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Connection and linearization coefficients of the Askey-Wilson polynomials. (English summary)

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Let $p_n(x;a,b,c,d|q)$ be the Askey-Wilson polynomial. The paper derives explicit expressions for the connection coefficients $C_m(n)$ and linearization coefficients $L_r(m,n)$, i.e., the coefficients in the following expansions:

$$p_n(x;a,b,c,d|q)=\sum_{m=0}^{n} C_m(n)p_m(x;\alpha,\beta,\gamma,\delta|q),$$

$$p_n(x;a_1,b_1,c_1,d_1|q)p_n(x;a_2,b_2,c_2,d_2|q)=\sum_{r=0}^{m+n} L_r(m,n)p_r(x;\alpha,\beta,\gamma,\delta|q).$$

The derivation starts with rewriting the second-order difference equation for the polynomials p_n in terms of two companion operators. Computations become feasible upon introducing a special basis (B_n) for these operators. In the first part of the paper the authors derive the coefficients of the three-term recurrence. The answer is known, but the method relies on the ideas introduced in this paper, and is new. Using the methods developed here, the authors then derive an expansion of the basis elements into a linear combination of the polynomials p_n (the inversion problem), and use this result to solve the connection problem. Then they derive a linearization formula for the basis (B_n) and finally solve the linearization problem for general Askey-Wilson polynomials. They also observe that their results can be used to find connection and linearization coefficients for any family of classical orthogonal polynomials using specialization and limiting, and illustrate this observation by deriving connection coefficients for the continuous q-Hahn, q-Racah, and Wilson polynomials.

Several important steps in the derivations are accomplished using computer algebra.

Reviewed by *Alexander M. Barg*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.