A package on formal power series

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Abstract:

Formal Laurent-Puiseux series of the form

$$f(x) = \sum_{k=k_0}^{\infty} a_k x^{k/n} \tag{1}$$

are important in many branches of mathematics. Whereas Mathematica supports the calculation of truncated series with its `Series` command, and the Mathematica package `SymbolicSum` that is shipped with Mathematica version 2 is able to convert formal series of type (1) in some instances to their corresponding generating functions, in [2]–[7] we developed an algorithmic procedure to do these conversions that is implemented by the author, A. Rennoch and G. Stölting in the Mathematica package `PowerSeries`. The implementation enables the user to reproduce most of the results of the extensive bibliography on series [Hansen [1], 1975].

Moreover a subalgorithm of its own significance generates differential equations satisfied by the input function.

1 Scope of the algorithm

In [2]–[4] three types of functions are covered by an algorithmic procedure for the conversion into their representing Laurent-Puiseux series (1) at the origin: functions of *rational type* which are rational, or have a rational derivative of some order, functions of *exp-like type* which satisfy a homogeneous linear differential equation with constant coefficients, and functions of *hypergeometric type* which have a representation (1) with coefficients satisfying a recurrence equation of the form

$$\begin{aligned} a_{k+m} &= R(k)\, a_k && \text{for } k \geq k_0 \\ a_k &= A_k && \text{for } k = k_0, k_0+1, \ldots, k_0+m-1 \end{aligned}$$

for some $m \in \mathbb{N}$, $A_k \in \mathbb{C}$ $(k = k_0+1, k_0+2, \ldots, k_0+m-1)$, $A_{k_0} \in \mathbb{C}\setminus\{0\}$, and some rational function R. The number m is then called the *symmetry number* of (the given representation of) f.

The most interesting case is formed by the functions of hypergeometric type as almost all transcendental elementary functions like `x^n`, `E^x`, `Log[x]`, `Sin[x]`, `Cos[x]`, `ArcSin[x]`, ArcTan[x], and all kinds of special functions like the Airy functions AiryAi[x], AiryBi[x], the Bessel functions BesselI[n,x], BesselJ[n,x], BesselY[n,x], and BesselK[n,x], the integral functions Erf[x], ExpIntegralEi[x], CosIntegral[x], and SinIntegral[x], the orthogonal polynomials JacobiP[n,a,b,x], GegenbauerC[n,m,x], ChebyshevT[n,x], ChebyshevU[n,x], LegendreP[n,x], LaguerreL[n,a,x], and HermiteH[n,x], and many more functions, are of that type. Some of the given examples have so-called logarithmic singularities which, however, can be covered by the given approach. As the calculation of the initial values needs the calculation of limits, in some instances it may be helpful to load the package Limit.m. This is the case e. g. with the example function ArcSech[x] for which in Mathematica's Version 1 one has the result

```
In[1]:= Series[ArcSech[x],{x,0,0}]

                                   2
Out[1]= (Log[2] - Log[x]) + O[x]
```

which cannot be obtained with Version 2.2.

It is essential for the development of our algorithm that functions of hypergeometric type satisfy a simple differential equation, see ([2], Theorems 2.1, and 8.1), i. e. a homogeneous linear differential equation with polynomial coefficients.

On the other hand, it is of a similar importance that almost every function that we may write down satisfies a simple differential equation. In [6] it is shown that all functions that can be algebraically constructed from the functions mentioned above by addition, multiplication, and by the composition with rational functions and rational powers, satisfy such a differential equation.

Rather than describing the conversion of functions in their representing Laurent-Puiseux expansions and vice versa in detail which is done elsewhere ([2], [4]) we give the following short description.

To find the representing Laurent-Puiseux expansion of a given expression f with respect to the variable x and point of development x_0 one uses the function call PowerSeries[f,x,x0] (short PS[f,x,x0], and PS[f,x] if $x_0 = 0$). The following steps are done internally:

(1) A homogeneous linear differential equation with polynomial coefficients for $f(x)$ is generated.

(2) This differential equation is converted to an equivalent recurrence equation for a_k.

(3) If the recurrence equation is of the hypergeometric type, then the coefficients can be explicitly calculated using a finite number of initial values.

In [3] we presented example results of our MATHEMATICA implementation which give insight in the underlying algorithmic procedure. We will present further examples in § 4 of this article.

To find the generating function of a given Laurent-Puiseux expansion (1), the process can be reversed. The function

```
Convert[Sum[a*x^(m*k+s),{k,k0,Infinity},x]
```

provides the following procedure to calculate the generating function of $\sum_{k=k_0}^{\infty} a_k x^{mk+s}$:

(1) A homogeneous linear recurrence equation with polynomial coefficients for a_k is generated.

(2) This recurrence equation is converted to an equivalent differential equation for the generating function $f(x)$.

(3) Finally the differential equation is solved using appropriate initial values.

For the last step our implementation uses the builtin MATHEMATICA function `DSolve`. Note that you may not have loaded the package `SymbolicSum` when using our package as in this case the `Sum` command is redefined to simplify automatically when available. We note further that some of the conversions are also obtained by the `SymbolicSum` package, whereas others are not. Our implementation, however, obviously is slower as it always uses `DSolve` taking usually some time, whereas in the `SymbolicSum` package hypergeometric functions are converted using the builtin MATHEMATICA pattern matching mechanism for hypergeometric functions. This obviously is fast but it works only if the conversion is embedded in MATHEMATICA's kernel.

2 Examples of the generation of differential equations

In this section we present some results in the form of direct MATHEMATICA output. The function `SimpleDE[f,x]` generates the homogeneous linear differential equation of least order satisfied by f. Here are some examples:

```
In[2]:= de[1]=SimpleDE[x^n*E^(alpha x),x]

Out[2]= (-n - alpha x) F[x] + x F'[x] == 0

In[3]:= de[2]=SimpleDE[((1+x)/(1-x))^n,x]

Out[3]= 2 n F[x] + (-1 + x) (1 + x) F'[x] == 0

In[4]:= de[3]=SimpleDE[ArcSin[x^5],x]

                   10                   11
Out[4]= (4 + x   ) F'[x] + (-x + x   ) F''[x] == 0

In[5]:= de[4]=SimpleDE[ArcSin[x],x]

                                2
Out[5]= x F'[x] + (-1 + x ) F''[x] == 0
```

As noted above, by addition, multiplication, and by the composition with rational functions and rational powers from functions satisfying this kind of differential equation new ones are created. We have for example:

```
In[6]:= de[5]=SimpleDE[ArcSin[x]^3,x]

                                  2
Out[6]= x F'[x] + (-4 + 7 x ) F''[x] +

                 2   (3)                 2 2  (4)
>      6 x (-1 + x ) F    [x] + (-1 + x )  F    [x] == 0

In[7]:= de[6]=SimpleDE[Sin[x]^5,x,6]

                                              (4)
Out[7]= 225 F[x] + 259 F''[x] + 35 F   [x] +

        (6)
>      F   [x] == 0

In[8]:= de[7]=SimpleDE[Exp[alpha x]*Sin[beta x],x]

               2       2
Out[8]= (alpha  + beta ) F[x] - 2 alpha F'[x] +

>      F''[x] == 0
```

By default `SimpleDE` searches for a simple differential equation of order up to 5. This setting can be changed entering the order as a third argument. In case one knows a priori that a differential equation of the given type exists, one may choose the order `Infinity`.

Note that there is no simple differential equation of order less than 5 for $\sin^5 x$.

In [7] a general method is described how orthogonal polynomials and special functions can be handled. Our implementation covers this approach. Therefore we can derive the usual differential equations, e. g.

```
In[9]:= de[8]=SimpleDE[LaguerreL[n,x],x]

Out[9]= n F[x] + (1 - x) F'[x] + x F''[x] == 0

In[10]:= de[9]=SimpleDE[ChebyshevT[n,x],x]

            2                          2
Out[10]= -(n  F[x]) + x F'[x] + (-1 + x ) F''[x] == 0

In[11]:= de[10]=SimpleDE[BesselY[n,x],x]

            2    2                     2
Out[11]= (-n  + x ) F[x] + x F'[x] + x  F''[x] == 0

In[12]:= de[11]=SimpleDE[AiryAi[x],x]

Out[12]= -(x F[x]) + F''[x] == 0
```

but by the above statement we know that much more complicated functions satisfy differential equations:[1]

```
In[13]:= de[12]=SimpleDE[Exp[alpha x]*BesselI[n,x],x]

            2              2        2  2
Out[13]= (-n  - alpha x - x  + alpha  x ) F[x] +

                    2           2
>     (x - 2 alpha x ) F'[x] + x  F''[x] == 0

In[14]:= de[13]=SimpleDE[Sin[m x]*BesselJ[n,x],x]

           2      4      6    2      2  2       2  2
Out[14]= (n  - 5 n  + 4 n  - x  - 2 m  x  + 22 n  x  -

             2  2  2       4  2       2  4  2    4
>        10 m  n  x  - 12 n  x  + 12 m  n  x  - x  +

            2  4      4  4       2  4       2  2  4
>        6 m  x  - 5 m  x  + 12 n  x  - 24 m  n  x  +

             4  2  4      6       2  6       4  6
>        12 m  n  x  - 4 x  + 12 m  x  - 12 m  x  +

            6  6
>        4 m  x ) F[x] +

           3      2  3      2  3       2  2  3
>     (-8 x  - 4 m  x  + 8 n  x  + 40 m  n  x  -

            5      4  5
>        8 x  + 8 m  x ) F'[x] +

           2       2  2      4  2      4      2  4
>     (-2 x  + 10 n  x  - 8 n  x  + 2 x  - 6 m  x  +

             2  4      6      4  6
>        16 n  x  - 8 x  + 8 m  x ) F''[x] +

           3       2  3      5      2  5   (3)
>     (-4 x  + 16 n  x  - 8 x  + 8 m  x ) F   [x] +

       4          2      2      2  2   (4)
>     x  (-1 + 4 n  - 4 x  + 4 m  x ) F   [x] == 0

In[15]:= de[14]=SimpleDE[LegendreP[n,x]^2,x]

                      2
Out[15]= (-4 n x - 4 n  x) F[x] +
```

[1] Note that the calculation of lines `In[14]` and `In[15]` is very time consuming!

```
                          2        2          2        2  2
>      (-2 + 4 n + 4 n   + 6 x   - 4 n x   - 4 n   x )

                                  2
>       F'[x] + 6 x (-1 + x ) F''[x] +

                  2 2  (3)
>      (-1 + x )  F    [x] == 0
```

In [8], e. g., the author studied the functions

$$F_n^{(\alpha)}(x) := e^{-x}\, L_n^{(\alpha)}(2x) \ ,$$

where $L_n^{(\alpha)}$ denotes the generalized Laguerre polynomial. For $F_n^{(\alpha)}$ we immediately deduce the differential equation

```
In[16]:= de[15]=SimpleDE[E^(-x)*LaguerreL[n,alpha,2x],x]

Out[16]= (1 + alpha + 2 n - x) F[x] +

>      (1 + alpha) F'[x] + x F''[x] == 0
```

We may observe that only in the case $\alpha = -1$ — which mainly is studied in [8] — the coefficient of F' in the differential equation vanishes.

3 Examples on the conversion of simple differential equations to recurrence equations

The function `DEtoRE[de,F,x,a,k]` converts the simple differential equation `de` in which F and x are used as symbols representing the function and the variable, respectively, to a recurrence equation for the coefficients a_k of a corresponding Laurent-Puiseux expansion of F. The output of `DEtoRE` is given in terms of the symbols a and k representing the coefficient sequence a_k. As examples we convert some of the differential equations that were derived in § 2.

The example

```
In[17]:= re[1]=DEtoRE[de[2],F,x]

Out[17]= k a[k] + 2 n a[1 + k] - (2 + k) a[2 + k] == 0
```

shows that the function $\left(\frac{1+x}{1-x}\right)^n$ is not of hypergeometric type, whereas the example

```
In[18]:= re[2]=DEtoRE[de[3],F,x]

          2
Out[18]= k  a[k] - (5 + k) (10 + k) a[10 + k] == 0
```

shows that the function $\arcsin(x^5)$ is of hypergeometric type with symmetry number $m = 10$. Similarly the Airy function is of hypergeometric type with symmetry number $m = 3$:

```
In[19]:= re[3]=DEtoRE[de[11],F,x]

Out[19]= -a[k] + (2 + k) (3 + k) a[3 + k] == 0
```

Under all functions of the form $e^{\alpha x}$ BesselI (n, x) $(\alpha \in \mathbb{R})$ only the cases $\alpha = 0$ and $\alpha = \pm 1$ represent functions of hypergeometric type as is seen from the recurrence equation

```
In[20]:= re[4]=DEtoRE[de[12],F,x]

Out[20]= (-1 + alpha) (1 + alpha) a[k] -

>     alpha (3 + 2 k) a[1 + k] +

>     (2 + k - n) (2 + k + n) a[2 + k] == 0
```

and under all functions of the form $\sin(mx)$ BesselJ (n, x) $(m \in \mathbb{R})$ only the cases $m = 0$ and $m = \pm 1$ represent functions of hypergeometric type as is seen from the recurrence equation

```
In[21]:= re[5]=DEtoRE[de[13],F,x]

                      3         3
Out[21]= 4 (-1 + m)  (1 + m)  a[k] +

>     (-1 + m) (1 + m)

                          2        2           2       2  2
>      (33 + 32 k + 8 k  + 27 m  + 32 k m  + 8 k  m  -

            2        2  2
>        12 n  + 12 m  n ) a[2 + k] +

                              2        3      4         2
>     (-297 - 402 k - 210 k  - 48 k  - 4 k  + 198 m  +

               2         2  2        3  2      4  2
>        362 k m  + 206 k  m  + 48 k  m  + 4 k  m  +

             2           2         2  2         2  2
>        246 n  + 120 k n  + 16 k  n  + 150 m  n  +

              2  2        4        2  4
>        40 k m  n  - 12 n  + 12 m  n ) a[4 + k] +

>     (5 + k - n) (6 + k - n) (5 + k + n) (6 + k + n)

>      (-1 + 2 n) (1 + 2 n) a[6 + k] == 0
```

The function `SimpleRE[f,x,a,k]` puts the two steps `SimpleDE`, and `DEtoRE` together, and calculates the recurrence equation of a Laurent-Puiseux series representation of f.

4 Examples on the calculation of Laurent-Puiseux series

In this section we present some of the results of the procedure `PowerSeries`. Obviously the program can handle the standard power series representations of the elementary functions:

```
In[22]:= ps[1]=PowerSeries[E^x,x]

              k
             x
Out[22]= Sum[--, {k, 0, Infinity}]
             k!

In[23]:= ps[2]=PowerSeries[Sin[x],x]

                 k  1 + 2 k
             (-1)  x
Out[23]= Sum[--------------, {k, 0, Infinity}]
               (1 + 2 k)!

In[24]:= ps[3]=PowerSeries[Cos[x],x]

                 k  2 k
             (-1)  x
Out[24]= Sum[----------, {k, 0, Infinity}]
               (2 k)!

In[25]:= ps[4]=PowerSeries[ArcSin[x],x]

              1 k  1 + 2 k       2
             (-)  x        (2 k)!
              4
Out[25]= Sum[---------------------, {k, 0, Infinity}]
                  2
                k!  (1 + 2 k)!

In[26]:= ps[5]=PowerSeries[ArcTan[x],x]

                 k  1 + 2 k
             (-1)  x
Out[26]= Sum[--------------, {k, 0, Infinity}]
                1 + 2 k

In[27]:= ps[6]=PowerSeries[Log[x],x,1]

                 k          1 + k
             (-1)  (-1 + x)
Out[27]= Sum[-------------------, {k, 0, Infinity}]
                    1 + k
```

In the last example the point of development is $x = 1$.

The following are some functions of hypergeometric type which may be unexpected.

```
In[28]:= ps[7]=PowerSeries[Exp[ArcSin[x]],x]

              k  2 k         5            2
             4  x    Product[- - 2 jj + jj , {jj, k}]
                             4
Out[28]= Sum[----------------------------------------,
                               (2 k)!

>     {k, 0, Infinity}] +

          k  1 + 2 k         1           2
         4  x        Product[- - jj + jj , {jj, k}]
                             2
>    Sum[------------------------------------------,
                          (1 + 2 k)!

>     {k, 0, Infinity}]

In[29]:= ps[8]=PowerSeries[Exp[ArcSinh[x]],x]

                    1  k  2 k
                 (-(-))  x    (2 k)!
                    4
Out[29]= x + Sum[-------------------, {k, 0, Infinity}]
                                2
                    (1 - 2 k) k!
```

If one is interested in the intermediate calculations one may set `psprint` which gives the global variable `PSPrintMessages` the value `True`.

```
In[30]:= psprint

In[31]:= ps[9]=PowerSeries[E^x-2 E^(-x/2) Cos[Sqrt[3]x/2-Pi/3],x]
ps-info: PowerSeries  version 1.00, Sep 17, 1993
ps-info: 3 step(s) for DE:
                  (3)
         -F[x] + F   [x] == 0
ps-info: RE for all k >= 0:
         a[3 + k] = a[k]/((1 + k)*(2 + k)*(3 + k))
ps-info: function of hypergeometric type
ps-info: a[0] = 0
ps-info: a[1] = 0
                3
ps-info: a[2] = -
                2
ps-info: a[3] = 0
ps-info: a[4] = 0

                      2 + 3 k
             9 (1 + k) x
Out[31]= Sum[------------------, {k, 0, Infinity}]
```

```
                (3 + 3 k)!
```

Note that the differential equation in this example is extremely simple.

We now suppress intermediate results, again, using the command `nopsprint` which sets `PSPrintMessages=False`.

The following is the generating function $f(x) = \sum_{k=0}^{\infty} a_k x^k$ of the Fibonacci numbers a_n that are defined by the recurrence

$$a_{n+1} = a_n + a_{n-1} \; , \qquad a_0 = 0 \; , \quad a_1 = 1 \; .$$

The call

```
In[32]:= nopsprint

In[33]:= ps[10]=PowerSeries[x/(1-x-x^2),x,0]

                  -2         k        -10     k   k
           (-(-----------)  + (--------) ) x
              1 + Sqrt[5]            3/2
                                  5 - 5
Out[33]= Sum[---------------------------------,
                          Sqrt[5]

>    {k, 0, Infinity}]
```

produces a well-known closed formula for the Fibonacci numbers. Moreover the statement `SimpleRE[x/(1-x-x^2),x,a,k]` generates the Fibonacci recursion

```
In[34]:= re[6]=SimpleRE[x/(1-x-x^2),x,a,k]

Out[34]= (1 + k) a[k] + (1 + k) a[1 + k] -

>     (1 + k) a[2 + k] == 0
```

again. Note that the common factor $(1+k)$ ensures that the recurrence equation holds for all $k \in \mathbb{Z}$.

Our implementation covers functions which correspond to hypergeometric type Laurent-Puiseux series rather than just power series. For examples see [3].

Here we want to emphasize on the case of special functions, again. In § 3 we discovered that the functions $e^x\,\mathrm{BesselI}\,(n,x)$ and $\sin(x)\,\mathrm{BesselJ}\,(n,x)$ are of the hypergeometric type. For $n = 0, 1$ we get the resulting series representations

```
In[35]:= ps[11]=PowerSeries[E^x*BesselI[0,x],x]

             1 k  k
            (-)  x  (2 k)!
             2
Out[35]= Sum[--------------, {k, 0, Infinity}]
                    3
```

```
                k!

In[36]:= ps[12]=PowerSeries[E^x*BesselI[1,x],x]

              1 k  1 + k
             (-)  x      (1 + 2 k)!
              2
Out[36]= Sum[----------------------, {k, 0, Infinity}]
                    2
                k!  (2 + k)!

In[37]:= ps[13]=PowerSeries[Sin[x]*BesselJ[0,x],x]

                1  k  1 + 2 k
             (-(-))  x        (1 + 4 k)!
                4
Out[37]= Sum[---------------------------,
                                  2
               (2 k)! (1 + 2 k)!

>    {k, 0, Infinity}]

In[38]:= ps[14]=PowerSeries[Sin[x]*BesselJ[1,x],x]

                1  k  2 + 2 k
             (-(-))  x        (3 + 4 k)!
                4
Out[38]= Sum[---------------------------,
                          2
              2 (1 + 2 k)!  (3 + 2 k)!

>    {k, 0, Infinity}]
```

Many more functions that are constructed by the given special functions are of hypergeometric type (see e. g. the extensive bibliography on series [1]), and their Laurent-Puiseux expansions so can be found by `PowerSeries`.

5 Examples on the calculation of generating functions

We give examples of the function `Convert` calculating generating functions.

```
In[39]:= conv[1]=Convert[Sum[(2k)!/k!^2*x^k,{k,0,Infinity}],x]

               1
Out[39]= -------------
         Sqrt[1 - 4 x]

In[40]:= conv[2]=Convert[Sum[k!^2/(2k)!*x^k,{k,0,Infinity}],x]
```

```
                                  Sqrt[x]
                4 Sqrt[x] ArcSin[-------]
        4                            2
Out[40]= ----- + -------------------------
        4 - x                 3/2
                         (4 - x)
```

Again, with psprint we get information about the intermediate steps.

```
In[41]:= psprint

In[42]:= conv[3]=Convert[Sum[k!/(2k)!x^k,{k,0,Infinity}],x]
ps-info: PowerSeries  version 1.00, Sep 17, 1993
ps-info: 1 step(s) for RE:
         a[-1 + k] + 2*(1 - 2*k)*a[k] == 0
ps-info: DE:
         f[x] + (-2 + x) f'[x] + (-4*x) f''[x] == 0
ps-info: Trying to solve DE ...
ps-info: DSolve`t encountered
ps-info: DSolve computes
         C[3] + E^(x/4)*x^(1/2)*

>     (C[1] + (C[2]*(-2 - E^(x/4)*Pi^(1/2)*x^(1/2)*Erf[x^(1/2)/2]))/

>        (E^(x/4)*x^(1/2)))
ps-info: expression rearranged:
         E^(x/4)*x^(1/2)*C[1] + C[3] + E^(x/4)*x^(1/2)*C[2]*Erf[x^(1/2)/2]
ps-info: Calculation of initial values...
ps-info: C[3] == 1
ps-info: C[1] == 0
ps-info: C[2]/Pi^(1/2) == 1/2

              x/4                     Sqrt[x]
             E    Sqrt[Pi] Sqrt[x] Erf[-------]
                                          2
Out[42]= 1 + ---------------------------------
                             2
```

Futher, we may work with special functions again, and try to find their generating functions (which, however, can only be successful, if the solution is representable by an elementary function, as otherwise MATHEMATICA will fail to solve the differential equation produced). We get, for example

```
In[43]:= conv[4]=Convert[Sum[ChebyshevT[k,x]z^k,{k,0,Infinity}],z]
ps-info: PowerSeries  version 1.00, Sep 17, 1993
ps-info: 2 step(s) for RE:
         a[-2 + k] - 2*x*a[-1 + k] + a[k] == 0
ps-info: DE:
         2 f[z] + 4*(-x + z) f'[z] + (1 - 2*x*z + z^2) f''[z] == 0
ps-info: Trying to solve DE ...
```

```
ps-info: DSolve computes
         (z*C[1] - C[2] + (z*C[2])/C[1])/(-1 + 2*x*z - z^2)
ps-info: expression rearranged:
         (z*C[1] + C[2])/(-1 + 2*x*z - z^2)
ps-info: Calculation of initial values...
ps-info: -C[2] == 1
ps-info: (-2*C[1] - 4*x*C[2])/2 == x

                  1                 x z
Out[43]= -(---------------) + ---------------
                        2                  2
           -1 + 2 x z - z     -1 + 2 x z - z

In[44]:= conv[5]=Convert[Sum[LaguerreL[k,a,x]z^k,{k,0,Infinity}],z]
ps-info: PowerSeries  version 1.00, Sep 17, 1993
ps-info: 2 step(s) for RE:
         (-1 + a + k)*a[-2 + k] + (1 - a - 2*k + x)*a[-1 + k] + k*a[k] == 0
ps-info: DE:
         (-1 - a + x + z + a*z) f[z] + ((-1 + z)^2) f'[z] == 0
ps-info: Trying to solve DE ...
ps-info: DSolve computes
         E^(x/(-1 + z))*(-1 + z)^(-1 - a)*C[1]
ps-info: expression rearranged:
         E^(x/(-1 + z))*(1 - z)^(-1 - a)*C[1]
ps-info: Calculation of initial values...
ps-info: C[1]/E^x == 1

          (x z)/(-1 + z)        -1 - a
Out[44]= E               (1 - z)
```

6 Examples on the calculation of recurrence equations

As described in § 1 the function `Convert` works as follows: A recurrence equation for the given coefficients is generated and converted to a differential equation for the unknown function that is solved. The first step to generate a recurrence equation for a given function a_n depending on an integer n is implemented in the MATHEMATICA function `FindRecursion[a,n]`. We give some examples for the use of this function some of which depend on special functions, again:

```
In[45]:= re[7]=FindRecursion[(1+(-1)^n)/n,n]

Out[45]= (2 - n) a[-2 + n] + n a[n] == 0

In[46]:= re[8]=FindRecursion[n+(-1)^n,n]

Out[46]= (1 - 2 n) a[-2 + n] - 2 a[-1 + n] +

>     (-3 + 2 n) a[n] == 0
```

```
In[47]:= re[9]=FindRecursion[(n+(-1)^n)/n^2,n]

                           2      3
Out[47]= (4 - 12 n + 9 n  - 2 n ) a[-2 + n] +

                        2                   2      3
>     (-2 + 4 n - 2 n ) a[-1 + n] + (-3 n  + 2 n ) a[n]

>      == 0

In[48]:= re[10]=FindRecursion[1/(2n+1)!,n]

Out[48]= a[-1 + n] - 2 n (1 + 2 n) a[n] == 0

In[49]:= re[11]=FindRecursion[ps[10][[1]],k]

            2
Out[49]= -(x  a[-2 + k]) - x a[-1 + k] + a[k] == 0

In[50]:= re[12]=FindRecursion[E^(-x)*LaguerreL[n,alpha,2x],n]

Out[50]= (-1 + alpha + n) a[-2 + n] +

>     (1 - alpha - 2 n + 2 x) a[-1 + n] + n a[n] == 0

In[51]:= re[13]=FindRecursion[n*LaguerreL[n,alpha,2x],n]

                                            2
Out[51]= (1 - alpha - 2 n + alpha n + n ) a[-2 + n] +

                                              2
>     (-2 + 2 alpha + 5 n - alpha n - 2 n  - 4 x +

                                             2
>        2 n x) a[-1 + n] + (2 - 3 n + n ) a[n] == 0

In[52]:= re[14]=FindRecursion[LaguerreL[n,alpha,2x]/n,n]

                                              2
Out[52]= (2 - 2 alpha - 3 n + alpha n + n )

>      a[-2 + n] + (-1 + alpha + 3 n - alpha n -

            2                                2
>        2 n  - 2 x + 2 n x) a[-1 + n] + n  a[n] == 0

In[53]:= re[15]=FindRecursion[LaguerreL[n,x]^2,n]

                           2      3                   2
Out[53]= (4 - 12 n + 9 n  - 2 n  + 4 x - 4 n x + n  x)
```

```
                                     2      3
>      a[-3 + n] + (-6 + 22 n - 21 n  + 6 n  - 14 x +

                     2        2        2    3
>        26 n x - 11 n  x - 7 x  + 6 n x  - x )

                                    2      3
>      a[-2 + n] + (2 - 10 n + 15 n  - 6 n  + 6 x -

                     2        2        2    3
>        18 n x + 11 n  x + 5 x  - 6 n x  + x )

                       2      3    2
>      a[-1 + n] + (-3 n  + 2 n  - n  x) a[n] == 0
```

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