Integral Proofs of Two Alternating Sign Binomial Coefficient Identities

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Abstract

An alternating sign binomial coefficient identity—available computationally from both a mainstream and specialist algebraic software package, and also as a special case of an analytic result due to P. Kirschenhofer—is proven in a novel way via integration. The method generates another, more complex, identity of the same type in a similar way and can be extended in principle to develop further results.

Introduction

Consider the sum

$$\sum_{k=0}^{n} \frac{(-1)^k}{(k+m)^2} \binom{n}{k} \tag{1}$$

for $m \geq 1$, $n \geq 0$ both integer. When converted to the hypergeometric series (in usual notation)

$$m^{-2}\,{}_3F_2\left(\begin{array}{c} m, m, -n \\ 1+m, 1+m \end{array} \middle| \, 1\right) \tag{2}$$

it cannot be evaluated to a closed form (involving elementary functions) via standard hypergeometric results, and a combination of Zeilberger's and Petkovšek's algorithms (implemented by the author W.A.K. using his specialist software package "hsum6.mpl",[1] see [1]) confirms this. The system "Mathematica" gives as output for the sum (1) the expression

$$\frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} \left[\psi_0(m+n+1) - \psi_0(m)\right] \tag{3}$$

in terms of the Gamma function $\Gamma(z)$ and a particular Polygamma function which is defined generally as

$$\psi_n(z) = \frac{d^{n+1}}{dz^{n+1}} \left(\ln[\Gamma(z)]\right), \qquad n \geq 0, \tag{4}$$

for complex z. Now,

$$\psi_n(z) = \frac{d^n}{dz^n}\left(\frac{d}{dz}\left(\ln\left[\Gamma(z)\right]\right)\right) = \frac{d^n}{dz^n}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \frac{d^n \psi_0(z)}{dz^n}, \tag{5}$$

where $\psi_0(z) = \Gamma'(z)/\Gamma(z)$ is termed the Digamma (or Psi) function, and it can be shown that, for integral $p \geq 2$ ($\psi_0(1) = -\gamma$),

$$\psi_0(p) = -\gamma + \sum_{k=1}^{p-1} \frac{1}{k}, \tag{6}$$

with γ the Euler-Mascheroni constant. Hence, noting that

$$\frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} = \frac{1}{m\binom{m+n}{n}} \tag{7}$$

[1]Visit the site **http://www.mathematik.uni-kassel.de/~koepf/Publikationen** for more information.

and, by (6),

$$\psi_0(m+n+1) - \psi_0(m) = \sum_{k=m}^{m+n} \frac{1}{k}, \tag{8}$$

we arrive at the following identity which does not appear in Gould's well known listing [2]:

Identity I For integer $m \geq 1$, $n \geq 0$,

$$m \binom{m+n}{n} \sum_{k=0}^{n} \frac{(-1)^k}{(k+m)^2} \binom{n}{k} = \sum_{k=m}^{m+n} \frac{1}{k}.$$

In 1996, Peter Kirschenhofer published a paper [3] in which he considered alternating sums of type

$$\sum_{k=a}^{N} (-1)^k f(k) \binom{N}{k}, \qquad 0 \leq a \leq N, \tag{9}$$

of which (1) describes a special case. Corollary 2.2 therein (p.4) states that, for K, m positive integers,

$$\begin{aligned} K(m-1)! \binom{N+K}{K} \sum_{k=0}^{N} \frac{(-1)^k}{(k+K)^m} \binom{N}{k} \\ = B_{m-1}\left(\ldots, (i-1)!\left(H_{N+K}^{(i)} - H_{K-1}^{(i)}\right), \ldots\right), \end{aligned} \tag{10}$$

where $H_r^{(i)} = \sum_{k=1}^{r} k^{-i}$ is a Harmonic number of order i and B_{m-1} is a Bell polynomial. For $L \geq 1$, then writing $\mathbf{x}_L = \{x_1, \ldots, x_i, \ldots, x_L\}$ the Bell polynomials $B_0 = 1, B_1(\mathbf{x}_1), B_2(\mathbf{x}_2), \ldots,$ appearing in (10) are among those generated according to the equation

$$\sum_{n=0}^{\infty} B_n(\mathbf{x}) \frac{t^n}{n!} = \exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k!}\right). \tag{11}$$

Kirschenhofer denotes by $B_m(\ldots, x_i, \ldots)$ the particular m-variate polynomial $B_m(x_1, \ldots, x_i, \ldots, x_m) = B_m(\mathbf{x}_m)$, where, explicitly, $B_1(x_1) = x_1$, $B_2(x_1, x_2) = x_1^2 + x_2$, $B_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3$, *etc.* Equation (10) has as its r.h.s. a Bell polynomial whose ith variable (of $m-1$ in total) is a linear combination of ith order Harmonic numbers, and recovers immediately Identity I for $m = 2$ with n, m then replacing N, K.

Noting that the traditional method of dealing with sums of the form (9) (which are said to arise frequently in connection with the so called "average-case" analysis of algorithms and data structures) is through complex contour integration, Kirschenhofer derived some powerful summation formulae from an adept manipulation of generating functions. In this paper we first present a new proof of Identity I via integration—more specifically, by treating an integral in two distinct ways. To this extent the technique echoes in style that of Larcombe *et al.* [4] in establishing the simpler result

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\begin{array}{c} n \\ k \end{array} \right) = \sum_{k=1}^{n} \frac{1}{k} = H_n^{(1)} \tag{12}$$

to which, we remark here (this observation was not made in [4]), Proposition 2.1 of [3, p.2] contracts upon setting $m = 1$, $N = n$ and $K = 0$. Before detailing this proof, we mention a quick computer-based confirmation of Identity I which is typical of a modern day approach to the validation of such a result. Denoting the r.h.s. sum as $\sum_{k=m}^{m+n} \frac{1}{k} = R_1(m,n)$, it is a trivial matter to write down the inhomogeneous recurrence (w.r.t. n)

$$1 = (m+n+1)[R_1(m,n+1) - R_1(m,n)] \tag{13}$$

satisfied by it. Replacing n with $n+1$ and combining the result with (13) yields the homogeneous recurrence

$$\begin{aligned} 0 \quad &= \quad (m+n+2)R_1(m,n+2) \\ &\quad - [2(m+n)+3]R_1(m,n+1) + (m+n+1)R_1(m,n) \end{aligned} \tag{14}$$

which Zeilberger's algorithm, when executed computationally, generates for the l.h.s. expression $L_1(m,n) = m\binom{m+n}{n}\sum_{k=0}^{n}[(-1)^k/(k+m)^2]\binom{n}{k}$ also; the identity is established after checking the initial conditions $L_1(m,0) = \frac{1}{m} = R_1(m,0)$ and $L_1(m,1) = \frac{2m+1}{m(m+1)} = R_1(m,1)$.

Proof of Identity I

Define the integral

$$I(m,n) = \int_0^\infty xe^{-mx}(1-e^{-x})^n \;\; dx \tag{I1}$$

for $m,n > 0$. Integrating by parts gives[2]

$$I(m,n) \quad = \quad -\frac{1}{m^2}\left[(1+mx)e^{-mx}(1-e^{-x})^n\right]_0^\infty$$

[2]Observe that, given $m,n > 0$, the expression evaluated at 0 and ∞ vanishes at each separate limit.

$$
\begin{aligned}
&\quad + \frac{n}{m^2}\int_0^\infty (1+mx)e^{-(m+1)x}(1-e^{-x})^{n-1}\ \ dx \\
= &\ \frac{n}{m^2}\int_0^\infty (1+mx)e^{-(m+1)x}(1-e^{-x})^{n-1}\ \ dx \\
= &\ \frac{n}{m}I(m+1,n-1) \\
&\quad + \frac{n}{m^2}\int_0^\infty e^{-(m+1)x}(1-e^{-x})^{n-1}\ \ dx. \qquad \text{(I2)}
\end{aligned}
$$

Now, applying the substitution $t(x) = e^{-x}$,

$$
\begin{aligned}
\int_0^\infty e^{-(m+1)x}(1-e^{-x})^{n-1}\ \ dx &= \int_0^1 t^m(1-t)^{n-1}\ \ dt \\
&= \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}, \qquad \text{(I3)}
\end{aligned}
$$

and (I2) becomes the recurrence

$$
I(m,n) = \frac{n}{m}I(m+1,n-1) + \frac{m!n!}{m^2(m+n)!} \qquad \text{(I4)}
$$

that drives this part of the proof. In the first instance, we use (I4) to write

$$
I(m+1,n-1) = \frac{n-1}{m+1}I(m+2,n-2) + \frac{m!(n-1)!}{(m+1)(m+n)!}, \qquad \text{(I5)}
$$

which, when back-substituted into it, gives

$$
\begin{aligned}
I(m,n) &= \frac{n(n-1)}{m(m+1)}I(m+2,n-2) \\
&\quad + \left(\frac{1}{m} + \frac{1}{m+1}\right)\frac{m!n!}{m(m+n)!}. \qquad \text{(I6)}
\end{aligned}
$$

In a likewise fashion,

$$
I(m+2,n-2) = \frac{n-2}{m+2}I(m+3,n-3) + \frac{(m+1)!(n-2)!}{(m+2)(m+n)!} \qquad \text{(I7)}
$$

from (I4), and so (I6) reads

$$
\begin{aligned}
I(m,n) &= \frac{n(n-1)(n-2)}{m(m+1)(m+2)}I(m+3,n-3) \\
&\quad + \left(\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2}\right)\frac{m!n!}{m(m+n)!}. \qquad \text{(I8)}
\end{aligned}
$$

Continuing this process a further $n-3$ times, it is found that

$$\begin{aligned} I(m,n) &= \frac{n!}{m(m+1)(m+2)\cdots(m+n-1)} I(m+n,0) \\ &\quad + \left(\frac{1}{m}+\frac{1}{m+1}+\cdots+\frac{1}{m+n-1}\right)\frac{m!n!}{m(m+n)!}, \end{aligned} \qquad \text{(I9)}$$

which yields (noting that $I(m+n,0)=1/(m+n)^2$), after a little manipulation,

$$\begin{aligned} I(m,n) &= \left(\frac{1}{m}+\frac{1}{m+1}+\cdots+\frac{1}{m+n}\right)\frac{m!n!}{m(m+n)!} \\ &= \frac{1}{m}\left(\begin{array}{c} m+n \\ n \end{array}\right)^{-1}\sum_{k=m}^{m+n}\frac{1}{k}, \qquad m>0, n\geq 0; \end{aligned} \qquad \text{(I10)}$$

the result holds at $n=0$, where both sides are $\frac{1}{m^2}$.

The remainder of the proof is elementary. Expanding $(1-e^{-x})^n$ binomially as

$$(1-e^{-x})^n = \sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n \\ k \end{array}\right) e^{-kx}, \qquad n \geq 0, \qquad \text{(I11)}$$

then, from the definition (I1) of $I(m,n)$,

$$\begin{aligned} I(m,n) &= \int_0^\infty xe^{-mx}\sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n \\ k \end{array}\right) e^{-kx}\ dx \\ &= \sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n \\ k \end{array}\right)\int_0^\infty xe^{-(k+m)x}\ dx \\ &= \sum_{k=0}^{n}\frac{(-1)^k}{(k+m)^2}\left(\begin{array}{c} n \\ k \end{array}\right), \qquad m>0, n\geq 0. \end{aligned} \qquad \text{(I12)}$$

Equating $I(m,n)$ as described in (I10) and (I12), the proof is complete.□

We now state another identity of the same type as Identity I, and outline its proof by means of the approach seen above, noting that once again the result is available from (10) due to Kirschenhofer with $N=n$, $K=m$, after setting $m=3$.

Identity II For integer $m\geq 1$, $n\geq 0$,

$$2m\left(\begin{array}{c} m+n \\ n \end{array}\right)\sum_{k=0}^{n}\frac{(-1)^k}{(k+m)^3}\left(\begin{array}{c} n \\ k \end{array}\right) = \left(\sum_{k=m}^{m+n}\frac{1}{k}\right)^2 + \sum_{k=m}^{m+n}\frac{1}{k^2}.$$

Proof of Identity II

Consider, for $m, n > 0$, the integral

$$\begin{aligned}
J(m,n) &= \int_0^\infty x^2 e^{-mx}(1-e^{-x})^n \;\; dx \\
&= \frac{n}{m}\int_0^\infty x^2 e^{-(m+1)x}(1-e^{-x})^{n-1} \;\; dx \\
&\qquad + \frac{2n}{m^2}\int_0^\infty x e^{-(m+1)x}(1-e^{-x})^{n-1} \;\; dx \\
&\qquad\qquad + \frac{2n}{m^3}\int_0^\infty e^{-(m+1)x}(1-e^{-x})^{n-1} \;\; dx \\
&= \frac{n}{m}J(m+1,n-1) + \frac{2n}{m^2}I(m+1,n-1) \\
&\qquad + \frac{2n}{m^3}\frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}, \qquad \text{(II1)}
\end{aligned}$$

having integrated by parts as appropriate and deployed (I3). Using (I10), the last two r.h.s. terms of (II1) combine to give

$$J(m,n) = \frac{n}{m}J(m+1,n-1) + \frac{2m!n!}{m^2(m+n)!}\sum_{k=m}^{m+n}\frac{1}{k}, \qquad \text{(II2)}$$

which, when applied $n-1$ times (as in the manner of the proof of Identity I), leads eventually to

$$J(m,n) = \frac{2}{m}\left(\begin{array}{c} m+n \\ n \end{array}\right)^{-1}\sum_{k=m}^{m+n}\frac{1}{k}\sum_{j=k}^{m+n}\frac{1}{j} \qquad \text{(II3)}$$

for $m > 0$, $n \geq 0$. By the previous expansion (I11) of $(1-e^{-x})^n$, then

$$J(m,n) = 2\sum_{k=0}^{n}\frac{(-1)^k}{(k+m)^3}\left(\begin{array}{c} n \\ k \end{array}\right) \qquad \text{(II4)}$$

also $(m > 0, n \geq 0)$, and, observing that

$$\begin{aligned}
\sum_{k=m}^{m+n}\frac{1}{k}\sum_{j=k}^{m+n}\frac{1}{j} &= \sum_{k=m}^{m+n}\frac{1}{k^2} + \sum_{k=m}^{m+n-1}\frac{1}{k}\sum_{j=k+1}^{m+n}\frac{1}{j} \\
&= \sum_{k=m}^{m+n}\frac{1}{k^2} + \frac{1}{2}\left[\left(\sum_{k=m}^{m+n}\frac{1}{k}\right)^2 - \sum_{k=m}^{m+n}\frac{1}{k^2}\right] \\
&= \frac{1}{2}\left[\left(\sum_{k=m}^{m+n}\frac{1}{k}\right)^2 + \sum_{k=m}^{m+n}\frac{1}{k^2}\right], \qquad \text{(II5)}
\end{aligned}$$

equating $J(m,n)$ in (II3) and (II4) establishes Identity II.$\Box$

We should remark that, so far as a computer verification is concerned, Zeilberger's algorithm within hsum6.mpl generates the third order (homogeneous) recurrence

$$\begin{aligned} 0 \quad = \quad & (m+n+3)^2 L_2(m,n+3) \\ & - [3(m^2+n^2)+6mn+15(m+n)+19] L_2(m,n+2) \\ & + 3(m+n+2)^2 L_2(m,n+1) \\ & - (m+n+1)(m+n+2) L_2(m,n) \end{aligned} \tag{15}$$

easily for the l.h.s. $L_2(m,n) = 2m\binom{m+n}{n}\sum_{k=0}^{n}[(-1)^k/(k+m)^3]\binom{n}{k}$ of Identity II. The computational route to showing that $R_2(m,n) = (\sum_{k=m}^{m+n}\frac{1}{k})^2 + \sum_{k=m}^{m+n}\frac{1}{k^2}$ satisfies a recursion which is consistent with (15) is, however, a lengthy one whose details are omitted here (the procedure is outlined in a footnote below[3]). Having done this, it remains but to check that $L_2(m,n) = R_2(m,n)$ for the six values $n = 0, \ldots, 5$.

To finish, we highlight an alternative means to obtain Identities I,II using a result from Gould. Identity Z.5 on p.82 of [2] states that, for any polynomial $f(x)$ in x of degree $\leq n$,

$$f(x+y) = y\begin{pmatrix} y+n \\ n \end{pmatrix}\sum_{k=0}^{n}(-1)^k\frac{f(x-k)}{y+k}\begin{pmatrix} n \\ k \end{pmatrix}. \tag{16}$$

Based on a special case of this, we show how Identity I is derived in the Appendix and leave the proof of Identity II from it as a reader exercise.

Remark For completeness we note that, whilst m is restricted to positive integer values in Identities I,II, it would appear from inspection that both are in fact valid for all complex $m \neq 0, -1, -2, \ldots, -n$ (as indeed it seems (10) holds for complex K excepting these same values).[4] Empirical numeric investigation conducted by the authors supports this statement.

[3]The actual homogeneous recursion for $R_2(m,n)$ that, given its representation in terms of Harmonic numbers, can be computed using linear algebra is a complex sixth order one of the form $0 = \sum_{i=0}^{6} f_i(m,n)R_2(m,n+i) = F(m,n)$, say. In order to demonstrate the compatibility of the functions $L_2(m,n)$ and $R_2(m,n)$ it is sufficient to substitute (15) into $F(m,n)$ iteratively to deduce zero after simplification. This corresponds to the fact that the r.h.s. of (15), considered as a recurrence operator polynomial, is a non-commutative divisor of the operator $F(m,n)$; the precise calculations can be downloaded as a "Maple" worksheet from **http://www.mathematik.uni-kassel.de/~koepf/LFK2002.mws**, which also includes a computer validation of Identity I.

[4]The same may also be said for equations (A1),(A5) of the Appendix, which, being themselves particular instances of (10), are identities of a similar type.

Summary

The integration technique employed here to prove Identities I,II is interesting, and can be adopted to produce further results of a similar nature by raising the integer power of x accordingly in the initial integral to be developed (we have seen the powers 1 and 2 associated with $I(m,n)$, $J(m,n)$, resp.). The method does, though, have the obvious limitation regarding its algebraic tractability, and of course is an *ad hoc* one compared with the elegant generality of (10) formulated by Kirschenhofer. Computer validations such as those discussed in the paper are also subject to problematic levels of complexity for extensions of the two identities studied.

Acknowledgements

The authors wish to thank Drs. Axel Riese and Carsten Schneider for together bringing Reference 3 to their attention.

Appendix

Consider (16) with $y = m$, which, upon choosing $f(x) = 1$, reads

$$1 = m \binom{m+n}{n} \sum_{k=0}^{n} \frac{(-1)^k}{k+m} \binom{n}{k}. \tag{A1}$$

Noting that, for α constant, $\frac{d}{dm}\{\Gamma(m+\alpha)\} = \Gamma(m+\alpha)\psi_0(m+\alpha)$, then differentiating (A1) partially w.r.t. m gives, after some routine algebra,

$$\begin{aligned} 0 &= -\, m \sum_{k=0}^{n} \frac{(-1)^k}{(k+m)^2} \binom{n}{k} \\ &\quad + \{1 + m[\psi_0(m+n+1) - \psi_0(m+1)]\} \sum_{k=0}^{n} \frac{(-1)^k}{k+m} \binom{n}{k}. \end{aligned} \tag{A2}$$

Now, concerning the r.h.s. of (A2), by (6)

$$\begin{aligned} 1 + m[\psi_0(m+n+1) - \psi_0(m+1)] &= 1 + m \left(\sum_{k=1}^{m+n} - \sum_{k=1}^{m} \right) \frac{1}{k} \\ &= 1 + m \sum_{k=m+1}^{m+n} \frac{1}{k} \end{aligned}$$

$$
\begin{aligned}
&= 1 + m\left(\sum_{k=m}^{m+n} \frac{1}{k} - \frac{1}{m}\right) \\
&= m\sum_{k=m}^{m+n} \frac{1}{k}, \qquad \text{(A3)}
\end{aligned}
$$

whilst

$$
\sum_{k=0}^{n} \frac{(-1)^k}{k+m} \left(\begin{array}{c} n \\ k \end{array}\right) = \frac{1}{m} \left(\begin{array}{c} m+n \\ n \end{array}\right)^{-1} \qquad \text{(A4)}
$$

directly from (A1), whence Identity I follows immediately.

We remark that partial differentiation of Identity I leads without difficulty to Identity II in an analogous fashion (the interested reader may care to check this), and further results can be built up. For example, differentiating Identity II the next one yielded is, for integer $m \geq 1$, $n \geq 0$,

$$
\begin{aligned}
&6m \left(\begin{array}{c} m+n \\ n \end{array}\right) \sum_{k=0}^{n} \frac{(-1)^k}{(k+m)^4} \left(\begin{array}{c} n \\ k \end{array}\right) \\
&\quad = \left(\sum_{k=m}^{m+n} \frac{1}{k}\right)^3 + 3\left(\sum_{k=m}^{m+n} \frac{1}{k}\right)\left(\sum_{k=m}^{m+n} \frac{1}{k^2}\right) + 2\sum_{k=m}^{m+n} \frac{1}{k^3}, \qquad \text{(A5)}
\end{aligned}
$$

the r.h.s. of which appears naturally as a consequence of the method (and co-incides with that delivered by (10))—this would not be the case via the approach by integration presented in the main part of the paper. As an aside, note that the underpinning equation (A1) can alternatively be derived from either (i) equation (10), or (ii) consideration of the integral $H(m,n) = \int_0^\infty e^{-mx}(1-e^{-x})^n \, dx$. It may also be found in Graham *et al.* [5, p.188] as equation (5.41), where it is obtained by applying the difference operator $\Delta f(m) = f(m+1) - f(m)$ n times to the function $f(m) = \frac{1}{m}$.

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