TWO FINITE HYPERGEOMETRIC SEQUENCES OF DISCRETE ORTHOGONAL POLYNOMIALS

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Abstract. Two finite hypergeometric sequences of symmetric orthogonal polynomials of a discrete variable are introduced and their standard properties, such as second order difference equations, explicit forms of the polynomials and three term recurrence relations are obtained. As a consequence of two specific Sturm-Liouville problems, it is proved that these polynomials are finitely orthogonal with respect to two symmetric weight functions.

1. Introduction

Orthogonal functions of a discrete variable may be solutions of a Sturm-Liouville problem in the form [2]

$$\Delta\left(K(x)\nabla y_n(x)\right)+\left(\lambda_n\rho(x)-q(x)\right)y_n(x)=0\qquad (K(x)>0,\rho(x)>0),\tag{1}$$

where

$$\Delta f(x)=\nabla f(x+1)=f(x+1)-f(x),$$

and (1) satisfies a set of boundary conditions as

$$\alpha_1 y_n(a)+\beta_1\nabla y_n(a)=0,\quad \alpha_2 y_n(b)+\beta_2\nabla y_n(b)=0,\tag{2}$$

in which α_1,α_2 and β_1,β_2, are given constants. This means that if $y_n(x)$ and $y_m(x)$ are two eigenfunctions of difference equation (1), they are orthogonal with respect to the weight function $\rho(x)$ on a discrete set [12] as

$$\sum_{x=a}^{b}\rho(x)\,y_n(x)\,y_m(x)=\Big(\sum_{x=a}^{b}\rho(x)\,y_n^2(x)\Big)\delta_{n,m}\qquad\text{where}\quad \delta_{n,m}=\begin{cases}0 & (n\neq m),\\ 1 & (n=m).\end{cases}\tag{3}$$

The following theorem has been recently presented in [6] by which one can extend ordinary Sturm-Liouville problems with symmetric solutions in discrete spaces.

Theorem 1. *Let $\phi_n(-x)=(-1)^n\phi_n(x)$ be a sequence of symmetric functions that satisfies the difference equation*

$$\begin{aligned}A(x)\Delta\nabla\phi_n(x)+\left(A(-x)-A(x)\right)\Delta\phi_n(x)&\\ +\left(\lambda_n C(x)+D(x)+\sigma_n E(x)\right)\phi_n(x)&=0,\end{aligned}\tag{4}$$

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where $\sigma_n = \frac{1-(-1)^n}{2}$ *and as usual,* $\Delta\nabla = \Delta - \nabla$. *If* $A(x)$ *is a free real function and* $(C(x) > 0)$, $D(x)$ *and* $E(x)$ *are even functions, then*

$$\sum_{x=-\theta}^{\theta} W^*(x)\phi_n(x)\phi_m(x) = \left(\sum_{x=-\theta}^{\theta} W^*(x)\phi_n^2(x)\right)\delta_{n,m},$$

in which

$$W^*(x) = C(x)W(x), \tag{5}$$

and $W(x)$ *satisfies the Pearson difference equation*

$$\Delta\big(A(x)W(x)\big) = \big(A(-x) - A(x)\big)W(x),$$

which is equivalent to

$$\frac{W(x+1)}{W(x)} = \frac{A(-x)}{A(x+1)}. \tag{6}$$

Moreover, the weight function defined in (5) must be even over one of the four following symmetric counter sets

i) $S_1 = \{-a-n, -a-n+1, \ldots, -a-1, -a, a, a+1, \ldots, a+n\}$, $a \in \mathbb{R}$,
ii) $S_2 = S_1 \cup \{0\}$ *(as any odd function is equal to zero at* $x = 0$*),*
iii) $S_3 = \{\ldots, -a-n, -a-n+1, \ldots, -a-1, -a, a, a+1, \ldots, a+n, \ldots\}$ *(an infinite set)*
iv) $S_4 = S_3 \cup \{0\}$,

and the function $A(x)W(x)$ *must vanish at* $x = \pm\theta$, *where* $[-\theta, \theta] \in \{S_1, S_2, S_3, S_4\}$.

Note that many special functions having valuable applications in physics and engineering are orthogonal solutions of a symmetric Sturm-Liouville problem, see e.g. [5, 8, 9, 10].

As a special case of equation (4), the following difference equation is defined in [7]:

$$\begin{aligned}(2x+1)(ax^3+bx^2+cx+d)\Delta\nabla\phi_n(x) - 2x\left(x^2(a+2b)+c+2d\right)\Delta\phi_n(x) \\ + \left(2n(an-2(a+b))\left(\frac{1}{4}-x^2\right) + \sigma_n\left(\frac{a}{2}+b+2c+4d\right)\right)\phi_n(x) = 0,\end{aligned} \tag{7}$$

with a general polynomial solution as

$$\begin{aligned}\phi_n(x) = x^{\sigma_n}\sum_{j=0}^{[n/2]}(-1)^j\binom{[n/2]}{j} \\ \times\left(\prod_{i=j}^{[n/2]-1}\frac{a(i+\sigma_n)^3 - b(i+\sigma_n)^2 + c(i+\sigma_n) - d}{a(i+[n/2]-\sigma_{n+1}) - b}\right)(\sigma_n - x)_j(\sigma_n + x)_j,\end{aligned} \tag{8}$$

in which $[x]$ denotes the integer part of x, $(A)_n = \prod_{j=0}^{n-1}(A+j) = \Gamma(A+n)/\Gamma(A)$ for $n \geq 1$ with $(A)_0 = 1$ and $\prod_{i=0}^{-1}(.) = 1$.

The monic type of polynomial (8) satisfies a three term recurrence relation as

$$\bar{\phi}_{n+1}(x) = x\bar{\phi}_n(x) - \gamma_n\bar{\phi}_{n-1}(x) \quad \text{with } \bar{\phi}_0(x) = 1,\ \bar{\phi}_1(x) = x, \tag{9}$$

where [7]

$$\gamma_n = \frac{\sum_{i=0}^{4} K_i(a,b,c,d)n^i}{32(b-a(n-2))(b-a(n-1))}, \tag{10}$$

and

$$\begin{aligned}
&K_4(a,b,c,d) = -2a^2,\\
&K_3(a,b,c,d) = 4a(3a+2b),\\
&K_2(a,b,c,d) = -8\left(3a^2+4ab+ac+b^2\right),\\
&K_1(a,b,c,d) = 2(3a+2b)(3a+4b+4c) - 2a(-1)^n(a+2b+4c+8d),\\
&K_0(a,b,c,d) = \left((-1)^n - 1\right)(3a+2b)(a+2b+4c+8d).
\end{aligned}$$

Since the recurrence relation (9) is explicitly known, according to the Favard theorem [3, 12], the complete form of the orthogonality relation of monic polynomial (8) is as

(11)

$$\sum_{x=-\theta}^{\theta} W\left(\begin{array}{cc} a & b \\ c & d \end{array} \Bigg| \ x\right) \bar{\phi}_n(x)\,\bar{\phi}_m(x) = \prod_{k=1}^{n}\gamma_k \left(\sum_{x=-\theta}^{\theta} W\left(\begin{array}{cc} a & b \\ c & d \end{array} \Bigg| \ x\right)\right)\delta_{n,m},$$

in which

$$W\left(\begin{array}{cc} a & b \\ c & d \end{array} \Bigg| \ x\right) = \left(\frac{1}{4} - x^2\right) W(x),$$

is the original weight function and $W(x)$ satisfies the difference equation

$$\frac{W(x+1)}{W(x)} = \frac{(1/2)-x}{(3/2)+x}\,\frac{-ax^3+bx^2-cx+d}{a(x+1)^3+b(x+1)^2+c(c+1)+d}. \tag{12}$$

By noting that $A(x) = (2x+1)(ax^3+bx^2+cx+d)$ for $|a|+|b| \neq 0$ in (7), two cases can generally happen for the parameter a in (12), i.e. when $a \neq 0$ and b arbitrary or $a = 0$ and $b \neq 0$.
In the first case, since any arbitrary polynomial of degree 3 has at least one real root, say $x = p \in \mathbb{R}$, the aforesaid $A(x)$ can be decomposed in three different forms, i.e.

$$\begin{aligned}
A(x) = {} & (2x+1)\,(x-p)\left(a\,x^2+u\,x+v\right) \\
& = \begin{cases} (2x+1)\,(x-p)\left(a\,x^2+u\,x+v\right), & (u^2 < 4av), \\ (2x+1)\,a(x-p)(x-q)(x-r), & (u^2 > 4av), \\ (2x+1)\,a(x-p)(x-q)^2, & (u^2 = 4av). \end{cases}
\end{aligned} \tag{13}$$

Similarly, in the second case when $a = 0$ and $b \neq 0$, $A(x)$ can be decomposed as

$$\begin{aligned}
A(x) = {} & (2x+1)\left(b\,x^2+c\,x+d\right) \\
& = \begin{cases} (2x+1)\left(b\,x^2+c\,x+d\right), & (c^2 < 4bd), \\ (2x+1)\,b(x-p)(x-q), & (c^2 > 4bd), \\ (2x+1)\,b(x-p)^2, & (c^2 = 4bd). \end{cases}
\end{aligned} \tag{14}$$

But there are two sub-cases in (13) and (14) that have not been considered in [7], i.e. for $A(x) = (2x+1)\,(x-p)\left(a\,x^2+u\,x+v\right)$ in (13) and $A(x) = (2x+1)\left(b\,x^2+c\,x+d\right)$ in (14), whose difference equations corresponding to (12) are respectively as

$$\frac{W(x+1)}{W(x)} = \frac{(1/2)-x}{(3/2)+x}\,\frac{p+x}{p-x-1}\,\frac{a\,x^2-u\,x+v}{a\,(x+1)^2+u\,(x+1)+v} \qquad (u^2<4av)\,, \tag{15}$$

and

$$\frac{W(x+1)}{W(x)} = \frac{(1/2)-x}{(3/2)+x}\,\frac{b\,x^2-c\,x+d}{b\,(x+1)^2+c\,(x+1)+d} \qquad (c^2<4bd)\,. \tag{16}$$

In this paper, by using an interesting property of the gamma function, we obtain all real solutions of difference equations (15) and (16) and show that they can be considered as the weight functions of two symmetric finite sequences of orthogonal polynomials of a discrete variable. For this purpose, we first note that the limit definition of the gamma function

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{\prod\limits_{k=0}^{n} z+k},$$

implies that [11]

$$\Gamma(p+iq)\,\Gamma(p-iq) = \Gamma^2(p)\prod_{k=0}^{\infty}\frac{(p+k)^2}{(p+k)^2+q^2}, \tag{17}$$

is always a real positive value for any $p>0$ and $q\in\mathbb{R}$. This result is frequently used throughout the paper. One of the consequences of (17) is that

$$(p+iq)_n\,(p-iq)_n = \prod_{k=0}^{n-1} q^2+(p+k)^2 \qquad (p,q\in\mathbb{R}), \tag{18}$$

is also a real positive value. Moreover, when $p=m\in\mathbb{N}$, relation (17) is simplified as

$$\Gamma(m+iq)\,\Gamma(m-iq) = \frac{q\,\pi}{\sinh q\pi}\prod_{k=1}^{m-1} q^2+(m-k)^2. \tag{19}$$

The question is now how to determine the parameter θ in orthogonality relation (11)? To answer this question, we should reconsider the main equation (7) and write it in a self-adjoint form to eventually obtain

$$\begin{aligned}\sum_{x=-\theta}^{\theta} \Delta\Big(A(x)W(x)\,(\phi_m(x)\nabla\phi_n(x)-\phi_n(x)\nabla\phi_m(x))\Big) \\ + (\lambda_n-\lambda_m)\sum_{x=-\theta}^{\theta}\left(\frac{1}{4}-x^2\right)W(x)\phi_n(x)\phi_m(x) \\ + \frac{(-1)^m-(-1)^n}{2}\left(\frac{a}{2}+b+2c+4d\right)\sum_{x=-\theta}^{\theta} W(x)\phi_n(x)\phi_m(x) = 0.\end{aligned} \tag{20}$$

On the other side, the identity

$$\phi_m(x)\nabla\phi_n(x)-\phi_n(x)\nabla\phi_m(x) = \phi_n(x)\phi_m(x-1)-\phi_m(x)\phi_n(x-1),$$

simplifies the first sum of (20) as

$$\begin{aligned}(21)\qquad &\sum_{x=-\theta}^{\theta} \Delta\Big(A(x)W(x)\left(\phi_m(x)\nabla\phi_n(x) - \phi_n(x)\nabla\phi_m(x)\right)\Big)\\ &\quad = A(x)W(x)\Big(\phi_n(x)\phi_m(x-1) - \phi_m(x)\phi_n(x-1)\Big)\Big|_{x=-\theta}^{x=\theta+1}\\ &\quad = A(\theta+1)W(\theta+1)\left(\phi_n(\theta+1)\phi_m(\theta) - \phi_m(\theta+1)\phi_n(\theta)\right)\\ &\qquad - A(-\theta)W(-\theta)\left(\phi_n(-\theta)\phi_m(-\theta-1) - \phi_m(-\theta)\phi_n(-\theta-1)\right).\end{aligned}$$

By taking into account that all weight functions are even, i.e. $W(-x) = W(x)$, the polynomials are symmetric, i.e. $\phi_n(x) = (-1)^n\phi_n(-x)$, and the Pearson difference equation (6) is also valid for $x = \theta$, i.e. $A(\theta+1)W(\theta+1) = A(-\theta)W(\theta)$, relation (21) is simplified as

$$\begin{aligned}(22)\qquad &\sum_{x=-\theta}^{\theta} \Delta\Big(A(x)W(x)\left(\phi_m(x)\nabla\phi_n(x) - \phi_n(x)\nabla\phi_m(x)\right)\Big)\\ &\quad = A(-\theta)W(\theta)\left(1 + (-1)^{n+m}\right)\left(\phi_m(\theta)\phi_n(\theta+1) - \phi_n(\theta)\phi_m(\theta+1)\right).\end{aligned}$$

Since $\phi_m(\theta)\phi_n(\theta+1) - \phi_n(\theta)\phi_m(\theta+1) \neq 0$, two cases can in general happen for the right hand side of (22):

i) If $n + m$ is odd then $1 + (-1)^{n+m} = 0$ and (22) is automatically zero. However, this case is clear as the sum of any odd summand on a symmetric counter set is equal to zero.
ii) If $\lim\limits_{\theta\to\infty} A(-\theta)W(\theta)\theta^{n+m} = 0$, then again (22) is equal to zero. This is a key condition for proving the finite orthogonality relations of the polynomials introduced in the next section.

2. Two finite hypergeometric sequences of discrete orthogonal polynomials

In this section, we introduce two new hypergeometric sequences of finite symmetric orthogonal polynomials of a discrete variable and obtain all basic properties corresponding to them. To define such real polynomials, relations (18) and (19) will be used, see polynomial definitions (23) and (35) in this regard.

2.1. **First sequence.**

For $p, q, r \in \mathbb{R}$, consider the equation

$$\begin{aligned}&(2x+1)(x-p)(x^2 - 2qx + q^2 + r^2)\Delta\nabla\phi_n(x)\\ &\quad - 2x\left((1-2p-4q)x^2 + 2pq + (q^2+r^2)(1-2p)\right)\Delta\phi_n(x)\\ &\qquad + \left(2n\big(n + 2(p+2q-1)\big)\left(\frac{1}{4} - x^2\right)\right.\\ &\qquad\qquad \left. - \frac{\sigma_n}{2}(2p-1)\big(4r^2 + (2q-1)^2\big)\right)\phi_n(x) = 0,\end{aligned}$$

having a monic polynomial solution, which can be represented as

$$\bar{\phi}_n(x;p,q,r)=\frac{(p+\sigma_n)_{[n/2]}(q+ir+\sigma_n)_{[n/2]}(q-ir+\sigma_n)_{[n/2]}}{([n/2]+p+2q-1+\sigma_n)_{[n/2]}} \times\ x^{\sigma_n}\,{}_4F_3\left(\begin{array}{c}-[n/2],\ [n/2]+p+2q-1+\sigma_n,\ \sigma_n-x,\ \sigma_n+x\\ p+\sigma_n,\ q+ir+\sigma_n,\ q-ir+\sigma_n\end{array}\middle|\ 1\right), \tag{23}$$

where ${}_4F_3$ is a special case of the generalized hypergeometric series [3, 4, 12]

$${}_pF_q\left(\begin{array}{cccc}a_1, & a_2, & \ldots & a_p\\ b_1, & b_2, & \ldots & b_q\end{array}\middle|\ z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\ldots(a_p)_k}{(b_1)_k(b_2)_k\ldots(b_q)_k}\frac{z^k}{k!},$$

in which z may be a complex variable. According to the ratio test [1], this type of series converges in $|z|<1$ for $p=q+1$, converges everywhere for $p<q+1$ and converges nowhere ($z\neq 0$) for $p>q+1$. Moreover, for $p=q+1$ it absolutely converges for $|z|=1$ if the condition

$$A^*=\operatorname{Re}\left(\sum_{j=1}^{q}b_j-\sum_{j=1}^{q+1}a_j\right)>0, \tag{24}$$

holds and is conditionally convergent for $|z|=1$ and $z\neq 1$ if $-1<A^*\leq 0$ and is finally divergent for $|z|=1$ and $z\neq 1$ if $A^*\leq -1$.

The polynomial (23) satisfies the recurrence relation

$$\bar{\phi}_{n+1}(x;p,q,r)=x\bar{\phi}_n(x;p,q,r)-\gamma_n(p,q,r)\,\bar{\phi}_{n-1}(x;p,q,r), \tag{25}$$

in which

(26)

$$\begin{aligned}&32(n+p+2q-2)(n+p+2q-1)\gamma_n(p,q,r)=\\ &-2n^4+4(3-2p-4q)n^3-8(p^2+5q^2+r^2+6pq-4p-8q+3)n^2\\ &+2\Big((3-2p-4q)\big(3-4p-8q(1-p)+4q^2+4r^2\big)+(-1)^n(2p-1)\big(4r^2+(2q-1)^2\big)\Big)n\\ &-\big(1-(-1)^n\big)(2p-1)(2p+4q-3)\big(4r^2+(2q-1)^2\big),\end{aligned}$$

and can be decomposed as

$$\begin{aligned}\gamma_n(p,q,r)=&-\frac{1}{16(n+p+2q-2)(n+p+2q-1)}\\ &\times\big(n+(2p-1)\sigma_n\big)\big(n+2q+2ri-1+(2p-1)(1-\sigma_n)\big)\\ &\times\big(n+2q-2ri-1+(2p-1)(1-\sigma_n)\big)\big(n+4q-2+(2p-1)\sigma_n\big).\end{aligned} \tag{27}$$

By noting (23)-(27), the orthogonality relation of the first sequence now takes the general form

$$\begin{aligned}&\sum_{x=-\theta}^{\theta}W\left(\begin{array}{cc}1 & -(p+2q)\\ 2pq+q^2+r^2 & -p(q^2+r^2)\end{array}\middle|\ x\right)\bar{\phi}_n(x;p,q,r)\,\bar{\phi}_m(x;p,q,r)\\ &=\prod_{k=1}^{n}\gamma_k(p,q,r)\left(\sum_{x=-\theta}^{\theta}W\left(\begin{array}{cc}1 & -(p+2q)\\ 2pq+q^2+r^2 & -p(q^2+r^2)\end{array}\middle|\ x\right)\right)\delta_{n,m},\end{aligned}$$

in which

$$W\left(\begin{array}{cc} 1 & -(p+2q) \\ 2pq+q^2+r^2 & -p(q^2+r^2) \end{array} \,\middle|\, x \right) = \Big(\frac{1}{4}-x^2\Big)W(x),$$

denotes the original weight function and $W(x)$ satisfies the difference equation

$$\frac{W(x+1)}{W(x)} = \frac{(1/2)-x}{(3/2)+x}\frac{-x-p}{x+1-p}\frac{-x-q-ir}{x+1-q-ir}\frac{-x-q+ir}{x+1-q+ir}. \tag{28}$$

There are 16 symmetric solutions for equation (28), which are listed below

$$W_1(x) = \Big(\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q-ir+x)\Gamma(1-q-ir-x) \times \Gamma(1-q+ir+x)\Gamma(1-q+ir-x)\Gamma(3/2+x)\Gamma(3/2-x)\Big)^{-1},$$

$$W_2(x) = \frac{\Gamma(p+x)\Gamma(p-x)}{\Gamma(1-q-ir+x)\Gamma(1-q-ir-x)\Gamma(1-q+ir+x)\Gamma(1-q+ir-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$W_3(x) = \frac{\Gamma(q+ir+x)\Gamma(q+ir-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q+ir+x)\Gamma(1-q+ir-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$W_4(x) = \frac{\Gamma(q-ir+x)\Gamma(q-ir-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q-ir+x)\Gamma(1-q-ir-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$W_5(x) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(q+ir+x)\Gamma(q+ir-x)}{\Gamma(1-q+ir+x)\Gamma(1-q+ir-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$W_6(x) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(q-ir+x)\Gamma(q-ir-x)}{\Gamma(1-q-ir+x)\Gamma(1-q-ir-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$W_7(x) = \frac{\Gamma(q+ir+x)\Gamma(q+ir-x)\Gamma(q-ir+x)\Gamma(q-ir-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$W_8(x) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-q-ir+x)\Gamma(1-q-ir-x)\Gamma(1-q+ir+x)\Gamma(1-q+ir-x)},$$

$$W_9(x) = \frac{\Gamma(q+ir+x)\Gamma(q+ir-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q+ir+x)\Gamma(1-q+ir-x)},$$

$$W_{10}(x) = \frac{\Gamma(q-ir+x)\Gamma(q-ir-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q-ir+x)\Gamma(1-q-ir-x)},$$

$$W_{11}(x) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(q+ir+x)\Gamma(q+ir-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-q+ir+x)\Gamma(1-q+ir-x)},$$

$$W_{12}(x) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(q-ir+x)\Gamma(q-ir-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-q-ir+x)\Gamma(1-q-ir-x)},$$

$$W_{13}(x) = \frac{\Gamma(q+ir+x)\Gamma(q+ir-x)\Gamma(q-ir+x)\Gamma(q-ir-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+x)\Gamma(1-p-x)},$$

$$W_{14}(x) = \frac{\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+x)\Gamma(1-p-x)\Gamma(1-q-ir+x)\Gamma(1-q-ir-x)\Gamma(1-q+ir+x)\Gamma(1-q+ir-x)},$$

$$W_{15}(x) = \frac{\Gamma(p+x)\Gamma(p-x)\Gamma(q+ir+x)\Gamma(q+ir-x)\Gamma(q-ir+x)\Gamma(q-ir-x)}{\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$\begin{aligned} W_{16}(x) &= \Gamma(p+x)\Gamma(p-x)\Gamma(q+ir+x)\Gamma(q+ir-x) \\ &\quad \times \Gamma(q-ir+x)\Gamma(q-ir-x)\Gamma(-1/2+x)\Gamma(-1/2-x). \end{aligned}$$

Note although $\{W_k(x)\}_{k=1}^{16}$ are all symmetric, only three weight functions W_1, W_2 and W_{14} are eligible to apply, as they are real-valued functions according to relation (17). On the other hand, an important part that one has to compute in norm square value of the orthogonality (11) is $\sum_{x=-\theta}^{\theta}\left(\frac{1}{4}-x^2\right)W(x)$. This sum can be simplified by using the two identities

$$\Gamma(p+x) = \Gamma(p)(p)_x \quad \text{and} \quad \Gamma(p-x) = \frac{\Gamma(p)(-1)^x}{(1-p)_x},$$

and this fact that

$$\sum_{x=-\theta}^{\theta}\left(\frac{1}{4}-x^2\right)W(x) = 2\sum_{x=0}^{\theta}\left(\frac{1}{4}-x^2\right)W(x) - \frac{1}{4}W(0).$$

Hence, the following sums can now be explicitly computed. For this purpose, if for simplicity we assume that

$$B(p,q,r) = 2\ {}_4F_3\left(\begin{array}{c} q+ir,\ q-ir,\ p,\ 1 \\ 1-q+ir,\ 1-q-ir,\ 1-p \end{array}\middle|\, 1\right) - 1,$$

then after some computations we obtain

$$\sum_{x=-\infty}^{\infty}\Big(\frac{1}{4}-x^2\Big)W_1(x)=\frac{B(p,q,r)}{\pi\,\Gamma^2(1-p)\Gamma^2(1-q-ir)\Gamma^2(1-q+ir)},$$
$$\sum_{x=-\infty}^{\infty}\Big(\frac{1}{4}-x^2\Big)W_2(x)=\frac{\Gamma^2(p)}{\pi\,\Gamma^2(1-q-ir)\,\Gamma^2(1-q+ir)}\,B(p,q,r),$$
$$\sum_{x=-\infty}^{\infty}\Big(\frac{1}{4}-x^2\Big)W_{14}(x)=\frac{\pi\,B(p,q,r)}{\Gamma^2(1-p)\Gamma^2(1-q-ir)\Gamma^2(1-q+ir)}.$$

The latter computations show that there exists a unique representation for the original weight function as

$$W(x;p,q,r)=\frac{(p)_x\,(q+ir)_x\,(q-ir)_x}{(1-p)_x\,(1-q+ir)_x\,(1-q-ir)_x}, \tag{29}$$

and by noting the identity

$$(p)_{-x}=\frac{(-1)^x}{(1-p)_x}, \tag{30}$$

it is symmetric on the support $(-\infty,\infty)$, i.e. using (30) one can prove that

$$W(-x;p,q,r)=W(x;p,q,r).$$

Now, to prove finite orthogonality, since in the first sequence

$$A(x)=(2x+1)(x-p)(x^2-2qx+q^2+r^2),$$

and

$$W(x)=\frac{W(x;p,q,r)}{(1/4)-x^2}=\frac{K}{1-4x^2}\,\frac{\Gamma(p+x)\,\Gamma(q+ir+x)\,\Gamma(q-ir+x)}{\Gamma(1-p+x)\,\Gamma(1-q+ir+x)\,\Gamma(1-q-ir+x)},$$

where

$$K=4\,\frac{\Gamma(1-p)\,\Gamma(1-q+ir)\,\Gamma(1-q-ir)}{\Gamma(p)\,\Gamma(q+ir)\,\Gamma(q-ir)},$$

the key condition

$$\lim_{\theta\to\infty}A(-\theta)W(\theta)\,\theta^{n+m}=0, \tag{31}$$

implies that we have

$$\lim_{\theta\to\infty}\frac{(2\theta-1)(\theta+p)(\theta^2+2q\theta+q^2+r^2)\Gamma(p+\theta)\Gamma(q+ir+\theta)\Gamma(q-ir+\theta)}{(1-4\theta^2)\Gamma(1-p+\theta)\Gamma(1-q+ir+\theta)\Gamma(1-q-ir+\theta)}\,\theta^{n+m}=0. \tag{32}$$

On the other side, since

$$\lim_{\theta\to\infty}\frac{(2\theta-1)(\theta+p)}{1-4\,\theta^2}=-\frac{1}{2},$$

and

$$\lim_{\theta\to\infty}\frac{1}{\ln\theta}\ln\frac{\Gamma(p+\theta)\Gamma(q+ir+\theta)\Gamma(q-ir+\theta)}{\Gamma(1-p+\theta)\Gamma(1-q+ir+\theta)\Gamma(1-q-ir+\theta)}=2p+4q-3,$$

relation (32) is equivalent to

$$\lim_{\theta\to\infty}\theta^{2+n+m+2p+4q-3}=0. \tag{33}$$

If in relation (33) we take $\max\{m,n\}=N$, then it yields

$$2N+2p+4q-1<0 \quad \text{or} \quad N<\frac{1}{2}-p-2q.$$

Corollary 1. *If*

$$\phi_n(x;p,q,r)=x^{\sigma_n}\,{}_4F_3\left(\begin{array}{c} -[n/2],\ [n/2]+p+2q-1+\sigma_n,\ \sigma_n-x,\ \sigma_n+x \\ p+\sigma_n,\ q+ir+\sigma_n,\ q-ir+\sigma_n \end{array}\middle|\, 1\right),$$

then the polynomial set $\{\phi_n(x;p,q,r)\}_{n=0}^{N}$ *where* $N<\frac{1}{2}-p-2q$ *is finitely orthogonal with respect to the weight function* (29) *on the support* $(-\infty,\infty)$ *so that we have*

$$\sum_{x=-\infty}^{\infty}\frac{(p)_x\,(q+ir)_x\,(q-ir)_x}{(1-p)_x\,(1-q+ir)_x\,(1-q-ir)_x}\,\phi_n(x;p,q,r)\,\phi_m(x;p,q,r)= \tag{34}$$

$$\frac{[n/2]!(p+2q-1+\sigma_n)_{[n/2]}\Big((p+2q-1+\sigma_n+[n/2])_{[n/2]}\Big)^2(2q)_{[n/2]}(p+q+ir)_{[n/2]}(p+q-ir)_{[n/2]}}{4^{n-\sigma_n}(p+\sigma_n)_{[n/2]}(p/2+q+\sigma_n)_{[n/2]}((p-1)/2+q)_{[n/2]}(p/2+q)_{[n/2]}((p+1)/2+q)_{[n/2]}(q+\sigma_n+ir)_{[n/2]}(q+\sigma_n-ir)_{[n/2]}}$$

$$\times\left(2\,{}_4F_3\left(\begin{array}{cccc} q+ir & q-ir & p & 1 \\ 1-q+ir & 1-q-ir & 1-p & \end{array}\middle|\, 1\right)-1\right)\delta_{n,m}\,.$$

Moreover, by noting (24), *orthogonality* (34) *is valid on* $(-\infty,\infty)$ *only if* $2q, p+2q\notin\mathbb{Z}^-$, $p+2q<1$, $p\in\ (0,1)$ *and* $r\in\mathbb{R}$.

For instance, the finite set $\{\phi_n(x;1/3,-31/3,2)\}_{n=0}^{N=20}$ *is orthogonal with respect to the weight function*

$$\frac{(1/3)_x((-31/3)+2i)_x((-31/3)-2i)_x}{(2/3)_x((34/3)+2i)_x((34/3)-2i)_x},$$

on the support $(-\infty,\infty)$.

2.2. **Second sequence.** For $p,q\in\mathbb{R}$, consider the equation

$$\begin{aligned}(2x+1)(x^2-2px+p^2+q^2)\Delta\nabla\phi_n^*(x)-4x\left(x^2+p^2+q^2-p\right)\Delta\phi_n^*(x)&\\ +\left(-4n\Big(\frac{1}{4}-x^2\Big)+\sigma_n\big((2p-1)^2+4q^2\big)\right)\phi_n^*(x)&=0,\end{aligned}$$

having a monic polynomial solution, which can be represented as

$$\begin{aligned}\bar{\phi}_n^*(x;p,q)&=(p+iq+\sigma_n)_{[n/2]}(p-iq+\sigma_n)_{[n/2]}\\ &\qquad\times x^{\sigma_n}\,{}_3F_2\left(\begin{array}{c} -[n/2],\ \sigma_n-x,\ \sigma_n+x \\ p+iq+\sigma_n,\ p-iq+\sigma_n \end{array}\middle|\, 1\right).\end{aligned} \tag{35}$$

The polynomial (35) satisfies a recurrence relation of type (9) with

$$\begin{aligned}\gamma_n^*(p,q)&=-\frac{1}{4}\Big(n^2+2n(2p-1)+\sigma_n\big(4q^2+(2p-1)^2\big)\Big)\\ &=-\frac{1}{4}\big(n+(2p-1-2qi)\sigma_n\big)\Big(n+3p+qi-\frac{3}{2}+(-1)^n\big(p-\frac{1}{2}-qi\big)\Big).\end{aligned}$$

Hence, the orthogonality relation corresponding to second sequence takes the general form

$$\sum_{x=-\theta}^{\theta} W\left(\begin{array}{cc|c} 0 & 1 & \\ -2p & p^2+q^2 & x \end{array}\right) \bar{\phi}_n^*(x;p,q)\,\bar{\phi}_m^*(x;p,q)$$

$$= \prod_{k=1}^{n} \gamma_k^*(p,q) \left(\sum_{x=-\theta}^{\theta} W\left(\begin{array}{cc|c} 0 & 1 & \\ -2p & p^2+q^2 & x \end{array}\right)\right) \delta_{n,m},$$

where

$$W\left(\begin{array}{cc|c} 0 & 1 & \\ -2p & p^2+q^2 & x \end{array}\right) = \left(\frac{1}{4} - x^2\right) W^*(x),$$

is the original weight function and $W(x)$ satisfies the difference equation

$$\frac{W^*(x+1)}{W^*(x)} = \frac{(1/2)-x}{(3/2)+x}\,\frac{-x-p-iq}{x+1-p-iq}\,\frac{-x-p+iq}{x+1-p+iq}. \tag{36}$$

There are 8 symmetric solutions for equation (36), which are respectively as follows:

$$W_{17}(x) = \frac{1}{\Gamma(1-p-iq+x)\Gamma(1-p-iq-x)\Gamma(1-p+iq+x)\Gamma(1-p+iq-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$W_{18}(x) = \frac{\Gamma(p+iq+x)\Gamma(p+iq-x)}{\Gamma(1-p+iq+x)\Gamma(1-p+iq-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$W_{19}(x) = \frac{\Gamma(p-iq+x)\Gamma(p-iq-x)}{\Gamma(1-p-iq+x)\Gamma(1-p-iq-x)\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$W_{20}(x) = \frac{\Gamma(p+iq+x)\Gamma(p+iq-x)\Gamma(p-iq+x)\Gamma(p-iq-x)}{\Gamma(3/2+x)\Gamma(3/2-x)},$$

$$W_{21}(x) = \frac{\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p-iq+x)\Gamma(1-p-iq-x)\Gamma(1-p+iq+x)\Gamma(1-p+iq-x)},$$

$$W_{22}(x) = \frac{\Gamma(p+iq+x)\Gamma(p+iq-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p+iq+x)\Gamma(1-p+iq-x)},$$

$$W_{23}(x) = \frac{\Gamma(p-iq+x)\Gamma(p-iq-x)\Gamma(-1/2+x)\Gamma(-1/2-x)}{\Gamma(1-p-iq+x)\Gamma(1-p-iq-x)},$$

$$W_{24}(x) = \Gamma(p+iq+x)\Gamma(p+iq-x)\Gamma(p-iq+x)\Gamma(p-iq-x)\Gamma(-1/2+x)\Gamma(-1/2-x).$$

Again, note although $\{W_k\}_{k=17}^{24}$ are all symmetric, only two weight functions W_{17} and W_{21} are real-valued functions according to relation (17). Hence, similar to the previous case, if for simplicity we assume that

$$C(p,q) = 2\ {}_3F_2\left(\begin{array}{c} p+iq,\ p-iq,\ 1 \\ 1-p+iq,\ 1-p-iq \end{array} \middle| \, 1 \right) - 1,$$

then we obtain

$$\sum_{x=-\infty}^{\infty} \left(\frac{1}{4} - x^2\right) W_{17}(x) = \frac{C(p,q)}{\pi\, \Gamma^2(1-p-iq)\, \Gamma^2(1-p+iq)},$$

and

$$\sum_{x=-\infty}^{\infty} \left(\frac{1}{4} - x^2\right) W_{21}(x) = \frac{\pi\, C(p,q)}{\Gamma^2(1-p-iq)\, \Gamma^2(1-p+iq)}.$$

These computations show that there exists a unique representation for the original weight function as

$$W(x;p,q) = \frac{(p+iq)_x\,(p-iq)_x}{(1-p+iq)_x\,(1-p-iq)_x} = W(-x;p,q), \tag{37}$$

which can be summed by the Dougall's bilateral sum [3] as

$$\begin{aligned} C(p,q) &= \sum_{x=-\infty}^{\infty} \frac{(p+iq)_x\,(p-iq)_x}{(1-p+iq)_x\,(1-p-iq)_x} \\ &= \frac{\Gamma^2(1-p+iq)\,\Gamma^2(1-p-iq)\,\Gamma(1-4p)}{\Gamma(1-2p+2iq)\,\Gamma(1-2p-2iq)\,\Gamma^2(1-2p)}, \end{aligned}$$

where $p < \frac{1}{4}$ and $q \in \mathbb{R}$.

Here we can apply once again the key condition (31) for

$$A(x) = (2x+1)(x^2 - 2px + p^2 + q^2),$$

and

$$W^*(x) = \frac{W(x;p,q)}{(1/4) - x^2} = \frac{K^*}{1-4x^2}\,\frac{\Gamma(p+iq+x)\,\Gamma(p-iq+x)}{\Gamma(1-p+iq+x)\,\Gamma(1-p-iq+x)},$$

where

$$K^* = 4\,\frac{\Gamma(1-p+iq)\Gamma(1-p-iq)}{\Gamma(p+iq)\Gamma(p-iq)}.$$

So, by noting that

$$\lim_{\theta\to\infty} \frac{1}{\ln\theta} \ln \frac{\Gamma(p+iq+\theta)\Gamma(p-iq+\theta)}{\Gamma(1-p+iq+\theta)\Gamma(1-p-iq+\theta)} = 4p-2,$$

the key condition (31) finally leads to the following corollary.

Corollary 2. *If*

$$\phi_n^*(x;p,q) = x^{\sigma_n}\, {}_3F_2\left(\begin{array}{c} -[n/2],\ \sigma_n - x,\ \sigma_n + x \\ p+iq+\sigma_n,\ p-iq+\sigma_n \end{array} \middle| \, 1 \right),$$

then the polynomial set $\{\phi_n^*(x;p,q)\}_{n=0}^N$ *where* $N < \frac{1}{2} - 2p$ *is finitely orthogonal with respect to the weight function* (37) *on the support* $(-\infty,\infty)$ *so that we have*

$$\sum_{x=-\infty}^{\infty} \frac{(p+iq)_x\,(p-iq)_x}{(1-p+iq)_x\,(1-p-iq)_x}\,\phi_n^*(x;p,q)\,\phi_m^*(x;p,q) = \tag{38}$$
$$\frac{[n/2]!(2p)_{[n/2]}}{(p+\sigma_n+iq)_{[n/2]}(p+\sigma_n-iq)_{[n/2]}}\,\frac{\Gamma^2(1-p+iq)\,\Gamma^2(1-p-iq)\,\Gamma(1-4p)}{\Gamma(1-2p+2iq)\,\Gamma(1-2p-2iq)\,\Gamma^2(1-2p)}\,\delta_{n,m}\,.$$

According to (24), *orthogonality* (38) *is valid only if* $p \notin \mathbb{Z}^-$, $p < \frac{1}{4}$ *and* $q \in \mathbb{R}$.

For instance, the finite set $\{\phi_n^*(x;-11/2,1)\}_{n=0}^{N=11}$ *is orthogonal with respect to the weight function*

$$\frac{((-11/2)+i)_x((-11/2)-i)_x}{((13/2)+i)_x((13/2)-i)_x},$$

on the support $(-\infty,\infty)$.

We finally point out that to compute the moments of a discrete orthogonal polynomial, usually different bases are used. For example, to compute the moments of Hahn discrete polynomials [3, 12], it is more convenient to use the Pochhammer basis $\{(-x)_n\}_{n\geq 0}$, instead of the canonical basis $\{x^n\}_{n\geq 0}$, to get

$$\sum_{x=0}^{N-1} \frac{\Gamma(N)\Gamma(\alpha+\beta+2)\Gamma(\alpha+N-x)\Gamma(\beta+x+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)\Gamma(N-x)\Gamma(x+1)}(-x)_n = (-1)^n\,\frac{(1-N)_n(\beta+1)_n}{(\alpha+\beta+2)_n}.$$

Following this approach, for the weight functions corresponding to two introduced hypergeometric sequences, we can explicitly compute the moments of the form

$$(\varrho_k)_n = \sum_{x=-\infty}^{\infty} \vartheta_n(x)\Big(\frac{1}{4}-x^2\Big)W_k(x),$$

where the symmetric basis $\vartheta_n(x)$ is defined as

$$\begin{aligned}\vartheta_n(x) &= (-1)^{[n/2]}x^{\sigma_n}(\sigma_n-x)_{[n/2]}(\sigma_n+x)_{[n/2]}\\ &= (-1)^{[n/2]}x^{\sigma_n}\prod_{k=0}^{[n/2]-1}\big((k+\sigma_n)^2-x^2\big) = (-1)^n\vartheta_n(-x).\end{aligned} \tag{39}$$

Since $\vartheta_{2n+1}(x)$ is an odd polynomial, all odd moments are clearly equal to zero. Moreover, from hypergeometric definitions (23) and (35) and using the orthogonality property of them, it can be proved by induction that the even moments corresponding to the first and second sequences respectively satisfy the following first order recurrence relations

$$(\varrho_k)_{2n} = -\frac{(p+n-1)\big(r^2+(q+n-1)^2\big)}{p+2q+n-1}(\varrho_k)_{2n-2},$$

and

$$(\varrho_j)_{2n} = -\big(q^2+(p+n-1)^2\big)(\varrho_j)_{2n-2}.$$

Hence, if the corresponding weight functions are normalized with the moment of order zero equal to one, then we eventually obtain

$$\sum_{x=-\infty}^{\infty}\Big(\frac{1}{4}-x^2\Big)W_k(x)\vartheta_{2m}(x)=\frac{(-1)^m(p)_m(q+ir)_m(q-ir)_m}{(p+2q)_m},$$

for the weight functions of the first sequence, and

$$\sum_{x=-\infty}^{\infty}\Big(\frac{1}{4}-x^2\Big)W_j(x)\vartheta_{2m}(x)=(-1)^m(p+iq)_m(p-iq)_m,$$

for the weight functions of the second sequence.

References

[1] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, **71**. Cambridge University Press, Cambridge, 1999.

[2] A. Jirari, *Second-order Sturm-Liouville difference equations and orthogonal polynomials.*, Mem. Am. Math. Soc. **542** (1995), 138 p.

[3] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010, With a foreword by Tom H. Koornwinder.

[4] W. Koepf: *Hypergeometric Summation – An Algorithmic Approach to Summation and Special Function Identities.* Second Edition, Springer Universitext. Springer, London, 2014.

[5] M. Masjed-Jamei, *A basic class of symmetric orthogonal polynomials using the extended Sturm-Liouville theorem for symmetric functions*, J. Math. Anal. Appl. **325** (2007), no. 2, 753–775.

[6] M. Masjed-Jamei and I. Area, *A symmetric generalization of Sturm-Liouville problems in discrete spaces*, J. Difference. Equ. Appl. **19** (2013) 1544–1562.

[7] M. Masjed-Jamei and I. Area, *A basic class of symmetric orthogonal polynomials of a discrete variable*, J. Math. Anal. Appl. **399** (2013) 291–305.

[8] M. Masjed-Jamei and M. Dehghan, *A generalization of Fourier trigonometric series*, Comput. Math. Appl. **56** (2008), no. 11, 2941–2947.

[9] M. Masjed-Jamei and W. Koepf, *On incomplete symmetric orthogonal polynomials of Jacobi type*, Integral Transforms Spec. Funct. **21** (2010), no. 9-10, 655–662.

[10] M. Masjed-Jamei and W. Koepf, *On incomplete symmetric orthogonal polynomials of Laguerre type.*, Appl. Anal. **90** (2011), no. 3-4, 769–775.

[11] M. Masjed-Jamei, G. V. Milovanović, *An extension of Pochhammer's symbol and its application to hypergeometric functions*, Filomat. **31** (2017) 207–215.

[12] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical orthogonal polynomials of a discrete variable*, Springer Series in Computational Physics, Springer-Verlag, Berlin, 1991.

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