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Solution properties of the de Branges differential recurrence equation. (English)

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Let S denote the class of analytic and univalent functions f of the form $f(z) = z + a_z^2 + \cdots + a_n z^n + \cdots$ in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ and let $k_n := \max|a_n(f)|$, $f \in \mathbb{S}$ ($\mathbb{S}$ is compact). In 1916 L. Bieberbach proved that $k_2 = 2$, with equality only for the Koebe function: $K(z) := z(1 - \varepsilon z)^{-2}$, $|\varepsilon| = 1$ and supposed that $k_n = n$ (the Bieberbach conjecture). Many years this conjecture attracted the attention of mathematicians. In particular, it was known that if the Milin conjecture is satisfied then the Bieberbach conjecture holds [*N. A. Lebedev* and *I. M. Milin*, Vestn. Leningr. Univ. 20, No. 19, (Ser. Math. Mekh. Astron. No. 4), 157–158 (1965; Zbl 0144.33303)].
In 1984 *L. de Branges* [Acta Math. 154, 137–152 (1985; Zbl 0573.30014)] verified the Milin conjecture, and in 1991 *L. Weinstein* [Int. Math. Res. Not. 1991, No. 5, 61–64 (1991; Zbl 0743.30021)] gave a different proof of it. Next it was proved [*P. G. Todorov*, Bull. Cl. Sci., VI. Sér., Acad. R. Belg. 3, No. 12, 335–346 (1992; Zbl 0806.30017); *H. S. Wilf*, Bull. Lond. Math. Soc. 26, No. 1, 61–63 (1994; Zbl 0796.30014)] that the de Branges functions $\tau_k^n(t)$ and Weinstein's functions $\lambda_k^n(t)$ essentially are the same ($\tau_k^n(t) = -k\lambda_k^n(t)$).
In this paper the authors study differential recurrence equations equivalent to de Branges' original ones and show that many solutions of these differential recurrence equations do not change sign so that the above inequality is not as surprising as expected. They also present a multiparameterized hypergeometric family of solutions of the de Branges differential recurrence equations showing that solutions are not rare at all.

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Keywords : Bieberbach conjecture; Milin conjecture; de Branges functions; Weinstein functions; hypergeometric functions

Classification :

$*$30C50 Coefficient problems for univalent and multivalent functions
33C20 Generalized hypergeometric series