Complex Variables, 1988, Vol. 10, pp. 349–357
Reprints available directly from the publisher
Photocopying permitted by license only

Printed in United States of America

Extremal Problems for Close-to-Convex Functions

WOLFRAM KOEPF

Freie Universität Berlin, I. Mathematisches Institut, Arnimallee 3, 1000 Berlin 33, Fed. Rep. Germany

Let $C(\beta)$, $\beta \geqslant 0$, denote the family of normalized close-to-convex functions of order β. For $\beta = 1$ this is the usual set of close-to-convex functions, which had been defined by Kaplan.

We study the family Sub $C(\beta)$ of functions which are subordinate to close-to-convex functions of order β. For $\beta \geqslant 1$ it is shown that the extreme points of the closed convex hull of Sub $C(\beta)$ are of the form

$$f(z) = \frac{w}{(\beta+1)(x+y)}\left(\left(\frac{1+xz}{1-yz}\right)^{\beta+1} - 1\right), \qquad |x| = |y| = |w| = 1, \quad x \neq -y.$$

Further for all $\beta \geqslant 0$ the coefficient problem is solved. Also for the family $C_m(\beta)$ of m-fold symmetric close-to-convex functions of order β an extreme point result is given, if $\beta \geqslant 1$. For all $\beta \geqslant 0$ and arbitrary $p \in \mathbb{R}$, the pth integral means of the derivatives are shown to be maximized by the function f with

$$f'(z) = \frac{(1+z^m)^\beta}{(1-z^m)^{\beta+2/m}}, \qquad f(0) = 0.$$

This shows in particular that f has a rectifiable boundary curve if $m > 2/(1-\beta)$. On the other hand it is shown that if $m > 4/(1-\beta)$ then f has furthermore a quasiconformal extension.

AMS No. 30C45, 30C50, 30C75, 30C80
Communicated: R. P. Gilbert
(Received August 22, 1986)

1. INTRODUCTION

Let S denote the family of univalent functions f of the unit disk $\mathbb{D}$, normalized by

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

We consider S as a subset of the family A of analytic functions of $\mathbb{D}$ endowed with the topology of locally uniform convergence. Let K denote the family of convex functions, i.e. univalent functions that have a convex range. A function f, normalized by (1), is called close-to-convex of order β, $\beta \geqslant 0$, if there is a convex function φ so that $f'/\varphi' = p^{\beta}$ for some function $p \in \tilde{P} := \{p \mid p(0) = 1, \exists\delta: \operatorname{Re}(e^{i\delta}p) > 0\}$. Let $C(\beta)$ denote the family of close-to-convex functions of order β. For $\beta \leqslant 1$ it turns out that a function is close-to-convex of order β, if and only if it maps $\mathbb{D}$ univalently onto a domain whose complement E is the union of rays, which are pairwise disjoint up to their tips, so that every ray is the bisector of a sector of angle $(1-\beta)\pi$ which wholly lies in E (see [13, p. 176]). Obviously $C(0)$ equals K.

Let B denote the family of analytic functions which fulfill the hypotheses of Schwarz's Lemma. We call g subordinate to f (denoted by $g \prec f$) if there is a function $\omega \in B$ such that $g = f \circ \omega$. If f is univalent, $g \prec f$ is equivalent to the statement that $g(\mathbb{D}) \subset f(\mathbb{D})$. Furthermore Schwarz's Lemma implies that $g(\{z \mid |z| < r\}) \subset f(\{z \mid |z| < r\})$. Let $\operatorname{Sub} C(\beta)$ denote the family of functions that are subordinate to some close-to-convex function of order β. In §2 for $\beta \geqslant 1$ the extreme points of the closed convex hull of $\operatorname{Sub} C(\beta)$—which is denoted by $\operatorname{co} \operatorname{Sub} C(\beta)$—are characterized, and for all $\beta \geqslant 0$ the coefficient problem is solved. Recall that an extreme point of a family is a function which does not have a proper convex representation within the family.

A function f is called m-fold symmetric ($m \in \mathbb{N}$) if it has the special form $f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \cdots$. By $C_m(\beta)$ we denote the family of m-fold symmetric close-to-convex functions of order β. In [12] we solved the coefficient problem when $\beta \geqslant 1 - 1/m$. In §3 the extreme points of $\operatorname{co} C_m(\beta)$ are characterized for $\beta \geqslant 1$. In §4 we consider integral means. Let

$$M_p(r, f) := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} \tag{2}$$

the pth integral mean. Then for an m-fold symmetric close-to-convex function of order β we get that $M_p(r, f') \leqslant M_p(r, k')$ holds for all $p \in \mathbb{R}$, where

$$k'(z) = \frac{(1+z^m)^{\beta}}{(1-z^m)^{\beta+2/m}}, \qquad k(0) = 0. \tag{3}$$

This shows in particular that $M_p(r, f')$ remains bounded as r tends to 1 (i.e. $f' \in H^p$) if $p \in]0, 1/(\beta + 2/m)[$. So if $m > 2/(1-\beta)$ then f has a rectifiable boundary curve. Moreover we show that if $m > 4/(1-\beta)$ then f has a quasiconformal extension.

2. ON FUNCTIONS SUBORDINATE TO CLOSE-TO-CONVEX FUNCTIONS

Hallenbeck and MacGregor ([8], see [9, Theorems 5.21 and 5.23]) characterized the extreme points of Sub K and Sub $C(1)$. With the aid of the result for $C(1)$ they got a new proof for the so-called Rogosinski conjecture for close-to-convex functions, originally proved by Robertson [14] (and now proved for all $f \in$ Sub S by de Branges [1]). We generalize their result to Sub $C(\beta)$. This will be done with the aid of the following general Lemma. An analytic function $f \in A$ is called a BCK-function if E co Sub$\{f\} = \{g \mid g(z) = f(xz), x \in \partial\mathbb{D}\}$ (see e.g. [11]). Well-known examples of BCK-functions are $\left(\frac{1+xz}{1-z}\right)^\alpha$ for $|x| \leqslant 1$ and $\alpha \geqslant 1$ ([2], see e.g. [9, Theorem 5.7]).

Lemma 1 *Let $F \subset A$ be a compact family of analytic functions normalized by* (1). *If E* co *F consists of BCK-functions, then the extreme points of* co Sub *F have the form $g(wz)$ for some $g \in E$* co *F and $w \in \partial\mathbb{D}$.*

Proof The argument given in [9, p. 65, after the proof of Lemma 5.20], shows that in the given situation an extreme point f of co Sub F must be subordinate to some $g \in E$ co F, or it equals the constant function 0.

In the latter case f is subordinate to each function of F, in particular to some $g \in E$ co F. So we have $f = g \circ \omega$ for some $\omega \in B$ and $g \in E$ co F. Suppose now there is no $w \in \partial\mathbb{D}$ such that $\omega(z) = wz$, then, because by hypothesis g is a BCK-function, it follows that f has a proper convex representation in Sub$\{g\}$ and so in Sub F. This gives the result. ■

A consequence is

Theorem 1 *Let $\beta \geqslant 1$, then the extreme points of* co Sub *$C(\beta)$ have the form*

$$f(z) = \frac{w}{(\beta+1)(x+y)}\left(\left(\frac{1+xz}{1-yz}\right)^{\beta+1} - 1\right), \qquad x, y, w \in \partial\mathbb{D}, \quad x \neq -y. \tag{4}$$

Proof For $\beta \geqslant 1$ it is well-known ([2], see e.g. [15, Theorem 2.22]) that an extreme point f of co $C(\beta)$ has the form (4) with $w = 1$. Because $\left(\frac{1+xz}{1-yz}\right)^{\beta+1}$ are BCK-functions, so is f, and an application of Lemma 1 gives the result by an easy change of variables. ■

We remark that the given argument does imply the result also in the case $\beta \in]0,1[$ if the corresponding extreme point result for $C(\beta)$ is true. As an application of the theorem one could deduce the coefficient result for Sub $C(\beta)$, $\beta \geqslant 1$, which had been shown in [7, Theorem 7]. By another approach we get it for all $\beta \geqslant 0$.

If $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$, then the coefficient domination $|a_n| \leqslant |b_n|$, $n \in \mathbb{N}_0$, is denoted by $f \ll g$.

THEOREM 2 *Let $\beta \geqslant 0$, $g \prec f$ and $f \in C(\beta)$. Then*

$$g' \ll h' := \frac{(1+z)^{\beta}}{(1-z)^{\beta+2}}. \tag{5}$$

Proof By hypotheses there are $\omega \in B$, $\varphi \in K$ and $p \in \tilde{P}$ such that $g = f \circ \omega$ and $f' = \varphi' \cdot p^{\beta}$. This gives

$$g'(z) = f'(\omega(z)) \cdot \omega'(z) = \varphi'(\omega(z)) \cdot \omega'(z) \cdot p(\omega(z))^{\beta}.$$

Now $\varphi'(\omega(z)) \cdot \omega'(z)$ is the derivative of some function subordinate to $\varphi \in K$, thus having a representation ([9, Theorem 5.21]) of the form $\int_{\partial\mathbb{D}^2} \frac{x\, d\mu(x,y)}{(1-yz)^2}$ for some Borel probability measure μ. Further $q(z) := p(\omega(z))$ has positive real part. So we have

$$g'(z) = \int_{(\partial\mathbb{D})^2} \frac{x\, d\mu(x,y)}{(1-yz)^2} \cdot q^{\beta}(z).$$

Now the same calculation as in $C(\beta)$ (see [15, Theorem 2.29]) gives the result since $|x| = 1$. ■

We remark that this is the adequate form of the statement of the Rogosinski conjecture for close-to-convex functions of order β. Furthermore the theorem shows that the functionals $|f^{(n)}(z)|$ are maximized in Sub $C(\beta)$ by the function h, given by (5).

3. ON m-FOLD SYMMETRIC CLOSE-TO-CONVEX FUNCTIONS

In [12] we solved the coefficient problem for m-fold symmetric close-to-convex functions of order β, if $\beta - 1/m \geqslant 1$, i.e. $f' \ll k'$, where k is defined by (3). Even though this statement is false, if $\beta - 1/m < 1$ [12], we show that corresponding distortion theorems hold for all $\beta \geqslant 0$.

THEOREM 3 *Let $\beta \geqslant 0$, $m \in \mathbb{N}$ and $f \in C_m(\beta)$. Then*

$$|f'(z)| \leqslant k'(|z|) = \frac{(1+|z|^m)^\beta}{(1-|z|^m)^{\beta+2/m}} \qquad \textit{and} \qquad |f(z)| \leqslant k(|z|).$$

Proof Let $f \in C_m(\beta)$. Then the function g, defined by $f'(z) = (g'(z^m))^{1/m}$, is close-to-convex of order $m\beta$ (see [12, Lemma]). Therefore

$$|f'(z)| = |(g'(z^m))^{1/m}| \leqslant \left(\frac{(1+|z|^m)^{m\beta}}{(1-|z|^m)^{m\beta+2}}\right)^{1/m} = k'(|z|)$$

(see e.g. [10]). An integration gives

$$|f(z)| = \left|\int_0^z f'(\zeta)\, d\zeta\right| \leqslant \int_0^{|z|} |f'(re^{i\theta})|\, dr \leqslant \int_0^{|z|} k'(r)\, dr. \qquad \blacksquare$$

From this we get a corresponding result for functions subordinate to odd close-to-convex functions.

COROLLARY 1 *Let $\beta \geqslant 0$ and $g \prec f \in C_2(\beta)$. Then*

$$|g'(z)| \leqslant \frac{(1+|z|^2)^\beta}{(1-|z|^2)^{\beta+1}} \qquad \textit{and} \qquad |g(z)| \leqslant k(|z|).$$

Proof Let $g = f \circ \omega$, $\omega \in B$. Then $g'(z) = f'(\omega(z)) \cdot \omega'(z)$, and the elementary inequality $(1-|z|^2)|\omega'(z)| \leqslant 1 - |\omega(z)|^2$ (see e.g. [5, p. 198]) and Theorem 3 imply that

$$|g'(z)| = |f'(\omega(z))||\omega'(z)| \leqslant \left(\frac{1+|\omega(z)|^2}{1-|\omega(z)|^2}\right)^\beta \frac{|\omega'(z)|}{1-|\omega(z)|^2}$$

$$\leqslant \left(\frac{1+|\omega(z)|^2}{1-|\omega(z)|^2}\right)^\beta \frac{1}{1-|z|^2}.$$

Now it follows from Schwarz's Lemma that

$$\frac{1+|\omega(z)|^2}{1-|\omega(z)|^2} \leqslant \frac{1+|z|^2}{1-|z|^2} =: H(|z|^2),$$

because H increases as $|z|$ increases, so that finally

$$|g'(z)| \leqslant \frac{(1+|z|^2)^\beta}{(1-|z|^2)^{\beta+1}} = k'(|z|).$$

The second statement follows as in the proof of Theorem 3. ■

Now we give an extreme point result for m-fold symmetric close-to-convex functions if $\beta \geqslant 1$. The case $\beta = 1$ is in [6, Theorem 8].

THEOREM 4 *Let $\beta \geqslant 1$ and $m \in \mathbb{N}$. Then the extreme points of* co $C_m(\beta)$ *have the form*

$$f(0) = 0, \qquad f'(z) = \frac{(1+xz^m)^\beta}{(1-yz^m)^{\beta+2/m}}, \qquad x, y \in \partial\mathbb{D}, \quad x \neq -y.$$

Proof Let $f \in C_m(\beta)$ have the representation $f'(z) = \varphi'(z) \cdot p^\beta(z^m)$, where $\varphi \in K_m$ and $p \in \tilde{P}$. So φ has a representation

$$\varphi'(z) = \int_{\partial\mathbb{D}} \frac{d\mu(w)}{(1-wz^m)^{2/m}} \tag{6}$$

(see [3, Theorem 3]) for some Borel probability measure μ. Since $\beta \geqslant 1$ there is a second representation

$$p^\beta(z) = \int_{(\partial\mathbb{D})^2} \left(\frac{1+xz^m}{1-yz^m}\right)^\beta d\nu(x, y)$$

[9, Theorem 5.7]. Now by the argument given in [2] (see [9, Theorem 5.11]), we deduce that there is a probability measure λ such that

$$f'(z) = \int_{\partial\mathbb{D}} \frac{d\mu(w)}{(1-wz^m)^{2/m}} \cdot \int_{(\partial\mathbb{D})^2} \left(\frac{1+xz^m}{1-yz^m}\right)^\beta d\nu(x, y)$$

$$= \int_{(\partial\mathbb{D})^2} \frac{(1+xz^m)^\beta}{(1-yz^m)^{\beta+2/m}} d\lambda(x, y).$$

So an extreme point is a kernel function. For $x = -y$ the kernel functions are convex, in particular starlike, but they are not extreme in the family of m-fold symmetric starlike functions (see [3, Theorem 3]), which is a subset of $C_m(\beta)$ for $\beta \geqslant 1$, so that they are not extreme in co $C_m(\beta)$. ■

We remark that our method also applies to the family $V_m(k)$ of m-fold symmetric functions with boundary rotation at most $k\pi$ for $k \geqslant 2m+2$, using the relation $V_m(k) \subset C((k/2-1)/m)$ (see [12, Theorem 2] and the fact that the extreme points of $\overline{\text{co}}\, C((k/2-1)/m)$ lie in $V_m(k)$, so that $\text{co}\, V_m(k) = \overline{\text{co}}\, C_m((k/2-1)/m)$.

4. INTEGRAL MEANS OF THE DERIVATIVE

Let the integral means $M_p(r, f')$ be defined by (2).

THEOREM 5 *Let $\beta \geqslant 0$, $m \in \mathbb{N}$ and $f \in C_m(\beta)$. Then*

$$M_p(r, f') \leqslant M_p(r, k') \qquad \text{for all} \quad p \in \mathbb{R}. \tag{7}$$

In particular: $f' \in H^p$ for all $p \in \,]0, 1/(\beta + 2/m)[$.

Proof Let $f \in C_m(\beta)$. Then the function g, defined by $f'(z) = (g'(z^m))^{1/m}$, is close-to-convex of order $m\beta$ (see [12, Lemma 1]). Therefore the result of Brown [4] implies ($z = re^{i\theta}$)

$$\frac{1}{2\pi}\int_0^{2\pi} |f'(z)|^p\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} |g'(z^m)|^{p/m}\, d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{(1+z^m)^{m\beta}}{(1-z^m)^{m\beta+2}}\right|^{p/m} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |k'(z)|^p\, d\theta,$$

where k is defined by (3), which shows (7). The last conclusion follows because $(1-z)^{-\alpha} \in H^p$ for $p \in \,]0, 1/\alpha[$. ■

We remark that the result for $\beta = 1$ seems to be new even for starlike functions. Further we have

COROLLARY 2 *Let $\beta < 1$, $m > 2/(1-\beta)$ and $f \in C_m(\beta)$. Then $f' \in H^1$, i.e. f has a rectifiable boundary.*

We note that the weaker statement that for $m > 2/(1-\beta)$ the functions are bounded follows immediately from the geometry: with a boundary point the function omits a sector of angle at least $(1-\beta)\pi$, and because of the symmetry there are at least m such sectors omitted. If the total value of the angles exceeds 2π, then obviously $f(\mathbb{D})$ is bounded.

5. QUASICONFORMAL EXTENSIONS

The following Theorem is due to Gall [6]:

THEOREM 6 *Let $\beta < 1$, $f \in C(\beta)$ with $f' = \varphi' \cdot p^{\beta}$, $\varphi \in K$ and $p \in \tilde{P}$. If*

$$\limsup_{r \to 1} \frac{\log\left(\max_{|z|=r} |\varphi'(z)|\right)}{\log \frac{1}{1-r}} < \frac{1-\beta}{2},$$

then f has a quasiconformal extension.

An application gives

COROLLARY 3 *Let $\beta < 1$, $m > 4/(1-\beta)$ and $f \in C_m(\beta)$. Then f has a quasiconformal extension.*

Proof Because $f \in C_m(\beta)$, there is a representation $f' = \varphi' \cdot p^{\beta}$ with an m-fold symmetric function $\varphi \in K_m$. From representation (6) one gets

$$|\varphi'(z)| \leqslant \frac{1}{(1-|z|^m)^{2/m}},$$

so that

$$\limsup_{r \to 1} \frac{\log\left(\max_{|z|=r} |\varphi'(z)|\right)}{\log \frac{1}{1-r}} \leqslant \frac{2}{m} < \frac{1-\beta}{2}$$

whenever $m > 4/(1-\beta)$, and the result follows from Theorem 6. ■

References

[1] L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.* **154** (1985), 137–152.

[2] D. A. Brannan, J. G. Clunie and W. E. Kirwan, On the coefficient problem for functions of bounded boundary rotation, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **523** (1973), 1–18.

[3] L. Brickman, D. J. Hallenbeck, T. H. MacGregor and D. R. Wilken, Convex hulls and extreme points of families of starlike and convex mappings, *Trans. Amer. Math. Soc.* **185** (1977), 413–428.

[4] J. E. Brown, Derivatives of close-to-convex functions, integral means and bounded mean oscillation, *Math. Z.* **178** (1981), 353–358.

[5] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1983.

[6] U. Gall, Über das Randverhalten von Bazilevic-Funktionen, Dissertation an der Technischen Universität Berlin, 1986.

[7] D. J. Hallenbeck, Convex hulls and extreme points of families of starlike and close-to-convex mappings, *Pacific J. Math.* **57** (1975), 167–176.

[8] D. J. Hallenbeck and T. H. MacGregor, Subordination and extreme-point theory, *Pacific J. Math.* **50** (1974), 455–468.

[9] D. J. Hallenbeck and T. H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Pitman, Boston-London-Melbourne, 1984.

[10] W. Koepf, Close-to-convex functions and linear-invariant families, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **8** (1983), 349–355.

[11] W. Koepf, Extrempunkte und Stützpunkte in Familien nichtverschwindender schlichter Funktionen, *Complex Variables* **8** (1987), 153–171.

[12] W. Koepf, On the coefficients of symmetric functions of bounded boundary rotation, to appear in *Proc. Amer. Math. Soc.*

[13] Ch. Pommerenke, On close-to-convex analytic functions, *Trans. Amer. Math. Soc.* **114** (1965), 176–186.

[14] M. S. Robertson, The generalized Bieberbach conjecture for subordinate functions, *Michigan Math. J.* **12** (1965), 421–429.

[15] G. Schober, Univalent functions — selected topics, *Springer Lecture Notes* No. 478, Springer-Verlag, Berlin-Heidelberg-New York, 1975.