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Coefficients of Multiplication Formulas for Classical Orthogonal Polynomials

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Abstract In this paper, using both an analytic and algorithmic approach, we derive the coefficients $D_m(n,a)$ of the multiplication formula

$$p_n(ax) = \sum_{m=0}^{n} D_m(n,a)\, p_m(x)$$

or the translation formula

$$p_n(x+a) = \sum_{m=0}^{n} D_m(n,a)\, p_m(x),$$

where $\{p_n\}_{n\geq 0}$ is an orthogonal polynomial set, including the classical continuous orthogonal polynomials, the classical discrete orthogonal polynomials, the q-classical orthogonal polynomials, as well as the classical orthogonal polynomials on a quadratic and a q-quadratic lattice. We give a representation of the coefficients $D_m(n,a)$ as a single, double or triple sum whereas in many cases we get simple representations.

Keywords Orthogonal polynomials · Hypergeometric representation · q-Hypergeometric representation · Inversion coefficients · multiplication coefficients · translation coefficients

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1 Introduction

Let $\mathscr{P}$ denote the linear space of polynomials with coefficients in $\mathbb{C}$, the field of complex numbers. A polynomial sequence $\{p_n\}_{n\geq 0}$ in $\mathscr{P}$ is called a polynomial set (or system) if p_n is of exact degree n for all nonnegative integers n. Given a polynomial set $\{p_n\}_{n\geq 0}$, the so-called *multiplication problem* associated to this family consists in finding the coefficients $D_m(n,a)$ in the expansion

$$p_n(ax) = \sum_{m=0}^{n} D_m(n,a)\, p_m(x), \tag{1}$$

where a designates a nonzero complex number. Such identities have applications in many problems in pure and applied mathematics, especially in combinatorial analysis. The multiplication formula (1) is sometime called *dilation formula*, see e.g. [37], [38].

Chaggara and Koepf [8], Ismail [21] and Rainville [30] used generating functions to solve the multiplication problem for some classical continuous orthogonal polynomials and classical discrete orthogonal polynomials (only for the specific case $a=-1$). In this work, we use the same approach as in [21, page 103], [30, page 209] and also an algorithmic approach to solve the multiplication problem for the classical continuous orthogonal polynomials (Jacobi, Gegenbauer, Laguerre, Hermite, Bessel polynomials), the classical discrete orthogonal polynomials (Hahn, Meixner, Krawtchouk, Charlier polynomials), the q-classical orthogonal polynomials (Big q-Jacobi, q-Hahn, Big q-Laguerre, etc.) as well as the classical orthogonal polynomials on a quadratic lattice (Wilson polynomials, etc.) and a q-quadratic lattice (Askey-Wilson polynomials, etc.). The major algorithmic tools for our development are Zeilberger's algorithm ([25], [29]), the Petkovšek-van-Hoeij algorithm [25], Algorithm 2.8 of [25] and it q-analogue. Our results recover and extend works by Chaggara and Koepf [8], Ismail [21] and Rainville [30]. Moreover, proceeding as in [21] and [30], we find the coefficients $D_m(n,a)$ of the translation formula

$$p_n(a+x) = \sum_{m=0}^{n} D_m(n,a)\, p_m(x)$$

of the Hahn, Meixner, Krawtchouk and Charlier polynomials. If we define the notion of degree of complexity as the number of summations of elementary terms in $D_m(n,a)$, then it is useful to note that we obtain nice results with degree of complexity zero whereas others are of degree of complexity one, two or three.

Classical orthogonal polynomials of a continuous, a discrete and a q-discrete variable, and on a quadratic or q-quadratic lattice $x=x(s)$ are known to satisfy, respectively, the following second-order differential, difference, q-difference and divided-difference equations (see e.g. [11], [23], [27]):

$$\begin{gathered}\sigma(x)\,\tfrac{d^2}{dx^2}\,y(x)+\tau(x)\,\tfrac{d}{dx}\,y(x)+\lambda_n\,y(x)=0,\\ \sigma(x)\,\Delta\nabla y(x)+\tau(x)\,\Delta\, y(x)+\lambda_n\, y(x)=0,\\ \sigma(x)\,D_q\,D_{\frac{1}{q}}\,y(x)+\tau(x)\,D_q\,y(x)+\lambda_{n,q}\,y(x)=0,\end{gathered} \tag{2}$$

$$\sigma(x(s))\mathbb{D}_x^2 y_n(x(s))+\tau(x(s))\mathbb{S}_x\mathbb{D}_x y_n(x(s))+\lambda_n y_n(x(s))=0, \tag{3}$$

where Δ, ∇, and D_q are, respectively, the forward, the backward and the Hahn operators defined by

$$\Delta f(x)=f(x+1)-f(x),\; \nabla f(x)=f(x)-f(x-1),\; D_q f(x)=\frac{f(qx)-f(x)}{(q-1)x},\; q\neq 1,\; x\neq 0,$$

with $D_q f(0) = \lim_{x\to 0} D_q f(x) = f'(0)$, provided that $f'(0)$ exists. $\mathbb{D}_x$ and $\mathbb{S}_x$ are the operators defined by [11]

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}, \quad \mathbb{S}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2},$$

$\sigma(x) = ax^2 + bx + c$, $\tau(x) = dx + e$, are polynomials of maximum degree 2 and 1 respectively, and λ_n, $\lambda_{n,q}$ are constants depending on the coefficients a and d of σ and τ.

The content of this paper is organized as follows: In Section 2, we compute the multiplication coefficients of classical orthogonal polynomial of a continuous variable. Section 3 is devoted to coefficients of the multiplication and translation formulas of classical discrete orthogonal polynomials. In Sections 4 and 5, the multiplication coefficients are given for q-classical orthogonal polynomials and classical orthogonal polynomials on a quadratic and q-quadratic lattice, respectively.

2 Multiplication Coefficients of Classical Continuous Orthogonal Polynomials

Note that by $P_n^{(\alpha,\beta)}(x)$, $C_n^{(\alpha)}(x)$, $L_n^{(\alpha)}(x)$, $H_n(x)$, $B_n^{(\alpha)}(x)$, we denote, respectively, the Jacobi, Gegenbauer (ultraspherical), Laguerre, Hermite and Bessel polynomials. Their hypergeometric representations are given as [23]

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\right|\,\frac{1-x}{2}\right),\ \alpha>-1,\ \beta>-1\\
&= (-1)^n\frac{(\beta+1)_n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n, n+\alpha+\beta+1\\ \beta+1\end{matrix}\,\right|\,\frac{1+x}{2}\right),\\
C_n^{(\alpha)}(x) &= \frac{(\alpha)_n 2^n x^n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n/2, -n/2+1/2\\ -n-\alpha+1\end{matrix}\,\right|\,\frac{1}{x^2}\right),\ \alpha>-\frac{1}{2} \text{ and } \alpha\neq 0,\\
L_n^{(\alpha)}(x) &= \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\left.\begin{matrix}-n\\ \alpha+1\end{matrix}\,\right|\,x\right),\ \alpha>-1,\\
H_n(x) &= 2^n x^n\,{}_2F_0\left(\left.\begin{matrix}-n/2, -n/2+1/2\\ -\end{matrix}\,\right|\,-\frac{1}{x^2}\right),\\
B_n^{(\alpha)}(x) &= {}_2F_0\left(\left.\begin{matrix}-n, n+\alpha+1\\ -\end{matrix}\,\right|\,-\frac{x}{2}\right),\ n=0,1,\ldots,N,\ \alpha<-2N-1.
\end{aligned}$$

In the above definitions,

$${}_pF_q\left(\left.\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\,\right|\,x\right) = \sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{x^m}{m!},$$

where $(a)_m$ denotes the *Pochhammer symbol (or shifted factorial)* defined by

$$(a)_0 = 1,\ (a)_m = a(a+1)(a+2)\cdots(a+m-1) \text{ if } m = 1,2,\ldots.$$

To get the multiplication formulas of the above classical orthogonal polynomials of a continuous variable, we use their inversion formulas, that is, a formula expanding a general basis $b_n(x)$ into a family of orthogonal polynomials $p_n(x)$

$$b_n(x) = \sum_{m=0}^{n} I_m(n) p_m(x). \tag{4}$$

Theorem 1 *The following representations of the powers in terms of the classical continuous orthogonal polynomials are valid:*

$$(1-x)^n = 2^n\Gamma(\alpha+n+1)\sum_{m=0}^{n}\frac{(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+m+1)}{\Gamma(\alpha+m+1)\Gamma(\alpha+\beta+n+m+2)}(-n)_m P_m^{(\alpha,\beta)}(x)$$

(see e.g. [21], [24]),

$$(1+x)^n = 2^n\Gamma(\beta+n+1)\sum_{m=0}^{n}(-1)^m(-n)_m\frac{(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+m+1)}{\Gamma(\beta+m+1)\Gamma(\alpha+\beta+n+m+2)}P_m^{(\alpha,\beta)}(x)$$

(see e.g. [5], [21], [24]),

$$x^n = \frac{n!}{(\alpha)_n 2^n}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-\frac{n}{2}-\frac{\alpha}{2}+1)_m(-n-\alpha)_m}{(-\frac{n}{2}-\frac{\alpha}{2})_m m!}(-1)^m C_{n-2m}^{\alpha}(x) = \frac{n!}{2^n}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n+\alpha-2m}{m!(\alpha)_{n+1-m}}C_{n-2m}^{(\alpha)}(x)$$

(see e.g. [24], [30]),

$$x^n = (1+\alpha)_n\sum_{m=0}^{n}\frac{(-n)_m}{(1+\alpha)_m}L_m^{(\alpha)}(x) = n!\sum_{m=0}^{n}\binom{n+\alpha}{n-m}(-1)^m L_m^{(\alpha)}(x)$$

(see e.g. [21], [24], [33]),

$$x^n = \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-\frac{n}{2})_m(-\frac{n}{2}+\frac{1}{2})_m}{m!2^{n-2m}}H_{n-2m}(x) = \frac{n!}{2^n}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{m!(n-2m)!}H_{n-2m}(x)$$

(see e.g. [21], [24]),

$$\begin{aligned} x^n &= \frac{(-2)^n}{(\alpha+2)_n}\sum_{m=0}^{n}\frac{(-n)_m(\alpha+1)_m(\frac{\alpha}{2}+\frac{3}{2})_m}{(n+2+\alpha)_m(\frac{\alpha}{2}+\frac{1}{2})_m m!}B_m^{(\alpha)}(x) \\ &= (-2)^n\sum_{m=0}^{n}(2m+\alpha+1)\frac{(-n)_m\Gamma(\alpha+m+1)}{m!\Gamma(n+m+\alpha+2)}B_m^{(\alpha)}(x) \end{aligned}$$

(see e.g. [24], [33]), where $\lfloor n/2\rfloor$ denotes the largest integer smaller or equal to $n/2$.

In the following theorem, using the hypergeometric representations of classical continuous orthogonal polynomials and their inversion formulas given in Theorem 1, we provide known multiplication formulas and moreover, we get new results for Jacobi and Gegenbauer polynomials.

Theorem 2 *The following multiplication formulas of the classical orthogonal polynomials of a continuous variable are valid:*

$$P_n^{(\alpha,\beta)}(ax) = \sum_{m=0}^{n}\sum_{j=0}^{n-m} \frac{(-a)^m(1-a)^j(-n)_{m+j}(\alpha+1)_n(n+\alpha+\beta+1)_{m+j}}{2^j\, n!\, j!(\alpha+1)_{m+j}(\alpha+\beta+m+1)_m} \times {}_2F_1\left(\begin{array}{c} \alpha+m+1,-j \\ \alpha+\beta+2m+2 \end{array}\middle| \frac{2a}{a-1}\right) P_m^{(\alpha,\beta)}(x),$$

$$C_n^{(\alpha)}(ax) = \Gamma(n+\alpha)a^n \sum_{m=0}^{\lfloor n/2\rfloor} \frac{n+\alpha-2m}{m!\Gamma(\alpha+n-m+1)}\, {}_2F_1\left(\begin{array}{c} -m, m-n-\alpha \\ -n-\alpha+1 \end{array}\middle| \frac{1}{a^2}\right) C_{n-2m}^{(\alpha)}(x),$$

$$L_n^{(\alpha)}(ax) = \sum_{m=0}^{n} \frac{(\alpha+1)_n a^m (1-a)^{n-m}}{(n-m)!(\alpha+1)_m} L_m^{(\alpha)}(x)$$

(see e.g. [8], [21], [30, p. 209], [34, Exercise 67, p. 387], compare [26]),

$$H_n(ax) = \sum_{m=0}^{\lfloor n/2\rfloor} \frac{a^n n!(1-a^{-2})^m}{(n-2m)!m!} H_{n-2m}(x)$$

(see e.g. [8], [21], compare [16]),

$$B_n^{(\alpha)}(ax) = \sum_{m=0}^{n} \frac{(-a)^m(-n)_m(\alpha+n+1)_m}{m!(\alpha+m+1)_m}\, {}_2F_1\left(\begin{array}{c} m-n, \alpha+m+n+1 \\ \alpha+2m+2 \end{array}\middle| a\right) B_m^{(\alpha)}(x)$$

(see e.g. [10], [26]).

Remark 3 1. The multiplication coefficients given above for the Laguerre and Hermite polynomials were already given by Rainville [30, p. 209, formula (5)] (only the Laguerre formula), Ismail [21, (4.6.32), (4.6.33)] and Chaggara and Koepf [8] using generating functions. Those of Jacobi are new, as far as we know. However, Chaggara and Koepf [8] expanded $P_n^{(\alpha,\beta)}(1-ax)$ in terms of $P_m^{(\alpha,\beta)}(1-x)$ and got

$$P_n^{(\alpha,\beta)}(1-ax) = \sum_{m=0}^{n} \frac{a^m(1+\alpha)_n(\alpha+\beta+1+n)_m}{(n-m)!(1+\alpha)_m(\alpha+\beta+1+m)_m}\, {}_2F_1\left(\begin{array}{c} m-n, m+n+\alpha+\beta+1 \\ 2m+\alpha+\beta+2 \end{array}\middle| a\right) P_m^{(\alpha,\beta)}(1-x).$$

From this formula, the multiplication formula for the Laguerre polynomials can be derived following the limit relation

$$L_n^{(\alpha)}(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2\beta^{-1}x).$$

2. In the proof of this theorem, we use Zeilberger's (see e.g. [25, Chap. 7][1]) and the Petkovšek-van-Hoeij algorithms[2] (see e.g. [25, Chap. 9]). Zeilberger's algorithm deals with the question of how to determine a holonomic recurrence equation (i.e., homogeneous and linear with polynomial coefficients) for sums $S_n = \sum_{m=-\infty}^{\infty} F(n,m)$ where $F(n,m)$ is a hypergeometric term, i.e., the term ratio $\frac{F(n+1,m)}{F(n,m)}$, $\frac{F(n,m+1)}{F(n,m)}$ represents a rational function of the variable n and m, respectively. The Petkovšek-van-Hoeij algorithm finds all hypergeometric term solutions of a holonomic recurrence equation.

[1] explicitly the Maple procedure `sumrecursion` of the accompanying `hsum` package

[2] explicitly the Maple procedure `LREtools[hypergeomsols]` written by van Hoeij [17]

Proof In the proof, we consider three cases. In every case, the coefficients $A_j(n)$ and $I_m(n)$ are, respectively, those of the hypergeometric representations and the inversion formulas.

1. Jacobi family $P_n^{(\alpha,\beta)}(x)$.
   We use the following variant of the binomial theorem

$$(1-ax)^j = \sum_{k=0}^{j} B_k(j,a)(1-x)^k \quad \text{with} \quad B_k(j,a) = a^k \binom{j}{k}(1-a)^{j-k}.$$

   Combining

$$P_n^{(\alpha,\beta)}(ax) = \sum_{j=0}^{n} A_j(n)(1-ax)^j, \quad (1-ax)^j = \sum_{k=0}^{j} B_k(j,a)(1-x)^k$$

   and

$$(1-x)^k = \sum_{m=0}^{k} I_m(k) P_m^{(\alpha,\beta)}(x)$$

   and interchanging the order of summation yields the representation

$$P_n^{(\alpha,\beta)}(ax) = \sum_{m=0}^{n} D_m(n,a) P_m^{(\alpha,\beta)}(x)$$

   with

$$D_m(n,a) = \sum_{j=0}^{n-m} \sum_{k=0}^{j} A_{j+m}(n) B_{m+k}(j+m,a) I_m(k+m).$$

   To complete the proof, we use Algorithm 2.8, p. 22 of [25][3] to convert the sum $\sum_{k=0}^{j} B_{m+k}(j+m,a) I_m(k+m)$ into hypergeometric notation.
2. Gegenbauer family $C_n^{(\alpha)}(x)$ and Hermite family $H_n(x)$.
   In the Gegenbauer and Hermite cases, we combine

$$P_n(ax) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} A_j(n,a) x^{n-2j} \quad \text{and} \quad x^j = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} I_m(j) P_{j-2m}(x)$$

   which yield

$$x^{n-2j} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor - j} I_m(n-2j) P_{n-2j-2m}(x)$$

   and substitute m by $m-j$ to obtain

$$P_n(ax) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} D_m(n,a) P_{n-2m}(x) \text{ with } D_m(n,a) = \sum_{j=0}^{m} A_j(n,a) I_{m-j}(n-2j).$$

   In the Hermite case, Zeilberger's algorithm finds a holonomic recurrence equation of first order with respect to m from which the result follows. But in the Gegenbauer case, we get a recurrence equation of order 2 which according to the Petkovšek-van-Hoeij algorithm doesn't have a hypergeometric term solution.

[3] explicitly the Maple procedure `Sumtohyper` of the `hsum` package

3. Laguerre family $L_n^{(\alpha)}(x)$ and Bessel family $B_n^{(\alpha)}(x)$
   In both cases, we combine

$$P_n(ax) = \sum_{j=0}^{n} A_j(n,a)x^j \quad \text{and} \quad x^j = \sum_{m=0}^{j} I_m(j)P_m(x)$$

   and interchange the order of summation to get

$$D_m(n,a) = \sum_{j=0}^{n-m} A_{j+m}(n,a)I_m(j+m).$$

   For the Laguerre family, Zeilberger's algorithm finds a recurrence equation of first order with respect to m from which the result follows. But in the Bessel case, we get a recurrence equation of order 2 which according to the Petkovšek-van-Hoeij algorithm doesn't have a hypergeometric term solution.

□

3 Coefficients of Multiplication and Translation Formulas of Classical Discrete Orthogonal Polynomials

3.1 Multiplication Coefficients of Classical Discrete Orthogonal Polynomials

Using generating functions, Chaggara and Koepf [8] solved the multiplication problem

$$p_n(ax) = \sum_{m=0}^{n} D_m(n,a)p_m(x)$$

for the Charlier, Meixner and Krawtchouk polynomials but only for the specific case $a = -1$. Recurrence relations satisfied by the multiplication coefficients were also given. Area et al. [4] presented an algorithmic approach to obtain these recurrence relations. Their approach was based on the so-called NaViMa algorithm. In this section the general multiplication problem for classical discrete orthogonal polynomials is solved.

We denote by $Q_n(x;\alpha,\beta,N)$, $M_n(x;\gamma,\mu)$, $K_n(x;p,N)$ and $C_n(x;\mu)$, the Hahn, Meixner, Krawtchouk and Charlier polynomials, respectively. Their hypergeometric representations are given in [23]

$$\begin{aligned}
Q_n(x;\alpha,\beta,N) &= {}_3F_2\left(\left.\begin{matrix}-n,-x,n+1+\alpha+\beta\\ \alpha+1,-N\end{matrix}\right|1\right),\\
&\qquad n,x=0,1,\dots,N,\ \alpha>-1 \text{ and } \beta>-1, \text{ or } \alpha<-N \text{ and } \beta<-N,\\
M_n(x;\gamma,\mu) &= {}_2F_1\left(\left.\begin{matrix}-n,-x\\ \gamma\end{matrix}\right|1-\frac{1}{\mu}\right),\ \gamma>0,\ 0<\mu<1,\ x=0,1,\dots,\\
K_n(x;p,N) &= {}_2F_1\left(\left.\begin{matrix}-n,-x\\ -N\end{matrix}\right|\frac{1}{p}\right),\ 0<p<1,\ n,x=0,1,\dots,N,\\
C_n(x;\mu) &= {}_2F_0\left(\left.\begin{matrix}-n,-x\\ -\end{matrix}\right|-\frac{1}{\mu}\right),\ \mu>0,\ x=0,1,\dots.
\end{aligned}$$

In the continuous case, the polynomials were represented in terms of the powers x^n. The corresponding choice in the discrete case is a representation in terms of the falling factorials

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1) = (-1)^n(-x)_n.$$

To get the multiplication formulas of the classical discrete orthogonal polynomials, we use their inversion formulas. Gasper [14], Koepf and Schmersau [24], Ronveaux et al. [32] (compare [1], [41]) proved that

Theorem 4 *The following representations for the falling factorials in terms of the classical discrete orthogonal polynomials are valid:*

$$\begin{aligned}
x^{\underline{n}} &= \sum_{m=0}^{n} \frac{(1+\alpha)_n(-N)_n(-1)^n}{(\alpha+\beta+2)_n} \frac{(\alpha+\beta+1+2m)}{(\alpha+\beta+1)} \frac{(-n)_m(\alpha+\beta+1)_m}{(n+2+\alpha+\beta)_m m!} Q_m(x;\alpha,\beta,N), \qquad (5)\\
x^{\underline{n}} &= \sum_{m=0}^{n} \frac{(-1)^n(\gamma)_n(\frac{\mu}{\mu-1})^n(-n)_m}{m!} M_m(x;\gamma,\mu),\\
x^{\underline{n}} &= \sum_{m=0}^{n} \frac{(-1)^n(-N)_n p^n(-n)_m}{m!} K_m(x;p,N),\\
x^{\underline{n}} &= \sum_{m=0}^{n} \frac{\mu^n(-n)_m}{m!} C_m(x;\mu).
\end{aligned}$$

Using the hypergeometric representations and the inversion formulas of the classical discrete orthogonal polynomials, we get

Theorem 5 *For the classical discrete orthogonal polynomials, the following multiplication relations are valid.*

$$\begin{aligned}
Q_n(ax;\alpha,\beta,N) = &\sum_{m=0}^{n} \frac{(\alpha+\beta+1+2m)(\alpha+\beta+1)_m}{(\alpha+\beta+1)m!} \sum_{j=0}^{n-m} \frac{(\alpha+1,-N)_{n-j}(-n+j)_m}{(\alpha+\beta+2)_{n-j}(\alpha+\beta+n-j+2)_m(n-j)!}\\
&\times \sum_{k=0}^{j} \frac{(-n,n+\alpha+\beta+1)_{k+n-j}}{(\alpha+1,-N)_{k+n-j}(k+n-j)!} \times \sum_{l=0}^{n-j} (-1)^l \binom{n-j}{l} (-al)_{k+n-j} Q_m(x;\alpha,\beta,N),
\end{aligned}$$

$$\begin{aligned}
M_n(ax;\gamma,\mu) = &\sum_{m=0}^{n} \sum_{j=0}^{n-m} \frac{(\gamma)_{n-j}(-n+j)_m}{m!(n-j)!} \times \sum_{k=0}^{j} \frac{(-n)_{n+k-j}}{(n+k-j)!(\gamma)_{n+k-j}} \left(\frac{\mu-1}{\mu}\right)^k\\
&\times \sum_{l=0}^{n-j} (-1)^l \binom{n-j}{l} (-al)_{n+k-j} M_m(x;\gamma,\mu),
\end{aligned}$$

$$\begin{aligned}
K_n(ax;p,N) = &\sum_{m=0}^{n} \sum_{j=0}^{n-m} \frac{(-N)_{n-j}(-n+j)_m}{m!(n-j)!}\\
&\times \sum_{k=0}^{j} \frac{(-n)_{n+k-j}}{p^k(-N)_{n+k-j}(n+k-j)!} \sum_{l=0}^{n-j} (-1)^l \binom{n-j}{l} (-al)_{n+k-j} K_m(x;p,N),
\end{aligned}$$

$$C_n(ax;\mu) = \sum_{m=0}^{n} \sum_{j=0}^{n-m} \frac{(-n+j)_m}{m!(n-j)!} \sum_{k=0}^{j} \frac{(-n)_{n+k-j}}{(n+k-j)!(-\mu)^k} \sum_{l=0}^{n-j} (-1)^l \binom{n-j}{l} (-al)_{n+k-j} C_m(x;\mu).$$

The proof of this theorem uses the following result, namely the discrete analog of the Taylor expansion.

Lemma 6 *(Compare [7, p. 35]) If f is a polynomial w.r.t. x of degree n, then*

$$f(x) = \sum_{k=0}^{n} f_k x^{\underline{k}}, \tag{6}$$

where

$$f_k = \frac{1}{k!} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} f(l).$$

Replacing $f(x)$ by $(ax)^{\underline{n}}$ in (6), we are led to

Corollary 7 *The following multiplication formula holds:*

$$(ax)^{\underline{n}} = \sum_{k=0}^{n} E_k(n,a) x^{\underline{k}} = \sum_{k=0}^{n} \frac{(-1)^{n+k}}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} (-al)_n x^{\underline{k}}.$$

Proof (of Theorem 5) Combining

$$p_n(ax) = \sum_{k=0}^{n} A_k(n)(ax)^{\underline{k}},\ (ax)^{\underline{k}} = \sum_{i=0}^{k} E_i(k,a) x^{\underline{i}} \text{ with } E_i(k,a) = \sum_{l=0}^{i} F_l(i,k,a),$$

and

$$x^{\underline{i}} = \sum_{m=0}^{i} I_m(i) p_m(x),$$

interchanging the order of summation and substituting i by $n-m-j$ yields the multiplication relation

$$p_n(ax) = \sum_{m=0}^{n} D_m(n,a) p_m(x)$$

with

$$D_m(n,a) = \sum_{j=0}^{n-m} \sum_{k=0}^{j} \sum_{l=0}^{n-j} A_{k+n-j}(n) F_l(n-j, k+n-j, a) I_m(n-j),$$

where the coefficients $A_k(n)$, $F_l(i,k,a)$ and $I_m(i)$ are, respectively, those of the hypergeometric representations, the multiplication formula of the above Corollary 7 and the inversion formulas. □

Remark 8 We remark here that the degree of complexity of the above multiplication coefficients is three, which is rather high. It could be necessary to find conditions on the parameters a to reduce it or to find another approach to simplify this degree. In fact if we give some specific values to the parameter a (for example $a=-1$ or $a=2$), then the last sum $\sum_{l=0}^{n-j} (-1)^l \binom{n-j}{l} (-al)_{n+k-j}$ simplifies to a hypergeometric term and the degree of complexity of the coefficients $D_m(n,a)$ decreases to two.

3.2 Coefficients of Translation Formulas of Classical Discrete Orthogonal Polynomials

In this section, proceeding as in [21, page 103], [30, page 209], we use generating functions of classical discrete orthogonal polynomials to find their translation formulas. The generating functions of the Charlier, Meixner and Krawtchouk polynomials are given below (see e.g. [23]), respectively, with some relevant relations they verify:

$$G_1(x,t) := e^t\left(1-\frac{t}{\mu}\right)^x = \sum_{n=0}^{\infty}\frac{C_n(x;\mu)}{n!}t^n,\ G_1(ax,t) = \left(1-\frac{t}{\mu}\right)^a G_1(x,at),$$

$$G_2(x,t) := \left(1-\frac{t}{\mu}\right)^x\left(1-t\right)^{-x-\gamma} = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!}M_n(x;\gamma,\mu)t^n,\ G_2(ax,t) = \left(1-\frac{t}{\mu}\right)^a\left(1-t\right)^{-a}G_2(x,at),$$

$$G_3(x,t) := \left(1-\frac{1-p}{p}t\right)^x(1+t)^{N-x} = \sum_{n=0}^{N}\binom{N}{n}K_n(x;p,N)t^n,\ G_3(ax,t) = \left(1-\frac{1-p}{p}t\right)^a\left(1-t\right)^{-a}G_3(x,at).$$

Using the above generating functions, we prove that

Theorem 9 *The following translation formulas are valid for the Charlier, Meixner, Krawtchouk and Hahn polynomials, respectively:*

$$\begin{aligned}
C_n(x+a;\mu) &= \sum_{m=0}^{n}\binom{n}{m}\frac{(-a)_{n-m}}{\mu^{n-m}}C_m(x;\mu),\\
M_n(x+a;\gamma,\mu) &= \sum_{m=0}^{n}\sum_{k=0}^{n-m}\frac{n!(\gamma)_m(-a)_k(a)_{n-m-k}}{m!k!(\gamma)_n\mu^k(n-m-k)!}M_m(x;\gamma,\mu),\\
K_n(x+a;p,N) &= \sum_{m=0}^{n}\sum_{k=0}^{n-m}\frac{(-1)^{n-m-k}\binom{N}{m}(-a)_k(1-p)^k(a)_{n-m-k}}{\binom{N}{n}k!p^k(n-m-k)!}K_m(x;p,M),\\
Q_n(a+x;\alpha,\beta,N) &= \sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-1)^m(m-n)_j(-n)_m(-a)_j(\alpha+\beta+n+1)_{m+j}}{j!m!(m-N)_j(\alpha+m+1)_j(\alpha+\beta+m+1)_m}\\
&\quad\times{}_3F_2\left(\left.\begin{array}{c}-j,m-N,\alpha+m+1\\ a+1-j,\alpha+\beta+2m+2\end{array}\right|1\right)Q_m(x;\alpha,\beta,N).
\end{aligned}$$

Proof The translation formulas for the Charlier, Meixner and Krawtchouk polynomials follow by equating coefficients of t^n in the relations between $G_i(ax,t)$ and $G_i(x,at)$, $i = 1,2,3$ using the binomial theorem

$$(1-z)^{-a} = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n,\ |z|<1.$$

For the Hahn polynomials, we combine the representation of $Q_n(x+a;\alpha,\beta,N)$ in terms of $(-x-a)_n$, the Chu–Vandermonde identity

$$(a+x)_n = \sum_{m=0}^{n}\binom{n}{m}(a)_{n-m}(x)_m$$

and the inversion formula (5) to get the result. ◻

4 Multiplication Coefficients of q-Classical Orthogonal Polynomials

The polynomial systems which are solution of the q-difference equation (2) form the q-Hahn tableau. They are represented as *basic hypergeometric series* ${}_r\phi_s$ defined by

$$ {}_r\phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array}\middle| q;z\right) = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}\frac{z^k}{(q;q)_k}, $$

where $(a_1,\ldots,a_r;q)_k := (a_1;q)_k\cdots(a_r;q)_k$, is the product of the q-Pochhammer symbol defined by

$$ (a;q)_0 = 1,\, (a;q)_k = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}) \text{ if } k = 1,2,3,\ldots, $$

and

$$ (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),\, 0<|q|<1. $$

The following systems are members of the q-Hahn tableau [23]:

1. the big q-Jacobi polynomials

$$ P_n(x;\alpha,\beta,\gamma;q) = {}_3\phi_2\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1},x \\ \alpha q,\gamma q \end{array}\middle| q;q\right), $$

which for $\alpha=\beta=1$ are the big q-Legendre polynomials

$$ P_n(x;\gamma;q) = {}_3\phi_2\left(\begin{array}{c} q^{-n},q^{n+1},x \\ q,\gamma q \end{array}\middle| q;q\right), $$

2. the q-Hahn polynomials

$$ Q_n(\bar{x};\alpha,\beta,N|q) = {}_3\phi_2\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1},\bar{x} \\ \alpha q,q^{-N} \end{array}\middle| q;q\right), \text{ with } \bar{x}=q^{-x} \text{ and } n=0,1,\ldots,N, $$

3. the big q-Laguerre polynomials

$$ P_n(x;\alpha,\beta;q) = {}_3\phi_2\left(\begin{array}{c} q^{-n},x,0 \\ \alpha q,\beta q \end{array}\middle| q;q\right), $$

4. the little q-Jacobi polynomials

$$ p_n(x;\alpha,\beta|q) = {}_2\phi_1\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1} \\ \alpha q \end{array}\middle| q;qx\right), $$

which for $\alpha=\beta=1$ are the little q-Legendre polynomials

$$ p_n(x|q) = {}_2\phi_1\left(\begin{array}{c} q^{-n},q^{n+1} \\ q \end{array}\middle| q;qx\right), $$

5. the q-Meixner polynomials

$$ M_n(\bar{x};\beta,\gamma;q) = {}_2\phi_1\left(\begin{array}{c} q^{-n},\bar{x} \\ \beta q \end{array}\middle| q;-\frac{q^{n+1}}{\gamma}\right), \text{ with } \bar{x}=q^{-x}, $$

6. the quantum q-Krawtchouk polynomials

$$K_n^{\mathrm{qtm}}(\bar{x};p,N;q)={}_2\phi_1\left(\begin{matrix}q^{-n},\bar{x}\\ q^{-N}\end{matrix}\,\middle|\,q;pq^{n+1}\right),\ \text{with } \bar{x}=q^{-x} \text{ and } n=0,1,\ldots,N,$$

7. the q-Krawtchouk polynomials

$$K_n(\bar{x};p,N;q)={}_3\phi_2\left(\begin{matrix}q^{-n},\bar{x},-pq^n\\ q^{-N},0\end{matrix}\,\middle|\,q;q\right)\ \text{with } \bar{x}=q^{-x} \text{ and } n=0,1,\ldots,N,$$

8. the affine q-Krawtchouk polynomials

$$K_n^{\mathrm{Aff}}(\bar{x};p,N;q)={}_3\phi_2\left(\begin{matrix}q^{-n},\bar{x},0\\ pq,q^{-N}\end{matrix}\,\middle|\,q;q\right)\ \text{with } \bar{x}=q^{-x} \text{ and } n=0,1,\ldots,N,$$

9. the little q-Laguerre / Wall polynomials

$$p_n(x;\alpha|q)={}_2\phi_1\left(\begin{matrix}q^{-n},0\\ \alpha q\end{matrix}\,\middle|\,q;qx\right),$$

10. the q-Laguerre polynomials

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix}q^{-n}\\ q^{\alpha+1}\end{matrix}\,\middle|\,q;-q^{n+\alpha+1}x\right),$$

11. the alternative q-Charlier or q-Bessel polynomials

$$y_n(x;\alpha;q)={}_2\phi_1\left(\begin{matrix}q^{-n},-\alpha q^n\\ 0\end{matrix}\,\middle|\,q;qx\right),$$

12. the q-Charlier polynomials

$$C_n(\bar{x};\alpha;q)={}_2\phi_1\left(\begin{matrix}q^{-n},\bar{x}\\ 0\end{matrix}\,\middle|\,q;-\frac{q^{n+1}}{\alpha}\right),\ \text{with } \bar{x}=q^{-x},$$

13. the Al-Salam-Carlitz I polynomials

$$U_n^{(\alpha)}(x;q)=(-\alpha)^n q^{\binom{n}{2}}\,{}_2\phi_1\left(\begin{matrix}q^{-n},x^{-1}\\ 0\end{matrix}\,\middle|\,q;\frac{qx}{\alpha}\right),$$

14. the Al-Salam-Carlitz II polynomials

$$V_n^{(\alpha)}(x;q)=(-\alpha)^n q^{-\binom{n}{2}}\,{}_2\phi_0\left(\begin{matrix}q^{-n},x\\ -\end{matrix}\,\middle|\,q;\frac{q^n}{\alpha}\right),$$

15. the Stieltjes-Wigert polynomials

$$S_n(x;q)=\frac{1}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix}q^{-n}\\ 0\end{matrix}\,\middle|\,q;-q^{n+1}x\right),$$

16. the discrete q-Hermite I polynomials

$$h_n(x;q) = q^{\binom{n}{2}}{}_2\phi_1\left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array}\middle| q; -qx\right),$$

the discrete q-Hermite I polynomials are the Al-Salam-Carlitz I polynomials with $\alpha = -1$ i.e. $h_n(x;q) = U_n^{(-1)}(x;q)$,

17. the discrete q-Hermite II polynomials

$$\tilde{h}_n(x;q) = i^{-n}q^{-\binom{n}{2}}{}_2\phi_0\left(\begin{array}{c} q^{-n}, ix \\ - \end{array}\middle| q; -q^n\right),$$

the discrete q-Hermite II polynomials are related to the Al-Salam-Carlitz II polynomials with $\alpha = -1$ by $\tilde{h}_n(x;q) = i^{-n}V_n^{(-1)}(ix;q)$.

The representation of the polynomials $p_n(x)$ belonging to the q-Hahn tableau as basic hypergeometric series suggests four natural bases $\{\mathcal{V}_m\}$ to obtain expansions of the form

$$p_n(x) = \sum_{m=0}^{n} A_m(n)\mathcal{V}_m(x).$$

These expansion bases are the q-shifted factorials (i.e. $\mathcal{V}_m(x) = (x;q)_m$), the powers of x (i.e. $\mathcal{V}_m(x) = x^m$), $\mathcal{V}_m(x) = (ix;q)_m$ and $\mathcal{V}_m(x) = (x-1)(x-q)\cdots(x-q^{m-1}) = (x^{-1};q)_m x^m$. These four bases can be generalized to the q-power basis [36]

$$(b\ominus a)_q^n = \begin{cases} (b-a)(b-aq)\cdots(b-aq^{n-1}), & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases}$$

where $a,\ b \in \mathbb{C}$. Indeed, we have

$$(x;q)_n = (1\ominus x)_q^n,\ x^n = (x\ominus 0)_q^n,\ (ix;q)_n = (1\ominus ix)_q^n \text{ and } (x^{-1};q)_n x^n = (x\ominus 1)_q^n.$$

For q-classical orthogonal polynomials (in short q-COP), we propose two different methods to solve the multiplication problem (1).

4.1 First Method

Here once more, we proceed as in [21, page 103], [30, page 209], using generating functions of some q-COP given below with some relevant relations they verify:

1. the discrete q-Hermite II polynomials

$$G(x,t) := \frac{(-xt;q)_\infty}{(-t^2;q^2)_\infty} = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(q;q)_n}\tilde{h}_n(x;q)t^n,\ G(ax,t) = \frac{(-a^2t^2;q^2)_\infty}{(-t^2;q^2)_\infty}G(x,at),$$

2. the discrete q-Hermite I polynomials

$$G(x,t) := \frac{(t^2;q^2)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty}\frac{h_n(x;q)}{(q;q)_n}t^n,\ G(ax,t) = \frac{(t^2;q^2)_\infty}{(a^2t^2;q^2)_\infty}G(x,at),$$

3. the Stieltjes-Wigert polynomials

$$G(x,t) := \frac{1}{(t;q)_\infty}\,{}_0\phi_1\left(\begin{matrix} - \\ 0 \end{matrix}\,\middle|\, q; -qxt\right) = \sum_{n=0}^{\infty} S_n(x;q)\,t^n,\; G(ax,t) = \frac{(at;q)_\infty}{(t;q)_\infty} G(x,at),$$

4. the Al-Salam-Carlitz II polynomials

$$G(x,t) := \frac{(xt;q)_\infty}{(t,\alpha t;q)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} V_n^{(\alpha)}(x;q),\; G(ax,t) = \frac{(at,\alpha at;q)_\infty}{(t,\alpha t;q)_\infty} G(x,at),$$

5. the Al-Salam-Carlitz I polynomials

$$G(x,t) := \frac{(t,\alpha t;q)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{U_n^{(\alpha)}(x;q)}{(q;q)_n} t^n,\; G(ax,t) = \frac{(t,\alpha t;q)_\infty}{(at,\alpha at;q)_\infty} G(x,at),$$

6. the q-Charlier polynomials

$$G(x,t) := \frac{1}{(t;q)_\infty}\,{}_0\phi_1\left(\begin{matrix} - \\ -\alpha^{-1}q \end{matrix}\,\middle|\, q; -\alpha^{-1}qxt\right) = \sum_{n=0}^{\infty} \frac{C_n(x;\alpha;q)}{(-\alpha^{-1}q,q;q)_n} t^n,\; G(ax,t) = \frac{(at;q)_\infty}{(t;q)_\infty} G(x,at),$$

7. the q-Laguerre polynomials

$$G(x,t) := \frac{1}{(t;q)_\infty}\,{}_0\phi_1\left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix}\,\middle|\, q; -q^{\alpha+1}xt\right) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_n} t^n,\; G(ax,t) = \frac{(at;q)_\infty}{(t;q)_\infty} G(x,at),$$

8. the little q-Laguerre / Wall polynomials

$$G(x,t) := \frac{(t;q)_\infty}{(xt;q)_\infty}\,{}_0\phi_1\left(\begin{matrix} - \\ \alpha q \end{matrix}\,\middle|\, q; \alpha qxt\right) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} p_n(x;\alpha|q)\,t^n,\; G(ax,t) = \frac{(t;q)_\infty}{(at;q)_\infty} G(x,at).$$

Using the above generating functions $G(x,t)$ and the relations between $G(ax,t)$ and $G(x,at)$ we prove

Theorem 10 *The following multiplication and translation formulas are valid for:*

1. *the discrete q-Hermite II polynomials*

$$\tilde{h}_n(ax;q) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m a^{n-2m} q^{\binom{n-2m}{2}} (q;q)_n (a^2;q^2)_m}{q^{\binom{n}{2}} (q^2;q^2)_m (q;q)_{n-2m}} \tilde{h}_{n-2m}(x;q),$$

2. *the discrete q-Hermite I polynomials*

$$h_n(ax;q) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{a^n (q;q)_n (a^{-2};q^2)_m}{(q^2;q^2)_m (q;q)_{n-2m}} h_{n-2m}(x;q),$$

3. *the Stieltjes-Wigert polynomials*

$$S_n(ax;q) = \sum_{m=0}^{n} \frac{a^m (a;q)_{n-m}}{(q;q)_{n-m}} S_m(x;q),$$

4. *the Al-Salam-Carlitz II polynomials*

$$V_n^{(\alpha)}(ax;q) = \sum_{m=0}^{n} \sum_{k=0}^{n-m} \frac{(-a)^m \alpha^k q^{\binom{m}{2}} (q;q)_n (a;q)_k (a;q)_{n-m-k}}{(-1)^n q^{\binom{n}{2}} (q;q)_k (q;q)_m (q;q)_{n-m-k}} V_m^{(\alpha)}(x;q),$$

5. *the Al-Salam-Carlitz I polynomials*

$$U_n^{(\alpha)}(ax;q) = \sum_{m=0}^{n}\sum_{k=0}^{n-m} \frac{a^n\alpha^k (q;q)_n (a^{-1};q)_k (a^{-1};q)_{n-m-k}}{(q;q)_m (q;q)_k (q;q)_{n-m-k}} U_m^{(\alpha)}(x;q),$$

6. *the q-Charlier polynomials*

$$C_n(q^{-(x+a)};\alpha;q) = \sum_{m=0}^{n} \frac{q^{-ma}(q^{-a};q)_{n-m}(-\alpha^{-1}q,q;q)_n}{(q;q)_{n-m}(-\alpha^{-1}q,q;q)_m} C_m(q^{-x};\alpha;q),$$

7. *the q-Laguerre polynomials*

$$L_n^{(\alpha)}(ax;q) = \sum_{m=0}^{n} \frac{a^m (a;q)_{n-m}(q^{\alpha+1};q)_n}{(q;q)_{n-m}(q^{\alpha+1};q)_m} L_m^{(\alpha)}(x;q),$$

8. *the little q-Laguerre / Wall polynomials*

$$p_n(ax;\alpha|q) = \sum_{m=0}^{n} \frac{(-1)^m(-a)^n q^{\binom{m}{2}}(q;q)_n (a^{-1};q)_{n-m}}{q^{\binom{n}{2}}(q;q)_m (q;q)_{n-m}} p_m(x;\alpha|q).$$

Proof The results follow by equating coefficients of t^n in the relations between $G(ax,t)$ and $G(x,at)$, using the q-binomial theorem (see e.g. [23, page 16])

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n,\ 0<|q|<1,\ |z|<1.$$

□

4.2 Second Method

In this method, for the remaining families of q-classical orthogonal polynomials we use the inversion formulas (4). It has been shown that

Theorem 11 **(See [28], [39], compare [2], [12])** *The inversion coefficients of the polynomial systems of the q-Hahn class are given in Table 1.*

Due to the bases in which the q-COP are represented, we consider here two cases.

4.2.1 Multiplication Coefficients of q-Orthogonal Polynomials Expanded in the Basis $\{x^n\}$

We suppose that the q-COP $p_n(x)$ are expanded in the basis x^n, i.e.

$$p_n(x) = \sum_{j=0}^{n} A_j(n)x^j,$$

so that

$$p_n(ax) = \sum_{j=0}^{n} A_j(n)a^j x^j.$$

We combine the latter expression with the inversion formula

$$x^j = \sum_{m=0}^{j} I_m(j)p_m(x)$$

Table 1 Inversion coefficients for q-COP

Family	Basis	$I_m(n)$
big q-Jacobi	$\{(x;q)_n\}_n$	$(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \frac{(\alpha q,\gamma q;q)_n(1-\alpha\beta q^{2m+1})}{(\alpha\beta q^{m+1};q)_n(1-\alpha\beta q^{n+m+1})}$
q-Hahn	$\{(x;q)_n\}_n$	$(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \frac{(\alpha q,q^{-N};q)_n(1-\alpha\beta q^{2m+1})}{(\alpha\beta q^{m+1};q)_n(1-\alpha\beta q^{n+m+1})}$
big q-Laguerre	$\{(x;q)_n\}_n$	$(-1)^m q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q (\alpha q,\beta q;q)_n$
q-Meixner	$\{(x;q)_n\}_n$	$(-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}(m+1)(m-2n)} \gamma^n (\beta q;q)_n$
affine q-Krawtchouk	$\{(x;q)_n\}$	$(-1)^m q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q (q^{-N};q)_n(pq;q)_n$
q-Krawtchouk	$\{(x;q)_n\}$	$(-1)^m q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(q^{-N};q)_n}{(-pq^m;q)_m(-pq^{2m+1};q)_{n-m}}$
quantum q-Krawtchouk	$\{(x;q)_n\}$	$(-1)^m q^{\frac{1}{2}(m+1)(m-2n)} p^{-n} \begin{bmatrix} n \\ m \end{bmatrix}_q (q^{-N};q)_n$
little q-Jacobi	$\{x^n\}_n$	$(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \frac{(\alpha q;q)_n(1-\alpha\beta q^{2m+1})}{(\alpha\beta q^{m+1};q)_n(1-\alpha\beta q^{n+m+1})}$
alternative q-Charlier	$\{x^n\}_n$	$(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{q^{\frac{m(m-1)}{2}}(1+\alpha q^{2m})}{(-\alpha q^{m+1};q)_n(1+\alpha q^m)}$

to get

$$p_n(ax) = \sum_{m=0}^{n} D_m(n,a)p_m(x) \quad \text{with} \quad D_m(n,a) = \sum_{j=0}^{n-m} a^{j+m} A_{j+m}(n) I_m(j+m).$$

We use the q-analogue of Algorithm 2.8 of [25, p. 22][4] to convert $D_k(n,a)$ into q-hypergeometric notation. It follows that

Theorem 12 *The multiplication formulae of the little q-Jacobi and alternative q-Charlier polynomials represented in the basis $\{x^n\}$ are given, respectively, by:*

$$p_n(ax;\alpha,\beta|q) = \sum_{m=0}^{n} \frac{(-a)^m q^{\frac{m(m+1)}{2}} (\alpha\beta q^{n+1};q)_m (q^{-n};q)_m}{(\alpha\beta q^{m+1};q)_m (q;q)_m} {}_2\phi_1\left(\begin{matrix} q^{m-n}, \alpha\beta q^{m+n+1} \\ \alpha\beta q^{2m+2} \end{matrix}\middle| q;aq\right) p_m(x;\alpha,\beta|q),$$

$$y_n(ax;\alpha;q) = \sum_{m=0}^{n} \frac{(-a)^m q^{\frac{m(m+1)}{2}} (q^{-n};q)_m (-\alpha q^n;q)_m}{(q;q)_m (-\alpha q^m;q)_m} {}_2\phi_1\left(\begin{matrix} q^{m-n}, -\alpha q^{m+n} \\ -\alpha q^{2m+1} \end{matrix}\middle| q;aq\right) y_m(x;\alpha;q).$$

4.2.2 Multiplication Coefficients of q-Orthogonal Polynomials Expanded in the Bases $\{(x;q)_n\}$ or $\{(ix;q)_n\}$

To solve the multiplication problem in these cases, we need the following result given in [15, Exercise 1.3].

Lemma 13 *The multiplication formula of the basis $\{(x;q)_n\}$ is given by*

$$(ax;q)_n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q a^m (a;q)_{n-m} (x;q)_m. \tag{7}$$

[4] explicitly the Maple procedure `sum2qhyper` of the accompanying `qsum` package

Proof Since

$$D_q(ax;q)_n = -a[n]_q(aqx;q)_{n-1},$$

it follows by iteration that

$$D_q^k(ax;q)_n = (-a)^k q^{\binom{k}{2}} \frac{(q^{n-k+1};q)_k}{(1-q)^k}(aq^k x;q)_{n-k}.$$

In order to obtain (7) we apply the operator D_q^k to both sides of the relation $(ax;q)_n = \sum\limits_{m=0}^{n} D_m(n,a)(x;q)_m$. This yields

$$a^k(q^{n-k+1};q)_k(aq^k x;q)_{n-k} = \sum_{m=k}^{n} D_m(n,a)(q^{m-k+1};q)_k(q^k x;q)_{m-k}.$$

For $x = q^{-k}$, since $(1;q)_k = 0,\ k \neq 0$, the latter equation gives the result. □

Remark 14 Another proof of (7) is by observing that the right hand side equals

$$(a;q)_n {}_2\phi_1\left(\begin{array}{c} q^{-n}, x \\ a^{-1}q^{1-n} \end{array} \middle| q;q\right)$$

which is evaluated by the q-Chu-Vandermonde formula

$${}_2\phi_1\left(\begin{array}{c} q^{-n}, a \\ c \end{array} \middle| q;q\right) = a^n \frac{(ca^{-1};q)_n}{(c;q)_n}.$$

This gives

$$(a;q)_n {}_2\phi_1\left(\begin{array}{c} q^{-n}, x \\ a^{-1}q^{1-n} \end{array} \middle| q;q\right) = (a;q)_n \frac{((ax)^{-1}q^{1-n};q)_n}{(a^{-1}q^{1-n};q)_n} = (ax;q)_n.$$

Using the representation of $p_n(x)$ in the basis $(x;q)_n$, Equation (7) and the inversion formulae of these polynomial systems, we prove

Theorem 15 *For the classical q-orthogonal polynomials represented in the basis $\{(x;q)_n\}$, the following multiplication and translation formulae are valid:*
1. q-Meixner

$$\begin{aligned} M_n(q^{-(x+a)};\beta,\gamma;q) &= \sum_{m=0}^{n}\sum_{j=0}^{n-m} \frac{(-1)^{m+j} q^{ma} q^{mn+nj+j}(q^{-a};q)_j(q^{-n};q)_m(q^{m-n};q)_j}{q^{\binom{m}{2}}\gamma^j(\beta q^{m+1};q)_j(q;q)_j(q;q)_m} \\ &\quad \times {}_2\phi_1\left(\begin{array}{c} q^{-j}, \beta q^{m+1} \\ q^{a-j+1} \end{array} \middle| q; -\frac{\gamma}{q^m}\right) M_m(q^{-x};\beta,\gamma;q), \end{aligned}$$

2. quantum q-Krawtchouk

$$\begin{aligned} K_n^{qtm}(ax;p,N;q) &= \sum_{m=0}^{n}\sum_{j=0}^{n-m} \frac{(-a)^m p^j q^{mn+nj+j}(q^{-n};q)_{j+m}(a;q)_j(q^{-N};q)_m}{q^{\binom{m}{2}}(q^{-N};q)_{j+m}(q;q)_j(q;q)_m} \\ &\quad \times {}_2\phi_1\left(\begin{array}{c} q^{-j}, q^{m-N} \\ \frac{q^{1-j}}{a} \end{array} \middle| q; \frac{1}{pq^m}\right) K_m^{qtm}(x;p,N;q), \end{aligned}$$

3. q-Krawtchouk

$$K_n(q^{-(x+a)};p,N;q) = \sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-q^{-a})^m q^{j+\frac{m(m+1)}{2}}(q^{-n};q)_{m+j}(-pq^n;q)_{m+j}(q^{-a};q)_j(q^{-N};q)_m}{(q^{-N};q)_{m+j}(q;q)_j(q;q)_m(-pq^m;q)_m} \times {}_3\phi_2\left(\begin{matrix} q^{-j},q^{m-N},0 \\ q^{1-j+a},-pq^{2m+1}\end{matrix}\middle| q;q\right)K_m(q^{-x};p,N;q),$$

4. big q-Laguerre

$$P_n(ax;\alpha,\beta;q) = \sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-a)^m q^{j+\frac{m(m+1)}{2}}(a;q)_j(q^{-n};q)_m(q^{m-n};q)_j}{(\alpha q^{m+1};q)_j(\beta q^{m+1};q)_j(q;q)_j(q;q)_m} \times {}_3\phi_2\left(\begin{matrix} q^{-j},\alpha q^{m+1},\beta q^{m+1} \\ \frac{q}{aq^j},0\end{matrix}\middle| q;q\right)P_m(x;\alpha,\beta;q),$$

5. affine q-Krawtchouk

$$K_n^{aff}(ax;p,N;q) = \sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-a)^m q^{j(N+1)+\frac{m(m+1)}{2}}(\frac{q}{q^{m+j}};q)_N(a;q)_j(q^{-n};q)_m(q^{m-n};q)_j}{(pq^{m+1};q)_j(q;q)_j(q;q)_m(q^m;q)_j(q^{1-m};q)_N} \times {}_3\phi_2\left(\begin{matrix} q^{-j},q^{m-N},pq^{m+1} \\ \frac{q}{aq^j},0\end{matrix}\middle| q;q\right)K_m^{aff}(x;p,N;q),$$

6. q-Hahn

$$Q_n(q^{-(x+a)};\alpha,\beta,N|q) = \sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-q^{-a})^m q^{j+\frac{m(m+1)}{2}}(q^{m-n};q)_j(q^{-n};q)_m(q^{-a};q)_j(\alpha\beta q^{n+1};q)_{m+j}}{(q;q)_j(q;q)_m(q^{m-N};q)_j(\alpha q^{m+1};q)_j(\alpha\beta q^{m+1};q)_m} \times {}_3\phi_2\left(\begin{matrix} q^{-j},q^{m-N},\alpha q^{m+1} \\ q^{a-j+1},\alpha\beta q^{2m+2}\end{matrix}\middle| q;q\right)Q_m(q^{-x};\alpha,\beta,N|q),$$

7. big q-Jacobi

$$P_n(ax;\alpha,\beta,\gamma;q) = \sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-a)^m q^{j+\frac{m(m+1)}{2}}(q^{m-n};q)_j(q^{-n};q)_m(a;q)_j(\alpha\beta q^{n+1};q)_{m+j}}{(q;q)_j(q;q)_m(\alpha q^{m+1};q)_j(\gamma q^{m+1};q)_j(\alpha\beta q^{m+1};q)_m} \times {}_3\phi_2\left(\begin{matrix} q^{-j},\alpha q^{m+1},\gamma q^{m+1} \\ \frac{q}{aq^j},\alpha\beta q^{2m+2}\end{matrix}\middle| q;q\right)P_m(x;\alpha,\beta,\gamma;q).$$

Proof Combining

$$p_n(ax)=\sum_{j=0}^{n}A_j(n)(ax;q)_j,\ (ax;q)_j=\sum_{k=0}^{j}B_k(j)(x;q)_k \text{ and } (x;q)_k=\sum_{m=0}^{k}I_m(k)p_m(x)$$

and interchanging the order of summation gives

$$p_n(ax)=\sum_{m=0}^{n}D_m(n,a)p_m(x) \text{ with } D_m(n,a)=\sum_{j=0}^{n-m}\sum_{k=0}^{j}A_{j+m}(n)B_{m+k}(j+m)I_m(k+m).$$

We use the q-analogue of Algorithm 2.8 of [25, p. 22] to convert $\sum_{k=0}^{j}B_{m+k}(j+m)I_m(k+m)$ into q-hypergeometric notation. □

Riese [31] developed an algorithm which finds recurrence equations for q-hypergeometric multiple sums and implemented it in *Mathematica* in his package *qMultisum*. Using this algorithm, we get

Proposition 16 *The following recurrence relations are satisfied by the multiplication coefficients of:*
1. q-Meixner polynomials

$$
\begin{aligned}
&-\gamma a^2 q^6 (q^n-q^m)\left(1-q^{1+m}\beta\right)\left(1-q^{2+m}\beta\right)D_m(n,a)-aq^3\left(1-q^{1+m}\right)\left(1-q^{2+m}\beta\right)\Big(q^{2m+4}\\
&-aq^{m+n+3}+a\gamma q^{m+1}(1+q)+\gamma q^{m+3}-a\gamma q^n(1+q+q^2)-\beta\gamma q^{2m+4}+a\beta\gamma q^{m+n+3}\Big)D_{m+1}(n,a)\\
&+q\left(1-q^{1+m}\right)\left(1-q^{2+m}\right)\Big((aq^{5+2m}-a^2q^{3+m+n}+a\gamma q^{m+3}-a\beta\gamma q^{2m+5}+a^2\beta\gamma q^{m+n+3})(1+q)\\
&-q^{8+3m}\beta+a^2q^{6+2m+n}\beta+a^2q^{2+m}\gamma-a^2\gamma q^n(1+q+q^2)\Big)D_{m+2}(n,a)\\
&-a\left(-1+q^{1+m}\right)\left(-1+q^{2+m}\right)\left(-1+q^{3+m}\right)\left(-q^{3+m}+aq^n\right)\left(q^{3+m}+\gamma\right)D_{3+m}(n,a)=0,
\end{aligned}
$$

2. quantum q-Krawtchouk polynomials

$$
\begin{aligned}
&a^2q^6(q^m-q^n)\left(q^{1+m}-q^N\right)\left(q^N-q^m\right)D_m(n,a)+aq^3\left(q^{1+m}-1\right)\left(q^{1+m}-q^N\right)\Big(apq^{3+m+n+N}\\
&+aq^{2+m+n}-q^{3+2m}+q^{3+m+N}-pq^{4+2m+N}-aq^{n+N}(1+q+q^2)+aq^{1+m+N}(1+q)\Big)D_{m+1}(n,a)\\
&+q^{1+N}\left(-1+q^{1+m}\right)\left(-1+q^{2+m}\right)\Big(-pq^{7+3m}+a^2pq^{5+2m+n}-a^2q^{2+m+N}+a^2q^{n+N}(1+q+q^2)\\
&+(aq^{4+2m}-a^2q^{2+m+n}-aq^{3+m+N}+apq^{5+2m+N}-a^2pq^{3+m+n+N})(1+q)\Big)D_{m+2}(n,a)\\
&-aq^{2N}\left(-1+q^{1+m}\right)\left(-1+q^{2+m}\right)\left(-1+q^{3+m}\right)\left(-1+pq^{3+m}\right)\left(-q^{3+m}+aq^n\right)D_{m+3}(n,a)=0,
\end{aligned}
$$

with the initial conditions $D_n(n,a)=a^n$, $D_{n+s}(n,a)=0,\ s=1,2,3.$

5 Multiplication Coefficients of Askey-Wilson and Wilson polynomials

In this section, we solve the multiplication problem for the Askey-Wilson and Wilson polynomials defined, respectively, by

$$
p_n(x;a,b,c,d|q)=\frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\begin{array}{c}q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta}\\ ab,ac,ad\end{array}\middle|\,q;q\right),\ x=\cos\theta,
$$

$$
W_n(x^2;a,b,c,d)=(a+b)_n(a+c)_n(a+d)_n{}_4F_3\left(\begin{array}{c}-n,n+a+b+c+d-1,a+ix,a-ix\\ a+b,a+c,a+d\end{array}\middle|\,1\right).
$$

Our results can be extended to other families of classical orthogonal polynomials on a quadratic or q-quadratic lattice by means of specialization and / or limiting processes following the Askey scheme and it q-analogue [23]. To illustrate this approach, we solve the multiplication problem for the q-Racah polynomials.

The basic hypergeometric resp. the hypergeometric representation of the Askey-Wilson resp. the Wilson polynomials suggests to use the basis $B_n(a,x)$ resp. $\vartheta_n(a,x)$ defined by $B_0(a,x)\equiv 1$,

$$
B_n(a,x)=(ae^{i\theta};q)_n(ae^{-i\theta};q)_n=(aq^s;q)_n(aq^{-s};q)_n=\prod_{k=0}^{n-1}(1-2axq^k+a^2q^{2k}),\ n\geq 1,\quad (8)
$$

where $x = x(s) = \cos\theta = \dfrac{q^s + q^{-s}}{2}$, $q^s = e^{i\theta}$;

$$\vartheta_n(a,x) = (a+ix)_n(a-ix)_n. \tag{9}$$

The operator $\mathbb{D}_x$ is appropriate for $B_n(a,x)$ whereas the corresponding operator for the basis $\{\vartheta_n(a,x)\}$ is the Wilson operator ([9], [22]) defined for an even function f by

$$\mathbf{D} f(x) = \frac{f\left(x+\frac{i}{2}\right) - f\left(x-\frac{i}{2}\right)}{2ix}. \tag{10}$$

5.1 Multiplication Formula of Askey-Wilson Polynomials

In order to get the multiplication coefficients $D_m(n,\alpha)$ of the relation

$$p_n(\alpha x; a,b,c,d|q) = \sum_{m=0}^{n} D_m(n,\alpha) p_m(x;a,b,c,d|q)$$

we need the multiplication coefficients of the basis family $(B_n(a,x))_n$ given by the following

Proposition 17 *For the Askey-Wilson polynomial basis $(B_n(a,x))_n$, the multiplication formula*

$$B_n(a,\alpha x) = \sum_{k=0}^{n} E_k(n) B_k(a,x) \tag{11}$$

holds with

$$E_k(n) = q^k \sum_{j=0}^{k} \frac{q^{-j^2} a^{-2j} \prod_{l=0}^{n-1} (1 - \alpha a^2 q^{l+j} - \alpha q^{l-j} + a^2 q^{2l})}{(q, a^2 q^{1+2j}; q)_{k-j} (q, a^{-2} q^{1-2j}; q)_j}.$$

The proof of this proposition uses Theorems 18 and 20 which follow below.

Theorem 18 (Expansion theorem, see e.g. [19], [20]) *Let f be a polynomial of degree n, then*

$$f(x) = \sum_{k=0}^{n} f_k B_k(a,x),$$

where

$$f_k = \frac{(q-1)^k}{(2a)^k (q;q)_k} q^{-\frac{k(k-1)}{4}} (\mathscr{D}_q^k f)(x_k) \tag{12}$$

with

$$x_k := \frac{1}{2}(a q^{k/2} + a^{-1} q^{-k/2}),$$

where $\mathscr{D}_q$ is the Askey-Wilson operator defined by (see e.g. [6])

$$\mathscr{D}_q f(x) = \frac{\breve{f}(q^{1/2} e^{i\theta}) - \breve{f}(q^{-1/2} e^{i\theta})}{\breve{e}(q^{1/2} e^{i\theta}) - \breve{e}(q^{-1/2} e^{i\theta})},$$

with $x = x(s) = \cos\theta = \dfrac{q^s + q^{-s}}{2}$, $q^s = e^{i\theta}$, where for a function f defined on $(-1,1)$ we have $\breve{f}(e^{i\theta}) := f(x)$, that is

$$\breve{f}(z) = f((z+1/z)/2), \quad z = e^{i\theta},$$

and $\breve{e}(x) = x$.

Remark 19 When we set $x = x(s) = \cos\theta = \dfrac{q^s + q^{-s}}{2}$, $q^s = e^{i\theta}$, we have $\mathbb{D}_x f(x(s)) = \mathscr{D}_q f(x)$.

Proof For $m = 0, 1, \ldots, k$, we apply $\mathscr{D}_q^m$ to both sides of $f(x) = \sum_{k=0}^{n} f_k B_k(a, x)$ to get

$$\mathscr{D}_q^m f(x) = \sum_{k=0}^{n} f_k \mathscr{D}_q^m B_k(a, x) = f_m \mathscr{D}_q^m B_m(a, x) + \sum_{k=m+1}^{n} f_k \mathscr{D}_q^m B_k(a, x). \tag{13}$$

By iteration of equation

$$\mathbb{D}_x B_n(a, x) = \frac{2a(1-q^n)}{q-1} B_{n-1}(a\sqrt{q}, x),$$

we have

$$\mathscr{D}_q^m B_k(a, x) = \frac{(2a)^m (q^{k-m+1}; q)_m q^{\frac{m(m-1)}{4}}}{(q-1)^m} B_{k-m}(aq^{\frac{m}{2}}, x). \tag{14}$$

For all $k \neq 0$, $B_k(aq^{\frac{m}{2}}, x) = 0 \Leftrightarrow x = x_m = \frac{1}{2}(aq^{m/2} + a^{-1}q^{-m/2})$. We substitute x by x_m in (13) and use (14) to get the result. □

We also need the following q-derivative rule due to Cooper [9] which is a generalization of Relation (14).

Theorem 20 ([9]) *The action of $\mathscr{D}_q^n$ on a function f is given by*

$$\mathscr{D}_q^n f(x) = \frac{2^n q^{\frac{n(1-n)}{4}}}{(q^{1/2} - q^{-1/2})^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(n-k)} z^{2k-n} \check{f}(q^{\frac{(n-2k)}{2}} z)}{(q^{1+n-2k} z^2; q)_k (q^{2k-n+1} z^{-2}; q)_{n-k}}, \tag{15}$$

where $x = \cos\theta = \frac{1}{2}(z + z^{-1})$ and $\check{f}(z) = f((z + 1/z)/2)$ with $z = e^{i\theta}$.

Proof (of Proposition 17) We have $x_k = \frac{1}{2}(aq^{\frac{k}{2}} + a^{-1}q^{\frac{-k}{2}}) = \frac{1}{2}(z + z^{-1})$ with $z = aq^{\frac{k}{2}}$. The combination of Equations (12) and (15) with $x = x_k$ and $z = aq^{\frac{k}{2}}$ yields

$$f_k = q^{k(1-k)} \sum_{j=0}^{k} \frac{q^{j(2k-j)} a^{2(j-k)} \check{f}(aq^{k-j})}{(q, a^2 q^{1+2(k-j)}; q)_j (q, a^{-2} q^{1-2(k-j)}; q)_{k-j}}.$$

If we substitute j by $k - j$, $f(x)$ by $B_n(a, \alpha x)$, the result follows. □

Finally we use the inversion formula of the Askey-Wilson polynomials.

Proposition 21 **(See e.g. [3], [13])** *The inversion formula of the Askey-Wilson orthogonal polynomial family is given by*

$$B_n(a, x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \frac{(-a)^m (abq^m, acq^m, adq^m; q)_{n-m}}{(abcdq^{m-1}; q)_m (abcdq^{2m}; q)_{n-m}} p_m(x; a, b, c, d|q). \tag{16}$$

We can now state and prove the multiplication formula of the Askey-Wilson polynomials.

Theorem 22 *The following multiplication formula is valid for the Askey-Wilson polynomials:*

$$p_n(\alpha x;a,b,c,d|q)=\sum_{m=0}^{n}D_m(n,\alpha)p_m(x;a,b,c,d|q), \tag{17}$$

with

$$D_m(n,\alpha)=\frac{(-1)^m a^{m-n}q^{\binom{m}{2}+2n}(ab,ac,ad;q)_n}{(abcdq^{m-1};q)_m}\sum_{s=0}^{n-m}\begin{bmatrix}n-s\\ m\end{bmatrix}_q\frac{q^{-2s}(abq^m,acq^m,adq^m;q)_{n-m-s}}{(abcdq^{2m};q)_{n-m-s}}\times$$
$$\sum_{i=0}^{n-s}\frac{a^{-2i}q^{-i^2}}{(q,a^2q^{2i+1};q)_{n-s-i}(q,\frac{q}{a^2q^{2i}};q)_i}\sum_{j=0}^{s}\frac{q^j\left(q^{-n},abcdq^{n-1};q\right)_{n+j-s}\prod_{l=0}^{n+j-s-1}(1-\alpha a^2q^{l+i}-\alpha q^{l-i}+a^2q^{2l})}{(q,ab,ac,ad;q)_{n+j-s}}.$$

Proof From the basic hypergeometric representation of the Askey-Wilson polynomials, since
$p_n(x;a,b,c,d|q)=\sum\limits_{j=0}^{n}A_j(n)B_j(a,x)$ with $B_j(a,x)$ defined by (8), we have

$$p_n(\alpha x;a,b,c,d|q)=\sum_{j=0}^{n}A_j(n)B_j(a,\alpha x).$$

From (11) and from the inversion formula (16), we have

$$B_j(a,\alpha x)=\sum_{k=0}^{j}E_k(j)B_k(a,x)\ \text{ and }\ B_k(a,x)=\sum_{m=0}^{k}I_m(k)p_m(x;a,b,c,d|q),$$

respectively. The combination of the above representations yields

$$p_n(\alpha x;a,b,c,d|q)=\sum_{m=0}^{n}D_m(n,\alpha)p_m(x;a,b,c,d|q)$$

with

$$D_m(n,\alpha)=\sum_{k=0}^{n-m}\sum_{j=0}^{n-m-k}I_m(k+m)A_{j+k+m}(n)E_{k+m}(j+k+m).$$

Since

$$E_k(n)=\sum_{i=0}^{k}F_i(k,n)\ \text{ with }\ F_i(k,n)=q^k\frac{q^{-i^2}a^{-2i}\prod_{l=0}^{n-1}(1-\alpha a^2q^{l+i}-\alpha q^{l-i}+a^2q^{2l})}{(q,a^2q^{1+2i};q)_{k-i}(q,a^{-2}q^{1-2i};q)_i},$$

it follows that

$$D_m(n,\alpha)=\sum_{k=0}^{n-m}\sum_{j=0}^{n-m-k}I_m(k+m)A_{j+k+m}(n)\sum_{i=0}^{k+m}F_i(k+m,j+k+m).$$

From the substitution $n-m-k=s$, we get

$$\begin{aligned}D_m(n,\alpha)&=\sum_{s=0}^{n-m}I_m(n-s)\sum_{j=0}^{s}\sum_{i=0}^{n-s}A_{j+n-s}(n)F_i(n-s,j+n-s)\\&=\sum_{s=0}^{n-m}I_m(n-s)\sum_{i=0}^{n-s}\sum_{j=0}^{s}A_{j+n-s}(n)F_i(n-s,j+n-s).\end{aligned}$$

□

5.2 Multiplication Formula of Wilson Polynomials

It is possible to proceed by a limiting process to get the multiplication formula for the Wilson polynomials from the Askey-Wilson multiplication formula. However, we use here the Wilson operator defined by Equation (10). Furthermore we need the following multiplication formula of the Wilson basis $(\vartheta_n(a,x))_n$.

Proposition 23 *The following multiplication formula is valid for the Wilson basis* $(\vartheta_n(a,x))_n$*:*

$$\vartheta_n(a,\alpha x) = \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(-k)_l}{k!l!}\frac{(2a+2l)(a-\alpha a-\alpha l)_n(a+\alpha a+\alpha l)_n}{(2a+l)_{k+1}}\vartheta_k(a,x). \tag{18}$$

The proof of this proposition needs the following theorems which are the analogues of Theorems 18 and 20.

Theorem 24 **(See e.g. [22])** *Let*

$$y_k = i\left(a+\frac{k}{2}\right),$$

and assume that $f(x)$ *is a polynomial of degree n in* x^2*. Then*

$$f(x) = \sum_{k=0}^{n} f_k\vartheta_k(a,x), \;\; \textit{with} \;\; f_k = \frac{1}{k!}(\mathbf{D}^k f)(y_k). \tag{19}$$

Proof Let $j = 0,1,\ldots,k$. We apply $\mathbf{D}^j$ to both sides of $f(x) = \sum\limits_{k=0}^{n} f_k\vartheta_k(a,x)$ and use the relation

$$\mathbf{D}^j\vartheta_k(a,x) = \frac{k!}{(k-j)!}\vartheta_{k-j}\left(a+\frac{j}{2},x\right), 0\le j\le k$$

to get

$$\mathbf{D}^j f(x) = f_j j! + \sum_{k=j+1}^{n} f_k\frac{k!}{(k-j)!}\vartheta_{k-j}(a+\frac{j}{2},x).$$

For $x = i\left(a+\frac{j}{2}\right)$, since $\vartheta_k(a,ai)=0,\ \forall k\ge 1$, we obtain

$$\mathbf{D}^j f\left(i\left(a+\frac{j}{2}\right)\right) = f_j j!.$$

This proves the theorem. □

Theorem 25 **(See [9])** *The action of* $\mathbf{D}^k$ *on an even function* f *is given by*

$$\mathbf{D}^k f(x) = \sum_{l=0}^{k}\frac{(-k)_l}{l!}\frac{(2ix-k+2l)}{(2ix-k+l)_{k+1}} f\left(x+\frac{k-2l}{2}i\right). \tag{20}$$

Proof (of Proposition 23) We combine (19) and (20) with $x = i(a+\frac{k}{2})$ to get

$$f_k = \sum_{l=0}^{k}\frac{(-k)_l}{l!k!}\frac{(-2a-2k+2l)}{(-2a-2k+l)_{k+1}} f(i(a+k-l)).$$

If we substitute l by $k-l$ and $f(x)$ by $\vartheta_n(a,\alpha x)$, using $(-1)^{k+1}(-2a-k-l)_{k+1} = (2a+l)_{k+1}$, the result follows. □

We combine the representation of the Wilson polynomial w.r.t. the basis $(\vartheta_n(a,x))_n$, the multiplication formula (18) and the inversion formula of the Wilson polynomials given by ([28], [39])

$$\vartheta_n(a,x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m (m+a+b, m+a+c, m+a+d)_{n-m}}{(a+b+c+d+m-1)_m (a+b+c+d+2m)_{n-m}} W_m(x^2; a,b,c,d).$$

Then we proceed as in the proof of Theorem 22 to get

Theorem 26 *The following multiplication formula is valid for the Wilson polynomials:*

$$W_n((\alpha x)^2; a,b,c,d) = \sum_{m=0}^{n} D_m(n,\alpha) W_m(x^2; a,b,c,d) \quad \textit{with}$$

$$D_m(n,\alpha) = \frac{(-1)^m (a+b,a+c,a+d)_n}{(a+b+c+d+m-1)_m} \sum_{s=0}^{n-m} \frac{\binom{n-s}{m} (m+a+b, m+a+c, m+a+d)_{n-m-s}}{(n-s)!(2m+a+b+c+d)_{n-m-s}}$$
$$\sum_{l=0}^{n-s} \frac{(-n+s)_l (2a+2l)}{l!(2a+l)_{n-s+1}} \sum_{j=0}^{s} \frac{(-n, n+a+b+c+d-1)_{n+j-s} (a-\alpha a-\alpha l, a+\alpha a+\alpha l)_{j+n-s}}{(j+n-s)!(a+b,a+c,a+d)_{j+n-s}}.$$

5.3 Multiplication Coefficients of the q-Racah Polynomials

The q-Racah polynomials defined by

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{matrix} \middle| q;q\right), \quad n = 0,1,\ldots,N,$$

where

$$\mu(x) := q^{-x} + \gamma\delta q^{x+1} \text{ and } \alpha q = q^{-N} \text{ or } \beta\delta q = q^{-N} \text{ or } \gamma q = q^{-N},$$

with a nonnegative integer N, are related to the Askey-Wilson polynomials in the following way. If we substitute [23, p. 421]

$$a^2 = \gamma\delta q, \; b^2 = \alpha^2\gamma^{-1}\delta^{-1} q, \; c^2 = \beta^2\gamma^{-1}\delta q, \; d^2 = \gamma\delta^{-1} q \text{ and } e^{2i\theta} = \gamma\delta q^{2x+1}$$

in the definition of the Askey-Wilson polynomials, we find

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = \frac{(\gamma\delta q)^{\frac{1}{2}n}}{(\alpha q, \beta\delta q, \gamma q; q)_n} p_n(\nu(x); (\gamma\delta q)^{\frac{1}{2}}, \alpha(\gamma\delta)^{-\frac{1}{2}} q^{\frac{1}{2}}, \beta\gamma^{-\frac{1}{2}}(\delta q)^{\frac{1}{2}}, (\gamma q)^{\frac{1}{2}}\delta^{-\frac{1}{2}}|q), \tag{21}$$

where

$$\nu(x) = \frac{1}{2}\gamma^{\frac{1}{2}}\delta^{\frac{1}{2}} q^{x+\frac{1}{2}} + \frac{1}{2}\gamma^{-\frac{1}{2}}\delta^{-\frac{1}{2}} q^{-x-\frac{1}{2}}.$$

The following multiplication formula for the q-Racah polynomial family is deduced from the multiplication formula for the Askey-Wilson polynomials using the above Relation (21) :

$$R_n(A\cdot\mu(x);\alpha,\beta,\gamma,\delta|q) = \sum_{m=0}^{n} D_m(n,A) R_m(\mu(x);\alpha,\beta,\gamma,\delta|q)$$

with

$$D_m(n,A) = \frac{(-1)^m q^{\binom{m}{2}+2n} (\alpha q, \beta\delta q, \gamma q; q)_m}{(\alpha\beta q^{m+1};q)_m} \sum_{s=0}^{n-m} \begin{bmatrix} n-s \\ m \end{bmatrix}_q \frac{(\alpha q^{m+1}, \beta\delta q^{m+1}, \gamma q^{m+1}; q)_{n-m-s}}{q^{2s}(\alpha\beta q^{2m+2};q)_{n-m-s}} \times$$
$$\sum_{i=0}^{n-s} \frac{(\gamma\delta q)^{-i} q^{-i^2}}{(q, \gamma\delta q^{2i+2};q)_{n-s-i} (q, \frac{q^{-2i}}{\gamma\delta};q)_i} \sum_{j=0}^{s} \frac{q^j (q^{-n}, \alpha\beta q^{n+1};q)_{n+j-s} \prod_{l=0}^{j+n-s-1} (1 - A\gamma\delta q^{l+i+1} - Aq^{l-i} + \gamma\delta q^{2l+1})}{(q, \alpha q, \beta\delta q, \gamma q; q)_{n+j-s}}.$$

6 Conclusion and Perspective

In this work, we have used both analytic and algorithmic approaches to compute the coefficients of the multiplication and translation formulas for all families of classical orthogonal polynomials of a continuous, a discrete and a q-discrete variable, and for the Askey-Wilson, the Wilson and the q-Racah polynomials. Outside the multiplication coefficients for the Hermite, the Laguerre and the Bessel polynomials which were already known, our results are new to the best of our knowledge. As perspective, it would be interesting to simplify multiplication coefficients appearing in double or triple summation or to see for which values of the parameter a the degree of complexity of those coefficients can decrease.

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