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Algorithms for m-fold Hypergeometric Summation

WOLFRAM KOEPF†

Konrad-Zuse-Zentrum für Informationstechnik Berlin, Germany

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Zeilberger's algorithm which finds holonomic recurrence equations for definite sums of hypergeometric terms $F(n, k)$ is extended to certain nonhypergeometric terms. An expression $F(n, k)$ is called hypergeometric term if both $F(n+1, k)/F(n, k)$ and $F(n, k+1)/F(n, k)$ are rational functions. Typical examples are ratios of products of exponentials, factorials, Γ function terms, binomial coefficients, and Pochhammer symbols that are integer-linear with respect to n and k in their arguments.

We consider the more general case of such ratios that are rational-linear with respect to n and k in their arguments, and present an extended version of Zeilberger's algorithm for this case, using an extended version of Gosper's algorithm for indefinite summation.

In a similar way the Wilf-Zeilberger method of rational function certification of integer-linear hypergeometric identities is extended to rational-linear hypergeometric identities.

The given algorithms on definite summation apply to many cases in the literature to which neither the Zeilberger approach nor the Wilf-Zeilberger method is applicable. Examples of this type are given by theorems of Watson and Whipple, and a large list of identities ("Strange evaluations of hypergeometric series") that were studied by Gessel and Stanton. Finally we show how the algorithms can be used to generate new identities.

1. Hypergeometric identities

In this paper we deal with hypergeometric identities. As usual, the notation of the generalized hypergeometric series ${}_pF_q$ defined by

$$ {}_pF_q\left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{array} \middle| x\right) := \sum_{k=0}^{\infty} A_k\, x^k = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdot (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdot (b_2)_k \cdots (b_q)_k\, k!} x^k \qquad (1.1) $$

is used, $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$ denoting the *Pochhammer symbol* or *shifted factorial*. The numbers a_k are called the *upper*, and b_k the *lower parameters* of ${}_pF_q$.

Consecutive terms $A_k x^k$ of the generalized hypergeometric series have the rational ratio

$$ \frac{A_{k+1}x^{k+1}}{A_k x^k} = \frac{(k+a_1)\cdot(k+a_2)\cdots(k+a_p)}{(k+b_1)\cdot(k+b_2)\cdots(k+b_q)(k+1)} x \quad (k \in \mathbb{N}) . $$

† E-mail: koepf@zib-berlin.de

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If a sequence a_k has a rational consecutive-term ratio a_{k+1}/a_k, we call a_k a *hypergeometric term* or *closed form*. Note that any hypergeometric term essentially has a representation as the ratio of shifted factorials (over $\mathbb{C}$), and its generating function is connected with a generalized hypergeometric series.

The classical reference concerning generalized hypergeometric series is the book of Bailey (1935) containing a huge amount of relations between hypergeometric series some of which represent the value of certain hypergeometric functions at a special point (mostly $x = 1$ or $x = -1$) by a single hypergeometric term. We will be concerned with this type of identities, and Table 1 is a complete list of all such hypergeometric identities found in Bailey's book.

Here $n \in \mathbb{N}$ is assumed to represent a positive integer so that the hypergeometric series with upper parameter $-n$ are terminating. All other parameters involved represent arbitrary complex variables such that none of the lower parameters corresponds to a nonpositive integer.

With a method due to Wilf and Zeilberger, and with an algorithm of Zeilberger, many of these hypergeometric identities can be checked. It turns out, however, that for some of these identities both methods fail. We give extensions of both the Wilf-Zeilberger approach, and the (fast) Zeilberger's algorithm with which all above identities can be handled as well as a large list of identities that were studied by Gessel and Stanton (1982).

Our extensions therefore unify the verification of hypergeometric identities.

2. Gosper's algorithm

In this section we recall Gosper's algorithm (Gosper, 1978), see also Graham, Knuth and Patashnik (1994), § 5.7.

Gosper's algorithm deals with the question to find an antidifference s_k for given a_k, i.e., a sequence s_k for which

$$a_k = s_k - s_{k-1} \tag{2.1}$$

in the particular case that s_k is a hypergeometric term, therefore

$$\frac{s_k}{s_{k-1}} \quad \text{is a rational function w.r.t. } k\,, \tag{2.2}$$

i.e., $s_k/s_{k-1} \in \mathbb{Q}(k)$. We call this *indefinite summation.*

Note that if a hypergeometric term antidifference s_k exists, we call the input function a_k *Gosper-summable* which then itself is a hypergeometric term since by (2.1) and (2.2)

$$\frac{a_k}{a_{k-1}} = \frac{s_k - s_{k-1}}{s_{k-1} - s_{k-2}} = \frac{\frac{s_k}{s_{k-1}} - 1}{1 - \frac{s_{k-2}}{s_{k-1}}} = \frac{u_k}{v_k} \in \mathbb{Q}(k)$$

is rational, i.e., $u_k, v_k \in \mathbb{Q}[k]$ are polynomials.

Whenever a_k is Gosper-summable then necessarily s_k is a rational multiple of a_k:

$$\frac{s_k}{a_k} = \frac{s_k}{s_k - s_{k-1}} = \frac{s_k}{s_{k-1}}\,\frac{1}{\frac{s_k}{s_{k-1}} - 1} = R_k \in \mathbb{Q}(k)\,.$$

Gosper's algorithm is a *decision procedure* which either returns "No closed form antidifference exists" or returns a closed form antidifference s_k of a_k, *provided one can decide the rationality of* a_k/a_{k-1}, i.e., one finds polynomials u_k, v_k such that $a_k/a_{k-1} = u_k/v_k$. In so far, Gosper's algorithm is an algorithm with input u_k and v_k rather than a_k.

Table 1. Bailey's hypergeometric database

page	Theorem	Identity
2–3	Vandermonde Gauß	$${}_2F_1\left(\begin{matrix} a\,,b \\ c \end{matrix}\,\middle\vert\, 1\right) = \frac{(c-b)_{-a}}{(c)_{-a}} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$
9	Saalschütz	$${}_3F_2\left(\begin{matrix} a\,,b\,,-n \\ c\,,1+a+b-c-n \end{matrix}\,\middle\vert\, 1\right) = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}$$
9	Kummer	$${}_2F_1\left(\begin{matrix} a\,,b \\ 1+a-b \end{matrix}\,\middle\vert\, -1\right) = \frac{(1+a)_{-b}}{(1+a/2)_{-b}} = \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)}$$
11	Gauß	$${}_2F_1\left(\begin{matrix} a\,,b \\ (a+b+1)/2 \end{matrix}\,\middle\vert\, \frac{1}{2}\right) = \frac{\Gamma(1/2)\Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)}$$
11	Bailey	$${}_2F_1\left(\begin{matrix} a\,,1-a \\ c \end{matrix}\,\middle\vert\, \frac{1}{2}\right) = \frac{\Gamma(c/2)\Gamma((c+1)/2)}{\Gamma((a+c)/2)\Gamma((1-a+c)/2)}$$
13	Dixon	$${}_3F_2\left(\begin{matrix} a\,,b\,,c \\ 1+a-b\,,1+a-c \end{matrix}\,\middle\vert\, 1\right) = \frac{(1+a)_{-c}\,(1+a/2-b)_{-c}}{(1+a/2)_{-c}\,(1+a-b)_{-c}} = \frac{\Gamma(1+a/2)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a/2-b-c)}{\Gamma(1+a)\Gamma(1+a/2-b)\Gamma(1+a/2-c)\Gamma(1+a-b-c)}$$
16	Watson Whipple	$${}_3F_2\left(\begin{matrix} a\,,b\,,c \\ (a+b+1)/2\,,2c \end{matrix}\,\middle\vert\, 1\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1+2c}{2})\Gamma(\frac{1+a+b}{2})\Gamma(\frac{1-a-b+2c}{2})}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1+b}{2})\Gamma(\frac{1-a+2c}{2})\Gamma(\frac{1-b+2c}{2})}$$
16	Whipple	$${}_3F_2\left(\begin{matrix} a,1-a,c \\ e,1+2c-e \end{matrix}\,\middle\vert\, 1\right) = \frac{\pi 2^{1-2c}\Gamma(e)\Gamma(1+2c-e)}{\Gamma(\frac{a+e}{2})\Gamma(\frac{a+1+2c-e}{2})\Gamma(\frac{1-a+e}{2})\Gamma(\frac{2+2c-a-e}{2})}$$
26	Dougall's Theorem	$${}_7F_6\left(\begin{matrix} a\,,1+a/2\,,b\,,c\,,d\,,1+2a-b-c-d+n\,,-n \\ a/2\,,1+a-b\,,1+a-c\,,1+a-d\,,b+c+d-a-n\,,1+a+n \end{matrix}\,\middle\vert\, 1\right) = \frac{(1+a)_n\,(1+a-b-c)_n\,(1+a-b-d)_n\,(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n\,(1+a-d)_n\,(1+a-b-c-d)_n}$$
25/27	Dougall	$${}_5F_4\left(\begin{matrix} a,1+a/2,c,d,e \\ a/2,1+a-c,1+a-d,1+a-e \end{matrix}\,\middle\vert\, 1\right) = \frac{(1+a)_{-e}(1+a-c-d)_{-e}}{(1+a-c)_{-e}(1+a-d)_{-e}} = \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)}$$
28	Whipple	$${}_4F_3\left(\begin{matrix} a,1+a/2,d,e \\ a/2,1+a-d,1+a-e \end{matrix}\,\middle\vert\, -1\right) = \frac{(1+a)_{-e}}{(1+a-d)_{-e}} = \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}$$
30	Bailey	$${}_3F_2\left(\begin{matrix} a\,,1+a/2\,,-n \\ a/2\,,w \end{matrix}\,\middle\vert\, 1\right) = \frac{(w-a-1-n)(w-a)_{n-1}}{(w)_n}$$
30	Bailey	$${}_3F_2\left(\begin{matrix} a\,,b\,,-n \\ 1+a-b\,,1+2b-n \end{matrix}\,\middle\vert\, 1\right) = \frac{(a-2b)_n(1+a/2-b)_n(-b)_n}{(1+a-b)_n(a/2-b)_n(-2b)_n}$$
30	Bailey	$${}_4F_3\left(\begin{matrix} a\,,1+a/2\,,b\,,-n \\ a/2\,,1+a-b\,,1+2b-n \end{matrix}\,\middle\vert\, 1\right) = \frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n}$$
30	Bailey	$${}_4F_3\left(\begin{matrix} a,1+a/2,b,-n \\ a/2,1+a-b,2+2b-n \end{matrix}\,\middle\vert\, 1\right) = \frac{(a-2b-1)_n(1/2+a/2-b)_n(-b-1)_n}{(1+a-b)_n(a/2-b-1/2)_n(-2b-1)_n}$$

Since without preprocessing, the user's input is a_k rather than the polynomials u_k and v_k, the success of an implementation depends heavily on an algorithm quickly and safely calculating (u_k, v_k) given a_k. In Algorithm 2.1, we present such a method. It turns out that none of the existing implementations of Gosper's algorithm uses such a method.

Gosper implemented his algorithm in the MACSYMA `nusum` command, an implementation of the algorithm is distributed with the `sum` command of the MAPLE V.3 system (to check its use set `infolevel[sum]:=5`), and one was delivered with MATHEMATICA Version 1.2 (`Algebra/GosperSum.m`). Another MATHEMATICA implementation was given by Paule and Schorn (1994).

Along the lines of Koornwinder (1993), together with Gregor Stölting I implemented Gosper's algorithm in REDUCE (Koepf, 1995) and MAPLE, using the decision procedure for rationality of hypergeometric terms described in Algorithm 2.1 below rather than internal simplification procedures (like MAPLE's `expand`), available in releases REDUCE 3.6 and MAPLE V.4 through the `zeilberg` and the `sumtools` packages.

In case Gosper's algorithm provides us with an antidifference s_k of a_k, any sum

$$\sum_{k=m}^{n} a_k = s_n - s_{m-1}$$

can be easily calculated by an evaluation of s_k at the boundary points like in the integration case. Note, however, that the sum

$$\sum_{k=0}^{n} \binom{n}{k} \tag{2.3}$$

e.g., is not of this type as the summand $\binom{n}{k}$ depends on the upper boundary point n explicitly. This is an example of a definite sum that we consider in § 3.

It is almost trivial but decisive that the following is a decision procedure for the rationality of a_k/a_{k-1} for input a_k (at least) of a special type:

ALGORITHM 2.1. (`simpcomb`)
The following algorithm decides the rationality of a_k/a_{k-1}:

1 Input: $a_k \not\equiv 0$ as ratio of products of rational functions, exponentials, factorials, Γ function terms, binomial coefficients, and Pochhammer symbols that are rational-linear in their arguments.

2 (`togamma`)
Build a_k/a_{k-1}, and convert all occurrences of factorials, binomial coefficients, and Pochhammer symbols to Γ function terms. The case of binomial coefficients is done by the rules

$$\binom{a}{k} \to \begin{cases} (-1)^k \frac{\Gamma(k-a)}{\Gamma(k+1)\,\Gamma(-a)} & \text{if } a \in \mathbb{Z},\ a<0 \\ 0 & \text{if } a-k \in \mathbb{Z},\ a-k<0 \\ \frac{\Gamma(a+1)}{\Gamma(k+1)\,\Gamma(a-k+1)} & \text{otherwise} \end{cases}$$

3 (`simplify_gamma`)
Recursively rewrite this expression according to the rule

$$\Gamma(a+j) = (a)_j \cdot \Gamma(a)$$

($(a)_j := a(a+1)\cdots(a+j-1)$ denoting the Pochhammer symbol) whenever the arguments a and $a+j$ of two representing Γ function terms have positive integer difference j. Reduce the final fraction cancelling common Γ terms.

4 (`simplify_power`)
Recursively rewrite the last expression according to the rule

$$b^{a+j} = b^j\, b^a$$

whenever the arguments a and $a+j$ of two representing exponential terms have positive integer difference j. Reduce the final fraction cancelling common exponential terms.

5 The expression a_k/a_{k-1} is rational if and only if the resulting expression in step 4 is rational of the form u_k/v_k, $u_k, v_k \in \mathbb{Q}[k]$.

6 Output: (u_k, v_k).

Note that this result follows immediately from the given form of a_k and therefore of the expression a_k/a_{k-1} considered.

As an example, the rationality of a_k/a_{k-1} for

$$a_k = \frac{\Gamma(2k)}{4^k\,\Gamma(k)\,\Gamma(k+1/2)}$$

is recognized by the given procedure, and from the resulting information ($a_k/a_{k-1} = 1$), by induction $a_k = 1/(2\sqrt{\pi})$ (Abramowitz and Stegun, 1964, (6.1.18)).

In most cases also sums of ratios of the described form can be treated by the same method. An important family of examples of this type will be considered next.

3. The Wilf–Zeilberger method

Examples for an application of Gosper's algorithm in connection with Algorithm 2.1 are given by the Wilf-Zeilberger method on *definite summation* (Wilf and Zeilberger, 1990), see also Wilf (1993).

The Wilf-Zeilberger method is a direct application of Gosper's algorithm to prove identities of the form

$$s_n := \sum_{k\in\mathbb{Z}} F(n,k) = 1 \tag{3.1}$$

for which $F(n,k)$ is a hypergeometric term w.r.t. both n and k, i.e.,

$$\frac{F(n,k)}{F(n-1,k)} \quad \text{and} \quad \frac{F(n,k)}{F(n,k-1)} \quad \text{are rational functions w.r.t. both } n \text{ and } k,$$

where n is assumed to be an integer, and the sum is to be taken over all integers $k \in \mathbb{Z}$. We moreover assume that $F(n,k)$ has *finite support*, i.e., is nonvanishing only for finitely many $k \in \mathbb{Z}$ for each fixed $n \in \mathbb{N}_0$.

To prove a statement of the form (3.1) by the WZ method†, one applies Gosper's algorithm to the expression

$$a_k := F(n,k) - F(n-1,k)$$

† Note that Wilf and Zeilberger use forward rather than backward differences, whereas we follow Gosper's original treatment. There is no theoretical difference between these two approaches, though.

w.r.t. the variable k. If successful, this generates $G(n,k)$ with

$$a_k = F(n,k) - F(n-1,k) = G(n,k) - G(n,k-1) , \tag{3.2}$$

and summing over all k leads to

$$s_n - s_{n-1} = \sum_{k\in\mathbb{Z}} \Big(F(n,k) - F(n-1,k)\Big) = \sum_{k\in\mathbb{Z}} \Big(G(n,k) - G(n,k-1)\Big) = 0$$

since the right hand side is telescoping, and $F(n,k)$, hence $G(n,k) = R(n,k)\,F(n,k)$ has finite support, see below. Therefore $s_n \equiv s_0$ is constant, and as $F(n,k)$ has finite support, we can prove $s_0 = 1$, and are done.

Since the WZ method only works if n is an integer, we can prove the statements of Bailey's list in Table 1 only if one of the upper parameters of the hypergeometric series involved is a negative integer. The extension to the general case is over the capabilities of the methods of this article, and must be handled by other means.

Note that the rationality of a_k/a_{k-1} for the WZ method is decided by Algorithm 2.1 since

$$\frac{a_k}{a_{k-1}} = \frac{F(n,k) - F(n-1,k)}{F(n,k-1) - F(n-1,k-1)} = \frac{F(n,k)}{F(n,k-1)} \cdot \frac{1 - \frac{F(n-1,k)}{F(n,k)}}{1 - \frac{F(n-1,k-1)}{F(n,k-1)}} .$$

Note moreover, that the application of Gosper's algorithm may be slow. But when Gosper's algorithm generates the function $G(n,k)$, it moreover finds the rational function

$$R(n,k) := \frac{G(n,k)}{F(n,k)} .$$

$R(n,k)$ is rational since $G(n,k)$ is a rational multiple of $a_k = F(n,k) - F(n-1,k)$, $G(n,k) = r(n,k) \cdot (F(n,k) - F(n-1,k))$, say, so that

$$R(n,k) = \frac{G(n,k)}{F(n,k)} = r(n,k)\frac{F(n,k) - F(n-1,k)}{F(n,k)} = r(n,k)\left(1 - \frac{F(n-1,k)}{F(n,k)}\right)$$

is rational. $R(n,k)$ is called the *rational certificate* of $F(n,k)$. Once the rational certificate of a hypergeometric expression $F(n,k)$ is known, it is a matter of pure rational arithmetic (which is fast) to decide the validity of (3.1) since the only thing that one has to show is (3.2) which after division by $F(n,k)$ is equivalent (modulo an application of Algorithm 2.1) to the purely rational identity

$$1 - R(n,k) + R(n,k-1)\frac{F(n,k-1)}{F(n,k)} - \frac{F(n-1,k)}{F(n,k)} = 0 . \tag{3.3}$$

As an example, to prove the Binomial Theorem in the form

$$s_n := \sum_{k=0}^{n} F(n,k) = \sum_{k=0}^{n} \frac{1}{2^n}\binom{n}{k} = 1 \tag{3.4}$$

by the WZ method, Algorithm 2.1 yields

$$\frac{a_k}{a_{k-1}} = \frac{F(n,k) - F(n-1,k)}{F(n,k-1) - F(n-1,k-1)} = \frac{(n-k+1)\,(n-2\,k)}{k\,(n-2\,k+2)}$$

so that Gosper's algorithm can be applied, and results in

$$G(n,k) = \frac{n-k}{2^n(2k-n)}\left(2\binom{n-1}{k} - \binom{n}{k}\right) .$$

This proves (3.4) since $s_0 = 1$.

The rational certificate function is

$$R(n,k) = \frac{G(n,k)}{F(n,k)} = \frac{k-n}{n},$$

and the verification of identity (3.4) is therefore reduced to simplify the rational expression

$$1-R(n,k)+R(n,k-1)\frac{F(n,k-1)}{F(n,k)}-\frac{F(n-1,k)}{F(n,k)} = 1-\frac{k-n}{n}+\frac{k-1-n}{n}\frac{k}{n+1-k}-\frac{2(n-k)}{n}$$

to zero.

Table 2. The WZ method

Theorem	n	$R(n,k)$
Vandermonde	$-a$	$-\frac{(b+k)(-n+k)}{n(c+n-1)}$
Saalschütz	n	$-\frac{(b+k)(-n+k)(a+k)}{n(c+n-1)(1+a+b-c-n+k)}$
Kummer	$-b$	$\frac{(a+k)(-n+k)}{n(a+2n)}$
Dixon	$-c$	$-\frac{(a+k)(-n+k)(b+k)}{n(a-b+n)(a+2n)}$
Watson Whipple	$-c$	$2\frac{(a+k)(-n+k)(b+k)}{(-1+a+b+2n)(-2n+1+k)(-2n+k)}$
Whipple	$-c$	$-\frac{(a+k)(a-1-k)(-n+k)}{n(2-2n-e+k)(1-2n-e+k)}$
Dougall	n	$\frac{(2a-b-c-d+2n)(a+k)(-n+k)(b+k)(c+k)(d+k)}{n(a+2k)(a-b-c+n-d-k)(a-d+n)(a-c+n)(a-b+n)}$
Dougall	$-e$	$-\frac{(a+k)(-n+k)(c+k)(d+k)}{n(a+2k)(a-c+n)(a-d+n)}$
Whipple	$-e$	$\frac{(d+k)(-n+k)(a+k)}{n(a+2k)(a-d+n)}$
Bailey	n	$-\frac{\left(a^2+2a-wa+na+2-2w-2kw+2ka+2k+2kn\right)(a+k)(-n+k)}{(-w+a+n)n(a+2k)(w+n-1)}$
Bailey	n	$-\frac{\left(-2b-2b^2+2nb+ab-1+n-k\right)(a+k)(-n+k)(b+k)}{nb(1+2b-n+k)(a-2b+2n-2)(a-b+n)}$
Bailey	n	$-\frac{(2b+ab+1-n+2kb+k)(b+k)(-n+k)(a+k)}{nb(a+2k)(1+2b-n+k)(a-b+n)}$
Bailey	n	$-\frac{(a+k)(-n+k)(b+k)}{nb(a+2k)(2+2b-n+k)(a-2b-3+2n)(a-b+n)} \cdot \left(-8b-4b^2+6nb-ab-2n^2+2nba-4+6n-2b^2a+a^2b-6k-8kb-4b^2k+4kn+4kbn+2kba-2k^2\right)$

Table 2 is a complete list of those identities of Bailey's list (Table 1) that can be treated by the given method together with their rational certificates with which the reader may verify them easily.

Note that neither the statements of Gauß and Bailey of argument $x = 1/2$ (p. 11) are accessible w.r.t. any of the parameters involved, nor can Watson's Theorem (p. 16) be proved by the WZ method w.r.t. Watson's original integer parameter a, nor can the method be applied to Whipple's Theorem (p. 16) concerning parameters a or b since in all these cases the term ratio a_k/a_{k-1} is not rational.

Our REDUCE and MAPLE implementations both generate the results of Table 2, and only the calculation of the rational certificate of Dougall's Theorem needs more than a few seconds.

In § 5, we consider a generalization of the WZ method. To be able to consider the most general case, we present an extended version of Gosper's algorithm next.

4. An extended version of Gosper's algorithm

Here we deal with the question, given a nonnegative integer m, to find a sequence s_k for given a_k satisfying

$$a_k = s_k - s_{k-m} \tag{4.1}$$

in the particular case that s_k is an *m-fold hypergeometric term*, i. e.

$$\frac{s_k}{s_{k-m}} \quad \text{is a rational function w.r.t. } k\,. \tag{4.2}$$

Note that in the given case the input function a_k itself is an m-fold hypergeometric term since by (4.1) and (4.2)

$$\frac{a_k}{a_{k-m}} = \frac{s_k - s_{k-m}}{s_{k-m} - s_{k-2m}} = \frac{\frac{s_k}{s_{k-m}} - 1}{1 - \frac{s_{k-2m}}{s_{k-m}}} = \frac{u_k}{v_k}$$

is rational, i. e., u_k and v_k can be chosen to be polynomials.

Assume first, given a_k, we have found s_k with $s_k - s_{k-m} = a_k$. Then we can easily construct an antidifference $\tilde{s}_k$ of a_k by

$$\tilde{s}_k := s_k + s_{k-1} + \cdots + s_{k-(m-1)} \tag{4.3}$$

since then $\tilde{s}_k - \tilde{s}_{k-1} = (s_k + \cdots + s_{k-(m-1)}) - (s_{k-1} + \cdots + s_{k-m}) = s_k - s_{k-m} = a_k$.

Since

$$\frac{s_k}{s_{k-m}} = \frac{s_k}{s_{k-1}} \cdot \frac{s_{k-1}}{s_{k-2}} \cdots \frac{s_{k-(m-1)}}{s_{k-m}}\,,$$

any hypergeometric term is also an m-fold symmetric hypergeometric term.

An m-fold antidifference always can be constructed by an application of Gosper's original algorithm in the following way:

ALGORITHM 4.1. (**extended_gosper**)
The following steps generate an m-fold antidifference:

1 Input: a_k, and $m \in \mathbb{N}$.
2 Define $b_k := a_{km}$.
3 Apply Gosper's algorithm to b_k w.r.t. k. Get the antidifference $T(k)b_k$ of b_k (where

$T(k)$ is a rational function), or the statement: "No hypergeometric term antidifference of b_k, and therefore no m-fold hypergeometric term antidifference of a_k exists."

4 The output $s_k := T(k/m)a_k$ is a solution of (4.1) with the property (4.2).

PROOF. Existence of an m-fold hypergeometric solution s_k of

$$s_k - s_{k-m} = a_k \tag{4.4}$$

is equivalent to the existence of a rational solution $S(k)$ of

$$S(k) - r(k)S(k-m) = 1 \tag{4.5}$$

where $r(k) = a_{k-m}/a_k$ and $S(k) = s_k/a_k$. Existence of a hypergeometric solution t_k of

$$t_k - t_{k-1} = a_{km} \tag{4.6}$$

is equivalent to the existence of a rational solution $T(k)$ of

$$T(k) - r(km)T(k-1) = 1 \tag{4.7}$$

where $T(k) = t_k/a_{km}$. Clearly (4.5) and (4.7) are either both solvable and have solutions such that $T(k) = S(km)$, or are both unsolvable. So either (4.6) has no hypergeometric solution and (4.4) has no m-fold hypergeometric solution, or (4.6) has a hypergeometric solution $t_k = T(k)a_{km}$ and (4.4) has an m-fold hypergeometric solution $s_k = S(k)a_k = T(k/m)a_k$. □

As an example, we consider $a_k := k\left(\frac{k}{2}\right)!$, and $m = 2$. Then $b_k = a_{2k} = 2k\,k!$, and Gosper's algorithm yields $t_k = 2(k+1)k!$. Therefore $s_k = t_{k/2} = (k+2)\left(\frac{k}{2}\right)!$ has the property that

$$s_k - s_{k-2} = a_k\,.$$

By (4.3), we moreover find the antidifference

$$\tilde{s}_k = s_k + s_{k-1} = (k+2)\left(\frac{k}{2}\right)! + (k+1)\left(\frac{k-1}{2}\right)!$$

of a_k.

As another example, for $a_k = \binom{n}{k/2} - \binom{n}{k/2-1}$, one gets

$$\begin{aligned}\tilde{s}_k &= \frac{(2n+3-k)(n+1-k)}{2(n+2-k)(n+1-k)}\left(\binom{n}{\frac{k-1}{2}} - \binom{n}{\frac{k-3}{2}}\right) \\ &\quad + \frac{(n+2-k)(2n+2-k)}{2(n+2-k)(n+1-k)}\left(\binom{n}{k/2} - \binom{n}{k/2-1}\right).\end{aligned}$$

Now, we give an algorithm that finds an appropriate nonnegative integer m for an arbitrary input function a_k given as ratio of products of rational functions, exponentials, factorials, Γ function terms, binomial coefficients, and Pochhammer symbols that are rational-linear in their arguments:

ALGORITHM 4.2. (`find_mfold`)
The following is an algorithm generating a successful choice for m for an application of Algorithm 4.1.

1 Input: a_k as ratio of products of rational functions, exponentials, factorials, Γ function terms, binomial coefficients, and Pochhammer symbols that are rational-linear in their arguments.
2 Build the list of all arguments. They are of the form $p_j/q_j\,k + \alpha_j$ with integer p_j and q_j, p_j/q_j in lowest terms, q_j positive.
3 Calculate $m := \operatorname{lcm}\{q_j\}$.

PROOF. It is clear that the procedure generates a representation for $b_k = a_{km}$ with the given choice of m which is integer-linear in the arguments involved. Since in this case b_k/b_{k-1} is rational, Algorithm 4.1 is applicable. □

In our example cases above, the given procedure yields the desired value $m = 2$.

5. Extension of the WZ method

In this section we will give an extended version of the WZ method which resolves the questions that remained open in Section 3 so that finally Bailey's complete list (Table 1) can be settled using a unifying approach.

Assume that for a hypergeometric identity the WZ method fails. This may happen either because a_k/a_{k-1} is not rational, or because there is no single formula for the result like in Andrews' statement

$$ {}_3F_2\left(\begin{matrix} -n\,,n+3a\,,a \\ 3a/2\,,(3a+1)/2 \end{matrix}\,\middle|\,\frac{3}{4}\right) = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod 3 \\ \dfrac{n!\,(a+1)_{n/3}}{(n/3)!\,(3a+1)_n} & \text{otherwise} \end{cases} \tag{5.1} $$

which—together with many similar statements—can be found in a paper of Gessel and Stanton (1982), Equation (1.1).

In such cases, we proceed as follows: To prove an identity of the form

$$ s_n := \sum_{k\in\mathbb{Z}} F(n,k) = \text{constant} \qquad (n \bmod m \text{ constant})\,, \tag{5.2} $$

m denoting a certain positive integer, $F(n,k)$ being an (m,l)*-fold hypergeometric term* w.r.t. (n,k), i.e.,

$$ \frac{F(n,k)}{F(n-m,k)} \quad \text{and} \quad \frac{F(n,k)}{F(n,k-l)} \quad \text{are rational functions w.r.t. both } n \text{ and } k, $$

with finite support, and n assuming to be an integer, we apply our extended version of Gosper's algorithm to find an l-fold antidifference of the expression

$$ a_k := F(n,k) - F(n-m,k) $$

w.r.t. the variable k. (In most cases $l = 1$, so that Gosper's original algorithm is applied.) If successful, this generates $G(n,k)$ with

$$ a_k = F(n,k) - F(n-m,k) = G(n,k) - G(n,k-l)\,, \tag{5.3} $$

and summing over all k leads to

$$ s_n - s_{n-m} = \sum_{k\in\mathbb{Z}} \Big(F(n,k) - F(n-m,k)\Big) = \sum_{k\in\mathbb{Z}} \Big(G(n,k) - G(n,k-l)\Big) = 0 $$

since the right hand side is telescoping, and $F(n,k)$ has finite support. Therefore s_n is constant mod m, and these constants can be calculated using suitable initial values. This can be accomplished as the series considered is terminating. Note, that again, the function

$$R(n,k) = \frac{G(n,k)}{F(n,k)} \tag{5.4}$$

acts as a rational certificate function.

As an example, we prove (5.1): In the given case, we set $m := 3$, $l := 1$, further

$$F(n,k) := \frac{(-n)_k\,(n+3a)_k\,(a)_k}{k!\,(3a/2)_k\,((3a+1)/2)_k}\,\frac{(n/3)!\,(3a+1)_n}{n!\,(a+1)_{n/3}}\left(\frac{3}{4}\right)^k ,$$

and notice that

$$\frac{F(n,k)}{F(n,k-1)} \quad \text{and} \quad \frac{F(n,k)}{F(n-3,k)}$$

are (complicated) rational functions (Algorithms 4.1 and 2.1). An application of Gosper's algorithm is successful, and leads to the rational certificate

$$R(n,k) = 3\frac{(a+k)\,(n-k)\,(3\,a+2\,n-3)}{(n+3\,a+k-2)\,(n+3\,a+k-1)\,n} .$$

Therefore

$$\sum_{k\in\mathbb{Z}} F(n,k) = \sum_{k=0}^{n} F(n,k) = \text{constant} \qquad (n \bmod 3 \text{ constant}) ,$$

and statement (5.1) follows using three trivial initial values.

Table 3 lists the hypergeometric identities of the Gessel-Stanton paper (note the misprint in Equation (1.4)), and Table 4 contains their rational certificates (5.4), calculated by our implementations, together with the certificates of Bailey's list (Table 1) to which the WZ method did not apply.

Note that in all cases considered, $l = 1$, so that the original Gosper algorithm is applied.

Note, moreover, that Gessel and Stanton were not able to present proofs for their statements (6.2), (6.3), (6.5), and (6.6): Table 5 contains proofs.

Finally, we give an example of an application for which $l \neq 1$. To prove the identity

$$-\sum_{k=0}^{n}(-2)^n \binom{n}{k}\cdot\binom{k/2}{n} = 1\,, \quad (n\in\mathbb{N})\,, \tag{5.5}$$

we apply our extended WZ method with $l = 2$, and $m = 1$, and get the rational certificate

$$R(n,k) = \frac{(-k+n-1)\,(-k+n)}{(n-1)\,(-k+2\,n-2)} ,$$

which proves (5.5).

6. Zeilberger's algorithm

In this section, we recall Zeilberger's algorithm (Zeilberger, 1990–1991), see also Graham, Knuth and Patashnik (1994), Section 5.8, with which one can not only verify hypergeometric identities but moreover in many cases definite sums can be calculated.

Table 3. Gessel and Stanton's hypergeometric identities

Eq.	Identity
(1.1)	${}_3F_2\left(\begin{matrix}-n\,, n+3a\,, a\\ 3a/2\,, (3a+1)/2\end{matrix}\,\middle\vert\,\frac{3}{4}\right) = \begin{cases}0 & \text{if } n \not\equiv 0 \pmod 3\\ \dfrac{n!\,(a+1)_{n/3}}{(n/3)!\,(3a+1)_n} & \text{otherwise}\end{cases}$
(1.2)	${}_5F_4\left(\begin{matrix}2a\,, 2b\,, 1-2b\,, 1+2a/3\,, -n\\ a-b+1\,, a+b+1/2\,, 2a/3\,, 1+2a+2n\end{matrix}\,\middle\vert\,\frac{1}{4}\right) = \dfrac{(a+1/2)_n\,(a+1)_n}{(a+b+1/2)_n\,(a-b+1)_n}$
(1.3)	${}_5F_4\left(\begin{matrix}a, b, a+1/2-b, 1+2a/3, -n\\ 2a+1-2b, 2b, 2a/3, 1+a+n/2\end{matrix}\,\middle\vert\,4\right) = \begin{cases}0 & \text{if } n \text{ odd}\\ \dfrac{n!\,(a+1)_{n/2}\,2^{-n}}{(\frac{n}{2})!(a-b+1)_{n/2}(b+\frac{1}{2})_{n/2}} & \text{otherwise}\end{cases}$
(1.4)	${}_3F_2\left(\begin{matrix}1/2+3a\,, 1/2-3a\,, -n\\ 1/2\,, -3n\end{matrix}\,\middle\vert\,\frac{3}{4}\right) = \dfrac{(1/2-a)_n\,(1/2+a)_n}{(1/3)_n\,(2/3)_n}$
(1.5)	${}_3F_2\left(\begin{matrix}1+3a\,, 1-3a\,, -n\\ 3/2\,, -1-3n\end{matrix}\,\middle\vert\,\frac{3}{4}\right) = \dfrac{(1+a)_n\,(1-a)_n}{(2/3)_n\,(4/3)_n}$
(1.6)	${}_3F_2\left(\begin{matrix}2a\,, 1-a\,, -n\\ 2a+2\,, -a-1/2-3n/2\end{matrix}\,\middle\vert\,1\right) = \dfrac{((n+3)/2)_n\,(n+1)(2a+1)}{(1+(n+2a+1)/2)_n\,(2a+n+1)}$
(1.7)	${}_7F_6\left(\begin{matrix}2a\,, 2b\,, 1-2b\,, 1+2a/3\,, a+d+n+1/2\,, a-d\,, -n\\ a-b+1\,, a+b+1/2\,, 2a/3\,, -2d-2n\,, 2d+1\,, 1+2a+2n\end{matrix}\,\middle\vert\,1\right) =$ $\dfrac{(2a+1)_{2n}\,(b+d+1/2)_n\,(d-b+1)_n}{(2d+1)_{2n}\,(a+b+1/2)_n\,(a-b+1)_n} = \dfrac{(a+1/2)_n\,(a+1)_n\,(b+d+1/2)_n\,(d-b+1)_n}{(a+b+1/2)_n\,(a-b+1)_n\,(d+1/2)_n\,(d+1)_n}$
(1.8)	${}_7F_6\left(\begin{matrix}a\,, b\,, a+1/2-b\,, 1+2a/3\,, 1-2d\,, 2a+2d+n\,, -n\\ 2a-2b+1\,, 2b\,, 2a/3\,, a+d+1/2\,, 1-d-n/2\,, 1+a+n/2\end{matrix}\,\middle\vert\,1\right) =$ $\begin{cases}0 & \text{if } n \text{ odd}\\ \dfrac{(b+d)_{n/2}\,(d-b+a+1/2)_{n/2}\,n!\,(a+1)_{n/2}\,2^{-n}}{(b+1/2)_{n/2}\,(a+d+1/2)_{n/2}\,(d)_{n/2}\,(n/2)!\,(a-b+1)_{n/2}} & \text{otherwise}\end{cases}$
(3.7)	${}_2F_1\left(\begin{matrix}-n\,, -2n-2/3\\ 4/3\end{matrix}\,\middle\vert\,-8\right) = \dfrac{(5/6)_n}{(3/2)_n}\,(-27)^n$
(5.21)	${}_3F_2\left(\begin{matrix}3a+1/2\,, 3a+1\,, -n\\ 6a+1\,, -n/3+2a+1\end{matrix}\,\middle\vert\,\frac{4}{3}\right) = \begin{cases}0 & \text{if } n \not\equiv 0 \pmod 3\\ \dfrac{(1/3)_{n/3}\,(2/3)_{n/3}}{(1+2a)_{n/3}\,(-2a)_{n/3}} & \text{otherwise}\end{cases}$
(5.22)	${}_2F_1\left(\begin{matrix}-n\,, 1/2\\ 2n+3/2\end{matrix}\,\middle\vert\,\frac{1}{4}\right) = \dfrac{(1/2)_n}{(2n+3/2)_n}\left(\dfrac{27}{4}\right)^n$
(5.23)	${}_2F_1\left(\begin{matrix}-n\,, -1/3-2n\\ 2/3\end{matrix}\,\middle\vert\,-8\right) = (-27)^n$
(5.24)	${}_2F_1\left(\begin{matrix}-n\,, n/2+1\\ 4/3\end{matrix}\,\middle\vert\,\frac{8}{9}\right) = \begin{cases}0 & \text{if } n \text{ odd}\\ \dfrac{(1/2)_{n/2}}{(7/6)_{n/2}}\,(-3)^{-(n/2)} & \text{otherwise}\end{cases}$
(5.25)	${}_2F_1\left(\begin{matrix}-n\,, 1/2\\ (n+3)/2\end{matrix}\,\middle\vert\,4\right) = \begin{cases}0 & \text{if } n \text{ odd}\\ \dfrac{(1/2)_{n/2}\,(3/2)_{n/2}}{(5/6)_{n/2}\,(7/6)_{n/2}} & \text{otherwise}\end{cases}$
(5.27)	${}_4F_3\left(\begin{matrix}1/3-n\,, -n/2\,, (1-n)/2\,, 22/21-3n/7\\ 5/6\,, 4/3\,, 1/21-3n/7\end{matrix}\,\middle\vert\,-27\right) = \dfrac{(-8)^n}{1-9n}$

Table 4. The extended WZ method

Bailey p.	n	m	$R(n,k)$
11, Gauß	$-a$	2	$-\frac{(b+k)(n-k)}{(-b+n-1-2k)n}$
11, Bailey	$-a$	2	$\frac{(2n-1)(n-k)}{(c+n-1)(n+k)}$
16, Watson	$-a$	2	$-2\frac{(c+k)(b+k)(n-k)}{(-1+n+2c)(-b+n-1-2k)n}$
16, Whipple	$-a$	2	$2\frac{(2n-1)(n-k)(c+k)}{(2c-e+n)(-1+n+e)(n+k)}$

G.-S. Eq.	m	$R(n,k)$
(1.1)	3	$3\frac{(a+k)(n-k)(3a+2n-3)}{(n+3a+k-2)(n+3a+k-1)n}$
(1.2)	1	$-\frac{(2a+k)(n-k)(2b-1-k)(2b+k)}{2n(3k+2a)(2a+2b-1+2n)(a-b+n)}$
(1.3)	2	$4\frac{(b+k)(a+k)(2a-2b+1+2k)(n-k)}{n(3k+2a)(2b-1+n)(2a-2b+n)}$
(1.4)	1	$3\frac{(n-k)(6a-1-2k)(6a+1+2k)}{(12n-4k)(3n-1-k)(3n-2-k)}$
(1.5)	1	$3\frac{(n-k)(3a-k-1)(3a+1+k)}{(3n-1-k)(3n-k)(3n-k+1)}$
(1.6)	2	$\frac{\left(-4a+4an+18n^2-20n+2-16nk\right)(n-k)(a-k-1)(2a+k)}{n(2a+1+3n-2k)(2a-1+3n-2k)(2a-3+3n-2k)(n-1)}$
(1.7)	1	$\frac{(2a-1+4n+2d)(a-d+k)(2a+k)(n-k)(2b-1-k)(2b+k)}{n(2a+3k)(2d+2n-k)(2d+2n-1-k)(2a+2b-1+2n)(a-b+n)}$
(1.8)	2	$8\frac{(2d-1-k)(b+k)(n-k)(a+k)(2a-2b+2k+1)(a+n+d-1)}{n(2a+3k)(-2+2d+n-2k)(2b-1+n)(2a-2b+n)(2a+2d+n+k-1)}$
(3.7)	1	$4\frac{(n-k)(6n+2-3k)(7n-1-3k)}{(3n+1)(1+2n)n}$
(5.21)	3	$2\frac{(3a+1+k)(6a+2k+1)(n-k)}{n(6a+n)(-n+6a+3+3k)}$
(5.22)	1	$\frac{(5+6k)(1+2k)(n-k)}{(24n+4)(6n-1)n}$
(5.23)	1	$4\frac{(21n-7-9k)(6n+1-3k)(n-k)}{(6n+1)(3n-1)n}$
(5.24)	2	$4\frac{n-k}{1+3n}$
(5.25)	2	$\frac{(4n-4k)(1+2k)(2+3k)}{n(3n-1)(1+3n)}$
(5.27)	1	$81\frac{(n-1-2k)(n-2k)(-1+3n-3k)}{n(3n-1)(-1+9n-21k)}$

Table 5. Gessel and Stanton's open problems

Eq.	Identity
(6.2)	$_7F_6\left(\begin{matrix} a+1/2\,,a\,,b\,,1-b\,,-n\,,(2a+1)/3+n\,,a/2+1 \\ 1/2\,,(2a-b+3)/3\,,(2a+b+2)/3\,,-3n\,,2a+1+3n\,,a/2 \end{matrix}\,\middle\vert\, 1\right) = \dfrac{((2a+2)/3)_n\,(2a/3+1)_n\,((1+b)/3)_n\,((2-b)/3)_n}{((2a-b)/3+1)_n\,((2a+b+2)/3)_n\,(2/3)_n\,(1/3)_n}$
(6.3)	$_5F_4\left(\begin{matrix} a+1/2\,,a\,,-n\,,(2a+1)/3+n\,,a/2+1 \\ 1/2\,,-3n\,,2a+1+3n\,,a/2 \end{matrix}\,\middle\vert\, 9\right) = \dfrac{((2a+2)/3)_n\,(2a/3+1)_n}{(2/3)_n\,(1/3)_n}$
(6.5)	$_2F_1\left(\begin{matrix} -n\,,-n+1/4 \\ 2n+5/4 \end{matrix}\,\middle\vert\, \frac{1}{9}\right) = \dfrac{(5/4)_{2n}}{(2/3)_n\,(13/12)_n}\left(\dfrac{2^6}{3^5}\right)^n$
(6.6)	$_2F_1\left(\begin{matrix} -n\,,-n+1/4 \\ 2n+9/4 \end{matrix}\,\middle\vert\, \frac{1}{9}\right) = \dfrac{(9/4)_{2n}}{(4/3)_n\,(17/12)_n}\left(\dfrac{2^6}{3^5}\right)^n$

Rational certificates

Eq.	m	$R(n,k)$
(6.2)	1	$6\dfrac{(a-1+3n)(a+k)(2a+2k+1)(n-k)(b-1-k)(b+k)}{(a+2k)(3n-k)(3n-1-k)(3n-2-k)(2a-b+3n)(2a+b-1+3n)}$
(6.3)	1	$-\dfrac{(6a-6+18n)(n-k)(2a+2k+1)(a+k)}{(a+2k)(3n-k)(3n-1-k)(3n-2-k)}$
(6.5)	1	$-\dfrac{\left(52n^2-13n-21-56k+16nk-32k^2\right)(n-k)(4n-1-4k)}{(108n-27)(3n-1)(1+12n)n}$
(6.6)	1	$-\dfrac{\left(52n^2+39n-55-84k+16nk-32k^2\right)(4n-1-4k)(n-k)}{(108n-27)(1+3n)(5+12n)n}$

Zeilberger's algorithm determines a *holonomic recurrence equation*

$$\sum_{j=0}^{J} P_j(n)\,\Sigma(n-j) = 0 \tag{6.1}$$

with polynomials P_j in n, for sums

$$\Sigma(n) := \sum_{k\in\mathbb{Z}} F(n,k) \tag{6.2}$$

for which $F(n,k)$ is a hypergeometric term w.r.t. both n and k. For rigorous descriptions, see Koornwinder (1993) and Graham, Knuth and Patashnik (1994), Section 5.8.

Implementations of the Zeilberger algorithm were given by Zeilberger (1990) and Koornwinder (1993) in MAPLE, and by Paule and Schorn (1994) in MATHEMATICA. Along the lines of Koornwinder (1993), we implemented Zeilberger's algorithm in REDUCE (Koepf, 1995) and MAPLE, available in releases REDUCE 3.6 and MAPLE V.4 through the `zeilberg` and the `sumtools` packages.

Like for the Wilf-Zeilberger method, Zeilberger's algorithm is accompanied by a rational *certification mechanism*. *Note that Zeilberger's algorithm can be applied to ratios of* products of rational functions, exponentials, factorials, Γ function terms, binomial coefficients, and Pochhammer symbols that are integer-linear in their arguments w.r.t. both n and k.

Next we will present a modified version of Zeilberger's algorithm that is applicable if the arguments of such expressions are rational-linear w.r.t. n and k.

7. An extended version of Zeilberger's algorithm

Our extended version of Zeilberger's algorithm deals with the question to determine a holonomic recurrence equation (6.1) for sums (6.2) for which $F(n,k)$ is an (m,l)-fold *hypergeometric term* w.r.t. (n,k), see Section 5.

In particular, this applies to all cases when the input function $F(n,k)$ is given as a ratio of products of rational functions, exponentials, factorials, Γ function terms, binomial coefficients, and Pochhammer symbols that are rational-linear in their arguments w.r.t. both n, and k.

First of all we mention that Zeilberger's algorithm may be applicable even though this is safely the case only if the arguments are integer-linear. An example of that type is

$$\Sigma(n) := {}_2F_1\left(\begin{array}{c} -n/2\,,\quad -n/2+1/2 \\ b+1/2 \end{array}\middle|\,1\right) = \sum_{k=0}^{\infty} \frac{(-n/2)_k\,(-n/2+1/2)_k}{k!\,(b+1/2)_k}\,,$$

for which an application of Zeilberger's algorithm yields the recurrence equation

$$(2b+n-1)\,\Sigma(n) - 2(b+n-1)\,\Sigma(n-1) = 0\,,$$

and therefore the explicit representation

$$\Sigma(n) = \frac{2^n\,(b)_n}{(2b)_n}\,.$$

Zeilberger's algorithm applies since $F(n,k)/F(n-1,k)$ and $F(n,k)/F(n,k-1)$ are rational even though the representing expression for $F(n,k)$ is not integer-linear w.r.t n.

On the other hand, not for every $F(n,k)$ given with rational-linear Γ-arguments, Zeilberger's algorithm is applicable. An example for this situation is the left hand side of Watson's theorem w.r.t. variable a (see Table 1).

We present now an algorithm which can be applied for arbitrary rational-linear input.

ALGORITHM 7.1. (`extended_sumrecursion`)
The following steps perform an algorithm to determine a holonomic recurrence equation (6.1) *for sums* (6.2).

1 Input: $F(n,k)$, given as a ratio of products of rational functions, exponentials, factorials, Γ function terms, binomial coefficients, and Pochhammer symbols with rational-linear arguments in n and k.
2 Build the list of all arguments. They are of the form $p_j/q_j\,n + s_j/t_j\,k + \alpha_j$ with integer p_j, q_j, s_j, t_j, p_j/q_j and s_j/t_j in lowest terms, q_j and t_j positive.
3 Calculate $m := \operatorname{lcm}\{q_j\}$ and $l := \operatorname{lcm}\{t_j\}$.
4 Define $\tilde{F}(n,k) := F(mn, kl)$. Then $\tilde{F}(n,k)$ is integer-linear in the arguments.

5 Apply Zeilberger's algorithm to $\tilde{F}(n,k)$. Get the recurrence equation

$$\sum_{j=0}^{J} P_j(n)\,\tilde{\Sigma}(n-j) = 0$$

with polynomials P_j in n, for the sum

$$\tilde{\Sigma}(n) := \sum_{k\in\mathbb{Z}} \tilde{F}(n,k)\,.$$

6 The output is the recurrence equation

$$\sum_{j=0}^{J} P_j(n/m)\,\Sigma(n-jm) = 0$$

for the sum

$$\Sigma(n) := \sum_{k\in\mathbb{Z}} F(n,k)\,.$$

PROOF. Obviously our construction provides us with $\tilde{F}(n,k)$ that is integer-linear in the arguments involved. Therefore Zeilberger's algorithm can be applied, and the result follows. □

Note that even though in general Zeilberger's algorithm does not find the recurrence equation of lowest order (which in practice rarely happens), one can prove that for *proper hypergeometric terms* $F(n,k)$ having a representation

$$F(n,k) = P(n,k)\,\frac{Q(n,k)}{R(n,k)} w^n z^k\,, \tag{7.1}$$

where $P(n,k)$ is a polynomial and $Q(n,k), R(n,k)$ are Γ-term products with integer-linear arguments, Zeilberger's algorithm terminates, see e.g. Knuth and Patashnik (1994), Section 5.8.

Therefore the above algorithm terminates for terms (7.1) where $P(n,k)$ is a polynomial and $Q(n,k), R(n,k)$ are Γ-term products with rational-linear arguments.

As a first example, we apply our algorithm to the Watson function

$$\Sigma(n) = {}_3F_2\left(\begin{array}{c} -n\,,b\,,c \\ (-n+b+1)/2\,,2c \end{array}\middle|\,1\right)$$

w.r.t. the variable n to which Zeilberger's algorithm does not apply. In this case, the algorithm determines $m=2$ and $l=1$, and leads to the two-fold recurrence equation

$$(b-2c-n+1)\,(n-1)\,\Sigma(n-2) - (b-n+1)\,(2c+n-1)\,\Sigma(n) = 0$$

from which the explicit right hand representation listed in Table 1 can be deduced since $\Sigma(0)=1$ and $\Sigma(1)=0$.

It turns out that our method is applicable to all identities considered in this paper to which Zeilberger's original approach does not apply.

For example, we consider one of the major identities of the paper of Gessel and Stanton

(1982): The evaluation of (1.8)

$$\begin{aligned}\Sigma(n) &:= {}_7F_6\left(\begin{array}{c} a\,,b\,,a+1/2-b\,,1+2a/3\,,1-2d\,,2a+2d+n\,,-n \\ 2a-2b+1\,,2b\,,2a/3\,,a+d+1/2\,,1-d-n/2\,,1+a+n/2 \end{array}\middle| 1\right) \\ &= \begin{cases} 0 & \text{if } n \text{ odd} \\ \dfrac{(b+d)_{n/2}\,(d-b+a+1/2)_{n/2}\,n!\,(a+1)_{n/2}}{(b+1/2)_{n/2}\,(a+d+1/2)_{n/2}\,(d)_{n/2}\,(n/2)!\,(a-b+1)_{n/2}} & \text{otherwise} \end{cases}\end{aligned}$$

cannot be handled w.r.t. n using Zeilberger's algorithm, but the extended version leads to the equivalent 2-fold recurrence equation

$$\begin{aligned} 0 &= (n-1+2d+2a)(2b-n-2a)(n-1+2b)(n-2+2d)\Sigma(n) \\ &\quad +(n-1+2d-2b+2a)(n-2+2d+2b)(2a+n)(n-1)\Sigma(n-2)\,. \end{aligned}$$

Finally, as an example with $l \neq 1$, we consider (5.5), again. Our algorithm generates $m = 1$ and $l = 2$, and the recurrence equations

$$\Sigma(n) - \Sigma(n-1) = 0 \qquad \text{and} \qquad 2\Sigma(n) + \Sigma(n-1) = 0$$

for

$$\Sigma(n) := (-2)^n \binom{n}{k} \cdot \binom{k/2}{n}\,, \qquad \text{and} \qquad \Sigma(n) := \binom{n}{k} \cdot \binom{k/2}{n}\,,$$

respectively.

8. Deduction of hypergeometric identities

In this section, we show that with a good implementation of Zeilberger's algorithm and our extension at hand, one can *discover* new identities. A kind of deductive strategy is: Applying Zeilberger's algorithm to the general ${}_2F_1$ polynomial

$$\Sigma(n) := {}_2F_1\left(\begin{array}{c} a\,,-n \\ b \end{array}\middle| x\right)\,,$$

e. g., leads to the recurrence equation

$$(b-1+n)\,\Sigma(n) + (-2n+xn+xa-b+2-x)\,\Sigma(n-1) - (x-1)\,(n-1)\,\Sigma(n-2) = 0\,.$$

It is hypergeometric in particular if the coefficient of $\Sigma(n-2)$ equals zero, i.e., if $x = 1$, implying Vandermonde's identity. Moreover, the coefficient of $\Sigma(n-1)$ can be made zero (equating coefficients) by choosing $x = 2$, and $b = 2a$, in which situation we get

$$(n+2a-1)\,\Sigma(n) - (n-1)\,\Sigma(n-2) = 0\,.$$

Therefore we have deduced the identity

$${}_2F_1\left(\begin{array}{c} a\,,-n \\ 2a \end{array}\middle| 2\right) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \dfrac{(1/2)_{n/2}}{(1/2+a)_{n/2}} & \text{otherwise} \end{cases}\,.$$

We see that this method, to some extent, can be a substitute for the ingenuity of people like Dougall, Bailey, Andrews, Gessel or Stanton to find hypergeometric sums which can be represented by single hypergeometric terms.

As a final example we try to find hypergeometric functions of the form

$$\Sigma(n) := {}_2F_1\left(\begin{array}{c} a\,,-n \\ n+b \end{array}\middle|\, x\right)$$

for which a, b and x are constants w.r.t. n, and for which a recurrence equation with only two terms $\Sigma(n-j)$ is valid.

The recurrence equation for $\Sigma(n)$ turns out to be

$$\begin{aligned} 0 \;=\; & -(x-1)^2\,(n-1)\,(n-1+b)\,(n-2+b)\,(xn+n-xa-x+bx)\,\Sigma(n-2) \\ & +(n-1+b)\,P(n,a,b,x)\,\Sigma(n-1) \\ & +x\,(2n+b-1)\,(2n+b-2)\,(n-a-1+b)\,(xn+n-xa-2x-1+bx)\,\Sigma(n)\,, \end{aligned}$$

where $P(n, a, b, x)$ denotes a very complicated polynomial of degree 2 in n. To obtain a recurrence equation for which only two terms $\Sigma(n-j)$ different from zero occur, we may set the coefficient lists with respect to n of any of the factors occurring to zero, and try to solve for a, b and x. Note that since the resulting equations are polynomial systems, by Gröbner bases methods these can be solved algorithmically.

In our case, we obtain either $x = 1$, or we are led to the Kummer identity, i.e., to the values $b = a + 1$ and $x = -1$. The only exception occurs when we set the coefficient list with respect to n of the factor $P(n, a, b, x)$ to zero, leading to the Kummer case again, and to the second solution set

$$\{a = 1/2, b = 3/2, x^2 - 6x + 1 = 0\}\,.$$

Therefore, for $x = 3 \pm 2\sqrt{2}$, we have the recurrence equation

$$-4\;(2n-1)\,(2n+1)\,\Sigma(n-2) + (4n-1)\,(4n+1)\,\Sigma(n) = 0$$

leading to the closed form representations

$$ {}_2F_1\left(\begin{array}{c} 1/2\,,-n \\ n+3/2 \end{array}\middle|\, 3\pm2\sqrt{2}\right) = \begin{cases} \dfrac{2\,(5/4)_{(n-1)/2}\,(7/4)_{(n-1)/2}}{5\,(11/8)_{(n-1)/2}\,(13/8)_{(n-1)/2}}\,(1\mp\sqrt{2}) & \text{if } n \text{ odd} \\[2ex] \dfrac{(3/4)_{n/2}\,(5/4)_{n/2}}{(7/8)_{n/2}\,(9/8)_{n/2}} & \text{otherwise} \end{cases}$$

Hence, for even n, the values at $x = 3 + 2\sqrt{2}$ and $x = 3 - 2\sqrt{2}$ are rational and equal:

$$ {}_2F_1\left(\begin{array}{c} 1/2,-2n \\ 2n+3/2 \end{array}\middle|\, 3\pm2\sqrt{2}\right) = \sum_{k=0}^{2n}(-1)^k\frac{\binom{2n}{k}\binom{2n+k+1}{k}}{\binom{4n+2k+2}{2k}}\left(3\pm2\sqrt{2}\right)^k = \frac{(3/4)_n\,(5/4)_n}{(7/8)_n\,(9/8)_n}.$$

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