The Sum $16^n \sum_{k=0}^{2n} 4^k \binom{\frac{1}{2}}{k}\binom{-\frac{1}{2}}{k}\binom{-2k}{2n-k}$: A Computer Assisted Proof of its Closed Form, and Some Generalised Results

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Abstract

We present a second proof of an interesting binomial coefficient identity which is computer assisted, together with two related generalised results also generated computationally.

Introduction

Consider the binomial coefficient sum $16^n \sum_{k=0}^{2n} 4^k \binom{\frac{1}{2}}{k}\binom{-\frac{1}{2}}{k}\binom{-2k}{2n-k}$. In [1] some powerful results concerning Dixon type sums and formulas—drawn together by Larsen [2, Chapter 9]—have been applied to this sum to produce the closed form $(4n+1)\binom{2n}{n}^2$. In this paper we present a computer assisted proof which is not without an element or two of interest, and which demonstrates the power of this type of modern day proof construction. This leads naturally on to two generalised identities (*i.e.*, hypergeometric evaluations) which recover the closed form of the sum (one result evaluates the sum as it stands, and the other the sum with the order of its terms reversed).

We first wish to convert the sum to hypergeometric form. Writing it as $16^n \sum_{k=0}^{2n} s(k;n)$, we see that [1, (4), p.4] the summand is expressible as

$$\begin{aligned} s(k;n) &= 4^k \binom{\frac{1}{2}}{k}\binom{-\frac{1}{2}}{k}\binom{-2k}{2n-k} \\ &= 4^k \frac{[\frac{1}{2}]_k}{k!}\frac{[-\frac{1}{2}]_k}{k!}\binom{-2k}{2n-k} \\ &= \frac{2}{(2n)!^2}(-1)^k[-2n]_k[1/2]_k[2n]_{2n-k}[2n-1]_{2n-k}\binom{2n}{k} \qquad (1) \end{aligned}$$

in terms of the usual falling factorial function $[u]_k = u(u-1)(u-2)(u-3)\cdots(u-k+1)$, with ratio

$$\begin{aligned} &\frac{s(k+1;n)}{s(k;n)} \\ &\quad = \frac{(-1)^{k+1}}{(-1)^k}\cdot\frac{[-2n]_{k+1}}{[-2n]_k}\cdot\frac{[1/2]_{k+1}}{[1/2]_k}\cdot \\ &\qquad\qquad \frac{[2n]_{2n-k-1}}{[2n]_{2n-k}}\cdot\frac{[2n-1]_{2n-k-1}}{[2n-1]_{2n-k}}\cdot\frac{\binom{2n}{k+1}}{\binom{2n}{k}} \\ &\quad = (-1)\cdot-(k+2n)\cdot-\left(k-\frac{1}{2}\right)\cdot\frac{1}{k+1}\cdot\frac{1}{k}\cdot-\frac{(k-2n)}{k+1} \\ &\quad = \frac{(k-2n)(k+2n)(k-\frac{1}{2})}{k(k+1)^2}. \qquad (2) \end{aligned}$$

Alternatively, by replacing the binomial coefficients by factorials (resp. Gamma terms) and cancelling factorials whose arguments differ by an integer, this ratio can be computed algorithmically (see Koepf [3, Algorithm 2.1, p.15]). The denominator factor k is clearly a problem, added to which the hypergeometric multiplier is $s(0;n) = 2[2n-1]_{2n}/(2n)!$ which is zero except at $n=0$. A slight change of approach is, therefore, required at this initial stage. The identity in question holds identically for $n=0$ (both l.h.s. and r.h.s. being 1). Since, for $n \geq 1$,

$$\begin{aligned} \sum_{k=0}^{2n} s(k;n) &= s(0;n) + \sum_{k=1}^{2n} s(k;n) \\ &= 0 + \sum_{k=1}^{2n} s(k;n) \end{aligned}$$

$$= \sum_{k=1}^{2n} s(k;n), \tag{3}$$

we prove an equivalent version of the identity which involves what may be described as the 'natural bounds' of the sum:

Theorem For integer $n \geq 1$,

$$S(n) = 16^n \sum_{k=1}^{2n} 4^k \binom{\frac{1}{2}}{k} \binom{-\frac{1}{2}}{k} \binom{-2k}{2n-k} = (4n+1) \binom{2n}{n}^2.$$

The Proof

Conversion of the sum $S(n)$ is now straightforward, an easy hand calculation yielding (we now consider instead $\sum_{k=1}^{2n} s(k;n) = \sum_{k=0}^{2n-1} s(k+1;n)$, with term ratio $s(k+2;n)/s(k+1;n)$ available directly from (2) and multiplier $s(1;n) = 2n$)

$$S(n) = 2^{4n+1} n\, {}_3F_2 \left(\begin{matrix} 1-2n, 1+2n, \frac{1}{2} \\ 2, 2 \end{matrix} \,\middle|\, 1 \right), \tag{P1}$$

whose evaluation we wish to secure. Zeilberger's algorithm[1] generates a recurrence equation for $S(n)$ which takes the form

$$0 = f_1(n)S(n) - f_2(n)S(n+1) + f_3(n)S(n+2), \tag{P2}$$

where

$$\begin{aligned} f_1(n) &= 256(24n^2+72n+53)n^2(2n+1)^2, \\ f_2(n) &= 4(768n^6+4608n^5+11872n^4 \\ &\qquad +16768n^3+13216n^2+5184n+729), \\ f_3(n) &= (24n^2+24n+5)(n+2)^2(2n+3)^2. \end{aligned} \tag{P3}$$

Since (P2) is not first order, we use an algorithm due to van Hoeij to detect any hypergeometric term solutions. The routine gives a completely automated means to determine whether or not $S(n)$ is a linear combination of hypergeometric terms,[2] and duly produces the answer

$$S(n) = As_1(n) + Bs_2(n), \tag{P4}$$

[1] Implemented through the software package "hsum9.mpl" of the author W.A.K., and accessible at **http://www.mathematik.uni-kassel.de/~koepf/Publikationen**.

[2] Whilst Petkovšek formulated an algorithm in the early 1990s to deal with such an issue [4], its inefficiency subsequently led van Hoeij [5] to develop an improved version which is presently embedded in the Maple 10 command "`LREtools/hypergeomsols`".

where

$$
\begin{aligned}
s_1(n) &= 16^n(1-4n)\frac{(n-1)!^2}{\Gamma^2(n+\frac{1}{2})}, \\
s_2(n) &= 16^n(1+4n)\frac{\Gamma^2(n+\frac{1}{2})}{n!^2}. \qquad \text{(P5)}
\end{aligned}
$$

Noting that $\Gamma(n+\frac{1}{2}) = 4^{-n}\sqrt{\pi}(2n)!/n!$, it remains but to firstly evaluate (P4) at $n=1$ which gives—with $s_1(1) = -48/\Gamma^2(\frac{3}{2}) = -192/\pi$, $s_2(1) = 80\Gamma^2(\frac{3}{2}) = 20\pi$ and $S(1) = 16[s(1;1)+s(2;1)] = 16[2-\frac{3}{4}] = 20$—the equation

$$20 = (-192/\pi)A + (20\pi)B, \qquad \text{(P6)}$$

and then at $n=2$ which generates in the same way the equation

$$324 = (-28672/9\pi)A + (324\pi)B. \qquad \text{(P7)}$$

The solution to (P6),(P7) by inspection is $A=0$, $B=1/\pi$,[3] whence to complete the proof we have simply that, from (P4) and (P5),

$$
\begin{aligned}
S(n) &= \frac{1}{\pi}s_2(n) \\
&= \frac{1}{\pi}\cdot 16^n(1+4n)\frac{4^{-2n}\pi(2n)!^2/n!^2}{n!^2} \\
&= (4n+1)\binom{2n}{n}^2, \qquad \text{(P8)}
\end{aligned}
$$

as required.□

Generalised Results

We have, from (P1),(P8), established the hypergeometric evaluation

$$ {}_3F_2\left(\left.\begin{matrix}1-2n, 1+2n, \frac{1}{2}\\ 2, 2\end{matrix}\right| 1\right) = \frac{1}{2^{4n+1}n}(4n+1)\binom{2n}{n}^2, \qquad (4)$$

which we note is a particular instance of the more general result

$$ {}_3F_2\left(\left.\begin{matrix}1-2n, 1+2c+2n, a+\frac{1}{2}\\ 2a+2, c+2\end{matrix}\right| 1\right) = \frac{1}{2\sqrt{\pi}}\frac{(2a+2c+4n+1)(c+1)}{(c+2n)(c-a+\frac{1}{2})}\frac{(c-a+\frac{1}{2})_n}{(a+1)_n(c+1)_n}\Gamma\left(n+\frac{1}{2}\right) \qquad (5)$$

[3]Of course any two values of $n \geq 1$ will produce simultaneous equations in the unknowns A, B; the interested reader might care to check that, for example, $n = 3, 4$ lead to $5200 = (-11534336/225\pi)A + (5200\pi)B$ and $83300 = (-201326592/245\pi)A + (83300\pi)B$, resp.

found by a repeat, and computationally non-trivial, application of the van Hoeij algorithm (the r.h.s. here involves the rising factorial function $(u)_k = u(u+1)(u+2)(u+3)\cdots(u+k-1)$)—setting $a = c = 0$ reduces (5) to simply

$$
\begin{aligned}
{}_3F_2\left(\begin{matrix}1-2n, 1+2n, \frac{1}{2} \\ 2, 2\end{matrix} \,\middle|\, 1\right) &= \frac{1}{2\sqrt{\pi}} \frac{(4n+1)}{n} \frac{(\frac{1}{2})_n}{(1)_n^2} \Gamma\left(n+\frac{1}{2}\right) \\
&= \frac{(4n+1)}{2n} \frac{(\frac{1}{2})_n^2}{n!^2}, \qquad (6)
\end{aligned}
$$

since $\Gamma(n+\frac{1}{2}) = \sqrt{\pi}(\frac{1}{2})_n$, and equation (4) is now immediate from $(\frac{1}{2})_n = 4^{-n}(2n)!/n!$.

That the evaluation of a generalised version of the ${}_3F_2(1)$ in (4) exists is not surprising since the latter is 'close' to being dealt with by either Watson's or Whipple's Theorem [6, (2.3.3.13),(2.3.3.14), p.54]. Alternatively, we can obtain the evaluation of $S(n)$ as a specialisation of the new identity

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix}a, -2n, c \\ 1+\frac{1}{2}a-n, 2c+1\end{matrix} \,\middle|\, 1\right) = \\
&\frac{1}{4^n} \frac{a(a-2c+2n)}{(a-2c)(a+2n)} (2n)! \frac{1}{(-\frac{1}{2}a)_n (c+\frac{1}{2})_n} \binom{-1-\frac{1}{2}a+c+n}{n} \qquad (7)
\end{aligned}
$$

which, for $a = 1-2n$, $c = -2n$, yields

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix}1-2n, -2n, -2n \\ \frac{3}{2}-2n, 1-4n\end{matrix} \,\middle|\, 1\right) \\
&\quad = \frac{1}{4^n} \frac{(1-2n)(1+4n)}{1+2n} (2n)! \frac{1}{(n-\frac{1}{2})_n (-2n+\frac{1}{2})_n} \binom{-\frac{3}{2}}{n} \\
&\quad = -(4n+1)(4n-1) \frac{(2n)!^6}{n!^4 (4n)!^2} \qquad (8)
\end{aligned}
$$

since

$$
\begin{aligned}
\left(n-\frac{1}{2}\right)_n &= \frac{1}{4^n} \frac{(2n-1)}{(4n-1)} \frac{n!(4n)!}{(2n)!^2}, \\
\left(-2n+\frac{1}{2}\right)_n &= \frac{(-1)^n}{4^n} \frac{n!(4n)!}{(2n)!^2}, \\
\binom{-\frac{3}{2}}{n} &= \frac{(-1)^n}{4^n} (2n+1) \frac{(2n)!}{n!^2}. \qquad (9)
\end{aligned}
$$

We then see that, on reversing the order of summation in $S(n)$,

$$
S(n) = 16^n \sum_{k=1}^{2n} 4^k \binom{\frac{1}{2}}{k} \binom{-\frac{1}{2}}{k} \binom{-2k}{2n-k}
$$

$$
\begin{aligned}
&= 16^{2n}\sum_{k=0}^{2n-1} 4^{-k}\binom{\frac{1}{2}}{2n-k}\binom{-\frac{1}{2}}{2n-k}\binom{2k-4n}{k} \\
&= 16^{2n}\binom{\frac{1}{2}}{2n}\binom{-\frac{1}{2}}{2n}\,{}_3F_2\left(\left.\begin{matrix}1-2n,-2n,-2n\\ \frac{3}{2}-2n,1-4n\end{matrix}\right|1\right) \\
&= -\frac{1}{(4n-1)}\frac{(4n)!^2}{(2n)!^4}\,{}_3F_2\left(\left.\begin{matrix}1-2n,-2n,-2n\\ \frac{3}{2}-2n,1-4n\end{matrix}\right|1\right) \\
&= (4n+1)\binom{2n}{n}^2 \qquad (10)
\end{aligned}
$$

by (8). The identity (7), like (5), was also revealed by an implementation of the van Hoeij algorithm.

Remark 1 As a minor point of interest, equating $S(n)$ in each of (P1) and the penultimate line of (10) gives a ${}_3F_2(1)$ transformation which is in fact available by other means independently (reader exercise).

Remark 2 Note that both generalisations (5) and (7) turn out to be special cases of a very general Watson type summation formula due to Lewanowicz [7] who evaluates the sum

$$
{}_3F_2\left(\left.\begin{matrix}a,b,c\\ \frac{1}{2}(a+b+i+1),2c+j\end{matrix}\right|1\right) \qquad (11)
$$

for integer values i,j (fixed j, arbitrary i—the classic Watson summation formula is delivered by setting $i=j=0$); our identities (5),(7) are each intances of the particular values $i=j=1$. Whereas the above sum (11) can be written in general as a linear combination of hypergeometric terms, in establishing a closed form for $S(n)$ we see that only one hypergeometric term survives.

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