) *Let $\beta < 1/2$, $m \geqq 4/(1-2\beta)$ and $f \in C_m(\beta)$. Then $f$ fulfills the Becker univalence criterion.*

For each close-to-convex function $f$ we define the *order of close-to-convexity* $\operatorname{ctc}(f)$ to be the smallest number $\beta$ such that $f$ is close-to-convex of order $\beta$, i.e.

$$\begin{aligned} \operatorname{ctc}(f) &:= \inf\{\beta \in \mathbb{R}^+ \mid f \in C(\beta)\} \\ &= -\frac{1}{\pi} \inf_{\substack{\theta_2-\theta_1\in[0,2\pi] \\ r\in[0,1[}} \int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1+re^{i\theta}\frac{f''}{f'}(re^{i\theta})\right)d\theta \end{aligned}$$

(see e.g. [48]). If the order of close-to-convexity is not greater than 1, then it has the geometrical meaning that the image domain is the complement $E$ of rays that are pairwise disjoint and whose symmetric sectors of angle $(1-\operatorname{ctc}(f))\pi$ lie in $E$, and that such a representation does not exist for any smaller number $\beta$.

The next theorem gives a result on the Nehari expression depending on the order of close-to-convexity.

**Theorem 2.11** (see [26]) *Let $\beta \in [0,1]$ and $f \in C(\beta)$. Then $\sigma(f;z) \leqq 2+4\beta$ with equality if $f$ has the form*

$$f'(z) = \frac{(1+z^2)^\beta}{(1-z^2)^{\beta+1}}\,, \quad f(0) = 0\,.$$

*Proof:* We apply our result about the functional $|a_3 - a_2^2|$ (see [26], Theorem 3): for $\beta \leqq 1$ and $f \in C(\beta)$ one has

$$6\left|a_3 - a_2^2\right| = \sigma(f;0) \leqq 2+4\beta\,.$$

As $C(\beta)$ is linearly invariant this gives the result. $\square$

The same procedure gives

**Theorem 2.12** (see [26]) *Let $\beta \geqq 1$ and $f \in C(\beta)$. Then $\sigma(f; z) \leqq 2\beta^2 + 4\beta$ with equality if $f$ has the form*

$$f(z) = \frac{1}{2(\beta+1)} \left( \left( \frac{1+z}{1-z} \right)^{\beta+1} - 1 \right) .$$

*Proof:* Here we apply our result about the functional $|a_3 - a_2^2|$ (see [26], Theorem 3) for $\beta \geqq 1$. □

As here the sharp functions are of bounded boundary rotation $\left(\frac{K}{2} - 1\right) \pi$ we get furthermore by (3.28)

**Corollary 2.9** *Let $K \geqq 4$ and $f \in V(K)$. Then $\sigma(f; z) \leqq \frac{1}{2}(K^2 - 4)$ with equality if $f$ has the form*

$$f(z) = \frac{1}{K} \left( \left( \frac{1+z}{1-z} \right)^{K/2} - 1 \right) .$$

Remark that this corollary had been proved by Lehto and Tammi [33], Theorem 2, where also a corresponding result for $K \in [2, 4]$ is given.

# 3. Coefficient results and extreme points

## 3.1 Successive coefficients of close-to-convex functions

Robertson conjectured in [51], that for $f \in C(1)$ with a representation (1.1) and for all $n, j \in \mathbb{N}_0$

$$\Big| n\,|a_n| - j\,|a_j| \Big| \leqq \left| n^2 - j^2 \right| = |nA_n - jA_j| \ ,$$

where $A_n$ are the coefficients of the Koebe function $z/(1-z)^2$, which would make the result sharp. Leung [36] verified this conjecture.

In another paper [35], Leung proved that

$$\Big| |a_n| - |a_{n-1}| \Big| \leqq 1$$

holds for normalized starlike functions $f \in St$. Because a function $f$ is convex, if and only if $zf'$ is starlike (see (1.23) and (1.24)), this implies

$$\Big| n\,|a_n| - (n-1)\,|a_{n-1}| \Big| \leqq 1$$

for normalized convex functions $f \in K = C(0)$, and it follows by induction that

$$\Big| n\,|a_n| - j\,|a_j| \Big| \leqq |n - j| = |nA_n - jA_j|$$

for $n, j \in \mathbb{N}_0$, where here $A_n$ denote the coefficients of the convex function $z/(1-z)$. Now we consider a similar problem for close-to-convex functions of order $\beta$.

Therefore we use the following lemma which is an essential result of Brannan, Clunie and Kirwan [8], Aharonov and Friedland [1] and Brannan [7] (see e.g. [53], Theorem 2.21). Furthermore we add the answer to the question when equality occurs in their equations. This will be of some interest in our further considerations.

If $f, g \in A$, then by $f << g$ we denote coefficient domination, i.e. $|a_n(f)| \leqq |a_n(g)|$ $(n \in \mathbb{N}_0)$.

**Lemma 3.1** *Let $p \in \widetilde{P}$, $\alpha \geqq 1$ and $p^\alpha(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then $p^\alpha << \left(\frac{1+z}{1-z}\right)^\alpha$. Equality holds for given $n \in \mathbb{N}$, i.e. $|p_n| = a_n\left(\left(\frac{1+z}{1-z}\right)^\alpha\right)$, if and only if*

(a): $\quad p(z) = \left(\frac{1+yz}{1-yz}\right)^\alpha \quad (y \in \partial\mathbb{D})$

*for $\alpha > 1$ and*

(b): $\quad p(z) = \sum\limits_{k=1}^{n} \mu_k \left(\frac{1+x_k z}{1-x_k z}\right), \quad \sum\limits_{k=1}^{n} \mu_k = 1,$

$$x_k = y \cdot e^{2\pi i k/n}, \quad y \in \partial\mathbb{D}, \quad \mu_k \geqq 0 \qquad (k = 1, \ldots, n)\,. \tag{3.1}$$

*for $\alpha = 1$.*

*Proof:* Let $p \in \widetilde{P}$ and $\alpha \geqq 1$. Then $p \prec \frac{1+xz}{1-z}$ for some $x \in \partial\mathbb{D}$ and by Brannan, Clunie and Kirwan's modification of Herglotz's theorem ([8], see e.g. [53], Theorem 2.20) $p^\alpha$ has a representation

$$p^\alpha(z) = \int\limits_{\partial\mathbb{D}} \left(\frac{1+xyz}{1-yz}\right)^\alpha d\mu(y)$$

with some Borel probability measure $\mu$ on $\partial\mathbb{D}$. Using the notation

$$P_n(x) := a_n\left(\left(\frac{1+xz}{1-z}\right)^\alpha\right)$$

we get

$$p_n = a_n\left(p^\alpha\right) = \int\limits_{\partial\mathbb{D}} P_n(x) y^n d\mu(y)\,,$$

and so

$$\begin{aligned} |p_n| &= \left|\int\limits_{\partial\mathbb{D}} P_n(x) y^n d\mu(y)\right| \\ &\leqq \int\limits_{\partial\mathbb{D}} |P_n(x)| d\mu(y) = |P_n(x)| \\ &\leqq P_n(1) = a_n\left(\left(\frac{1+z}{1-z}\right)^\alpha\right)\,. \end{aligned}$$

Here the second inequality was proven by Aharonov and Friedland [1] who showed furthermore that equality occurs for some $n \in \mathbb{N}$ only if $x = 1$. On the other hand in the first inequality we have equality only if $\mu$ is such that the integrand has constant argument on the support of $\mu$. This implies that $\mu = \sum_{k=1}^{n} \mu_k \delta_{x_k}$, $x_k \in \partial\mathbb{D}$, $\mu_k \geqq 0$, $\sum_{k=1}^{n} \mu_k = 1$, and so (3.1) follows.

For $\alpha = 1$ it is easily seen that all those functions in fact give equality implying (b). Let now $\alpha > 1$. Then by a result of Hallenbeck and MacGregor [19], Theorem 8, it follows that nontrivial finite convex combinations of the type (3.1) do not lie in $\widetilde{P}^{\alpha}$ which gives (a). □

The next lemma is the main tool to solve the case of equality in the following theorem.

**Lemma 3.2** *Let $n \in \mathbb{N}$, $\beta_k > 0$, $p_k \in \widetilde{P}$ $(k = 1, \ldots, n)$, let further $p := \prod_{k=1}^{n} p_k^{\beta_k}$ and $\alpha := \sum_{k=1}^{n} \beta_k > 1$. Then $p \in \widetilde{P}^{\alpha}$,*

$$p << \left(\frac{1+z}{1-z}\right)^{\alpha}, \tag{3.2}$$

*and equality holds for some $m \in \mathbb{N}$, i.e. $|a_m(p)| = a_m\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$, if and only if for all $k = 1, \ldots, n$*

$$p_k(z) = \frac{1+xz}{1-xz} \quad \textit{for some fixed} \quad x \in \partial\mathbb{D}\,.$$

*Proof:* Consider $N := \{f \in A \mid f(0) = 1 \text{ and } 0 \notin f(\mathbb{D})\}$ as a real linear space with the nonstandard addition and scalar multiplication ($f, g \in N$, $\lambda \in \mathbb{R}$)

$$f \oplus g := f \cdot g \quad \text{and} \quad \lambda \odot f := f^{\lambda}\,.$$

The functional $L$ with $L(f) = a_1(f)$ is linear with respect to this linear structure, and continuous with respect to locally uniform convergence. The function $p_0 = \frac{1+z}{1-z}$ is the only solution of the extremal problem $\operatorname{Re} L(p_0) = \max_{f \in \widetilde{P}} \operatorname{Re} L(f)$ so that $p_0$ is an extreme point of $\widetilde{P}$ with respect to this nonstandard structure.

Suppose now $n \in \mathbb{N}$, $\beta_k > 0$, $p_k \in \widetilde{P}$ $(k = 1, \ldots, n)$, $p = \prod_{k=1}^{n} p_k^{\beta_k}$ and $\alpha = \sum_{k=1}^{n} \beta_k > 1$. Clearly $p^{1/\alpha} \in \widetilde{P}$, and by Lemma 3.1 we get (3.2). Let now $|a_m(p)| = a_m\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$ hold for some $m \in \mathbb{N}$. Then again by Lemma

3.1 we get that $p(z) = p_0(xz)$, $x \in \partial\mathbb{D}$ as $\alpha > 1$. On the other hand the definition of $p$

$$p_0(z) = p(\overline{x}z) = \beta_1 \odot p_1(\overline{x}z) \oplus \cdots \oplus \beta_n \odot p_n(\overline{x}z) ,$$

gives a convex representation of the extreme point $p_0$ of $\widetilde{P}$. So this convex representation must be trivial giving the conclusion. □

**Theorem 3.1** *Let $\beta > 0$, $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in C(\beta)$ and*

$$F(z) = \frac{1}{2(\beta+1)}\left(\left(\frac{1+z}{1-z}\right)^{\beta+1} - 1\right) = \sum_{n=1}^{\infty} A_n z^n . \tag{3.3}$$

*Then for all $n, j \in \mathbb{N}_0$ for which $n - j$ is even holds*

$$\Big| n\,|a_n| - j\,|a_j| \Big| \leqq |nA_n - jA_j| .$$

*Equality holds for given $n, j \in \mathbb{N}_0$ if and only if $f(z) = F(xz)$ for some $x \in \partial\mathbb{D}$.*

*Proof:* First observe that the result is sharp, because $F \in C(\beta)$. The proof is on the lines of Robertson [51], Theorem 3. Let $f$ be close-to-convex of order $\beta$. By (1.36) there are $h \in St$ and $p \in \widetilde{P}$ such that

$$f'(z) = \frac{h(z)}{z} \cdot p^{\beta}(z) .$$

Because of an easy compactness argument we may assume that both $f$ and $h$ are analytic in the closed disk. The function $h$ is starlike, so that it is in particular starlike in the direction of its diametral line (see [51]). Thus it has a representation of the form

$$\frac{h(z)}{z} = \frac{q(z)}{1 - \zeta z^2}$$

for some $\zeta \in \partial\mathbb{D}$ and some $q \in P$. From this we get

$$(1 - \zeta z^2)f'(z) = q(z) \cdot p^{\beta}(z) . \tag{3.4}$$

Observe that $(q \cdot p^{\beta})^{\frac{1}{1+\beta}} \in \widetilde{P}$. By Lemma 3.1 we get

$$q \cdot p^{\beta} << \left(\frac{1+z}{1-z}\right)^{1+\beta} = (1 - z^2)F'(z) , \tag{3.5}$$

and therefore

$$(1-\zeta z^2)f'(z) << (1-z^2)F'(z)\,,$$

which is equivalent to the statement

$$|(n+1)a_{n+1}-\zeta(n-1)a_{n-1}| \leqq (n+1)A_{n+1}-(n-1)A_{n-1} \quad (n\in\mathbb{N})\,.$$

Finally we have

$$\begin{aligned}\Big|(n+1)\,|a_{n+1}|-(n-1)\,|a_{n-1}|\Big| &\leqq |(n+1)a_{n+1}-\zeta(n-1)a_{n-1}| \\ &\leqq (n+1)A_{n+1}-(n-1)A_{n-1} \quad (n\in\mathbb{N})\,.\end{aligned}$$

The general case follows from this by induction.

Now suppose that for given $n, j \in \mathbb{N}_0$ equality holds. Observe that the family of functions $f$ for which there is a representation (3.4) with some $\zeta \in \partial\mathbb{D}$, $p \in \widetilde{P}$ and $q \in P$ is compact. So such a representation holds for all $f \in C(\beta)$, and not only if $h$ and $p$ are analytic in the closed disk. Therefore equality in (3.5) implies that the functions $p$ and $q$ corresponding to $f$ coincide and are a certain rotation of $\frac{1+z}{1-z}$ by Lemma 3.2. This gives the conclusion. □

We want to remark that by induction this gives also a new proof for the coefficient result (3.23) in $C(\beta)$.

Whereas for $\beta = 1$ the fact that $n-j$ is even is not essential as Leung's result shows, for small $\beta$ it is. To prove this, suppose $f$ is normalized by (1.1), $t \in [0,1]$ and

$$f'(z) = \frac{1}{(1-z)^2}\cdot\left(t\frac{1+z}{1-z}+(1-t)\frac{1+z^2}{1-z^2}\right)^{\beta}\,.$$

Then obviously $f \in C(\beta)$ and

$$3a_3-2a_2 = 1+2\beta+2\beta t+2\beta(\beta-1)t^2 =: H(t)\,.$$

A simple calculation shows, that $H$ takes its maximum value at an interior point $t \in\, ]0,1[$ if $\beta \in\, ]0,1/2[$. For those values of $\beta$ the theorem cannot be generalized to odd differences $n-j$. On the other hand, we get for $\beta \geqq 1/2$:

**Theorem 3.2** *Let $\beta \geqq 1/2$, $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in C(\beta)$ and $A_n$ be defined by* (3.3). *Then*

$$\Big|3|a_3|-2|a_2|\Big| \leqq 3A_3-2A_2 = 1+2\beta+2\beta^2\,.$$

*Proof:* Let $f \in C(\beta)$, then there are functions $h(z) = z + c_2 z^2 + c_3 z^3 + \cdots \in St$, $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$, and a real number $\alpha$ such that

$$f'(z) = \frac{h(z)}{z} \cdot e^{-i\alpha} \left( \cos \alpha \bigl(p(z)\bigr)^{\beta} + i \sin \alpha \right) . \tag{3.6}$$

Since $h$ is starlike $zh'/h \in P$, and the second coefficient of $zh'/h$ is bounded in modulus by 2 (see e.g. [49]) implying

$$\left| c_3 - \frac{1}{2} c_2^2 \right| \leqq 1 . \tag{3.7}$$

Comparing coefficients in (3.6) gives

$$\begin{aligned} 2a_2 &= c_2 + e^{-i\alpha} \cos \alpha (\beta p_1) , \\ 3a_3 &= c_3 + e^{-i\alpha} \cos \alpha \left( \beta p_1 c_2 + \beta p_2 + \frac{\beta(\beta - 1)}{2} p_1^2 \right) , \end{aligned} \tag{3.8}$$

so that

$$3a_3 - c_2 a_2 = \left( c_3 - \frac{c_2^2}{2} \right) + e^{-i\alpha} \cos \alpha \left( \frac{\beta p_1 c_2}{2} + \beta p_2 + \frac{\beta(\beta - 1)}{2} p_1^2 \right) .$$

With the aid of (3.7) and $|c_2| \leqq 2$ we get

$$\begin{aligned} |3a_3 - c_2 a_2| &\leqq 1 + \beta |p_1| + \beta \left| p_2 - \frac{p_1^2}{2} \right| + \frac{\beta^2}{2} |p_1|^2 \\ &\leqq 1 + \beta |p_1| + \beta \left( 2 - \frac{|p_1|^2}{2} \right) + \frac{\beta^2}{2} |p_1|^2 , \end{aligned}$$

where we used Lemma 5.1 from Chapter 5. It is easily verified, that if $\beta \geqq 1/2$ then the right hand term has no maximum for $|p_1| \in \, ]0, 2[$ so that the maximal value is attained at $|p_1| = 2$, and it follows again using $|c_2| \leqq 2$ that

$$3|a_3| - 2|a_2| \leqq \left| 3a_3 - \frac{c_2}{2} \cdot 2a_2 \right| \leqq 1 + 2\beta + 2\beta^2 .$$

On the other hand, if $\beta > 1$ then, using $|a_2| \leqq 1 + \beta$ (see (3.23)), we have

$$2|a_2| - 3|a_3| \leqq 2|a_2| \leqq 2(1 + \beta) < 1 + 2\beta + 2\beta^2 ,$$

and if $\beta \in [\frac{1}{2}, 1]$, then the functions are univalent so that

$$|a_2| - |a_3| \leqq 1$$

(see e.g. [13], Theorem 3.11), implying

$$2|a_2| - 3|a_3| \leqq 2(|a_2| - |a_3|) \leqq 2 < 1 + 2\beta + 2\beta^2 ,$$

which finishes the proof. □

Whereas by the above result the functional $\Big|3|a_3| - 2|a_2|\Big|$ is maximized by the function $F$ defined by (3.3) for all $\beta \geqq 1/2$, the same fails for the successive coefficient functional $\Big||a_3| - |a_2|\Big|$. Here the odd function $G$ defined by

$$G'(z) = \frac{(1+z^2)^\beta}{(1-z^2)^{\beta+1}} , \quad G(z) = \sum_{n=1}^{\infty} B_n z^n , \tag{3.9}$$

gives the functional a larger value if $\beta \in ]0,1[$. For $\beta \geqq 1$ we show now that the successive coefficient functional is maximized by $F$ which is stronger than Theorem 3.2. For $\beta = 1$ this result contrasts that one in $S$:

$$\max_{f\in S} \Big||a_3| - |a_2|\Big| = 1.029\ldots .$$

$G$ is shown to give the maximum for $|a_3| - |a_2|$ if $\beta \leqq 8/9$.

**Theorem 3.3** (see [26]) *Let* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in C(\beta)$ *and* $A_n, B_n$ *defined by* (3.3) *and* (3.9). *Then*

(a) *for* $\beta \geqq 1$

$$\Big||a_3| - |a_2|\Big| \leqq A_3 - A_2 = \frac{\beta}{3}(1+2\beta) , \tag{3.10}$$

(b) *and for* $\beta \in [0, 8/9]$

$$|a_3| - |a_2| \leqq B_3 - B_2 = \frac{1}{3}(1+2\beta) . \tag{3.11}$$

*Proof:* (a): At first we show

$$|a_3| - |a_2| \leqq \frac{1}{3}(2\beta^2 + \beta) . \tag{3.12}$$

We use our results about the *Fekete-Szegö functional* $|a_3 - \lambda a_2^2|$ $(\lambda \in \mathbb{C})$ (see [26], Corollary 1, Theorem 1 and Theorem 3): for $\beta \geqq 1$ and $f \in C(\beta)$ hold

$$\left|a_3 - \frac{1}{3}a_2^2\right| \leqq \frac{1}{3}(\beta^2 + 2\beta + 2) , \tag{3.13}$$

$$\left|a_3 - \frac{2}{3}a_2^2\right| \leqq \frac{1}{3}(1+2\beta)\,, \tag{3.14}$$

and

$$\left|a_3 - a_2^2\right| \leqq \frac{1}{3}(\beta^2+2\beta)\,. \tag{3.15}$$

Equation (3.13) yields

$$\begin{aligned} |a_3| - |a_2| &\leqq \left|a_3 - \frac{1}{3}a_2^2\right| + \frac{1}{3}|a_2|^2 - |a_2| \\ &\leqq \frac{1}{3}(\beta^2+2\beta+2) + \frac{1}{3}|a_2|^2 - |a_2| =: U(|a_2|)\,. \end{aligned}$$

Because $U$ defines a convex parabola, it takes its maximum value at the boundary of its interval of definition. Furthermore the relation

$$U(|a_2|) = \frac{1}{3}(2\beta^2+\beta)$$

implies that $|a_2| = 2-\beta$ or $|a_2| = 1+\beta$, so that

$$U(|a_2|) \leqq \frac{1}{3}(2\beta^2+\beta) \qquad \text{for} \qquad |a_2| \in [2-\beta, 1+\beta]\,. \tag{3.16}$$

From this equation (3.12) follows if $\beta \geqq 2$, because $|a_2| \leqq 1+\beta$ in $C(\beta)$. Let now $\beta \in [1,2[$. Then with the aid of (3.14) we have furthermore that

$$\begin{aligned} |a_3| - |a_2| &\leqq \left|a_3 - \frac{2}{3}a_2^2\right| + \frac{2}{3}|a_2|^2 - |a_2| \\ &\leqq \frac{1}{3}(1+2\beta) + \frac{2}{3}|a_2|^2 - |a_2| =: V(|a_2|)\,. \end{aligned} \tag{3.17}$$

The same procedure as above shows that

$$V(|a_2|) \leqq \frac{1}{3}(2\beta^2+\beta) \qquad \text{for} \qquad |a_2| \in [1-\beta, 1/2+\beta]\,,$$

which, together with (3.16), gives (3.12).

Now we shall show that

$$|a_2| - |a_3| \leqq \frac{1}{3}(2\beta^2+\beta)\,,$$

which is trivially true if $|a_2| \in [0, (2\beta^2+\beta)/3]$. This gives the result for $\beta \geqq (\sqrt{7}+1)/2$. For $\beta < (\sqrt{7}+1)/2$ let now $|a_2|$ lie in the remaining interval $[(2\beta^2+\beta)/3, 1+\beta]$. Then (3.15) gives

$$\begin{aligned} |a_2| - |a_3| &= |a_2|^2 - |a_3| - |a_2|^2 + |a_2| \leqq \left|a_2^2 - a_3\right| - |a_2|^2 + |a_2| \\ &\leqq \frac{1}{3}(\beta^2+2\beta) - |a_2|^2 + |a_2| =: W(|a_2|)\,. \end{aligned}$$

$W$ takes its global maximum at $|a_2| = 1/2$ which does not lie in the interval considered. Thus, for $|a_2| \in [(2\beta^2+\beta)/3, 1+\beta]$, $W$ is decreasing, and it remains to show that

$$W\Big((2\beta^2+\beta)/3\Big) \leqq \frac{1}{3}(2\beta^2+\beta)\,,$$

i.e. $3\beta^2 + 6\beta \leqq (2\beta^2+\beta)^2$, which obviously holds for $\beta \geqq 1$.
(b): We shall use the relations $p := |p_1| \leqq 2, c := |c_2| \leqq 2$, Lemma 5.1 and the relation

$$\left|c_3 - \frac{3}{4}c_2^2\right| \leqq 1 - \frac{1}{4}|c_2|^2\,, \tag{3.18}$$

which holds for starlike functions (see e.g. [25], Lemma). With notation (3.6) and (3.8) we get

$$\begin{aligned}
3\left(|a_3|-|a_2|\right) &\leqq \left|3a_3 - \frac{3}{2}c_2a_2\right| \\
&= \left|\left(c_3-\frac{3}{4}c_2^2\right) + e^{-i\alpha}\cos\alpha\left(\frac{1}{4}\beta p_1c_2 + \beta\left(p_2-\frac{1}{2}p_1^2\right) + \frac{\beta^2}{2}p_1^2\right)\right| \\
&\leqq \left|c_3 - \frac{3}{4}c_2^2\right| + \frac{1}{4}\beta pc + \beta\left|p_2 - \frac{1}{2}p_1^2\right| + \frac{\beta^2}{2}p^2 \\
&\leqq 1 - \frac{c^2}{4} + \frac{1}{4}\beta pc + \beta\left(2 - \frac{1}{2}p^2\right) + \frac{\beta^2}{2}p^2 \\
&= 1 + 2\beta - \frac{c^2}{4} + \frac{1}{4}\beta pc - \frac{\beta}{2}(1-\beta)p^2 =: H(p,c)\,.
\end{aligned}$$

We shall show that $H$ takes its maximum value for $(p,c) \in [0,2]^2 =: Q$ at the point $(0,0)$ if $\beta \in [0, 8/9]$ which gives the result.

Suppose there is a local maximum of $H$ at $(p_0,c_0) \in\, ]0,2[^2$. Then

$$4\frac{\partial H}{\partial c}(p_0,c_0) = -2c_0 + \beta p_0 = 0 \tag{3.19}$$

and

$$\frac{4}{\beta}\cdot\frac{\partial H}{\partial p}(p_0,c_0) = c_0 - 4(1-\beta)p_0 = 0\,. \tag{3.20}$$

These both equations lead to

$$\beta = \frac{8}{9} \qquad \text{and} \qquad c_0 = \frac{4}{9}p_0\,,$$

so that $H(p_0, c_0) \equiv 1 + 2\beta = H(0,0)$. For this number $\beta = 8/9$ the line $c = \frac{4}{9}p$ in $[0,2]^2$ is a saddle line of $H$, and for other values of $\beta$ a local maximum of $H$ does not exist.

Now we explore $H$ on the sides of $Q$. It is easily seen that on $\{p = 0\}$ and $\{c = 0\}$ the value $H(p,c)$ is not greater that $H(0,0)$. Let now $c = 2$. Here

$$H(p,2) = 2\beta + \frac{\beta}{2}p - \frac{\beta}{2}(1-\beta)p^2 ,$$

and for a local maximum $p_0 \in ]0,2[$ it follows that $p_0 = \frac{1}{2(1-\beta)}$, and so

$$H(p_0, 2) = 2\beta + \frac{\beta}{8(1-\beta)} \leqq 1 + 2\beta ,$$

where the last inequality follows because $\beta \leqq 8/9$. On the other hand,

$$H(2,c) = 1 + 2\beta^2 - \frac{c^2}{4} + \frac{1}{2}\beta c .$$

For a local maximum $c_0$ it follows that $c_0 = \beta$, and so

$$H(2, c_0) = 1 + \frac{9}{4}\beta^2 \leqq 1 + 2\beta$$

for $\beta \leqq 8/9$ which finishes the proof. □

We are not able to show that $|a_4| - |a_3| \leqq A_4 - A_3$ for $\beta \geqq 1$, but give a weaker result in this direction.

**Theorem 3.4** (see [25]) *Let* $\beta \geqq 1$, $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in C(\beta)$. *Then*

$$\Big||a_4| - |a_2|\Big| \leqq A_4 - A_2 = \frac{(1+\beta)}{3}(\beta^2 + 2\beta) .$$

*Proof:* First we use Lemma 3.3 implying that with $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in C(\beta)$ the function $h(z) = z + b_3z^3 + b_5z^5 + \cdots$, defined by $h'(z) = (f'(z^2))^{1/2}$, $h(0) = 0$, is an odd close-to-convex function of order $\beta/2$. Now, because $\beta \geqq 1$, we can use the coefficient domination theorem for such functions (Theorem 3.5) and get

$$|b_7| = \frac{1}{7}\left|2a_4 - \frac{3}{2}a_2a_3 + \frac{a_2^3}{2}\right| \leqq \frac{(1+\beta)}{42}(\beta^2 + 2\beta + 6) . \tag{3.21}$$

For $b_5$ we have

$$|b_5| = \frac{3}{10}\left|a_3 - \frac{1}{3}a_2^2\right| \leqq \frac{\beta^2 + 2\beta + 2}{10} . \tag{3.22}$$

Now we get with the aid of (3.21) and (3.22) that

$$\begin{aligned}
|a_4| - |a_2| &\leqq \left|a_4 - \frac{3}{4}a_2a_3 + \frac{a_2^3}{4}\right| + \frac{3}{4}|a_2|\left|a_3 - \frac{1}{3}a_2^2\right| - |a_2| \\
&\leqq \frac{(1+\beta)}{12}(\beta^2 + 2\beta + 6) + \frac{3}{4}|a_2|\frac{(\beta^2 + 2\beta + 2)}{3} - |a_2| \\
&\leqq \frac{(1+\beta)}{12}(\beta^2 + 2\beta + 6) + \frac{|a_2|}{4}(\beta^2 + 2\beta - 2)\,.
\end{aligned}$$

Because $(\beta^2 + 2\beta - 2) \geqq 0$ it follows now from $|a_2| \leqq 1 + \beta$ that

$$|a_4| - |a_2| \leqq \frac{(1+\beta)}{3}(\beta^2 + 2\beta)\,.$$

On the other hand, for $\beta \geqq 1$

$$|a_2| - |a_4| \leqq |a_2| \leqq \frac{(1+\beta)}{3} \cdot 3 \leqq \frac{(1+\beta)}{3}(\beta^2 + 2\beta)$$

is trivially true. □

## 3.2 Coefficients of symmetric close-to-convex functions

Brannan, Clunie and Kirwan had used their crucial result (3.1) on functions in $\widetilde{P}$ to solve the coefficient problem for functions $f \in C(\beta)$, namely

$$f' \ll \frac{(1+z)^\beta}{(1-z)^{\beta+2}} \tag{3.23}$$

(see [8], [7], [1] and [53], Theorem 2.29).

Here we generalize this result to $m$-fold symmetric functions. A first step in this direction was done by Pommerenke [45] whose asymptotic results give support to the conjecture that if $\beta > 1 - 2/m$, then the coefficients of a function $f \in C_m(\beta)$ given by (1.2) are dominated in modulus by the corresponding coefficients of the function $F_m$ for which

$$F_m'(z) = \frac{(1+z^m)^\beta}{(1-z^m)^{\beta+2/m}}\,, \quad F_m(z) = \sum_{n=1}^{\infty} A_n z^n\,. \tag{3.24}$$

For $m = 1$ this is Brannan, Clunie and Kirwan's result and for $\beta = 1$ it had been proved by Pommerenke ([45], Theorem 3). The latter statement includes the truth of the Littlewood-Paley conjecture (see e.g. [13], section 3.8) for odd close-to-convex functions (of order one).

We shall now prove the above statement for $\beta \geqq 1 - 1/m$, whereas for $0 < \beta < 1 - 1/m$ the statement is false as examples show, so that the number $1 - 1/m$ is sharp. However, for $\beta = 0$, i.e. for convex functions, the statement is again true, as was shown by Robertson ([50], p. 380).

**Theorem 3.5** (see [29]) *Let $m \in \mathbb{N}$, $\beta \geqq 1 - 1/m$ and $f \in C_m(\beta)$. Then*

$$f' \ll \frac{(1+z^m)^\beta}{(1-z^m)^{\beta+2/m}} .$$

*Proof:* Let $f$ be an $m$-fold symmetric close-to-convex function of order $\beta$. Then there exist $h \in St_m$ and $p \in \widetilde{P}$ such that

$$f'(z) = \frac{h(z)}{z} \cdot p^\beta(z^m)$$

(see (1.36)). For $h$ there is a representation of the form

$$h(z) = \int\limits_{\partial\mathbb{D}} \frac{z}{(1 - xz^m)^{2/m}} d\mu(x) \tag{3.25}$$

(see [9], Theorem 3), where $\mu$ is a Borel probability measure on the unit circle. Thus we have

$$\begin{aligned} f'(z) &= \int\limits_{\partial\mathbb{D}} \frac{d\mu(x)}{(1-xz^m)^{2/m}} \cdot p^\beta(z^m) \\ &= \int\limits_{\partial\mathbb{D}} \frac{d\mu(x)}{(1-x^2z^{2m})^{1/m}} \cdot \left(\frac{1+xz^m}{1-xz^m}\right)^{1/m} \cdot p^\beta(z^m) . \end{aligned}$$

For fixed $x \in \partial\mathbb{D}$ the function

$$\left(\left(\frac{1+xz^m}{1-xz^m}\right)^{1/m} \cdot p^\beta(z^m)\right)^{\frac{1}{\beta+1/m}} =: q_x(z^m)$$

is of the form (2.42) and lies in $\widetilde{P}$. Therefore by the Brannan-Clunie-Kirwan lemma (3.1) it follows that

$$q_x^{\beta+1/m}(z^m) \ll \left(\frac{1+z^m}{1-z^m}\right)^{\beta+1/m} , \tag{3.26}$$

because $\beta + 1/m \geqq 1$. Thus we get

$$
\begin{aligned}
f'(z) &= \int_{\partial\mathbb{D}} \frac{d\mu(x)}{(1-x^2z^{2m})^{1/m}} \cdot q_x^{\beta+1/m}(z^m) & (3.27)\\
&= \sum_{j=0}^{\infty} \binom{j-1+1/m}{j} z^{2mj} \left( \int_{\partial\mathbb{D}} x^{2j} q_x^{\beta+1/m}(z^m) d\mu(x) \right) \\
&\ll \sum_{j=0}^{\infty} \binom{j-1+1/m}{j} z^{2mj} \left(\frac{1+z^m}{1-z^m}\right)^{\beta+1/m} = \frac{(1+z^m)^\beta}{(1-z^m)^{\beta+2/m}} ,
\end{aligned}
$$

because $\mu$ has total mass one and all numbers $\binom{j-1+1/m}{j}$ are nonnegative. □

As the method used to prove Theorem 3.4 shows, this result is somewhat stronger than the original domination theorem (3.23), especially the statement that

$$
|a_{2n+1}| \leqq \begin{cases} \dfrac{1}{2n+1}\left( \dbinom{n/2+1/2}{n/2} + \dbinom{n/2-1/2}{n/2-1} \right) & \text{if } n \text{ is even} \\ \dfrac{2}{2n+1} \dbinom{n/2}{n/2-1/2} & \text{if } n \text{ is odd} \end{cases}
$$

holds for $f \in C_2(1/2)$ and improves $|a_n| \leqq n$ $(n \in \mathbb{N})$ for $f \in C$.

Because for all $k \in \mathbb{N}_0$ the numbers $A_k$ are nonnegative, it follows from Theorem 3.5 that the functional $\left|f^{(n)}(z)\right|$ is maximized over $C_m(\beta)$ by $F_m$ for all $n \in N_0$ and $z \in \mathbb{D}$ if $\beta \geqq 1 - 1/m$.

Now we show that the result cannot be generalized to the case when $0 < \beta < 1 - 1/m$, not even for the third nonvanishing coefficient. Therefore, suppose $f$ is normalized by (2.42), $t \in [0,1]$ and

$$
f'(z) = \frac{1}{(1-z^m)^{2/m}} \cdot \left( t\frac{1+z^m}{1-z^m} + (1-t)\frac{1+z^{2m}}{1-z^{2m}} \right)^\beta ,
$$

then $f \in C_m(\beta)$ and

$$
(2m+1)a_{2m+1} = 2\beta\left(1 + (\beta-1)t^2\right) + \frac{4\beta t}{m} + \frac{1}{m}\left(1 + \frac{2}{m}\right) =: H(t) .
$$

It is easily seen that $H$ has a local maximum at the point $t_0 = \frac{1}{m(1-\beta)}$, which lies in the interval $]0,1[$ if $0 < \beta < 1/m$ and is greater than the corresponding coefficient of $F_m$.

On the other hand we shall show now that for $m = 2$ a certain linear combination of the coefficients is dominated by $F_m$ for all $\beta \geqq 0$.

**Theorem 3.6** *Let $\beta \geqq 0$, $f(z) = z + a_3z^3 + a_5z^5 + \cdots \in C_2(\beta)$ and $F_m$ be given by* (3.24) *with $m = 2$. Then for all $n \in \mathbb{N}$*

$$(2n+1)|a_{2n+1}| + (2n-1)|a_{2n-1}| \leqq (2n+1)A_{2n+1} + (2n-1)A_{2n-1} \ .$$

*Proof:* Because $f$ is odd and close-to-convex of order $\beta$, there is an odd starlike function $h$ and $p \in \widetilde{P}$ with

$$f'(z) = \frac{h(z)}{z} \cdot p^\beta(z^2) \ .$$

Because $h$ is odd – and so $h(\mathbb{D})$ is symmetric with respect to the origin – and starlike, it follows that for every $\zeta \in \partial\mathbb{D}$ there is a function $q_\zeta \in \widetilde{P}$ such that

$$(1-\zeta z)(1+\zeta z)\frac{h(z)}{z} = q_\zeta(z^2)$$

(see e.g. [13], p. 248, Lemma 1, and its proof). Thus we get

$$(1-\zeta^2z^2)f'(z) = q_\zeta(z^2)\cdot p^\beta(z^2) << \left(\frac{1+z^2}{1-z^2}\right)^{1+\beta} = (1+z^2)F_m'(z) \ ,$$

where the assertion about domination follows in the usual way with the Brannan-Clunie-Kirwan lemma. This leads to

$$\left|(2n+1)a_{2n+1} - \zeta^2(2n-1)a_{2n-1}\right| \leqq (2n+1)A_{2n+1} + (2n-1)a_{2n-1} \ ,$$

which holds for all $\zeta \in \partial\mathbb{D}$ and $n \in \mathbb{N}$ implying the conclusion. □

Also for all $\beta \geqq 0$ a distortion theorem holds:

**Theorem 3.7** (see [28]) *Let $\beta \geqq 0$, $m \in \mathbb{N}$ and $f \in C_m(\beta)$. Then*

$$|f'(z)| \leqq F_m'(|z|) = \frac{(1+|z|^m)^\beta}{(1-|z|^m)^{\beta+2/m}}$$

*and*

$$|f(z)| \leqq F_m(|z|) \ .$$

*Proof:* Let $f \in C_m(\beta)$. Then the function $g$ defined by $f'(z) = \left(g'(z^m)\right)^{1/m}$ is close-to-convex of order $m\beta$ (Lemma 3.3 (b)). Therefore

$$|f'(z)| \leqq \left|\left(g'(z^m)\right)^{1/m}\right| \leqq \left(\frac{(1+|z|^m)^{m\beta}}{(1-|z|^m)^{m\beta+2}}\right)^{1/m} = F_m'(|z|)$$

where we used the domination theorem for close-to-convex functions (3.23). An integration gives moreover

$$|f(z)| = \left|\int_0^z f'(\zeta)d\zeta\right| \leqq \int_0^{|z|} \left|f'(re^{i\theta})\right| dr \leqq \int_0^{|z|} F_m'(r)dr \ .$$

□

## 3.3 Coefficients of symmetric functions of bounded boundary rotation

Here we extend the inclusion relation between functions of bounded boundary rotation and close-to-convex functions

$$V(K) \subset C(K/2-1)\,, \quad K \geqq 2\,. \tag{3.28}$$

(see [53], Theorem 2.26) to $m$-fold symmetric functions. Using the corresponding result for close-to-convex functions of the last section this leads to sharp coefficient bounds for $m$-fold symmetric functions of bounded boundary rotation at most $K\pi$ when $K \geqq 2m$. Moreover it shows that an $m$-fold symmetric function of bounded boundary rotation at most $(2m+2)\pi$ is close-to-convex and thus univalent.

**Lemma 3.3** (see [29]) *Let $m \in \mathbb{N}$, $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and $h(z) = z + b_{m+1}z^{m+1} + b_{2m+1}z^{2m+1} + \cdots$ have the property*

$$h'(z) = \left(f'(z^m)\right)^{1/m}.$$

*Then*

(a): $f \in V_1(K) \iff h \in V_m(K)$
*and*
(b): $f \in C_1(\beta) \iff h \in C_m(\beta/m)$.

*Proof:* (a): Let $f \in V_1(K)$. As

$$1 + z^m \frac{f''(z^m)}{f'(z^m)} = 1 + z \frac{h''(z)}{h'(z)},$$

we get $h \in V_m(K)$. The converse follows in the same way because for an $m$-fold symmetric function of bounded boundary rotation the corresponding functions with positive real part can be chosen to be of form (2.42).
(b): If $f \in C_1(\beta)$, then there are $g \in K$ and $p \in \widetilde{P}$ such that

$$f'(z) = g'(z) \cdot p^{\beta}(z).$$

Now

$$h'(z) = \left(f'(z^m)\right)^{1/m} = \left(g'(z^m)\right)^{1/m} \cdot p^{\beta/m}(z^m) = g_m(z) \cdot p^{\beta/m}(z^m).$$

The function $g_m$ represents an $m$-fold symmetric convex function, because of (a) – remember that $K = V(2)$ – so that $h \in C_m(\beta/m)$. Here also the converse follows from the fact that for an $m$-fold close-to-convex function the corresponding function in $\widetilde{P}$ can be chosen to be of form (2.42). □

An application of the lemma together with (3.28) gives

**Theorem 3.8** (see [29]) *Let $m \in \mathbb{N}$ and $K \geqq 2$. Then*

$$V_m(K) \subset C_m\left((K/2-1)/m\right).$$

Now the result of the last section can be applied and leads to

**Theorem 3.9** (see [29]) *Let $m \in \mathbb{N}$, $K \geqq 2m$ and $f \in V_m(K)$. Then*

$$f' << \frac{(1+z^m)^{\frac{1}{m}(\frac{K}{2}-1)}}{(1-z^m)^{\frac{1}{m}(\frac{K}{2}+1)}}.$$

This follows from Theorem 3.5. Observe that the result is sharp, because the function $F_m$ defined by (3.24) with $\beta = (K/2-1)/m$ is in $V_m(K)$ as

$$1 + z\frac{F''_m}{F'_m}(z) = \left(\frac{K}{4}+\frac{1}{2}\right) \cdot \frac{1+z^m}{1-z^m} - \left(\frac{K}{4}-\frac{1}{2}\right) \cdot \frac{1-z^m}{1+z^m}.$$

For $m = 2$ and $K = 6$ we have the statement of the Littlewood-Paley conjecture.

A further consequence of Theorem 3.8 is

**Theorem 3.10** (see [29]) *Let $m \in \mathbb{N}$. Then $V_m(2m+2)$ consists of close-to-convex and thus univalent functions.*

## 3.4 Extreme points of symmetric close-to-convex functions and symmetric functions of bounded boundary rotation

As in Section 3.2 was shown, the function $F_m$ dominates the coefficients over $C_m(\beta)$ for $\beta \geqq 1 - 1/m$. $F_m$ has a sector of angle $(1+\beta)\pi$ as Riemann image surface. Now we show that for $\beta \geqq 1$ the extreme points of the closed convex hull of $C_m(\beta)$ are exactly the functions with this geometric property.

**Theorem 3.11** (see [28]) *Let $\beta \geqq 1$ and $m \in \mathbb{N}$. Then an extreme point $f$ of $\overline{\mathrm{co}}\, C_m(\beta)$ has the form*

$$f'(z) = \frac{(1+xz^m)^\beta}{(1-yz^m)^{\beta+2/m}}\,, \quad f(0)=0\,, \qquad x, y \in \partial\mathbb{D}\,,\ x \neq -y\,. \tag{3.29}$$

*Proof:* Let $f \in C_m(\beta)$ be represented by $f'(z) = \frac{h(z)}{z} \cdot p^\beta(z^m)$, where $h \in St_m$ and $p \in \tilde{P}$. Then $h$ has a representation (3.25)

$$\frac{h(z)}{z} = \int\limits_{\partial\mathbb{D}} \frac{d\mu(w)}{(1-wz^m)^{2/m}}$$

with a Borel probability measure $\mu$ on $\partial\mathbb{D}$. Since $\beta \geqq 1$ by Brannan, Clunie and Kirwan's modification of Herglotz's theorem ([8], see e.g. [53], Theorem 2.20) $p$ has a representation

$$p^\beta(z) = \int\limits_{(\partial\mathbb{D})^2} \left(\frac{1+xz^m}{1-yz^m}\right)^\beta d\nu(x,y)\,,$$

where $\nu$ is a Borel probability measure on $(\partial\mathbb{D})^2$. Now by the argument given in [8] (see [20], Theorem 5.11), we deduce that there is a Borel probability measure $\lambda$ such that

$$f'(z) = \int\limits_{\partial\mathbb{D}} \frac{d\mu(w)}{(1-wz^m)^{2/m}} \cdot \int\limits_{(\partial\mathbb{D})^2} \left(\frac{1+xz^m}{1-yz^m}\right)^\beta d\nu(x,y) = \int\limits_{(\partial\mathbb{D})^2} \frac{(1+xz^m)^\beta}{(1-yz^m)^{\beta+2/m}} d\lambda(x,y).$$

So an extreme point is a kernel function. For $x = -y$ the kernel functions are convex, in particular starlike, but they are not extreme in the family of $m$-fold symmetric starlike functions (see [9], Theorem 3), which is a subset of $C_m(\beta)$ for $\beta \geqq 1$, so that they are not extreme in $\overline{\mathrm{co}}\, C_m(\beta)$. □

We remark that the method also applies to the family $V_m(K)$ for $K \geqq 2m+2$ using Theorem 3.8, i.e. the inclusion relation $V_K \subset C\Big((K/2-1)/m\Big)$ and the fact that the functions (3.29) lie in $V_m(K)$, so that we have

**Corollary 3.1** *Let $m \in \mathbb{N}$ and $K \geqq 2m+2$. Then*

$$\overline{\mathrm{co}}\, V_m(K) = \overline{\mathrm{co}}\, C_m\Big((K/2-1)/m\Big)$$

## 3.5 Coefficients of functions subordinate to close-to-convex functions

Brannan, Clunie and Kirwan proved the coefficient result for close-to-convex functions of order $\beta$. For $\beta = 1$ this is the result of the *Bieberbach conjecture* which finally had been proven by de Branges [6] and holds for all univalent functions $f \in S$. De Branges' theorem includes moreover the truth of the *Rogosinski conjecture* which states that the same coefficient result holds for all functions subordinate to some function $f \in S$.

Now we consider the similar problem for close-to-convex functions of order $\beta$.

**Theorem 3.12** (see [28]) *Let $\beta \geqq 0$, $g \prec f$ and $f \in C(\beta)$. Then*

$$g' << F' := \frac{(1+z)^\beta}{(1-z)^{\beta+2}}\,. \tag{3.30}$$

*Proof:* By hypotheses there are $\omega \in B$, $\varphi \in K$ and $p \in \widetilde{P}$ such that $g = f \circ \omega$ and $f' = \varphi' \cdot p^\beta$. This gives

$$g'(z) = f'\Big(\omega(z)\Big) \cdot \omega'(z) = \varphi'\Big(\omega(z)\Big) \cdot \omega'(z) \cdot p\Big(\omega(z)\Big)^\beta\,.$$

Now $\varphi'\Big(\omega(z)\Big) \cdot \omega'(z)$ is the derivative of some function subordinate to $\varphi \in K$, thus having a representation of the form $\int\limits_{(\partial\mathbb{D})^2} \frac{x d\mu(x,y)}{(1-yz)^2}$ for some Borel probability measure $\mu$ (see [9], Theorem 5.21). Further $q(z) := p\Big(\omega(z)\Big)$ lies in $\widetilde{P}$. So we have

$$g'(z) = \int\limits_{(\partial\mathbb{D})^2} \frac{x d\mu(x,y)}{(1-yz)^2} \cdot q^\beta(z)\,.$$

Now the same proof as in Theorem 3.5 (for $m = 1$) leads to the result since $|x| = 1$. □

We remark that this is the adequate form of a Rogosinski type conjecture for close-to-convex functions of order $\beta$. Furthermore the theorem shows that the functionals $\left|f^{(n)}(z)\right|$, $(n \in \mathbb{N}, z \in \mathbb{D})$ are maximized in $\operatorname{Sub} C(\beta)$ by the function $F$ given by (3.30).

Also there is a corresponding result for functions of bounded boundary rotation by (3.28).

**Corollary 3.2** *Let $K \geqq 2$, $g \prec f$ and $f \in V(K)$. Then*

$$g' << \frac{(1+z)^{\frac{K}{2}-1}}{(1-z)^{\frac{K}{2}+1}} .$$

The following is a distortion theorem for functions subordinate to odd close-to-convex functions.

**Theorem 3.13** (see [28]) *Let $\beta \geqq 0$ and $g \prec f \in C_2(\beta)$. Then*

$$|g'(z)| \leqq F_2'(|z|) = \frac{(1+|z|^2)^\beta}{(1-|z|^2)^{\beta+1}}$$

*and*

$$|g(z)| \leqq F_2(|z|) .$$

*Proof:* Let $g = f \circ \omega$, $\omega \in B$. Then $g'(z) = f'\big(\omega(z)\big) \cdot \omega'(z)$, and the elementary inequality $(1-|z|^2)\,|\omega'(z)| \leqq 1-|\omega(z)|^2$ (see e.g. [13], p. 918) together with Theorem 3.7 implies that

$$\begin{aligned} |g'(z)| &= \left|f'\big(\omega(z)\big)\right| |\omega'(z)| \leqq \left(\frac{1+|\omega(z)|^2}{1-|\omega(z)|^2}\right)^\beta \cdot \frac{|\omega'(z)|}{1-|\omega(z)|^2} \\ &\leqq \left(\frac{1+|\omega(z)|^2}{1-|\omega(z)|^2}\right)^\beta \cdot \frac{1}{1-|z|^2} . \end{aligned}$$

Now it follows from Schwarz's Lemma that

$$\frac{1+|\omega(z)|^2}{1-|\omega(z)|^2} \leqq \frac{1+|z|^2}{1-|z|^2} =: H\left(|z|^2\right) ,$$

because $H$ increases as $|z|$ increases, so that finally

$$|g'(z)| \leqq \frac{(1+|z|^2)^\beta}{(1-|z|^2)^{\beta+1}} = F_2'(|z|) .$$

The second statement follows as in the proof of Theorem 3.7. □

## 3.6 Extreme points of functions subordinate to close-to-convex functions

The following lemma is the essential tool to get the extreme points of $\overline{\mathrm{co}}\,\mathrm{Sub}\,C(\beta)$ from the extreme points of $\overline{\mathrm{co}}\,C(\beta)$.

An analytic function $f \in A$ is called a *BCK-function* if each function $g \prec f$ has a representation of the form $\int\limits_{\partial\mathbb{D}} f(xz)d\mu(x)$ for some probability measure $\mu$ on $\partial\mathbb{D}$.

The well-known examples of BCK-functions $\left(\frac{1+xz}{1-z}\right)^{\alpha}$ for $|x| \leqq 1$ and $\alpha \geqq 1$ are due to Brannan, Clunie and Kirwan (see [8]).

**Lemma 3.4** (see [28]) *Let $F \subset A$ be a compact family of analytic functions $f$ with $f(0) = 0$. If $\mathrm{E}\,\overline{\mathrm{co}}\,F$ consists of BCK-functions, then the extreme points of $\overline{\mathrm{co}}\,\mathrm{Sub}\,F$ have the form $g(wz)$ for some $g \in \mathrm{E}\,\overline{\mathrm{co}}\,F$ and $w \in \partial\mathbb{D}$.*

*Proof:* First we show that in the given situation an extreme point $f$ of $\overline{\mathrm{co}}\,\mathrm{Sub}\,F$ must be subordinate to some $g \in \mathrm{E}\,\overline{\mathrm{co}}\,F$ (see [20]).

Let $f \in \mathrm{E}\,\overline{\mathrm{co}}\,\mathrm{Sub}\,F$. Then by a general result of Milman (see e.g. [53], Appendix A) $f \in \mathrm{Sub}\,F$, because with $F$ automatically $\mathrm{Sub}\,F$ and $\overline{\mathrm{co}}\mathrm{Sub}\,F$ are compact (see e.g. [40], p. 365 - 366). So $f = g \circ \omega$ for some $g \in F$ and $\omega \in B$.

Suppose now $g \notin \mathrm{E}\,\overline{\mathrm{co}}\,F$, then there is a representation

$$g = tg_1 + (1-t)g_2,\ t \in\, ]0,1[,\ g_{1,2} \in \overline{\mathrm{co}}\,F,\ g_1 \neq g_2\,.$$

By the Krein-Milman theorem $g_{1,2} \in \overline{\mathrm{co}}\,(\mathrm{E}\,\overline{\mathrm{co}}\,F)$, so that

$$f_{1,2} := g_{1,2} \circ \omega \in \overline{\mathrm{co}}\left((\mathrm{E}\,\overline{\mathrm{co}}\,F) \circ \omega\right) \subset \overline{\mathrm{co}}\,(F \circ \omega) \subset \overline{\mathrm{co}}\,(\mathrm{Sub}\,F)\,.$$

So $f = tf_1 + (1-t)f_2$ is a proper convex representation of $f$ within $\overline{\mathrm{co}}(\mathrm{Sub}\,F)$ which contradicts the assumption.

So we have $f = g \circ \omega$ with $\omega \in B$, and $g \in \mathrm{E}\,\overline{\mathrm{co}}\,F$. Suppose now there is no $w \in \partial\mathbb{D}$ such that $\omega(z) = wz$, then, because by hypothesis $g$ is a BCK-function, it follows that $f$ has a proper convex representation in $\mathrm{Sub}\,\{g\}$ and so in $\mathrm{Sub}\,F$, which gives the result. □

As a consequence we have

**Theorem 3.14** (see [28]) *Let $\beta \geqq 1$, then an extreme point $f$ of $\overline{\mathrm{co}}\,\mathrm{Sub}C(\beta)$ has the form*

$$f(z) = \frac{w}{(\beta+1)(x+y)}\left(\left(\frac{1+xz}{1-yz}\right)^{\beta+1} - 1\right), \quad x, y, w \in \partial\mathbb{D},\ x \neq -y\,. \quad (3.31)$$

*Proof:* For $\beta \geqq 1$ it is known (see [20], Theorem 2.22) that an extreme point of $\overline{\mathrm{co}}\, C(\beta)$ has the form (3.31) with $w = 1$. Because $\left(\frac{1+xz}{1-yz}\right)^{\beta+1}$ are BCK-functions, so is $f$, and an application of Lemma 3.4 gives the result by an easy change of variables. □

We remark that the given argument implies the result also for $\beta < 1$ if the corresponding extreme point result for $C(\beta)$ is true.

# 4. Results about integral means

## 4.1 Integral means

For $f \in A$ and $r \in [0, 1[$ let

$$
\begin{aligned}
M_p(r, f) &:= \left( \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{1/p} \qquad (p \in\, ]0, \infty[)\,, \qquad (4.1)\\
M_\infty(r, f) &:= \max_{\theta \in [0, 2\pi]} \left| f(re^{i\theta}) \right|
\end{aligned}
$$

denote the $p$-th integral means. For $p \in\, ]0, \infty]$ let $H^p$ denote the family of functions $f$ for which $M_p(r, f)$ remains bounded as $r \to 1$.

$M_p(r, f)$ turns out to be a nondecreasing function of $r$ and also nondecreasing as function of $p$. For $f \in H^p$ the radial limit

$$f(e^{i\theta}) := \lim_{r \to 1} f(re^{i\theta})$$

turns out to exist for almost all $\theta \in [0, 2\pi]$ and is in $L^p([0, 2\pi])$, and

$$M_p(1, f) := \left( \frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^p d\theta \right)^{1/p} = \lim_{r \to 1} M_p(r, f)\,. \qquad (4.2)$$

The *Littlewood subordination theorem* states that $f \prec F$ implies that $M_p(r, f) \leqq M_p(r, F)$ for all $p \in\, ]0, \infty]$ and all $r \in [0, 1]$.

If the derivative $f'$ of some function $f \in A$ is in $H^p$ for some $p \in\, ]0, \infty]$, then so is $f$, i.e.

$$f' \in H^p \quad \Longrightarrow \quad \begin{cases} f \in H^\infty & \text{if } p \geqq 1 \\ f \in H^{\frac{p}{1-p}} & \text{otherwise} \end{cases}\,. \qquad (4.3)$$

Moreover if $f \in S$ maps $\mathbb{D}$ onto some bounded Jordan domain, then

$$f' \in H^1 \quad \Longleftrightarrow \quad \partial f(\mathbb{D}) \text{ is rectifiable}\,.$$

For functions $f$ which are in $H^p$ for some $p \in \,]0,\infty]$ we define the *Hardy-dimension of $f$* by

$$\dim_{H^p}(f) := \sup\{p \in \,]0,\infty] \mid f \in H^p\} \ .$$

(References: [39], [12].)

## 4.2 Polygons

If $f$ is a polygonal mapping normalized by (1.1), then by the Schwarz-Christoffel formula (1.16) one has

$$f'(z) = \prod_{k=1}^{n} \frac{1}{(1-\overline{x_k}z)^{2\mu_k}} \ , \quad x_k \in \partial\mathbb{D} \quad (k=1,\ldots,n) \ , \quad \sum_{k=1}^{n} \mu_k = 1 \ . \tag{4.4}$$

From this representation one can see at once that $f' \in H^p$ for some $p > 0$ (namely for all $p < 1/2$, see e.g. [20], p. 80), so that $f'(e^{i\theta})$ exists for almost all $\theta \in [0, 2\pi]$ and

$$\lim_{r\to 1} \int_0^{2\pi} \left| f'(re^{i\theta}) \right|^p d\theta = \int_0^{2\pi} \left| f'(e^{i\theta}) \right|^p d\theta \ . \tag{4.5}$$

For to get a sharp $H^p$-result for a polygonal mapping $f$ depending only on the parameters of the Schwarz-Christoffel formula, hence on the geometry of the image surface of $f$, we assume without loss of generality that $\mu_k > 0$ $(k = 1,\ldots,m)$ and $\nu_k := -\mu_{k+m} > 0$ $(k = 1,\ldots,n-m)$ and write $y_k := x_{k+m}$ $(k = 1,\ldots,n-m)$. Then

$$f'(z) = \prod_{k=1}^{n} \frac{1}{(1-\overline{x_k}z)^{2\mu_k}} = \frac{\prod\limits_{k=1}^{n-m} (1-\overline{y_k}z)^{2\nu_k}}{\prod\limits_{k=1}^{m} (1-\overline{x_k}z)^{2\mu_k}} \ ,$$

and so

$$|f'(z)| \leqq \frac{\prod\limits_{k=1}^{n-m} 2^{2\nu_k}}{\prod\limits_{k=1}^{m} |1-\overline{x_k}z|^{2\mu_k}} \ .$$

As $\sum\limits_{k=1}^{n-m} \nu_k < \sum\limits_{k=1}^{n} |\mu_k| =: \frac{K}{2}$ we get

$$|f'(z)| < 2^K \frac{1}{\prod\limits_{k=1}^{m} |1 - \overline{x_k} z|^{2\mu_k}} . \tag{4.6}$$

By (4.5) we have to check the finiteness of

$$\int_0^{2\pi} \frac{d\theta}{\prod\limits_{k=1}^{m} |1 - \overline{x_k} e^{i\theta}|^{2\mu_k p}} . \tag{4.7}$$

Therefore suppose without loss of generality that $x_k$ $(k = 1, \ldots, m)$ are ordered successively on $\partial\mathbb{D}$ and define $(x_{m+1} := x_1)$

$$d := \min \{\operatorname{dist}(x_k, x_{k+1}) \mid k = 1, \ldots, m\} . \tag{4.8}$$

Clearly $d > 0$ as the points $x_k$ $(k = 1, \ldots, m)$ are isolated. (On the other hand the value of $d$ depends heavily on $n$ and for all sequences $(x_k)_{k \in \mathbb{N}}$ of unimodular numbers $d \to 0$ as $n \to \infty$.) Now we decompose the integral (4.7) in $m$ components. Choose $t_k := \frac{1}{2}(\arg(x_k) + \arg(x_{k-1}))$ $(k = 1, \ldots, m)$ and observe that

$$\left|1 - \overline{x_k} e^{i\theta}\right| = \left|e^{i\theta} - x_k\right| > \frac{d}{2} \qquad (k = 1, \ldots, m) \tag{4.9}$$

for $\theta \notin [t_{k-1}, t_k]$, $(t_{m+1} := t_1)$. Now it follows for $j = 1, \ldots, m$ that

$$\int_{t_j}^{t_{j+1}} \frac{d\theta}{\prod\limits_{k=1}^{m} |1 - \overline{x_k} e^{i\theta}|^{2\mu_k p}} < \left(\frac{2}{d}\right)^{Kp} \cdot \int_{t_j}^{t_{j+1}} \frac{d\theta}{|1 - \overline{x_j} e^{i\theta}|^{2\mu_j p}} \leqq \left(\frac{2}{d}\right)^{Kp} \cdot \int_0^{2\pi} \frac{d\theta}{|1 - \overline{x_j} e^{i\theta}|^{2\mu_j p}},$$

which is finite if and only if $p < \frac{1}{2\mu_j}$. So (4.7) is finite iff $p < \frac{1}{2\mu_{\max}^+}$, where $\mu_{\max}^+ = \max \{\mu_k \mid k = 1, \ldots, n\}$. This gives

**Lemma 4.1** *Let $f$ be a Schwarz-Christoffel mapping. Then $f' \in H^p$ for all $p < \frac{1}{2\mu_{\max}}$, and this bound is sharp, i.e.*

$$\dim_{H^p}(f') = \frac{1}{2\mu_{\max}^+} .$$

Analogously one gets for $1/f'$

**Lemma 4.2** *Let $f$ be a Schwarz-Christoffel mapping. Then $1/f' \in H^p$ for all $p < \frac{1}{\alpha_{\max}-1}$, and this bound is sharp, i.e.*

$$\dim_{H^p}(1/f') = \frac{1}{\alpha_{\max} - 1} .$$

*Proof:* The same procedure as above shows that $1/f' \in H^p$ for all $p < \frac{1}{2\nu_{\max}}$ where $\nu_{\max} := \max\{\nu_k \mid k = 1, \ldots, n-m\}$. By (1.17) it follows that $2\nu_{\max} = -(1 - \alpha_{\max})$. □

## 4.3 Functions of bounded boundary rotation

For functions of bounded boundary rotation $K\pi$ we have the usual representation (2.7)

$$\frac{f''}{f'}(z) = -2 \int\limits_{\partial \mathbb{D}} \frac{d\mu(x)}{z - x}$$

for some signed measure $\mu$ with Lebesgue decomposition $\mu = \mu_{\text{disc}} + \mu_{\text{cont}}$. Then $\mu_{\text{disc}} = \sum\limits_{k=1}^{\infty} \mu_k \delta_{x_k}$ for $x_k \in \partial\mathbb{D}$ $(k \in \mathbb{N})$ and $\sum\limits_{k=1}^{\infty} |\mu_k| \leqq \frac{K}{2}$. Let now $\varepsilon > 0$ be given and choose $m \in \mathbb{N}$ large enough that

$$\sum_{k=m+1}^{\infty} |\mu_k| \leqq \varepsilon \tag{4.10}$$

and that the maximal value $\mu_{\max} = |\mu_{k_0}|$ is attained for $k_0 \leqq m$. We write $y_k := x_k$, $(k > m)$ and get

$$\frac{f''}{f'}(z) = -2 \sum_{k=1}^{m} \frac{\mu_k}{z - x_k} - 2 \sum_{k=m+1}^{\infty} \frac{\mu_k}{z - y_k} - 2 \int\limits_{\partial\mathbb{D}} \frac{d\mu_{\text{cont}}(x)}{z - x} .$$

For the last expression we write

$$-2 \int\limits_{\partial\mathbb{D}} \frac{d\mu_{\text{cont}}(x)}{z - x} =: \frac{k''}{k'}(z) , \tag{4.11}$$

so that an integration gives (without loss of generality $f$ is always assumed to be normalized by (1.1)),

$$f'(z) = \prod_{k=1}^{m} \frac{1}{(1 - \overline{x_k} z)^{2\mu_k}} \cdot \prod_{k=m+1}^{\infty} \frac{1}{(1 - \overline{y_k} z)^{2\mu_k}} \cdot k'(z) .$$

Now we go on as in the case of polygonal functions. Suppose without loss of generality that $x_k$ $(k = 1, \dots, m)$ are ordered successively on $\partial\mathbb{D}$ and define (now set $x_{m+1} := x_1$) $d > 0$ by (4.8). Choose $t_k := \frac{1}{2}(\arg(x_k) + \arg(x_{k-1}))$ $(k = 1, \dots, m)$ so that (4.9) holds for $\theta \notin [t_{k-1}, t_k]$, $(t_{m+1} := t_1)$.

Suppose now that $k' = 1$. Then it follows for $j = 1, \dots, m$ that

$$\begin{aligned}
\int_{t_j}^{t_{j+1}} \left|f'(e^{i\theta})\right|^p d\theta &= \int_{t_j}^{t_{j+1}} \frac{d\theta}{\prod\limits_{k=1}^{m} |1 - \overline{x_k}e^{i\theta}|^{2\mu_k p} \cdot \prod\limits_{k=m+1}^{\infty} |1 - \overline{y_k}e^{i\theta}|^{2\mu_k p}} \\
&< \left(\frac{2}{d}\right)^{Kp} \cdot \int_{t_j}^{t_{j+1}} \frac{d\theta}{\left|1 - \overline{x_j}e^{i\theta}\right|^{2\mu_j p} \cdot \prod\limits_{k=m+1}^{\infty} |1 - \overline{y_k}e^{i\theta}|^{2\mu_k p}} \\
&\leqq \left(\frac{2}{d}\right)^{Kp} \cdot \int_{0}^{2\pi} \frac{d\theta}{\left|1 - \overline{x_j}e^{i\theta}\right|^{2\mu_j p} \cdot \prod\limits_{k=m+1}^{\infty} |1 - \overline{y_k}e^{i\theta}|^{2\mu_k p}} \\
&\leqq \left(\frac{2}{d}\right)^{Kp} \cdot \int_{0}^{2\pi} \frac{d\theta}{|1 - \overline{x_j}e^{i\theta}|^{2p(\mu_j + \varepsilon)}}
\end{aligned}$$

by (4.10) (for the last step see also [20], p. 80) which is finite if and only if $p < \frac{1}{2(\mu_j + \varepsilon)}$. As $\varepsilon$ was arbitrary we see that $f' \in H^p$ for all $p < \frac{1}{2\mu^+_{\max}}$ where $\mu^+_{\max} := \max\limits_{k \in \mathbb{N}} \mu_k$ as in the polygonal case. This gives

**Theorem 4.1** *Let $f \in V(K)$ with $f(\mathbb{D}) = F$ such that $\partial F$ is linear except of a countable number of vertices $w_k = f(x_k)$ of outer angle $2\mu_k\pi$ $(k \in \mathbb{N})$. Then*

$$\dim_{H^p}(f') = \frac{1}{2\mu^+_{\max}} \tag{4.12}$$

*and*

$$\dim_{H^p}(1/f') = \frac{1}{\alpha_{\max} - 1}\,. \tag{4.13}$$

The result given here holds also if the function $k'$ defined by (4.11) is bounded in $\mathbb{D}$. We conjecture that (4.12) - (4.13) hold for all functions of bounded boundary rotation. Theorem 4.1 should be compared with results of Warschawski and Schober who showed the validity of (4.12) and (4.13) firstly for bounded univalent functions of bounded boundary rotation whose boundary curves $\partial f(\mathbb{D})$ are furthermore of bounded arc length-chord length ratio and secondly for bounded univalent functions whose ranges have only a finite number of vertices and for which some further technical conditions hold ([56], Theorems 2 and 3). We remark that our result does not at all depend on boundedness or univalence.

## 4.4 Convex functions with vanishing second coefficient

For convex functions the results of the last section apply. Moreover we get for functions with vanishing second coefficient

**Theorem 4.2** *Let $m \in \mathbb{N}$ and $f_m \in K$ of form* (2.39). *Then $f_m' \in H^p$ for all $p < \frac{m}{2}$. This result is sharp for the convex function $f$ with $f'(z) = \frac{1}{(1-z^m)^{2/m}}$.*

*Proof:* By (2.41) in the given situation $f_m'(z) \prec \frac{1}{(1-z)^{2/m}} =: F'(z)$, so that the result follows by the Littlewood subordination theorem as $\frac{1}{(1-z)^\alpha} \in H^p$ for all $p < \frac{1}{\alpha}$.

For $f'(z) = \frac{1}{(1-z^m)^{2/m}} = F'(z^m)$ we have

$$\int_0^{2\pi} \left|f'(re^{i\theta})\right|^p d\theta = \int_0^{2\pi} \left|F'(re^{im\theta})\right|^p d\theta = \int_0^{2\pi} \left|F'(re^{i\theta})\right|^p d\theta$$

where the last equation follows by the substitution $\theta \to m\theta$ and from the periodicity of the exponential function, so that the result is sharp. □

As a corollary we have a generalization of Theorem 2.8 (c).

**Corollary 4.1** *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in K$ with $a_2 = a_3 = 0$. Then $f' \in H^1$ and $f(\mathbb{D})$ has a rectifiable boundary.*

*Proof:* The theorem shows that $f' \in H^1$. As $f$ is bounded by Theorem 2.8 (c) (or by (4.3)) and $f(\mathbb{D})$ therefore is a Jordan domain, we get the conclusion. □

We remark that the theorem is a special case of our conjecture as functions of the given form satisfy $2\mu_{max}^+ \leqq \frac{2}{m}$ (see Theorem 2.8 (a)).

## 4.5 Close-to-convex functions

Brown [10] showed that for $f \in C(\beta)$ one has $M_p(r, f') \leqq M_p(r, F')$ and $M_p(r, 1/f') \leqq M_p(r, 1/F')$ for all $p \in ]0, \infty]$ where $F$ is the generalized Koebe function (3.3). We modify this to $m$-fold symmetric functions.

**Theorem 4.3** (see [28]) *Let $\beta \geqq 0$, $m \in \mathbb{N}$ and $f \in C_m(\beta)$. Then*

$$M_p(r, f') \leqq M_p(r, F') \tag{4.14}$$

*and*

$$M_p(r, 1/f') \leqq M_p(r, 1/F') \tag{4.15}$$

*for all $p \in ]0, \infty]$ and $r \in ]0, 1[$ where*

$$F'(z) = \frac{(1 + z^m)^\beta}{(1 - z^m)^{\beta + 2/m}}\,, \quad F(0) = 0\,. \tag{4.16}$$

*In particular: $f' \in H^p$ for all $p < \frac{1}{\beta + 2/m}$, i.e.*

$$\dim_{H^p}(f') \geqq \frac{1}{\beta + 2/m}\,.$$

*Proof:* Let $f \in C_m(\beta)$. Then the function $g$ defined by $f'(z) = \big(g'(z^m)\big)^{1/m}$ is close-to-convex of order $m\beta$ by Lemma 3.3 (b). Therefore the result of Brown implies that ($z = re^{i\theta}$)

$$\int_0^{2\pi} |f'(z)|^p \, d\theta = \int_0^{2\pi} |g'(z^m)|^{\frac{p}{m}} \, d\theta \leqq \int_0^{2\pi} \left| \frac{(1 + z^m)^{m\beta}}{(1 - z^m)^{m\beta + 2}} \right|^{\frac{p}{m}} d\theta = \int_0^{2\pi} |F'(z)|^p \, d\theta,$$

where $F$ is defined by (4.16), which shows (4.14).

The same procedure gives (4.15). □

We remark that the result for $\beta = 1$ seems to be new even for starlike functions. The Hardy-dimension for $1/f'$ which follows from (4.15) does not depend on $m$ and is so the same as for $m = 1$.

From Theorem 4.3 it follows

**Corollary 4.2** (see [28]) *Let $\beta < 1$, $m > \frac{2}{1-\beta}$ and $f \in C_m(\beta)$. Then $f' \in H^1$ and $f(\mathbb{D})$ has a rectifiable boundary.*

*Proof:* By the theorem in the given situation $f' \in H^1$ and in particular $f \in H^\infty$, so $f$ is bounded. (This can be proved also purely geometrically: with a boundary point the function omits a sector of angle at least $(1-\beta)\pi$, and because of the symmetry there are at least $m$ symmetric sectors of the same angle omitted. If the total angles exceed $2\pi$, then obviously $f(\mathbb{D})$ is bounded.) By a result of Pommerenke ([48], Theorem 2(a)) close-to-convex functions of order $\beta$ with $\beta < 1$ have a continuous extension to $\overline{\mathbb{D}}$, so that $f(\mathbb{D})$ is a bounded Jordan domain. From $f' \in H^1$ it follows then that $\partial f(\mathbb{D})$ is rectifiable. □

On the other hand the theorem implies

**Corollary 4.3** *Let $\beta \geqq 0$, $m \in \mathbb{N}$ and if $\beta < 1$ then $m \leqq \frac{2}{1-\beta}$. Then for $f \in C_m(\beta)$ we have*

$$\dim_{HP}(f) \geqq \frac{1}{\beta + 2/m - 1} \,.$$

At the end of this section we give a sufficient condition for quasiconformal extension.

**Theorem 4.4** (see [28]) *Let $\beta < 1$, $m > \frac{4}{1-\beta}$ and $f \in C_m(\beta)$. Then $f$ has a quasiconformal extension to $\mathbb{C}$.*

*Proof:* As $f \in C_m(\beta)$, there is a representation $f'(z) = \frac{h(z)}{z} \cdot p^\beta(z)$ with an $m$-fold symmetric function $h \in St_m$. From representation (3.25) one gets that

$$\left|\frac{h(z)}{z}\right| \leqq \frac{1}{(1-|z|^m)^{2/m}} \,,$$

so that

$$\limsup_{r\to 1} \frac{\ln\left(\max\limits_{|z|=r}|h(z)|\right)}{\ln\frac{1}{1-r}} \leqq \frac{2}{m} < \frac{1-\beta}{2}$$

whenever $m > \frac{4}{1-\beta}$, and the result follows from a general condition on quasiconformal extensibility for Bazilevič functions due to Gall [15]. □

## 4.6 Weakly linearly accessible domains

A domain $F$ is called *weakly (angularly) accessible of order $\beta$* ($\beta \in [0,1]$) if it is the complement of the union of rays, such that every ray is the bisector of a sector of angle $(1-\beta)\pi$ which wholly lies in the complement of $F$.

Clearly this is weaker than the (strong) accessibility of order $\beta$ as we do not suppose that the rays are pairwise disjoint.

It turns out that a $H^p$ result for $f$ (rather that for $f'$) is available from this geometrical description, which depends on the following lemma.

**Lemma 4.3** (see [24]) *Let $\beta \in [0,1]$ and $F$ be weakly accessible of order $\beta$. Then for each $w_0 \in \partial F$ there is some sector $S_{w_0}$ of angle $(1+\beta)\pi$ with vertex in $w_0$ such that $F \subset S_{w_0}$.*

*Proof:* Let $w_0$ be an arbitrary boundary point of $F$. By hypothesis $\mathbb{C} \setminus F$ is the union of sectors of angle $(1-\beta)\pi$, and so $w_0$ lies in one of them. By a parallel motion we find a sector lying in $\mathbb{C} \setminus F$ with vertex $w_0$ whose complement $S_{w_0}$ is the sector searched for. □

From this it follows

**Theorem 4.5** *Let $\beta \in [0,1]$, $m \in \mathbb{N}$ and if $\beta < 1$ then $m \leqq \frac{2}{1-\beta}$. Then for an $m$-fold symmetric weakly accessible function $f$ of order $\beta$ we have*

$$\dim_{H^p}(f) \geqq \frac{1}{\beta + 2/m - 1}\,.$$

*Proof:* From the lemma it follows that $f(\mathbb{D})$ lies in some sector of angle $(1+\beta)\pi$ with vertex at some boundary point $w_0 \in \partial f(\mathbb{D})$. From the $m$-fold symmetry it follows that the same holds $m$-fold symmetrically, so that $f(\mathbb{D}) \subset a \cdot F(\mathbb{D})$ for some $a \in \mathbb{C}$ where $F'(z) = \frac{(1+z^m)^\beta}{(1-z^m)^{\beta+2/m}}$, $F(0) = 0$, and so $f \prec a \cdot F$ by the subordination principle. The Littlewood subordination theorem then implies the result. □

The proof shows that the result is implied by the geometry of $f(\mathbb{D})$. For close-to-convex functions the statement was deduced from the corresponding result for the derivative in Corollary 4.3.

# 5. Functions with positive real part

## 5.1 Uniqueness statements

It is an easy consequence of Schwarz' Lemma that $p \in P$ implies $|p_1| \leqq 2$ with equality iff $p(z) = \frac{1+xz}{1-xz}$ $(x \in \partial\mathbb{D})$. This includes the uniqueness statement that

$$p \in P,\ p_1 = 2x \quad (x \in \partial\mathbb{D}) \quad \Longrightarrow \quad p(z) = \frac{1+xz}{1-xz}\,.$$

We shall now give a generalization of this statement.

**Theorem 5.1** *Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$ and $n \in \mathbb{N}$. Suppose that for all $j = 1, \ldots, n$ holds*

$$p_j = 2\sum_{k=1}^{n} t_k x_k^j, \quad \sum_{k=1}^{n} t_k = 1, \quad t_k > 0,\ x_k \in \partial\mathbb{D} \quad (k = 1, \ldots, n), \tag{5.1}$$

*then* (5.1) *holds for all $j \in \mathbb{N}$, i.e.*

$$p(z) = \sum_{k=1}^{n} t_k \left(\frac{1+x_k z}{1-x_k z}\right)\,.$$

*Proof:* The proof is an easy consequence of the Carathéodory-Toeplitz-Fejér theory on positive harmonic functions. Observe that

$$D_n := \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ \overline{p_1} & 2 & p_1 & \cdots & p_{n-1} \\ \overline{p_2} & \overline{p_1} & 2 & \cdots & p_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{p_n} & \overline{p_{n-1}} & \overline{p_{n-2}} & \cdots & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2\sum_{k=1}^{n} t_k & 2\sum_{k=1}^{n} t_k x_k & 2\sum_{k=1}^{n} t_k x_k^2 & \cdots & 2\sum_{k=1}^{n} t_k x_k^n \\ 2\sum_{k=1}^{n} t_k \overline{x_k} & 2\sum_{k=1}^{n} t_k & 2\sum_{k=1}^{n} t_k x_k & \cdots & 2\sum_{k=1}^{n} t_k x_k^{n-1} \\ 2\sum_{k=1}^{n} t_k \overline{x_k^2} & 2\sum_{k=1}^{n} t_k \overline{x_k} & 2\sum_{k=1}^{n} t_k & \cdots & 2\sum_{k=1}^{n} t_k x_k^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2\sum_{k=1}^{n} t_k \overline{x_k^n} & 2\sum_{k=1}^{n} t_k \overline{x_k^{n-1}} & 2\sum_{k=1}^{n} t_k \overline{x_k^{n-2}} & \cdots & 2\sum_{k=1}^{n} t_k \end{vmatrix}$$

$$= \left(2\sum_{k=1}^{n} t_k\right)^{n+1} \cdot \begin{vmatrix} 1 & x_k & x_k^2 & \cdots & x_k^n \\ \overline{x_k} & 1 & x_k & \cdots & x_k^{n-1} \\ \overline{x_k^2} & \overline{x_k} & 1 & \cdots & x_k^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{x_k^n} & \overline{x_k^{n-1}} & \overline{x_k^{n-2}} & \cdots & 1 \end{vmatrix} = 0 ,$$

as the last determinant vanishes for all $k = 1, \ldots, n$ which is easily seen by induction. So by [11], Theorem VI, it follows that $D_j = 0$ for $j > n$, which establishes the result. □

In particular we have

**Corollary 5.1** *Let* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$ *with* $p_1/2 \in \mathbb{D}$ *arbitrarily. Let furthermore* $t \in [0,1]$ *and* $x, y \in \partial\mathbb{D}$ *such that* $p_1 = 2\big(tx + (1-t)y\big)$ *(in fact for each* $t \in ]0,1[$ *such a representation exists).*
*If now* $p_2$ *has a representation* $p_2 = 2\big(tx^2 + (1-t)y^2\big)$*, then* $p$ *is uniquely determined and*

$$p(z) = t\left(\frac{1+xz}{1-xz}\right) + (1-t)\left(\frac{1+yz}{1-yz}\right) . \tag{5.2}$$

The functions of form (5.2) are the extremals also for the next problem.

**Lemma 5.1** *Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$. Then*

$$\left| p_2 - \frac{1}{2} p_1^2 \right| \leqq 2 - \frac{1}{2} |p_1|^2 \tag{5.3}$$

*with equality if and only if $p$ is of form* (5.2) *for some $x, y \in \partial\mathbb{D}$ and $t \in [0,1]$.*

*Proof:* For $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$ we have $\omega(z) := \frac{1}{z} \cdot \frac{p(z)-1}{p(z)+1} = \omega_0 + \omega_1 z + \omega_2 z^2 + \cdots \in B$, and it follows that $|\omega_1| \leqq 1 - |\omega_0|^2$ with equality if and only if $\omega(z) = w^2 \frac{z+a}{1+\overline{a}z}$ for some $w \in \partial\mathbb{D}$ and $a \in \mathbb{D}$ (see e.g. [17], Kapitel VIII, Satz 2). This inequality is equivalent to (5.3). If equality occurs in (5.3), then

$$p(z) = \frac{1 + z\omega(z)}{1 - z\omega(z)} = \frac{1 + w^2 z \frac{z+a}{1+\overline{a}z}}{1 - w^2 z \frac{z+a}{1+\overline{a}z}} = \frac{1 + z(\overline{a} + w^2 a) + w^2 z^2}{1 + z(\overline{a} - w^2 a) - w^2 z^2}$$

$$= \frac{1 + wz(2 \operatorname{Re} aw) + w^2 z^2}{1 - wz(2i \operatorname{Im} aw) - w^2 z^2},$$

and so by writing $b := aw$ it follows

$$p(\overline{w}z) = \frac{1 + z(b + \overline{b}) + z^2}{1 - z(b - \overline{b}) - z^2}. \tag{5.4}$$

Obviously there is no loss of generality to show the result for the rotated function $p(\overline{w}z)$.

Observe that the zeros of the denominator of the right hand fraction in (5.4) (as well as the zeros of its numerator) have unit modulus. So we have the partial fraction decomposition

$$\begin{aligned} p(\overline{w}z) &= \frac{1 + z(b + \overline{b}) + z^2}{1 - z(b - \overline{b}) - z^2} = -1 + \frac{A}{1 - xz} + \frac{B}{1 - yz} \\ &= \frac{-1 + z(b - \overline{b}) + z^2 + A(1 - yz) + B(1 - xz)}{1 - z(b - \overline{b}) - z^2} \end{aligned} \tag{5.5}$$

for some $x, y \in \partial\mathbb{D}$ with $(1 - z(b - \overline{b}) - z^2) = (1 - xz)(1 - yz)$. Equating the coefficients of denominators and numerators leeds to the equations

$$xy = -1, \tag{5.6}$$

$$x + y = b - \overline{b}\,, \tag{5.7}$$

$$A + B = 2\,, \tag{5.8}$$

$$yA + xB = -2\overline{b}\,. \tag{5.9}$$

From these equations we conclude (using that $y \in \partial\mathbb{D}$) that

$$yA \overset{(5.9)}{=\!=} -2\overline{b} - xB \overset{(5.6),(5.8)}{=\!=} -2\overline{b} + \overline{y}(2 - A)\,,$$

and so

$$A = \frac{2(\overline{y} - \overline{b})}{y + \overline{y}}\,,$$

and by (5.8) it follows further that

$$B = \frac{2(\overline{b} + y)}{y + \overline{y}}\,.$$

As $\operatorname{Im} y = \operatorname{Im} b$ by (5.6) and (5.7), and as $\operatorname{Re} b < \operatorname{Re} y = \sqrt{1 - (\operatorname{Im} b)^2}$ we get finally that

$$A = 1 - \frac{\operatorname{Re} b}{\operatorname{Re} y}$$

and

$$B = 1 + \frac{\operatorname{Re} b}{\operatorname{Re} y}$$

are nonnegative real numbers whose sum is 2. Setting now $t := \frac{A}{2} \in [0, 1]$, we get from (5.5)

$$p(\overline{w}z) = t\left(\frac{1 + xz}{1 - xz}\right) + (1 - t)\left(\frac{1 + yz}{1 - yz}\right)$$

as desired. On the other hand a calculation shows that the functions of form (5.2) with $x, y \in \partial\mathbb{D}$ and $t \in [0, 1]$ give actually equality in (5.3). □

We remark that Corollary 5.1 also follows from Lemma 5.1.

## 5.2 The coefficients of the logarithmic derivative and an application

In the Introduction we gave a dense subset of $\widetilde{P}$. As an application of the solution of the coefficient problem for the logarithmic derivative we get a family of inequalities for sets of consecutive points on the unit circle.

**Theorem 5.2** *Let $p \in \widetilde{P}$ and $z\frac{p'}{p}(z) = \sum\limits_{j=1}^{\infty} \gamma_j z^j$ . Then for all $m \in \mathbb{N}$ we have $|\gamma_m| \leqq 2m$, and this is sharp as $p(z) = \frac{1+xz^m}{1-xz^m}$ for $x \in \partial\mathbb{D}$ shows.*

*Proof:* Let $p \in \widetilde{P}$. Then there is a number $x \in \partial\mathbb{D}$ such that $p \prec \frac{1+xz}{1-z}$, so that $\ln p \prec \ln(1+xz) + (-\ln(1-z))$. The last function on the right hand side has the expansion

$$G(z) := -\ln(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k} ,$$

so that for each $g \prec G$ and each $m \in \mathbb{N}$ holds

$$|a_m(g)| \leqq a_1(G) = 1 , \tag{5.10}$$

as the coefficients $a_m(G)$ form a decreasing and convex sequence of positive real numbers (see e.g. [39], Theorem 216). For $f(z) \prec F(z) := \ln(1+xz)$ it also follows that

$$|a_m(f)| \leqq a_1(G) = 1 \tag{5.11}$$

as $F(z) = -G(-xz)$ and so we have (with some $\omega \in B$)

$$|a_m(\ln p)| = |a_m(F \circ \omega) + a_m(G \circ \omega)| \leqq |a_m(f)| + |a_m(g)| \leqq 2 ,$$

implying the result. For the function $p(z) = \frac{1+xz^m}{1-xz^m}$ equality holds as is easily verified, which finishes the proof. □

Applying the theorem to the dense subset of $\widetilde{P}$ of Lemma 1.3 leads to

**Corollary 5.2** *Let $n \in \mathbb{N}$ be given and $x_k, y_k \in \partial\mathbb{D}$ $(k = 1, \ldots, n)$ have the property*

$$\arg x_1 < \arg y_1 < \arg x_2 < \arg y_2 < \cdots < \arg x_n < \arg y_n < \arg x_1 + 2\pi ,$$

*then for all $m \in \mathbb{N}$*

$$\left| \sum_{k=1}^{n} (x_k^m - y_k^m) \right| \leqq 2m .$$

*Equality occurs for given $m \in \mathbb{N}$ if $n = m$, $x_k = e^{2\pi i k/m} x_0$ and $y_k = e^{\pi i/m} x_k$ $(k = 1, \ldots, m)$ for some $x_0 \in \partial\mathbb{D}$.*

We remark that for $m = 1$ the Corollary is a statement about the sum of the lengths of the vectors $x_k - y_k$, which can be proven also by geometrical means. In this sense Corollary 5.2 is a geometrical statement.

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