# Mixed recurrence equations and interlacing properties for zeros of sequences of classical $q$-orthogonal polynomials

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**Abstract**

Using the $q$-version of Zeilberger's algorithm, we provide a procedure to find mixed recurrence equations satisfied by classical $q$-orthogonal polynomials with shifted parameters. These equations are used to investigate interlacing properties of zeros of sequences of $q$-orthogonal polynomials. In the cases where zeros do not interlace, we give some numerical examples to illustrate this.

*Keywords:* classical $q$-orthogonal polynomials, Mixed recurrence equations, Interlacing of zeros
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## 1. Introduction

Let $0 < q < 1$. The classical $q$-orthogonal polynomials were introduced by Hahn [8] and can be written in terms of basic hypergeometric series, as introduced by Heine [9] in 1847. These polynomials are associated especially to quantum groups (cf. [16, 18, 19]), as introduced in [4, 26]. We list the systems of monic $q$-orthogonal polynomials considered in this paper (cf. [15]).

1. Askey-Wilson polynomials

$$\tilde{p}_n(x;a,b,c,d|q) = \frac{(ab,ac,ad;q)_n}{(2a)^n(abcdq^{n-1};q)_n}\,{}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\,\middle|\, q;q\right),\ x=\cos\theta, \quad (1)$$

with $a,b,c,d$ either real, or they occur in complex conjugate pairs, and $\max(|a|,|b|,|c|,|d|) < 1$, $x \in (-1,1)$;

2. $q$-Racah polynomials

$$\tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = \frac{(\alpha q,\beta\delta q,\gamma q;q)_n}{(\alpha\beta q^{n+1};q)_n}\,{}_4\phi_3\left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{matrix}\,\middle|\, q;q\right), \mu(x) = q^{-x}+\gamma\delta q^{x+1}, \quad (2)$$

$n \in \{0,1,\ldots,N\}$, $\alpha q = q^{-N}$ or $\beta\delta q = q^{-N}$ or $\gamma q = q^{-N}$, $N$ a nonnegative integer;

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3. Big $q$-Jacobi polynomials

$$\tilde{P}_n(x;\alpha,\beta,\gamma;q) = \frac{(\alpha q;q)_n(\gamma q;q)_n}{(\alpha\beta q q^n;q)_n}\,{}_3\phi_2\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1},x \\ \alpha q,\gamma q \end{array}\middle| q;q\right), \tag{3}$$

with $0 < \alpha q < 1$, $0 \leq \beta q < 1$ and $\gamma < 0$, $x \in (\gamma q, \alpha q)$;

4. $q$-Hahn polynomials

$$\tilde{Q}_n(\bar{x};\alpha,\beta,N|q) = \frac{(\alpha q;q)_n(q^{-N};q)_n}{(\alpha\beta q q^n;q)_n}\,{}_3\phi_2\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1},\bar{x} \\ \alpha q,q^{-N} \end{array}\middle| q;q\right),$$

with $\bar{x} = q^{-x}$, $n \in \{0,1,\ldots,N\}$, $0 < \alpha q < 1$ and $0 < \beta q < 1$ or $\alpha > q^{-N}$ and $\beta > q^{-N}$, $\bar{x} \in (1, q^{-N})$;

5. Little $q$-Jacobi polynomials

$$\tilde{p}_n(x;\alpha,\beta|q) = (-1)^n q^{\binom{n}{2}} \frac{(\alpha q;q)_n}{(\alpha\beta q q^n;q)_n}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1} \\ \alpha q \end{array}\middle| q;qx\right),\ 0 < \alpha q < 1,\ \beta q < 1, x \in (0,1);$$

6. $q$-Meixner polynomials

$$\tilde{M}_n(\bar{x};\beta,\gamma;q) = (-1)^n q^{-n^2}\gamma^n(\beta q;q)_n\,{}_2\phi_1\left(\begin{array}{c} q^{-n},\bar{x} \\ \beta q \end{array}\middle| q;-\frac{q^{n+1}}{\gamma}\right),$$

with $\bar{x} = q^{-x}$, $0 \leq \beta q < 1$, $\gamma > 0$, $\bar{x} \in (1,\infty)$;

7. $q$-Krawtchouk polynomials

$$\tilde{K}_n(\bar{x};p,N;q) = \frac{(q^{-N};q)_n}{(-pq^n;q)_n}\,{}_3\phi_2\left(\begin{array}{c} q^{-n},\bar{x},-pq^n \\ q^{-N},0 \end{array}\middle| q;q\right),$$

with $\bar{x} = q^{-x}$ and $n \in \{0,1,\ldots,N\}$, $p > 0$, $\bar{x} \in (1,q^{-N})$;

8. $q$-Laguerre polynomials

$$\tilde{L}_n^{(\alpha)}(x;q) = \frac{(-1)^n(q^{\alpha+1};q)_n}{q^{n(n+\alpha)}}\,{}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array}\middle| q;-q^{n+\alpha+1}x\right),\ \alpha > -1,\ x \in (0,\infty);$$

9. Alternative $q$-Charlier or $q$-Bessel polynomials

$$\tilde{y}_n(x;\alpha;q) = \frac{(-1)^n q^{\binom{n}{2}}}{(-\alpha q^n;q)_n}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},-\alpha q^n \\ 0 \end{array}\middle| q;qx\right),\ \alpha > 0,\ x \in (0,1);$$

10. Al-Salam-Carlitz I polynomials

$$\tilde{U}_n^{(\alpha)}(x;q) = (-\alpha)^n q^{\binom{n}{2}}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},x^{-1} \\ 0 \end{array}\middle| q;\frac{qx}{\alpha}\right),\ \alpha < 0,\ x \in (\alpha,1);$$

11. Al-Salam-Carlitz II polynomials

$$\tilde{V}_n^{(\alpha)}(x;q) = (-\alpha)^n q^{-\binom{n}{2}} {}_2\phi_0\left( \begin{array}{c} q^{-n}, x \\ - \end{array} \middle| q; \frac{q^n}{\alpha} \right), \; 0 < \alpha q < 1, \; x \in (1, \infty).$$

In the above definitions, the basic hypergeometric series ${}_r\phi_s$ is given by

$${}_r\phi_s\left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| q; z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left( (-1)^k q^{\binom{k}{2}} \right)^{1+s-r} \frac{z^k}{(q;q)_k},$$

where the $q$-Pochhammer symbol $(a_1, a_2, \ldots, a_k; q)_n$ is defined by

$$(a_1, \ldots, a_r; q)_k := (a_1;q)_k \cdots (a_r;q)_k, \quad \text{with} \;\; (a_i;q)_k = \begin{cases} \prod_{j=0}^{k-1} (1 - a_i q^j) & \text{if } \; k = 1, 2, 3, \ldots \\ 1 & \text{if } \; k = 0. \end{cases}$$

If $\{p_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials, the zeros of $p_n$ are real and simple and it is well known that the zeros of $p_n$ and $p_{n-1}$ interlace, i.e., if $x_{n,1} < x_{n,2} < \ldots < x_{n,n}$ denote the zeros of $p_n$, then

$$x_{n,1} < x_{n-1,1} < x_{n,2} < \cdots < x_{n,n-1} < x_{n-1,n-1} < x_{n,n}.$$

If polynomials $p_n$ and $q_n$ are of the same degree, the zeros are said to interlace if either

$$x_{n,1} < y_{n,1} < x_{n,2} < y_{n,2} \cdots < x_{n,n} < y_{n,n}$$

or

$$y_{n,1} < x_{n,1} < y_{n,2} < x_{n,2} \cdots < y_{n,n} < x_{n,n},$$

where $\{y_{n,k}\}_{k=1}^{n}$ denote the zeros of $q_n$.

The separation of the zeros of different sequences of Hahn polynomials of the same or adjacent degree was first studied by Levit [20] in 1967, and similar interlacing results followed for Jacobi polynomials [1, 5], Krawtchouk polynomials [3, 11] and Meixner and Meixner-Pollaczek polynomials [11]. The different sequences were obtained by integer shifts of the parameters and in order to prove these results, recurrence equations, following from the contiguous relations for hypergeometric polynomials [22, p. 71], were used. In the case of Gauss' hypergeometric function (cf. [24, Equation 4.21.3]), a useful algorithm in this regard is available as a computer package [25].

Interlacing results for the zeros of different sequences of $q$-orthogonal sequences with shifted parameters are given for $q$-Laguerre polynomials in [12, 21], for Al-Salam-Chihara, $q$-Meixner-Pollaczek and $q$-ultraspherical polynomials in [12] and for ${}_2\phi_1$ hypergeometric polynomials, associated with the little $q$-Jacobi polynomials, in [7]. The recurrence equations necessary to prove these results were obtained respectively from relationships between polynomials orthogonal to a positive measure $d\Psi(x)$ and those orthogonal to $x d\Psi(x)$ (cf. [14]), from the generating functions of the appropriate polynomials and from the contiguous function relations satisfied by the basic hypergeometric function (cf. [9]). In order to determine the specific order of the interlacing zeros, Markov's monotonicity theorem (or a consequence of it), is used (cf. [24, Theorems 6.12.1,6.12.2] or [10, Theorem 7.1.1]).

In this paper, we use mixed recurrence equations, satisfied by different sequences of the appropriate $q$-orthogonal polynomial systems, to study interlacing properties of the zeros of sequences of orthogonal systems on the $q$-linear lattice and the $q$-quadratic lattice, mentioned above, as well as of the zeros of $q$-orthogonal systems obtained by limit relations of these polynomials, namely the big $q$-Laguerre, quantum $q$-Krawtchouk, affine $q$-Krawtchouk, little $q$-Laguerre (or Wall), $q$-Charlier, continuous dual $q$-Hahn, Al-Salam Chihara, continuous big $q$-Hermite, continuous $q$-Hahn polynomials and the dual $q$-Hahn polynomials. The necessary equations are obtained using an algorithmic approach, whereas one may also use contiguous function relations for the basic hypergeometric series (see e.g. [7, 9, 23]) to get some of these recurrence equations. We use an adaption of the $q$-version of Zeilberger's algorithm which is an extension of Gosper's algorithm. Gosper's algorithm deals with the question how to find an anti-difference $s_k$ for given $a_k$, i.e., a sequence $s_k$ for which $a_k = \Delta s_k = s_{k+1} - s_k$, in a particular case that $s_k$ is a hypergeometric term, i.e., $\dfrac{s_{k+1}}{s_k} \in \mathbb{Q}(k)$. Given $F(n,k)$, Zeilberger's algorithm provides a recurrence equation for $s_n = \sum\limits_{k=-\infty}^{\infty} F(n,k)$, where $F(n,k)$ is a hypergeometric term with respect to both $n$ and $k$. We set $a_k = F(n,k) + \sum\limits_{j=1}^{J} \sigma_j(n) F(n+j,k)$ with undetermined variables $\sigma_j(n)$ and apply Gosper's algorithm to $a_k$. If successful, Gosper's algorithm finds $G(n,k)$ with $G(n,k+1) - G(n,k) = a_k$ and at the same time $\sigma_j(n)$, $j \in \{1,2,\ldots,J\}$. By summation, we have

$$s_n + \sum_{j=1}^{J} \sigma_j(n) s_{n+j} = 0.$$

We refer the reader to [17] and references there-in for more details about the algorithms of Gosper and Zeilberger and their $q$-analogues.

In our case, if we set, for example, $a_k = F(n,k,\alpha) + \sum\limits_{j=1}^{J} \sigma_j(n) F(n+j,k,\alpha+1)$, we obtain a recurrence equation of the form

$$s_n(\alpha) + \sum_{j=1}^{J} \sigma_j(n) s_{n+j}(\alpha+1) = 0, \; s_n(\alpha) = \sum_{k=-\infty}^{\infty} F(n,k,\alpha).$$

The $q$-analogues of Gosper's and Zeilberger's algorithms are implemented in the Maple `qsum` package [17] which can be downloaded at http://www.mathematik.uni-kassel.de/~koepf/Publikationen. By applying an adaption of the `qsumdiffeq` [17, p. 210] procedure of the `qsum` package, we wrote codes to derive recurrence equations of type (5) for the $q$-orthogonal polynomial systems considered in the sequel.

In section 2, the preliminary results that are used to prove our interlacing results, are listed. These results are generalizations of [2, Theorem 3], but we prove them here for polynomial systems with interlacing zeros and not necessarily polynomial systems that belong to the same orthogonal sequence. In each case, our results illustrate the monotonicity of the zeros, that can be obtained by the theorem of Markov [24, Theorem 6.12.2].

## 2. Preliminary results

**Lemma 1** (cf. [2, 7, 12])**.** *Let $(c,d)$ be a finite or infinite interval and $p_n$ and $q_n$ polynomials (not necessarily orthogonal) of degree $n$, with zeros $c < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < d$ and $c < y_{n,1} < y_{n,2} < \cdots < y_{n,n} < d$, respectively, satisfying the interlacing property*

$$x_{n,1} < y_{n-1,1} < x_{n,2} < y_{n-1,2} < \cdots < x_{n,n-1} < y_{n-1,n-1} < x_{n,n}. \tag{4}$$

*Let $a$ and $b$ be continuous functions on $(c,d)$ and assume that $f_n$ is a polynomial of degree $n$, with zeros $c < z_{n,1} < z_{n,2} < \cdots < z_{n,n} < d$, satisfying the equation*

$$f_n(x) = a(x)p_n(x) + b(x)q_{n-1}(x). \tag{5}$$

*Then,*

(a) *if $b$ has constant sign on $(c,d)$, the zeros of $f_n$ and $p_n$ interlace;*
(b) *if $a$ has constant sign on $(c,d)$, $f_n$ has an odd number of zeros between any two zeros of $q_{n-1}$.*

*Proof.* Assume that $f_n$ has degree $n$ with zeros $z_{n,1} < z_{n,2} < \cdots < z_{n,n}$.

(a) We evaluate (5) at $x_{n,k}$ and $x_{n,k+1}$, $k \in \{1,2,\ldots,n-1\}$, two consecutive zeros of $p_n(x)$. Then

$$f_n(x_{n,k})f_n(x_{n,k+1}) = b(x_{n,k})b(x_{n,k+1})q_{n-1}(x_{n,k})q_{n-1}(x_{n,k+1}).$$

By (4) the zeros of $p_n$ and $q_{n-1}$ interlace, therefore $q_{n-1}$ will differ in sign at $x_{n,k}$ and $x_{n,k+1}$, $k \in \{1,2,\ldots,n-1\}$, which implies $q_{n-1}(x_{n,k})q_{n-1}(x_{n,k+1}) < 0$. Since $b(x)$ has constant sign on $(c,d)$, we have $b(x_{n,k})b(x_{n,k+1}) > 0$ and therefore $f_n(x_{n,k})f_n(x_{n,k+1}) < 0$. $f_n$ must therefore have an odd number of zeros in each interval with endpoints $x_{n,k}$ and $x_{n,k+1}$, $k \in \{1,2,\ldots,n-1\}$, and the interlacing result follows.

(b) We evaluate (5) at $y_{n-1,k}$ and $y_{n-1,k+1}$, $k \in \{1,2,\ldots,n-2\}$, two consecutive zeros of $q_{n-1}(x)$. Then

$$f_n(y_{n-1,k})f_n(y_{n-1,k+1}) = a(y_{n-1,k})a(y_{n-1,k+1})p_n(y_{n-1,k})p_n(y_{n-1,k+1}).$$

From (4) we know that the zeros of $p_n$ and $q_{n-1}$ interlace, therefore $p_n$ will differ in sign at $y_{n-1,k}$ and $y_{n-1,k+1}$, $k \in \{1,2,\ldots,n-2\}$, and $p_n(y_{n-1,k})p_n(y_{n-1,k+1}) < 0$. Since $a(x)$ has constant sign on $(c,d)$, we have $a(y_{n-1,k})a(y_{n-1,k+1}) > 0$ and therefore $f_n(y_{n-1,k})f_n(y_{n-1,k+1}) < 0$, which implies that $f_n$ must have an odd number of zeros in each interval with endpoints $y_{n-1,k}$ and $y_{n-1,k+1}$, $k \in \{1,2,\ldots,n-2\}$.

□

If a polynomial $p_n$ is monic, then $\lim_{x\to\infty} p_n(x) = +\infty$. In the following result, which follows from Lemma 1, we assume that polynomials $p_n$ and $q_n$ are monic.

**Corollary 2** (cf. [2, 7, 12, 13]). *Let $(c,d)$ be a finite or infinite interval and assume that $p_n$ and $q_n$ are monic polynomials (not necessarily orthogonal) of degree $n$, with zeros $c < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < d$ and $c < y_{n,1} < y_{n,2} < \cdots < y_{n,n} < d$, respectively, satisfying the interlacing property (4). Assume that $a$ and $b$ are continuous and have constant sign on $(c,d)$ and that $f_n$ is a polynomial of degree $n$ with zeros $c < z_{n,1} < z_{n,2} < \cdots < z_{n,n} < d$, satisfying (5). Then, for each $k \in \{1,2,\ldots,n-1\}$,*

(a) *if $a(x)$ and $b(x)$ have the same sign on $(c,d)$, $z_{n,k} < x_{n,k} < y_{n-1,k} < z_{n,k+1} < x_{n,k+1}$;*
(b) *if $a(x)$ and $b(x)$ differ in sign on $(c,d)$, $x_{n,k} < z_{n,k} < y_{n-1,k} < x_{n,k+1} < z_{n,k+1}$.*

*Proof.* Assume that $f_n$ has degree $n$ and both $a$ and $b$ have constant sign on $(c,d)$. Then both results of Lemma 1 are true. From Lemma 1(a), the zeros of $f_n$ and $p_n$ interlace and either $z_{n,k} < x_{n,k}$ or $x_{n,k} < z_{n,k}$ for each $k \in \{1,2,\ldots,n\}$.

Evaluating (5) at $y_{n-1,n-1}$ and $x_{n,n}$, we obtain

$$f_n(x_{n,n})f_n(y_{n-1,n-1}) = a(y_{n-1,n-1})b(x_{n,n})p_n(y_{n-1,n-1})q_{n-1}(x_{n,n}). \tag{6}$$

Since, by assumption, $p_n$ and $q_{n-1}$ are monic polynomials with interlacing zeros, $p_n(y_{n-1,n-1}) < 0$ and $q_{n-1}(x_{n,n}) > 0$.

(a) Assume $a$ and $b$ have the same sign on $(c,d)$. Then $a(y_{n-1,n-1})b(x_{n,n}) > 0$ and, since $p_n(y_{n-1,n-1})q_{n-1}(x_{n,n}) < 0$, we deduce from (6) that $f_n(x_{n,n})f_n(y_{n-1,n-1}) < 0$. This implies $f_n$ has an odd number of zeros in the interval $(y_{n-1,n-1}, x_{n,n})$.
Suppose $z_{n,k} < x_{n,k}, k \in \{1,2,\ldots,n\}$. From (4) we deduce that $z_{n,1} < x_{n,1} < y_{n-1,1}$ and thus one zero of $f_n$ lies to the left of $y_{n-1,1}$. From Lemma 1(b), we know there is an odd number of zeros of $f_n$ in each of the $n-2$ intervals $(y_{n-1,k}, y_{n-1,k+1}), k \in \{1,2,\ldots,n-2\}$. If each of the $n-2$ intervals between the first and the last zero of $q_{n-1}$ has exactly one zero of $f_n$, we have $n-1$ zeros accounted for. There is only one zero remaining (since $f_n$ has $n$ zeros), and we deduce that only one zero of $f_n$ lies in $(y_{n-1,n-1}, x_{n,n})$, which leads to the configuration

$$z_{n,1} < x_{n,1} < y_{n-1,1} < z_{n,2} < \cdots < x_{n,n-1} < y_{n-1,n-1} < z_{n,n} < x_{n,n}.$$

Suppose $x_{n,k} < z_{n,k}, k \in \{1,2,\ldots,n\}$. From (4), we deduce that $y_{n-1,n-1} < x_{n,n} < z_{n,n}$. This contradicts the fact that $f_n$ must have an odd number of zeros in the interval $(y_{n-1,n-1}, x_{n,n})$.

(b) Assume $a$ and $b$ have different signs on $(c,d)$. Then $a(y_{n-1,n-1})b(x_{n,n}) < 0$ and we deduce from (6) that $f_n(x_{n,n})f_n(y_{n-1,n-1}) > 0$, thus $f_n$ has either 0 or an even number of zeros in the interval $(y_{n-1,n-1}, x_{n,n})$.
Suppose $x_{n,k} < z_{n,k}, k \in \{1,2,\ldots,n\}$. From (4) we deduce that $y_{n-1,n-1} < x_{n,n} < z_{n,n}$ and the only option, counting the zeros, is that

$$x_{n,1} < z_{n,1} < y_{n-1,1} < x_{n,2} < \cdots < z_{n,n-1} < y_{n-1,n-1} < x_{n,n} < z_{n,n}.$$

Suppose $z_{n,k} < x_{n,k}, k \in \{1,2,\ldots,n\}$. From (4) we deduce that $z_{n,1} < x_{n,1} < y_{n-1,1}$ and thus one zero of $f_n$ lies to the left of $y_{n-1,1}$. From Lemma 1(b), we know there is an odd number of zeros of $f_n$ in each of the $n-2$ intervals $(y_{n-1,k}, y_{n-1,k+1}), k \in \{1,2,\ldots,n-2\}$. If each of the $n-2$ intervals between the first and the last zero of $q_{n-1}$ has exactly one zero of $f_n$, we have $n-1$ zeros accounted for. There is only one zero remaining (since $f_n$ has $n$ zeros). The one remaining zero therefore must lie to the right of $y_{n-1,n-1}$, such that $y_{n-1,n-1} < z_{n,n} < x_{n,n}$, which contradicts the fact that $f_n$ must have either 0 or an even number of zeros in the interval $(y_{n-1,n-1}, x_{n,n})$.

□

From Corollary 2 we remark that, once we have a relation of type (5), it is sufficient to know the sign of $a$ and $b$ to prove our interlacing results.

## 3. Classical orthogonal polynomials on a $q$-linear lattice

In this section, we consider interlacing properties of zeros of different sequences of orthogonal polynomials on a $q$-linear lattice.

### 3.1. *The big q-Jacobi polynomials*

The sequence of big $q$-Jacobi polynomials $\{\tilde{P}_n(x;\alpha,\beta,\gamma;q)\}$ is orthogonal for $0 < \alpha q < 1$, $0 \le \beta q < 1$ and $\gamma < 0$ with respect to a continuous weight function, on the interval $(\gamma q, \alpha q)$. As the parameter $\alpha$ decreases to $\alpha q$, the interval in which the zeros lie decreases to $(\gamma q, \alpha q^2)$ and we can deduce that the zeros of $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ decrease as $\alpha$ decreases to $\alpha q$. Similarly, as $\gamma$ increases to $\gamma q$, the zeros will increase, since the interval in which the zeros lie reduces to $(\gamma q^2, \alpha q)$.

The following recurrence equations will be used to prove our results and can be downloaded from http://www.mathematik.uni-kassel.de/~koepf/Publikationen.

**Proposition 3.**

$$\tilde{P}_n(x;\alpha,\beta,\gamma;q) = \tilde{P}_n(x;\alpha q,\beta,\gamma;q) + \frac{\alpha\, q\,(q^n-1)\,(\beta\, q^n-1)\,(\gamma\, q^n-1)}{(\alpha\beta\, q^{2\,n+1}-1)\,(\alpha\,\beta\, q^{2\,n}-1)}\tilde{P}_{n-1}(x;\alpha q,\beta,\gamma;q); \tag{7a}$$

$$\tilde{P}_n(x;\alpha,\beta,\gamma;q) = \tilde{P}_n(x;\alpha,\beta q,\gamma;q) - \frac{\alpha\,\beta\, q^{n+1}\,(\alpha\, q^n-1)\,(\gamma\, q^n-1)\,(q^n-1)}{(\alpha\beta\, q^{2\,n+1}-1)\,(\alpha\,\beta\, q^{2\,n}-1)}\tilde{P}_{n-1}(x;\alpha,\beta q,\gamma;q); \tag{7b}$$

$$\tilde{P}_n(x;\alpha,\beta q,\gamma;q) = \tilde{P}_n(x;\alpha q,\beta,\gamma;q) + \frac{\alpha q(q^n-1)(\gamma q^n-1)}{\alpha\beta q^{2n+1}-1}\tilde{P}_{n-1}(x;\alpha q,\beta q,\gamma;q). \tag{7c}$$

**Theorem 4.** *Let $0 < \alpha q < 1$, $0 \le \beta q < 1$, $\gamma < 0$ and denote the zeros of $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ by $\gamma q < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < \alpha q$, the zeros of $\tilde{P}_n(x;\alpha q,\beta,\gamma;q)$ by $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$, the zeros of $\tilde{P}_n(x;\alpha,\beta q,\gamma;q)$ by $z_{n,1} < z_{n,2} < \cdots < z_{n,n}$ and the zeros of $\tilde{P}_n(x;\alpha q,\beta q,\gamma;q)$ by $t_{n,1} < t_{n,2} < \cdots < t_{n,n}$. Then, for each $i \in \{1,2,\ldots,n-1\}$,*

(a) $y_{n,i} < x_{n,i} < y_{n-1,i} < y_{n,i+1} < x_{n,i+1}$ ,

(b) $x_{n,i} < z_{n,i} < z_{n-1,i} < x_{n,i+1} < z_{n,i+1}$ ,

(c) $y_{n,i} < x_{n,i} < z_{n,i} < t_{n-1,i} < y_{n,i+1} < x_{n,i+1} < z_{n,i+1}$.

*Proof.* Let $0 < \alpha q < 1$, $0 \leq \beta q < 1$, $\gamma < 0$. Since $0 < q < 1$, it follows that $q^n - 1 < 0$, $\alpha q^n - 1 < 0$, $\beta q^n - 1 < 0$, $\alpha\beta q^{2n} - 1 < 0$, $\alpha\beta q^{2n+1} - 1 < 0$ and $\gamma q^n - 1 < 0$.

(a) Since $\tilde{P}_n(x; \alpha q, \beta, \gamma; q)$ and $\tilde{P}_{n-1}(x; \alpha q, \beta, \gamma; q)$ belong to the same orthogonal sequence, their zeros interlace and the interlacing property (4) is satisfied. (7a) is in the form of (5) with $a(x) = 1$ and, taking into consideration the restrictions on the parameters, $b(x)$ is a negative constant and the interlacing follows from Corollary 2 (b).

(b) The polynomials $\tilde{P}_n(x; \alpha, \beta q, \gamma; q)$ and $\tilde{P}_{n-1}(x; \alpha, \beta q, \gamma; q)$ belong to the same orthogonal sequence and their zeros satisfy (4). (7b) is in the form of (5) with $a(x) = 1$ and taking into consideration the restrictions on the parameters, $b(x)$ is a positive constant. The result follows from Corollary 2 (a).

(c) In (b) we have proved that the zeros of $\tilde{P}_n(x; \alpha, \beta, \gamma; q)$ and $\tilde{P}_{n-1}(x; \alpha, \beta q, \gamma; q)$ interlace for all $\alpha$ such that $0 < \alpha q < 1$, from which we can deduce that the zeros of $\tilde{P}_n(x; \alpha q, \beta, \gamma; q)$ and $\tilde{P}_{n-1}(x; \alpha q, \beta q, \gamma; q)$ interlace, satisfying (4). Equation (7c) is in the form of (5) with $a(x) = 1$ and taking into consideration the restrictions on the parameters, $b(x)$ is a negative constant. Applying Corollary 2 (b), we obtain $y_{n,i} < z_{n,i} < t_{n-1,i} < y_{n,i+1} < z_{n,i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$. Furthermore, $y_{n,i} < x_{n,i} < z_{n,i}$ for each $i \in \{1, 2, \ldots, n\}$ (from (a) and (b)), and the required combined interlacing follows.

□

**Corollary 5.** *For each $i \in \{1, 2, \ldots, n-1\}$,*

(a) $x_{n,i} < y_{n-1,i} < x_{n-1,i} < x_{n,i+1}$ ,

(b) $x_{n,i} < x_{n-1,i} < z_{n-1,i} < x_{n,i+1}$.

*Proof.* We obtain the results by combining the interlacing of the zeros of $\tilde{P}_n(x; \alpha, \beta, \gamma; q)$ and $\tilde{P}_{n-1}(x; \alpha, \beta, \gamma; q)$ with the results proved in Theorem 4 (a) and (b), respectively. □

**Remark 6.** *(i) In general, the zeros of $\tilde{P}_n(x; \alpha, \beta, \gamma; q)$ do not interlace with the zeros of $\tilde{P}_n(x; \alpha, \beta, \gamma q; q)$ or with the zeros of $\tilde{P}_{n-1}(x; \alpha, \beta, \gamma q; q)$. For example, when $n = 4, \alpha = 1, \beta = 3, \gamma = -5, q = 0.14$, the zeros of $\tilde{P}_n(x; \alpha, \beta, \gamma; q)$ are $\{-0.6993, -0.1066, 0.0198, 0.1353\}$, the zeros of $\tilde{P}_n(x; \alpha, \beta, \gamma q; q)$ are $\{-0.0992, 0.0000, 0.0071, 0.1407\}$ and the zeros of $\tilde{P}_{n-1}(x; \alpha, \beta, \gamma q; q)$ are $\{-0.0978, 0.0056, 0.1399\}$;*

*(ii) When $\beta = 0$ in the definition of the monic big q-Jacobi polynomials, we obtain the monic big q-Laguerre polynomials, i.e., $\tilde{P}_n(x; \alpha, 0, \gamma; q) = \tilde{P}_n(x; \alpha, \gamma; q)$ [15, Equation 14.5.13]. The interlacing property of the zeros of the big q-Laguerre polynomials, as $\alpha$ decreases to $\alpha q$, can thus be obtained from the result obtained for the big q-Jacobi polynomials. Furthermore, we have $\tilde{P}_n(x; \alpha, \beta; q) = \tilde{P}_n(x; \beta, \alpha; q)$ and the interlacing property as $\beta$ increases to $\beta q$ follows directly. The interlacing results of Theorem 4 and Corollary 5 are therefore valid, where $x_{n,i}, y_{n,i}, z_{n,i}, t_{n,i}, i \in \{1, 2, \ldots, n\}$ are the zeros of $\tilde{P}_n(x; \alpha, \gamma; q)$, $\tilde{P}_n(x; \alpha q, \gamma; q)$, $\tilde{P}_n(x; \alpha, \gamma q; q)$ and $\tilde{P}_n(x; \alpha q, \gamma q; q)$, respectively.*

*3.2. The q-Hahn polynomials*

**Proposition 7.**

$$\tilde{Q}_n(x;\alpha,\beta,N|q) = \tilde{Q}_n(x;\alpha q,\beta,N|q) + \frac{\alpha\,(q^n-1)\,(\beta\,q^n-1)\left(q^n-q^{N+1}\right)}{q^N\,(\alpha\beta\,q^{2\,n+1}-1)\,(\alpha\,\beta\,q^{2\,n}-1)}\tilde{Q}_{n-1}(x;\alpha q,\beta,N|q); \tag{8a}$$

$$\tilde{Q}_n(x;\alpha,\beta,N|q) = \tilde{Q}_n(x;\alpha,\beta q,N|q) + \frac{\alpha\,\beta\,q^{n-N}\left(q^{N+1}-q^n\right)(\alpha\,q^n-1)\,(q^n-1)}{(\alpha\beta\,q^{2\,n+1}-1)\,(\alpha\,\beta\,q^{2\,n}-1)}\tilde{Q}_{n-1}(x;\alpha,\beta q,N|q); \tag{8b}$$

$$\tilde{Q}_n(x;\alpha,\beta q,N|q) = \tilde{Q}_n(x;\alpha q,\beta,N|q) + \frac{\alpha(q^n-1)(q^n-q^{N+1})}{q^N(\alpha\beta q^{2n+1}-1)}\tilde{Q}_{n-1}(x;\alpha q,\beta q,N|q); \tag{8c}$$

$$\begin{aligned}\tilde{Q}_n\,(x;\alpha,\beta,N|q) &= \frac{\left(\alpha\beta q^{2\,n+1}-1\right)\left(\alpha\beta q^{N+2}(\alpha q^{n+1}-1)-\alpha q^{N-n+2}(q^n-1)-(\alpha q-1)\right)}{(\alpha q^{n+1}-1)\,(\alpha\beta q^{n+1}-1)\,(\alpha\beta q^{n+N+2}-1)}\tilde{Q}_n(x;\alpha q,\beta,N|q)\\ &+\frac{\alpha q\,(\beta q^n-1)\left(q^n-q^{1+N}\right)(q^n-1)\left(\alpha q^2-x\right)}{q^n\,(\alpha q^{n+1}-1)\,(\alpha\beta q^{n+1}-1)\,(\alpha\beta q^{n+N+2}-1)}\tilde{Q}_{n-1}(x;\alpha q^2,\beta,N|q).\end{aligned} \tag{8d}$$

**Theorem 8.** *Let $0<\beta q<1$, $0<\alpha q<1$, $n\in\{0,1,\ldots,N\}$. We denote the zeros of $\tilde{Q}_n(x;\alpha,\beta,N|q)$ by $1<x_{n,1}<x_{n,2}<\cdots<x_{n,n}<q^{-N}$, the zeros of $\tilde{Q}_n(x;\alpha q,\beta,N|q)$ by $y_{n,1}<y_{n,2}<\cdots<y_{n,n}$, the zeros of $\tilde{Q}_n(x;\alpha,\beta q,N|q)$ by $z_{n,1}<z_{n,2}<\cdots<z_{n,n}$ and the zeros of $\tilde{Q}_n(x;\alpha q,\beta q,N|q)$ by $t_{n,1}<t_{n,2}<\cdots<t_{n,n}$. Then, for $i\in\{1,2,\ldots,n-1\}$,*

(a) $x_{n,i}<y_{n,i}<y_{n-1,i}<x_{n,i+1}<y_{n,i+1}$,

(b) $z_{n,i}<x_{n,i}<z_{n-1,i}<z_{n,i+1}<x_{n,i+1}$,

(c) $z_{n,i}<x_{n,i}<y_{n,i}<t_{n-1,i}<z_{n,i+1}<x_{n,i+1}<y_{n,i+1}$.

*Proof.* Let $0<\beta q<1$, $0<\alpha q<1$, $n\in\{0,1,\ldots,N\}$. Since $0<q<1$, it follows that $q^n-1<0$, $\beta q^n-1<0$, $\alpha\beta q^{2n}-1<0$ and $\alpha\beta q^{2n+1}-1<0$. Furthermore, $q^m<q^n$ for $m>n$ and consequently $q^{N+1}-q^n<0$.

The two polynomials on the right-hand side of each of the equations (8a) and (8b) belong to the same orthogonal sequence, therefore their zeros interlace and satisfy the interlacing property (4). Each of these equations are thus are in the form of (5) with $a(x)=1$. Furthermore,

(a) in (8a), $b(x)>0$ on $(1,q^{-N})$ and the required interlacing follows from Corollary 2 (a);

(b) $b(x)$ in (8b) is a negative constant and the result follows from Corollary 2 (b);

(c) From the interlacing of the zeros of $\tilde{Q}_n(x;\alpha,\beta q,N|q)$ and $\tilde{Q}_{n-1}(x;\alpha,\beta q,N|q)$ for all $\alpha$ such that $0<\alpha q<1$ (from (b)), the interlacing of the zeros of $\tilde{Q}_n(x;\alpha q,\beta,\gamma;q)$ and $\tilde{Q}_{n-1}(x;\alpha q,\beta q,\gamma;q)$ follows directly. Equation (8c) is in the form of (5) with $a(x)=1$ and taking into consideration the restrictions on the parameters, $b(x)$ is a positive constant. Applying Corollary 2 (a), we obtain $z_{n,i}<y_{n,i}<t_{n-1,i}<z_{n,i+1}<y_{n,i+1}$ for each $i\in\{1,2,\ldots,n-1\}$. Furthermore, it follows from (a) and (b) that $z_{n,i}<x_{n,i}<y_{n,i}$ for each $i\in\{1,2,\ldots,n\}$, and the required combined interlacing follows.

□

**Corollary 9.** *For each $i \in \{1, 2, \ldots, n-1\}$,*

(a) $x_{n,i} < x_{n-1,i} < y_{n-1,i} < x_{n,i+1}$,

(b) $x_{n,i} < z_{n-1,i} < x_{n-1,i} < x_{n,i+1}$.

*Proof.* We obtain the results by combining the interlacing of the zeros of $\tilde{Q}_n(x;\alpha,\beta,N|q)$ and $\tilde{Q}_{n-1}(x;\alpha,\beta,N|q)$ with the results proved in Theorem 8 (a) and (b), respectively. □

**Remark 10.** *(i) When we let $\beta = 0$ in the definition of the monic q-Hahn polynomials, we obtain the monic affine q-Krawtchouk polynomials [15, Section 14.16] $\tilde{K}_n^{Aff}(\bar{x};\alpha,N;q)$, orthogonal on $(1, q^{-N})$ if $0 < \alpha q < 1$. The interlacing results in Theorem 8 (a) and Corollary 9 (a) follow from (8a) (with $\beta = 0$), where $x_{n,i}$ and $y_{n,i}, i \in \{1, 2, \ldots, n\}$ are the zeros of $\tilde{K}_n^{Aff}(\bar{x};\alpha,N;q)$ and $\tilde{K}_n^{Aff}(\bar{x};\alpha q,N;q)$, respectively. Furthermore, when we let $\beta = 0$ in (8d), we find that*

$$x_{n,i} < y_{n,i} < Y_{n-1,i} < x_{n,i+1} < y_{n,i+1},$$

*for each $i \in \{1, 2, \ldots, n-1\}$, where $Y_{n,i}, i \in \{1, 2, \ldots, n\}$ are the zeros of $\tilde{K}_n^{Aff}(\bar{x};\alpha q^2,N;q)$;*

*(ii) Since $\lim_{\alpha\to\infty} \tilde{Q}_n(\bar{x};\alpha,\beta,N|q) = \tilde{K}_n^{qtm}(\bar{x};\beta,N;q)$ [15, Section 14.14], we obtain from (8b), the equation*

$$\tilde{K}_n^{qtm}(\bar{x};\beta,N;q) = \tilde{K}_n^{qtm}(\bar{x};\beta q,N;q) + \frac{\left(q^{N+1} - q^n\right)(q^n - 1)}{\beta q^{2n+N+1}} \tilde{K}_{n-1}^{qtm}(\bar{x};\beta q,N;q),$$

*from which the interlacing results in Theorem 8 (b) and Corollary 9 (b) follow directly, where $x_{n,i}$ and $z_{n,i}, i \in \{1, 2, \ldots, n\}$ are the zeros of the monic quantum q-Krawtchouk polynomials $\tilde{K}_n^{qtm}(\bar{x};\beta,N;q)$ and $\tilde{K}_n^{qtm}(\bar{x};\beta q,N;q)$, respectively.*

### *3.3. The little q-Jacobi polynomials*

**Proposition 11.**

$$\tilde{p}_n(x;\alpha,\beta|q) = \tilde{p}_n(x;\alpha q,\beta|q) + \frac{\alpha q^n (q^n - 1)(\beta q^n - 1)}{(\alpha\beta q^{2n+1} - 1)(\alpha\beta q^{2n} - 1)} \tilde{p}_{n-1}(x;\alpha q,\beta|q); \tag{9a}$$

$$\tilde{p}_n(x;\alpha,\beta|q) = \tilde{p}_n(x;\alpha,\beta q|q) - \frac{\alpha\beta q^{2n} (q^n - 1)(\alpha q^n - 1)}{(\alpha\beta q^{2n+1} - 1)(\alpha\beta q^{2n} - 1)} \tilde{p}_{n-1}(x;\alpha,\beta q|q); \tag{9b}$$

$$\begin{aligned} \tilde{p}_n(x;\alpha,\beta|q) &= \frac{(\alpha q - 1)\left(\alpha\beta q^{2n+1} - 1\right)\tilde{p}_n(x;\alpha q^2,\beta|q)}{(\alpha q^{n+1} - 1)(\alpha\beta q^{n+1} - 1)} \\ &+ \frac{\alpha q (q^n - 1)(\beta q^n - 1)\left(\left(\alpha\beta q^{2n+2} - 1\right)x + q^n(\alpha q - 1)\right)\tilde{p}_{n-1}(x;\alpha q^2,\beta|q)}{(\alpha q^{n+1} - 1)(\alpha\beta q^{2n+2} - 1)(\alpha\beta q^{n+1} - 1)}; \end{aligned} \tag{9c}$$

$$\begin{aligned} \tilde{p}_n(x;\alpha,\beta|q) &= -\frac{\left(\alpha\beta q^{n+1}(q^n - 1) + 1 - \beta q^{n+1}\right)\left(\alpha\beta q^{2n+1} - 1\right)\tilde{p}_n(x;\alpha,\beta q|q)}{(\beta q^{n+1} - 1)(\alpha\beta q^{n+1} - 1)} \\ &+ \frac{\alpha\beta q^{2n}\left(\beta q^2 x - 1\right)(q^n - 1)(\alpha q^n - 1)\tilde{p}_{n-1}(x;\alpha,\beta q^2|q)}{(\beta q^{n+1} - 1)(\alpha\beta q^{n+1} - 1)} \quad \textit{(cf. [7, Equation 9])}; \end{aligned} \tag{9d}$$

$$\tilde{p}_n(x;\alpha,\beta q|q) = \tilde{p}_n(x;\alpha q,\beta|q) + \frac{\alpha q^n (q^n - 1)}{\beta\alpha q^{2n+1} - 1} \tilde{p}_{n-1}(x;\alpha q,\beta q|q) \quad \textit{(cf. [7, Equation 10])}. \tag{9e}$$

**Theorem 12.** *Let $0 < \alpha q < 1$ and $\beta q < 1$ and denote the zeros of $\tilde{p}_n(x; \alpha, \beta|q)$ by $0 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < 1$, the zeros of $\tilde{p}_n(x; \alpha q, \beta|q)$ by $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$, the zeros of $\tilde{p}_n(x; \alpha q^2, \beta|q)$ by $Y_{n,1} < Y_{n,2} < \cdots < Y_{n,n}$, the zeros of $\tilde{p}_n(x; \alpha, \beta q|q)$ by $z_{n,1} < z_{n,2} < \cdots < z_{n,n}$, the zeros of $\tilde{p}_n(x; \alpha, \beta q^2|q)$ by $Z_{n,1} < Z_{n,2} < \cdots < Z_{n,n}$.Then, for $i \in \{1, 2, \ldots, n-1\}$,*

(a) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$,
(b) $z_{n,i} < x_{n,i} < z_{n-1,i} < z_{n,i+1} < x_{n,i+1}$ *if* $\beta > 0$ *and* $x_{n,i} < z_{n,i} < z_{n-1,i} < x_{n,i+1} < z_{n,i+1}$ *if* $\beta < 0$,
(c) $x_{n,i} < y_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < y_{n,i+1} < Y_{n,i+1}$,
(d) $x_{n,i} < z_{n,i} < Z_{n-1,i} < x_{n,i+1} < z_{n,i+1}$ *if* $\beta < 0$,
(e) $z_{n,i} < y_{n,i} < t_{n-1,i} < z_{n,i+1} < y_{n,i+1}$ *if* $\beta < 0$.

*Proof.* Let $0 < \alpha q < 1$ and $\beta q < 1$. We note that, since $0 < q < 1$, $q^n - 1 < 0$, $\alpha q^n - 1 < 0$, $\beta q^n - 1 < 0$, $\alpha\beta q^n - 1 < 0$, for all positive integers $n$.

The polynomials on the right-hand side of each of the equations (9a) and (9b) belong to the same orthogonal sequence, therefore their zeros interlace and satisfy the property (4). Each of these equations are thus are in the form of (5) with $a(x) = 1$. Furthermore,

(a) $b(x)$ in (9a) is a positive constant and the result follows from Corollary 2 (a);
(b) taking into consideration the restrictions on the parameters, $b(x)$ in (9b) is a positive constant if $\beta < 0$ and $b(x)$ is negative when $\beta > 0$. The result follows from applying Corollary 2 to the different situations.
(c) The polynomials $\tilde{p}_n(x; \alpha q^2, \beta|q)$ and $\tilde{p}_{n-1}(x; \alpha q^2, \beta|q)$ belong to the same orthogonal sequence and their zeros satisfy (4). (9c) is in the form of (5) and taking into consideration the restrictions on the parameters, $a(x)$ is a positive constant.

$$\begin{aligned} b(x) &= \frac{\alpha\, q\, (q^n - 1)\, (\beta q^n - 1)}{(\alpha\, q^{n+1} - 1)\, (\alpha\, \beta q^{2\,n+2} - 1)\, (\alpha\, \beta\, q^{n+1} - 1)} \left(\left(\alpha\, \beta q^{2\,n+2} - 1\right) x + q^n\, (\alpha\, q - 1)\right) \\ &= -k^2 \left(\left(\alpha\, \beta q^{2\,n+2} - 1\right) x + q^n\, (\alpha\, q - 1)\right),\ k \in \mathbb{R}, \end{aligned}$$

represents a linear function with gradient $-k^2(\alpha\, \beta q^{2\,n+2} - 1) > 0$, intersecting the $x$-axis at $x = \frac{-q^n(\alpha\, q - 1)}{(\alpha q)\,(\beta q) q^{2\,n} - 1} < 0$ for $\beta q < 1$. $b(x)$ is thus positive on $(0, 1)$ and from Corollary 2 (a) we deduce that $x_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < Y_{n,i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$. Furthermore, by replacing $\alpha$ with $\alpha q$ in (9a), we obtain $y_{n,i} < Y_{n,i} < Y_{n-1,i} < y_{n,i+1} < Y_{n,i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$ and by combining these two interlacing results with the fact that $x_{n,i} < y_{n,i}$ for each $i \in \{1, 2, \ldots, n\}$, the required interlacing follows.
(d) Let $\beta < 0$. By replacing $\beta$ with $\beta q$ in (9b), we obtain $z_{n,i} < Z_{n,i} < Z_{n-1,i} < z_{n,i+1} < Z_{n,i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$ and equation (9d) is in the form of (5). Under the condition that $\beta < 0$, the coefficient of $\tilde{p}_n(x; \alpha, \beta q|q)$ is a positive constant. The coefficient of $\tilde{p}_{n-1}(x; \alpha, \beta q^2|q)$ is

$$b(x) = \frac{\alpha\, \beta\, q^{2\,n}\, (q^n - 1)\, (\alpha\, q^n - 1)}{(\beta\, q^{n+1} - 1)\, (\alpha\, \beta\, q^{n+1} - 1)} \left(\beta\, q^2 x - 1\right) = -k^2 \left(\beta q^2 x - 1\right),\ k \in \mathbb{R},$$

that represents a linear function with positive gradient, intersecting the negative $x$-axis and $b(x)$ is thus positive on $(0, 1)$. The result follows from Corollary 2 (a).

(e) Assume $\beta < 0$. From (b) we know that the zeros of $\tilde{p}_n(x;\alpha,\beta|q)$ and $\tilde{p}_{n-1}(x;\alpha,\beta q|q)$ interlace. By replacing $\alpha$ by $\alpha q$, it follows that $y_{n,i} < t_{n-1,i} < y_{n,i+1}$ for each $i \in \{1,2,\cdots,n-1\}$. Equation (9e) is in the form of (5) with $a(x) = 1$ and, taking into consideration the restrictions on the parameters, $b(x)$ is a positive constant. The result follows from Corollary 2 (a).

□

**Corollary 13.** *For each* $i \in \{1,2,\ldots,n-1\}$*,*

(a) $x_{n,i} < x_{n-1,i} < y_{n-1,i} < x_{n,i+1}$*,*

(b) $x_{n,i} < z_{n-1,i} < x_{n-1,i} < x_{n,i+1}$ *if* $\beta > 0$ *and* $x_{n,i} < x_{n-1,i} < z_{n-1,i} < x_{n,i+1}$ *if* $\beta < 0$*,*

(c) $x_{n,i} < z_{n,i} < y_{n,i} < x_{n,i+1} < z_{n,i+1} < y_{n,i+1}$ *if* $\beta < 0$*.*

*Proof.* (a) We combine the interlacing of the zeros of $\tilde{p}_n(x;\alpha,\beta|q)$ and $\tilde{p}_{n-1}(x;\alpha,\beta|q)$ with the results proved in Theorem 12 (a) to obtain the required interlacing.

(b) We combine the interlacing of the zeros of $\tilde{p}_n(x;\alpha,\beta|q)$ and $\tilde{p}_{n-1}(x;\alpha,\beta|q)$ with the result of Theorem 12 (b).

(c) Let $\beta < 0$. This result follows from the interlacing proved in Theorem 12 (a), (b) and (e).

□

**Remark 14.** *(i) We note that our results differ from the interlacing results for the little q-Jacobi polynomials, given in [7, Section 3]. In [7, Theorem 2], the values of x, given as the zeros of the polynomial* $p_n(x;\alpha,\beta|q)$*, are in actual fact the zeros y of the polynomial* $p_n(q^y;\alpha,\beta|q)$*. The same is true for the interlacing results in [7, Theorems 4,5,6 and 7];*

*(ii) When* $\beta = 0$ *in the definition of the little q-Jacobi polynomials, we obtain the little q-Laguerre (or Wall) polynomials* $\tilde{p}_n(x;\alpha|q)$*, that are orthogonal on* $(0,1)$ *when* $0 < \alpha q < 1$*. The interlacing results in Theorem 12 (a) and (c) and Corollary 12 (a) follow from* (9a) *and* (9c) *(with* $\beta = 0$*), where* $x_{n,i}, y_{n,i}$ *and* $Y_{n,i}, i \in \{1,2,\ldots,n\}$ *are the zeros of* $\tilde{p}_n(x;\alpha|q), \tilde{p}_n(x;\alpha q|q)$ *and* $\tilde{p}_n(x;\alpha q^2|q)$*, respectively.*

### *3.4. The q-Meixner polynomials*

We note that in the definition of the $q$-Meixner polynomials, we let $\bar{x} = q^{-x}$, i.e., $x = \frac{ln\bar{x}}{-lnq}$ and as $x$ increases on $(0,\infty)$, $\bar{x}$ will increase on $(1,\infty)$. The variable $x$ in our equations thus represents $\bar{x}$ in the definition of the polynomials and for $0 < \beta q < 1$ and $\gamma > 0$, the polynomial $\tilde{M}_n(x;\beta,\gamma;q)$ is orthogonal on $(1,\infty)$.

**Proposition 15.**

$$\tilde{M}_n(x;\beta,\gamma q;q) = \tilde{M}_n(x;\beta,\gamma;q) + \gamma\, q^{-2n+1}\,(q^n-1)\,(\beta q^n-1)\,\tilde{M}_{n-1}(x;\beta,\gamma;q); \tag{10a}$$

$$\tilde{M}_n(x;\beta,\gamma q^2;q) = -\frac{(\beta\gamma q\,(q^n-1) - q\gamma - 1)\,q^n\tilde{M}_n(x;\beta,\gamma;q)}{\gamma q + q^n} + \frac{\gamma\, q^{-n+1}\,(\beta q^n-1)\,(q^n-1)\,(\gamma\beta q + q^n x + \gamma q + 1)\,\tilde{M}_{n-1}(x;\beta,\gamma;q)}{\gamma q + q^n}; \tag{10b}$$

$$\tilde{M}_n(x;\beta,\gamma;q) = \frac{(\gamma\beta q + q^n x)\,\tilde{M}_n(x;\beta q,\gamma;q)}{q^n(\gamma\beta q + x)} - \frac{\gamma\beta\,(q^n+\gamma)\,(q^n-1)}{q^{3n-2}(\gamma\beta q + x)}\tilde{M}_{n-1}(x;\beta q,\gamma;q); \tag{10c}$$

$$\tilde{M}_n(x;\beta,\gamma q;q) = \tilde{M}_n(x;\beta q,\gamma;q) - \gamma q^{-2n+1}\,(q^n-1)\,\tilde{M}_{n-1}(x;\beta q,\gamma;q). \tag{10d}$$

**Theorem 16.** *Let $0 < \beta q < 1$ and $\gamma > 0$ and denote the zeros of $\tilde{M}_n(x;\beta,\gamma;q)$ by $1 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < \infty$, the zeros of $\tilde{M}_n(x;\beta q,\gamma;q)$ by $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$, the zeros of $\tilde{M}_n(x;\beta,\gamma q;q)$ by $z_{n,1} < z_{n,2} < \cdots < z_{n,n}$ and the zeros of $\tilde{M}_n(x;\beta,\gamma q^2;q)$ by $Z_{n,1} < Z_{n,2} < \cdots < Z_{n,n}$. Then, for $i \in \{1, 2, \ldots, n-1\}$,*

(a) $z_{n,i} < x_{n,i} < x_{n-1,i} < z_{n,i+1} < x_{n,i+1}$,
(b) $Z_{n,i} < x_{n,i} < x_{n-1,i} < Z_{n,i+1} < x_{n,i+1}$,
(c) $z_{n,i} < x_{n,i} < y_{n,i} < y_{n-1,i} < z_{n,i+1} < x_{n,i+1} < y_{n,i+1}$.

*Proof.* Let $0 < \beta q < 1$ and $\gamma > 0$. Since $0 < q < 1$, it follows that $q^n - 1 < 0$ and $\beta q^n - 1 < 0$.

The polynomials on the right-hand side of each of the equations (10a) - (10d) belong to the same orthogonal sequence, therefore their zeros interlace and satisfy the property (4). Each of these equations thus is in the form of (5) with

(a) $a(x) = 1$ and $b(x) > 0$ in (10a) and the required interlacing follows from Corollary 2 (a);
(b) $a(x) > 0$ in (10b) and, taking in consideration the restrictions on the parameters,

$$b(x) = \frac{\gamma\,(\beta q^n - 1)\,(q^n - 1)}{q^{n-1}(\gamma q + q^n)}\,(q^n x + \gamma\beta\, q + \gamma q + 1)$$

is a linear function with positive gradient and is positive on $(1, \infty)$. The interlacing follows from Corollary 2 (a);
(c) Taking into consideration the restrictions on the parameters, the coefficients of both polynomials on the righthand side of (10c) are positive on $(1, \infty)$, and following Corollary 2 (a), $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$. Furthermore, the coefficients of both polynomials on the righthand side of (10d) are positive constants and applying Corollary 2 (a) for a second time, we obtain $z_{n,i} < y_{n,i} < y_{n-1,i} < z_{n,i+1} < y_{n,i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$. It is known, from (a), that $z_{n,i} < x_{n,i}$ for each $i \in \{1, 2, \ldots, n\}$, and the required combined interlacing follows.

□

**Corollary 17.** *For $i \in \{1, 2, \ldots, n-1\}$,*

(a) $z_{n,i} < z_{n-1,i} < x_{n-1,i} < y_{n-1,i} < z_{n,i+1}$,
(b) $Z_{n,i} < Z_{n-1,i} < x_{n-1,i} < Z_{n,i+1}$,
(c) $z_{n,i} < x_{n,i} < x_{n-1,i} < y_{n-1,i} < z_{n,i+1} < x_{n,i+1}$.

*Proof.* (a) The result follows from Theorem 16 (c) and the interlacing of the zeros of $\tilde{M}_n(x;\beta,\gamma q;q)$ and $\tilde{M}_{n-1}(x;\beta,\gamma q;q)$.
(b) The result follows from Theorem 16 (b) and the interlacing of the zeros of $\tilde{M}_n(x;\beta,\gamma q^2;q)$ and $\tilde{M}_{n-1}(x;\beta,\gamma q^2;q)$.
(c) We combine the interlacing of the zeros of $\tilde{M}_n(x;\beta,\gamma;q)$ and $\tilde{M}_{n-1}(x;\beta,\gamma;q)$ with the result of Theorem 16 (c) to obtain the required interlacing.

□

**Remark 18.** *(i) In general, the zeros of* $\tilde{M}_n(x;\beta,\gamma;q)$ *and* $\tilde{M}_{n-1}(x;\beta,\gamma q;q)$ *do not interlace. These polynomials satisfy*

$$\tilde{M}_{n-1}(x;\beta,\gamma q;q) = -\frac{q^{2n-1}\tilde{M}_n(x;\beta,\gamma;q)}{\gamma q + q^n} + \frac{b(x)\tilde{M}_{n-1}(x;\beta,\gamma;q)}{q(\gamma q + q^n)}$$

*with* $b(x) = q^{2n}x + \gamma q(\beta q^n + q^n - 1)$, *which represents a linear function that changes sign on* $(1,\infty)$ *for* $0 < \beta q < 1$ *and* $\gamma > 0$*. For example, when* $n = 2, \beta = 1, \gamma = 5, q = 0.1$*, the zeros of* $\tilde{M}_n(x;\beta,\gamma;q)$ *are* $\{42.15, 5413.85\}$ *and the zero of* $\tilde{M}_{n-1}(x;\beta,\gamma q;q)$ *is* $\{5.50\}$*;*

*(ii) When* $\beta = 0$ *in the definition of the q-Meixner polynomials, we obtain the q-Charlier polynomials* $\tilde{C}_n(x;\gamma;q)$*. The interlacing results in Theorem 16 (a) and (b) and Corollary 17 (b) follow from (10a) and (10b) (with* $\beta = 0$*), where* $x_{n,i}, y_{n,i}$ *and* $Z_{n,i}, i \in \{1,2,\dots,n\}$*, are the zeros of* $\tilde{C}_n(x;\gamma;q), \tilde{C}_n(x;\gamma q;q)$ *and* $\tilde{C}_n(x;\gamma q^2;q)$*, respectively.*

*3.5. The q-Krawtchouk polynomials*

**Proposition 19.**

$$\tilde{K}_n(x;p,N;q) = \tilde{K}_n(x;pq,N;q) + \frac{pq^n\left(q^{N+1}-q^n\right)(q^n-1)}{q^N(1+pq^{2n})(q+pq^{2n})}\tilde{K}_{n-1}(x;pq,N;q); \tag{11a}$$

$$\begin{aligned}\tilde{K}_n(x;p,N;q) &= \frac{\left(pq^{2n}+1\right)\left(pq^{N+1}+1\right)\tilde{K}_n(x;pq^2,N;q)}{(pq^n+1)(pq^{n+N+1}+1)} \\ &+ \frac{p(q^n-1)\left(q^{N+1}-q^n\right)\left(q^N\left(pq^{2n+1}+1\right)x + q^n\left(pq^{1+N}+1\right)\right)\tilde{K}_{n-1}(x;pq^2,N;q)}{q^N(pq^n+1)(pq^{n+N+1}+1)(pq^{2n+1}+1)}.\end{aligned} \tag{11b}$$

**Theorem 20.** *Let* $p > 0$*,* $n \in \{0,1,\dots,N\}$ *and denote the zeros of* $\tilde{K}_n(x;p,N;q)$ *by* $1 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < q^{-N}$*, the zeros of* $\tilde{K}_n(x;pq,N;q)$ *by* $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ *and the zeros of* $\tilde{K}_n(x;pq^2,N;q)$ *by* $Y_{n,1} < Y_{n,2} < \cdots < Y_{n,n}$*. Then, for each* $i \in \{1,2,\dots,n-1\}$*,*

(a) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$,
(b) $x_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < Y_{n,i+1}$.

*Proof.* Let $p > 0$, $n \in \{0,1,\dots,N\}$. We note that $q^n - 1 < 0$ and since $q^m < q^n$ for $m > n$, $q^{N+1} - q^n < 0$. Since the polynomials on the righthand-side of both equations (11a) and (11b) belong to the same orthogonal sequences, their zeros interlace and both these equations are in the form of (5). The required interlacing follows from Corollary 2 (a), since

(a) both $a(x)$ and $b(x)$ in (11a) are positive constants;
(b) taking into account the restrictions on the parameters, it is clear that $a(x)$ is a positive constant and $b(x) > 0$ represents a linear function that does not change sign on $(1, q^{-N})$.

□

**Corollary 21.** *For* $i \in \{1,2,\dots,n-1\}$*,*

(a) $x_{n,i} < x_{n-1,i} < y_{n-1,i} < x_{n,i+1}$,

(b) $x_{n,i} < y_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < y_{n,i+1} < Y_{n,i+1}$.

*Proof.* (a) The result follows directly from Theorem 20 (a) and the interlacing of the zeros of $\tilde{K}_n(x; p, N; q)$ and $\tilde{K}_{n-1}(x; p, N; q)$.

(b) When we replace $p$ by $pq$ in (11a), we obtain, using the same argument as in the proof of Theorem 20 (a), that $y_{n,i} < Y_{n,i} < Y_{n-1,i} < y_{n,i+1} < Y_{n,i+1}$, for each $i \in \{1, 2, \ldots, n-1\}$. We combine this with the interlacing results in Theorem 20 (a) and (b), which leads to the required result.

□

*3.6. The q-Laguerre polynomials*

In [21], relations between different sequences of $q$-Laguerre polynomials are provided and interlacing results between the zeros of different sequences of these polynomials are given in [12, 21].

**Proposition 22.**

$$\tilde{L}_n^{(\alpha)}(x; q) = \tilde{L}_n^{(\alpha+1)}(x; q) - q^{-2n-\alpha}(q^n - 1)\,\tilde{L}_{n-1}^{(\alpha+1)}(x; q) \quad \textit{cf. [21, Eq (4.12)]}; \tag{12a}$$

$$\tilde{L}_n^{(\alpha)}(x; q) = \frac{\left(q^{\alpha+1} - 1\right) q^n \tilde{L}_n^{(\alpha+2)}(x; q)}{q^{n+\alpha+1} - 1} + \frac{\left(q^{n+\alpha+1}x - q^{\alpha+1} + 1\right)(q^n - 1)\,\tilde{L}_{n-1}^{(\alpha+2)}(x; q)}{q^{n+\alpha+1}(q^{n+\alpha+1} - 1)}. \tag{12b}$$

**Theorem 23.** *Let $\alpha > -1$. We denote the zeros of $\tilde{L}_n^{(\alpha)}(x; q)$ by $0 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < \infty$, the zeros of $\tilde{L}_n^{(\alpha+1)}(x; q)$ by $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ and the zeros of $\tilde{L}_n^{(\alpha+2)}(x; q)$ by $Y_{n,1} < Y_{n,2} < \cdots < Y_{n,n}$. Then, for $i \in \{1, 2, \ldots, n-1\}$,*

(a) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$ *(cf. [21, Theorem 3]),*

(b) $x_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < Y_{n,i+1}$.

*Proof.* Let $\alpha > -1$. We note that $q^n - 1 < 0$ and $q^{n+\alpha} - 1 < 0$.

(a) Since $\tilde{L}_n^{(\alpha+1)}(x; q)$ and $\tilde{L}_{n-1}^{(\alpha+1)}(x; q)$ belong to the same orthogonal sequence, the interlacing property (4) is satisfied and (12a) is in the form of (5). Both $a(x)$ and $b(x)$ are positive constants and the result follows from Corollary 2 (a).

(b) The polynomials $\tilde{L}_n^{(\alpha+2)}(x; q)$ and $\tilde{L}_{n-1}^{(\alpha+2)}(x; q)$ belong to the same orthogonal sequence, which implies (4) is satisfied and equation (12b) is in the form of (5). For the given values of the parameters, $a(x)$ is a positive constant and

$$b(x) = \frac{q^n - 1}{q^{n+\alpha+1}(q^{n+\alpha+1} - 1)}\left(q^{n+\alpha+1}x - q^{\alpha+1} + 1\right) > 0$$

on $(0, \infty)$ and the interlacing follows from Corollary 2 (a).

□

**Corollary 24.** *For $i \in \{1, 2, \ldots, n-1\}$,*

(a) $x_{n,i} < x_{n-1,i} < y_{n-1,i} < Y_{n-1,i} < x_{n,i+1}$,

(b) $x_{n,i} < y_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < y_{n,i+1} < Y_{n,i+1}$.

*Proof.* (a) See [12, Theorem 5.1].

(b) When we replace $\alpha$ by $\alpha + 1$ in (12a), we obtain, using the same argument as in the proof of Theorem 23 (a), that $y_{n,i} < Y_{n,i} < Y_{n-1,i} < y_{n,i+1} < Y_{n,i+1}$, for each $i \in \{1, 2, \dots, n-1\}$. We combine this with the results in Theorem 23 (a) and (b) to obtain the required result.

□

**Remark 25.** *In [12], the result in Corollary (24) (a) is extended to also include a continuous shift of the parameter $\alpha$. Furthermore, examples are provided to show that, in general, interlacing breaks down between the zeros of: $\tilde{L}_n^{(\alpha)}(x; q)$ and $\tilde{L}_n^{(\alpha+3)}(x; q)$, $\tilde{L}_n^{(\alpha)}(x; q)$ and $\tilde{L}_{n-1}^{(\alpha+3)}(x; q)$ and $\tilde{L}_n^{(\alpha+1)}(x; q)$ and $\tilde{L}_{n-1}^{(\alpha)}(x; q)$.*

### 3.7. *The alternative q-Charlier or q-Bessel polynomials*

**Proposition 26.**

$$\tilde{y}_n(x; \alpha; q) = \tilde{y}_n(x; \alpha q; q) - \frac{\alpha\, q^{2n} (q^n - 1)}{(q + \alpha\, q^{2n})(1 + \alpha\, q^{2n})} \tilde{y}_{n-1}(x; \alpha q; q); \tag{13a}$$

$$\tilde{y}_n(x; \alpha; q) = \frac{\left(\alpha q^{2n} + 1\right) \tilde{y}_n(x; \alpha q^2; q)}{\alpha\, q^n + 1} - \frac{\alpha\, q^n (q^n - 1) \left((\alpha\, q^{2n+1} + 1)x + q^n\right) \tilde{y}_{n-1}(x; \alpha q^2; q)}{(\alpha\, q^{2n+1} + 1)(\alpha\, q^n + 1)}. \tag{13b}$$

**Theorem 27.** *Let $\alpha > 0$. We denote the zeros of $\tilde{y}_n(x; \alpha; q)$ by $0 < x_{n,1} < x_{n,2} < \dots < x_{n,n} < 1$, the zeros of $\tilde{y}_n(x; \alpha q; q)$ by $z_{n,1} < z_{n,2} < \dots < z_{n,n}$ and the zeros of $\tilde{y}_n(x; \alpha q^2; q)$ by $Z_{n,1} < Z_{n,2} < \dots < Z_{n,n}$. Then, for $i \in \{1, 2, \dots, n-1\}$,*

(a) $x_{n,i} < z_{n,i} < z_{n-1,i} < x_{n,i+1} < z_{n,i+1}$,

(b) $x_{n,i} < Z_{n,i} < Z_{n-1,i} < x_{n,i+1} < Z_{n,i+1}$.

*Proof.* Let $\alpha > 0$. The polynomials on the right-hand side of each of the equations (13a) and (13b) belong to the same orthogonal sequence and their zeros satisfy (4), therefore these equations are in the form of (5). Taking into consideration the values of the parameters,

(a) both $a(x)$ and $b(x)$ in (13a) are positive constants and the result follows from Corollary 2 (a).

(b) $a(x)$ in (13b) is a positive constant and $b(x)$ represents a linear function that does not change sign on $(0, 1)$ and the interlacing follows from Corollary 2 (a).

□

**Corollary 28.** *For $i \in \{1, 2, \dots, n-1\}$,*

(a) $x_{n,i} < x_{n-1,i} < z_{n-1,i} < Z_{n-1,i} < x_{n,i+1}$,

(b) $x_{n,i} < z_{n,i} < Z_{n,i} < Z_{n-1,i} < x_{n,i+1} < z_{n,i+1} < Z_{n,i+1}$.

*Proof.* (a) The result follows from Theorem 27 (a) and (b) and the interlacing of the zeros of $\tilde{y}_n(x; \alpha; q)$ and $\tilde{y}_{n-1}(x; \alpha; q)$.

(b) When we replace $\alpha$ by $\alpha q$ in (13a), we deduce that $z_{n,i} < Z_{n,i} < Z_{n-1,i} < z_{n,i+1} < Z_{n,i+1}$ and we combine this with the interlacing results in Theorem 27 (a) and (b) to obtain the required result.

□

### 3.8. *The Al-Salam-Carlitz I polynomials*

**Proposition 29.**

$$\tilde{U}_n^{(\alpha)}(x;q) = \tilde{U}_n^{(\alpha q)}(x;q) + \alpha(q^n - 1)\tilde{U}_{n-1}^{(\alpha q)}(x;q). \tag{14}$$

**Theorem 30.** *Let $\alpha < 0$ and denote the zeros of $\tilde{U}_n^{(\alpha)}(x;q)$ by $\alpha < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < 1$ and the zeros of $\tilde{U}_n^{(\alpha q)}(x;q)$ by $\alpha q < y_{n,1} < y_{n,2} < \cdots < y_{n,n} < 1$. Then, for $i \in \{1, 2, \ldots, n-1\}$, $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$.*

*Proof.* Let $\alpha < 0$. Since $\tilde{U}_n^{(\alpha q)}(x;q)$ and $\tilde{U}_{n-1}^{(\alpha q)}(x;q)$ belong to the same orthogonal sequence, the interlacing property (4) is satisfied and (14) is in the form of (5). Taking into consideration the values of the parameters, $a(x) > 0$ and $b(x) > 0$ are constants on $(\alpha, 1)$ and the result follows from Corollary 2 (a). □

**Corollary 31.** *For $i \in \{1, 2, \ldots, n-1\}$, $x_{n,i} < x_{n-1,i} < y_{n-1,i} < x_{n,i+1}$.*

*Proof.* The result follows from Theorem 30 and the interlacing of the zeros of $\tilde{U}_n^{(\alpha)}(x;q)$ and $\tilde{U}_{n-1}^{(\alpha)}(x;q)$. □

In general, the zeros of $\tilde{U}_n^{(\alpha)}(x;q)$ do not interlace with the zeros of $\tilde{U}_n^{(\alpha q^2)}(x;q)$ or with the zeros of $\tilde{U}_{n-1}^{(\alpha q^2)}(x;q)$. For example, when $n = 2, \alpha = -16$ and $q = 0.9$, the zeros of $\tilde{U}_n^{(\alpha)}(x;q)$ are $\{-15.77, -12.78\}$, the zeros of $\tilde{U}_n^{(\alpha q^2)}(x;q)$ are $\{-12.64, -10.08\}$ and the zero of $\tilde{U}_{n-1}^{(\alpha q^2)}(x;q)$ is $\{-11.96\}$.

### 3.9. *The Al-Salam-Carlitz II polynomials*

**Proposition 32.**

$$\tilde{V}_n^{(\alpha q)}(x;q) = \tilde{V}_n^{(\alpha)}(x;q) - \alpha\, q(q^n - 1)q^{-n}\tilde{V}_{n-1}^{(\alpha)}(x;q); \tag{15a}$$

$$\tilde{V}_n^{(\alpha q^2)}(x;q) = \left(\alpha q^{n+1} + 1 - \alpha\, q\right)\tilde{V}_n^{(\alpha)}(x;q) - \alpha q^{-n+1}\,(q^n - 1)\,(q^n x + 1 - \alpha\, q)\,\tilde{V}_{n-1}^{(\alpha)}(x;q). \tag{15b}$$

**Theorem 33.** *Let $0 < \alpha q < 1$. Denote the zeros of $\tilde{V}_n^{(\alpha)}(x;q)$ by $1 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < \infty$, the zeros of $\tilde{V}_n^{(\alpha q)}(x;q)$ by $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ and the zeros of $\tilde{V}_n^{(\alpha q^2)}(x;q)$ by $Y_{n,1} < Y_{n,2} < \cdots < Y_{n,n}$. Then, for $i \in \{1, 2, \ldots, n-1\}$,*

(a) $y_{n,i} < x_{n,i} < x_{n-1,i} < y_{n,i+1} < x_{n,i+1}$,
(b) $Y_{n,i} < x_{n,i} < x_{n-1,i} < Y_{n,i+1} < x_{n,i+1}$.

*Proof.* Let $0 < \alpha q < 1$. Since $\tilde{V}_n^{(\alpha)}(x;q)$ and $\tilde{V}_{n-1}^{(\alpha)}(x;q)$ belong to the same orthogonal sequence, the interlacing property (4) is satisfied and both (15a) and (15b) are in the form of (5).

(a) Taking into consideration the values of the parameters, both the coefficients of $\tilde{V}_n^{(\alpha)}(x;q)$ and $\tilde{V}_{n-1}^{(\alpha)}(x;q)$ in (15a) are positive constants and the result follows from Corollary 2 (a).
(b) Taking into consideration the restrictions on the parameters, $a(x)$ in (15b) is a positive constant and $b(x) = \frac{\alpha(1-q^n)}{q^{n-1}}\,(q^n x - \alpha\, q + 1)$ represents a linear function with positive values on $(1, \infty)$. The result follows from Corollary 2 (a).

□

**Corollary 34.** *For* $i \in \{1, 2, \ldots, n-1\}$,

(a) $y_{n,i} < y_{n-1,i} < x_{n-1,i} < y_{n,i+1}$,
(b) $Y_{n,i} < y_{n,i} < x_{n,i} < x_{n-1,i} < Y_{n,i+1} < y_{n,i+1} < x_{n,i+1}$,
(c) $Y_{n,i} < Y_{n-1,i} < x_{n-1,i} < Y_{n,i+1}$.

*Proof.* (a) The result follows from Theorem 33 (a) and the interlacing of the zeros of $\tilde{V}_n^{(\alpha q)}(x;q)$ and $\tilde{V}_{n-1}^{(\alpha q)}(x;q)$.

(b) By replacing $\alpha$ with $\alpha q$ in (15a), we obtain $Y_{n,i} < y_{n,i} < y_{n-1,i} < Y_{n,i+1} < y_{n,i+1}$. We combine this with the interlacing results in Theorem 33 (a) and (b) to obtain the required result.

(c) The result follows directly from Theorem 33 (b) and the interlacing of the zeros of $\tilde{V}_n^{(\alpha q^2)}(x;q)$ and $\tilde{V}_{n-1}^{(\alpha q^2)}(x;q)$.

□

**Remark 35.** *In general, the zeros of* $\tilde{V}_n^{(\alpha)}(x;q)$ *and* $\tilde{V}_{n-1}^{(\alpha q)}(x;q)$ *do not interlace. These polynomials satisfy*

$$\tilde{V}_{n-1}^{(\alpha q)}(x;q) = -q^{n-1}\tilde{V}_n^{(\alpha)}(x;q) + b(x)\tilde{V}_{n-1}^{(\alpha)}(x;q)$$

*with* $b(x) = q^{-1}(q^n x - \alpha q)$, *a function that changes sign on* $(1,\infty)$ *for* $0 < \alpha q < 1$. *However, when we restrict* $\alpha$ *in such a way that* $0 < \alpha q < q^n < 1$, *the zeros interlace as follows:* $x_{n,i} < y_{n-1,i} < x_{n-1,i} < x_{n,i+1}$ *for each* $i \in \{1, 2, \ldots, n-1\}$.

## 4. Classical orthogonal polynomials on a *q*-quadratic lattice

This section is devoted to the study of interlacing properties of the zeros of different sequences of orthogonal polynomials on a $q$-quadratic lattice.

### *4.1. Askey-Wilson polynomials*

The weight function of the Askey-Wilson polynomials

$$w(x;a,b,c,d|q) = \frac{1}{\sqrt{1-x^2}}\left|\frac{(e^{2i\theta};q)_\infty}{(ae^{i\theta},be^{i\theta},ce^{i\theta},de^{i\theta};q)_\infty}\right|^2, \tag{16}$$

is clearly independent of the order of the parameters $a, b, c$ and $d$ and by shifting $b$ to $bq$, $c$ to $cq$ or $d$ to $dq$, we obtain the same interlacing results as by shifting $a$ to $aq$.

**Proposition 36.**

$$\begin{aligned}\tilde{p}_n(x;a,b,c,d|q) &= \tilde{p}_n(x;aq,b,c,d|q) \\ &\quad - \frac{a(1-q^n)(1-cdq^{n-1})(1-bdq^{n-1})(1-bcq^{n-1})}{2(1-abcdq^{2n-1})(1-abcdq^{2n-2})}\tilde{p}_{n-1}(x;aq,b,c,d|q),\end{aligned} \tag{17a}$$

$$\tilde{p}_n(x;a,bq,c,d|q) = \tilde{p}_n(x;aq,b,c,d|q) + \frac{(b-a)(1-q^n)(1-cdq^{n-1})}{2(1-abcdq^{2n-1})}\tilde{p}_{n-1}(x;aq,bq,c,d|q). \tag{17b}$$

**Theorem 37.** *Suppose $a, b, c, d$ are real and $\max(|a|, |b|, |c|, |d|) < 1$. Denote the zeros of $\tilde{p}_n(x; a, b, c, d|q)$ by $-1 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < 1$, the zeros of $\tilde{p}_n(x; aq, b, c, d|q)$ by $-1 < x^{(a)}_{n,1} < x^{(a)}_{n,2} < \cdots < x^{(a)}_{n,n} < 1$, the zeros of $\tilde{p}_n(x; a, bq, c, d|q)$ by $-1 < x^{(b)}_{n,1} < x^{(b)}_{n,2} < \cdots < x^{(b)}_{n,n} < 1$, the zeros of $\tilde{p}_n(x; aq, bq, c, d|q)$ by $-1 < x^{(a,b)}_{n,1} < x^{(a,b)}_{n,2} < \cdots < x^{(a,b)}_{n,n} < 1$. Then,*

(a) *if $-1 < a < 0$, $x_{n,i} < x^{(a)}_{n,i} < x^{(a)}_{n-1,i} < x_{n,i+1} < x^{(a)}_{n,i+1}$, and*
*if $0 < a < 1$, $x^{(a)}_{n,i} < x_{n,i} < x^{(a)}_{n-1,i} < x^{(a)}_{n,i+1} < x_{n,i+1}$;*
(b) *if $b - a > 0$, $x^{(b)}_{n,i} < x^{(a)}_{n,i} < x^{(a,b)}_{n-1,i} < x^{(b)}_{n,i+1} < x^{(a)}_{n,i+1}$, and*
*if $b - a < 0$, $x^{(a)}_{n,i} < x^{(b)}_{n,i} < x^{(a,b)}_{n-1,i} < x^{(a)}_{n,i+1} < x^{(b)}_{n,i+1}$.*

*Proof.* Suppose $a, b, c, d$ are real and $\max(|a|, |b|, |c|, |d|) < 1$. Then $\max(|ac|, |ad|, |bc|, |bd|, |cd|, |abcd|) < 1$ and, for $n \in \mathbb{N}$, $1 - acq^n > 0$, $1 - bcq^n > 0$, $1 - bdq^n > 0$, $1 - cdq^n > 0$ and $1 - abcdq^n > 0$. Since $\tilde{p}_n(x; aq, b, c, d|q)$ and $\tilde{p}_{n-1}(x; aq, b, c, d|q)$ belong to the same orthogonal sequence, their zeros interlace and (17a) is in the form of (5), with $a(x) = 1$ and

(a) $b(x) > 0$ if $-1 < a < 0$ and the result follows from Corollary 2 (a) and $b(x) < 0$ if $0 < a < 1$ and the result follows from Corollary 2 (b).
(b) Since by shifting $b$ to $bq$, we obtain the same interlacing results as by shifting $a$ to $aq$ and we have $x_{n,i} < x^{(b)}_{n,i} < x^{(b)}_{n-1,i} < x_{n,i+1} < x^{(b)}_{n,i+1}$ if $-1 < b < 0$, and $x^{(b)}_{n,i} < x_{n,i} < x^{(b)}_{n-1,i} < x^{(b)}_{n,i+1} < x_{n,i+1}$ if $0 < b < 1$. By replacing $a$ by $aq$, it follows that $x^{(a)}_{n,i} < x^{(a,b)}_{n-1,i} < x^{(a)}_{n,i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$. Equation (17b) is in the form of (5), with $a(x) = 1$, $b(x) < 0$ if $b - a < 0$ and the result follows from Corollary 2 (b), $b(x) > 0$ if $b - a > 0$ and the result follows from Corollary 2 (a).

□

The following result follows directly:

**Corollary 38.** *For $i \in \{1, 2, \ldots, n-1\}$,*

(a) *if $-1 < a < 0$ and $0 < b < 1$, $x^{(b)}_{n,i} < x_{n,i} < x^{(a)}_{n,i} < x^{(a,b)}_{n-1,i} < x^{(b)}_{n,i+1} < x_{n,i+1} < x^{(a)}_{n,i+1}$;*
(b) *if $-1 < b < 0$ and $0 < a < 1$, $x^{(a)}_{n,i} < x_{n,i} < x^{(b)}_{n,i} < x^{(a,b)}_{n-1,i} < x^{(a)}_{n,i+1} < x_{n,i+1} < x^{(b)}_{n,i+1}$.*

**Remark 39.** *The following systems of polynomials follow from the Askey-Wilson polynomials:*

*(i) By setting $d = 0$, we obtain the monic continuous dual $q$-Hahn polynomials:*

$$\tilde{p}_n(x; a, b, c|q) = \frac{(ab, ac; q)_n}{(2a)^n} {}_3\phi_2\left( \begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac \end{matrix} \middle| q; q \right), \quad x = \cos\theta,$$

*orthogonal on $(-1, 1)$ with respect to $w(x; a, b, c, 0|q)$ in (16) where $a, b, c$ are real and $\max(|a|, |b|, |c|) < 1$;*

*(ii) By setting $c = d = 0$, we obtain the monic Al-Salam Chihara polynomials:*

$$\tilde{Q}_n(x; a, b|q) = \frac{(ab; q)_n}{(2a)^n} {}_3\phi_2\left( \begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix} \middle| q; q \right), \quad x = \cos\theta,$$

*are orthogonal on $(-1, 1)$ with respect to $w(x; a, b, 0, 0|q)$ in (16) where $a, b$ are real and $\max(|a|, |b|) < 1$;*

*(iii)* *By setting* $b = c = d = 0$, *we obtain the monic continuous big q-Hermite polynomials:*

$$\tilde{H}_n(x; a|q) = (2a)^{-n} {}_3\phi_2 \left( \begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ 0, 0 \end{matrix} \middle| q; q \right), \; x = \cos\theta,$$

*are orthogonal on* $(-1, 1)$ *with respect to* $w(x; a, 0, 0, 0|q)$ *in* (16) *where a is real and* $|a| < 1$.

*(iv)* *By the substitutions* $\theta \to \theta + \phi$, $a \to ae^{i\phi}$, $b \to be^{i\phi}$, $c \to ce^{-i\phi}$ *and* $d \to de^{-i\phi}$ *we obtain the monic continuous q-Hahn polynomials:*

$$\tilde{p}_n(x; a, b, c, d; q) = \frac{(abe^{2i\phi}, ac, ad; q)_n}{(2ae^{i\phi})^n (abcdq^{n-1}; q)_n} {}_4\phi_3 \left( \begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i(\theta+2\phi)}, ae^{-i\theta} \\ abe^{2i\phi}, ac, ad \end{matrix} \middle| q; q \right), \; x = \cos(\theta + \phi),$$

*orthogonal on* $(-\pi, \pi)$ *with respect to*

$$w(\cos(\theta + \phi); a, b, c, d|q) = \left| \frac{(e^{2i(\theta+\phi)}; q)_\infty}{(ae^{i(\theta+\phi)}, be^{i(\theta+\phi)}, ce^{i\theta}, de^{i\theta}; q)_\infty} \right|^2,$$

*if* $c = a$ *and* $d = b$ *and, if a and b are real and* $\max(|a|, |b|) < 1$, *or if* $b = \overline{a}$ *and* $|a| < 1$. *Using the above substitution in* (17a), *we obtain*

$$\begin{aligned} \tilde{p}_n(x; a, b, c, d; q) &= \tilde{p}_n(x; aq, b, c, d; q) \\ &\quad - \frac{a(1 - q^n)(e^{i\phi} - cdq^{n-1}e^{-i\phi})(1 - bdq^{n-1})(1 - bcq^{n-1})}{2(1 - abcdq^{2n-1})(1 - abcdq^{2n-2})} \tilde{p}_{n-1}(x; aq, b, c, d; q) \end{aligned}$$

*and it is clear that we can not apply our method to deduce the interlacing properties of the zeros of* $\tilde{p}_n(x; a, b, c, d; q)$, *since it is not possible to determine if* $e^{i\phi} - cdq^{n-1}e^{-i\phi}$ *is positive or negative.*

**Corollary 40.** *Suppose* $a, b, c, d$ *are real and* $\max(|a|, |b|, |c|, |d|) < 1$. *Then for each of the systems in (i) - (iii) above, we have, for* $i \in \{1, 2, \ldots, n-1\}$,

(a) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$ *if* $-1 < a < 0$*;*
(b) $y_{n,i} < x_{n,i} < y_{n-1,i} < y_{n,i+1} < x_{n,i+1}$ *if* $0 < a < 1$,

*where* $-1 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < 1$ *are the zeros of the polynomial* $\tilde{p}_n(x; a, b, c|q)$ *in (i)* ($\tilde{Q}_n(x; a, b|q)$, $\tilde{H}_n(x; a|q)$), *and* $-1 < y_{n,1} < y_{n,2} < \cdots < y_{n,n} < 1$ *are the zeros of the polynomial with a shifted to aq, i.e.,* $\tilde{p}_n(x; aq, b, c|q)$ ($\tilde{Q}_n(x; aq, b|q)$, $\tilde{H}_n(x; aq|q)$).

*4.2. The q-Racah polynomials*

The $q$-Racah polynomials are orthogonal on $(0, N)$ if $\alpha q = q^{-N}$ or $\beta\delta q = q^{-N}$ or $\gamma q = q^{-N}$, and $N$ a nonnegative integer. In order to prove interlacing results, we make some assumptions on the parameters.

**Proposition 41.**

$$\tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = \tilde{R}_n(\mu(x);\alpha q,\beta,\gamma,\delta|q) - \frac{\alpha q(1-q^n)(1-\beta q^n)(1-\gamma q^n)(1-\beta\delta q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}\tilde{R}_{n-1}(\mu(x);\alpha q,\beta,\gamma,\delta|q); \tag{18a}$$

$$\tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = \tilde{R}_n(\mu(x);\alpha,\beta q,\gamma,\delta|q) + \frac{\beta q(1-q^n)(1-\alpha q^n)(1-\gamma q^n)(\alpha q^n-\delta)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}\tilde{R}_{n-1}(\mu(x);\alpha,\beta q,\gamma,\delta|q). \tag{18b}$$

**Theorem 42.** *We denote the zeros of $\tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$ by $\mu(0) < \mu_{n,1} < \mu_{n,2} < \cdots < \mu_{n,n} < \mu(N)$, the zeros of $\tilde{R}_n(\mu(x);\alpha q,\beta,\gamma,\delta|q)$ by $\mu(0) < \mu^{(\alpha)}_{n,1} < \mu^{(\alpha)}_{n,2} < \cdots < \mu^{(\alpha)}_{n,n} < \mu(N)$ and the zeros of $\tilde{R}_n(\mu(x);\alpha,\beta q,\gamma,\delta|q)$ by $\mu(0) < \mu^{(\beta)}_{n,1} < \mu^{(\beta)}_{n,2} < \cdots < \mu^{(\beta)}_{n,n} < \mu(N)$ and we assume that $(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1}) > 0$, $\gamma q < 1$ and $0 < \delta q < 1$.*

(a) *Let $\alpha q = q^{-N} > 1$. If $\beta q < 1$ and $\beta\delta q < 1$, then, for $i \in \{1,2,\dots,n-1\}$, $\mu^{(\alpha)}_{n,i} < \mu_{n,i} < \mu^{(\alpha)}_{n-1,i} < \mu^{(\alpha)}_{n,i+1} < \mu_{n,i+1}$;*

(b) *Let $\beta\delta q = q^{-N} > 1$. If $\alpha q < 1$ and $\alpha q^n < \delta$, then, for $i \in \{1,2,\dots,n-1\}$, $\mu^{(\beta)}_{n,i} < \mu_{n,i} < \mu^{(\beta)}_{n-1,i} < \mu^{(\beta)}_{n,i+1} < \mu_{n,i+1}$.*

*Proof.* The polynomials on the right hand side of both equations (18a) and (18b) belong to the same orthogonal sequences, their zeros interlace and these equations are both in the form of (5), with $a(x) = 1$.

(a) Let $\alpha q = q^{-N} > 1$ and assume that $\beta q < 1$ and $\beta\delta q < 1$. Then $1 - \beta\delta q^n > 0$, i.e., $b(x) < 0$ and the result follows from Corollary 2 (b);

(b) Let $\beta\delta q = q^{-N} > 1$ and assume that $\alpha q < 1$ and $\alpha q^n < \delta$. Then $b(x)$ is a negative constant and the result follows from Corollary 2 (b).

□

**Remark 43.** *When we take $\beta = 0, \gamma q = q^{-N}$ and $\delta \to \alpha\delta q^{N+1}$ in the definition of the $q$-Racah polynomials, we obtain the monic dual $q$-Hahn polynomials, i.e.,*

$$\tilde{R}_n(\mu(x);\alpha,0,q^{-N-1},\alpha\delta q^{N+1}|q) = \tilde{R}_n(\mu(x);\alpha,\delta,N|q), n \in \{0,1,\dots,N\},$$

*with $\mu(x) = q^{-x} + \alpha\delta q^{x+1}$, and (18a) becomes*

$$\tilde{R}_n(\mu(x);\alpha,\delta,N|q) = \tilde{R}_n(\mu(x);\alpha q,\delta,N|q) - \alpha q(1-q^n)(1-q^{n-N-1})\tilde{R}_{n-1}(\mu(x);\alpha q,\delta,N|q),$$

*with $0 < \alpha q < 1$ and $0 < \delta q < 1$, and, since $-\alpha q(1-q^n)(1-q^{n-N-1}) > 0$, the zeros interlace as follows:*

$$\mu_{n,i} < \mu^{(\alpha)}_{n,i} < \mu^{(\alpha)}_{n-1,i} < \mu_{n,i+1} < \mu^{(\alpha)}_{n,i+1}, i \in \{1,2,\dots,n-1\},$$

*where $\mu_{n,i}$ are the zeros of $\tilde{R}_n(\mu(x);\alpha,\delta,N|q)$ and $\mu^{(\alpha)}_{n,i}$, $i \in \{1,2,\dots,n\}$, the zeros of $\tilde{R}_n(\mu(x);\alpha q,\delta,N|q)$.*

For the monic dual $q$-Hahn polynomials we also get the following results.

**Proposition 44.**

$$\tilde{R}_n(\mu(x);\alpha,\delta q,N|q) = \tilde{R}_n(\mu(x);\alpha q,\delta,N|q) + \alpha q^{-N}(1-q^n)(q^n-q^{N+1})\tilde{R}_{n-1}(\mu(x);\alpha q,\delta,N|q); \quad (19a)$$

$$\tilde{R}_n(\mu(x);\alpha,\delta q,N|q) = \frac{1-\alpha\delta\, q^{x+N+3-n}}{1-\alpha\delta q^{x+N+3}}\tilde{R}_n(\mu(x);\alpha q,\delta q,N|q) \quad (19b)$$
$$+\frac{\alpha\,(1-q^n)\left(q^n-q^{N+1}\right)\left(1-\delta\, q^{N+2-n}\right)}{q^N(1-\alpha\,\delta q^{x+N+3})}\tilde{R}_{n-1}(\mu(x);\alpha q,\delta q,N|q).$$

**Theorem 45.** *Let $n \in \{0,1,\ldots,N\}$, $0 < \alpha q < 1$ and $0 < \delta q < 1$, and denote the zeros of $\tilde{R}_n(\mu(x);\alpha,\delta q,N|q)$ by $\mu(0) < \mu_{n,1}^{(\delta)} < \mu_{n,2}^{(\delta)} < \cdots < \mu_{n,n}^{(\delta)} < \mu(N)$, the zeros of $\tilde{R}_n(\mu(x);\alpha q,\delta,N|q)$ by $\mu(0) < \mu_{n,1}^{(\alpha)} < \mu_{n,2}^{(\alpha)} < \cdots < \mu_{n,n}^{(\alpha)} < \mu(N)$ and the zeros of $\tilde{R}_n(\mu(x);\alpha q,\delta q,N|q)$ by $\mu(0) < \mu_{n,1}^{(\alpha,\delta)} < \mu_{n,2}^{(\alpha,\delta)} < \cdots < \mu_{n,n}^{(\alpha,\delta)} < \mu(N)$. Then, for $i \in \{1,2,\cdots,n-1\}$,*

(a) $\mu_{n,i}^{(\delta)} < \mu_{n,i}^{(\alpha)} < \mu_{n-1,i}^{(\alpha)} < \mu_{n,i+1}^{(\delta)} < \mu_{n,i+1}^{(\alpha)}$,
(b) $\mu_{n,i}^{(\delta)} < \mu_{n,i}^{(\alpha,\delta)} < \mu_{n-1,i}^{(\alpha,\delta)} < \mu_{n,i+1}^{(\delta)} < \mu_{n,i+1}^{(\alpha,\delta)}$.

*Proof.* Let $0 < \alpha q < 1$ and $0 < \delta q < 1$. Then $1-\alpha\delta q^j > 0$, $1-\alpha q^j > 0$, $1-\delta q^j > 0$ if $j > 0$ and equations (19a)–(19b) are in the form of (5) and under the given assumptions, the results follows from Corollary 2 (a). □

## 5. Appendix

In this section, some comments on how to use our Maple codes that can be downloaded from `http://www.mathematik.uni-kassel.de/~koepf/Publikationen` are given. The first program called `qMixRec1(F,q,k,S(n),a,s)` finds a recurrence equation of the form

$$S(n,a) = \sum_{j=0}^{J}\sigma_j S(n-j,aq^s),\, J \in \{1,2,\ldots\},$$

where $S(n,a) = \sum_{k=-\infty}^{\infty} F$, $F$ is a function of $k$, $n$ and $a$, and $s$ is a positive integer. The second one denoted by `qMixRec2(F,q,k,S(n),a,`$s_0$`,b,`$s_1$`,`$s_2$`,r)` finds a recurrence equation of the form

$$S(n,a,bq^{s_1}) = \sum_{j=0}^{J}\sigma_j S(n-j,aq^{s_0},bq^{s_2+rj}),\ J \in \{1,2,\ldots\},\ r \in \{0,1\},$$

where $S(n,a,b) = \sum_{k=-\infty}^{\infty} F$, $F$ is a function of $k$, $n$, $a$ and $b$, and $s_0,\ s_1,\ s_2$ are positive integers.

For example, for the big $q$-Jacobi polynomials, $\tilde{P}_n(x;\alpha,\beta,\gamma;q) = \sum_{k=-\infty}^{\infty} F$, where, by (3),

$$F := \frac{(\alpha q,\gamma q;q)_n(q^{-n},\alpha\beta q^{n+1},x;q)_k q^k}{(\alpha\beta q^{n+1};q)_n(\alpha q,\gamma q,q;q)_k},$$

and equations (7a), (7b) and (7c) are obtained using, `qMixRec1(F,q,k,P(n),alpha,1)`, `qMixRec1(F,q,k,P(n),beta,1)` and `qMixRec2(F,q,k,P(n),alpha,1,beta,1,0,1)`, respectively.

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