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A high-school algebra, "Formal calculus", proof of the Bieberbach conjecture [after L. Weinstein]. (English)

Barcelo, Helene (ed.) et al., Jerusalem combinatorics '93: an international conference in combinatorics, May 9-17, 1993, Jerusalem, Israel. Providence, RI: American Mathematical Society. Contemp. Math. 178, 113-115 (1994). [ISBN 0-8218-0294-1/pbk; ISSN 0271-4132]

In this article, the author shows how parts of L. Weinstein's celebrated proof version of the Bieberbach conjecture [Inter. Math. Res. Notices 3, 61-64 (1991; Zbl 743.30021)] can be principally computerized. The main argument is the following: The coefficients of the function

$$\frac{1}{\sqrt{1-z\big(2x^2+(1-z^2)(w+1/w)\big)+z^2}}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(n-k)!}{(n+k)!}(1-x^2)^k C_{k,n}(x)(w^k+w^{-k})z^n$$

are polynomials $C_{k,n}(x)\in[x]$ with rational coefficients. To prove that these form the squares of another system of polynomials

$$D_{k,n}(x)^2=C_{k,n}(x)*$$

– which indeed is one of the major steps in Weinstein's proof – the author suggests to calculate the first polynomials $D_{k,n}(x)$ for $0\le k\le n\le 20$, then to "guess" a holonomic recurrence equation w.r.t. n

$$\sigma_2 D_{k,n+2}(x)+\sigma_1 D_{k,n+1}(x)+\sigma_0 D_{k,n}(x)=0$$

with polynomials $\sigma_j\in[k,n]$ of fixed degrees w.r.t. k and n, and to use linear algebra to find $\sigma_0,\sigma_1,\sigma_2$. Unfortunately, the author refers to the Maple packacke gfun of Salvy and Zimmermann [ACM Transactions on Mathematical Software 20, 163-177 (1994)] with regard to this step. Although this is a very nice and mighty program, it cannot be used in the current context, since it uses generating functions. With an own Maple implementation the reviewer was able to carry out this step, however. It turns out that

$$(n-k+2)D_{k,n+2}(x)-(2n+3)xD_{k,n+1}(x)+(n+k+1)D_{k,n}(x)=0,**$$

and only the initial values for $0\le k\le n\le 6$ were necessary to find this equation. Assume, $E_{k,n}(x)$ is the solution of (**) with the appropriate initial values. Then by another application of linear algebra this recurrence equation can be "squared", i.e. it is possible to calculate the recurrence equation $\mathcal{R}$ of third order valid for $E_{k,n}(x)^2$. This step can, indeed, be accomplished by the gfun packacke. Then, finally, by an implementation of the author, using the so-called WZ method [Invent. Math. 103, No. 3, 575-634 (1992; Zbl 782.05009)] one can indeed show that $\mathcal{R}$ is valid for $C_{k,n}(x)$. This gives an a posteriori proof of the guessed (*). The method presented here differs from the argument given in the paper slightly. The author asks the reader to find a holonomic recurrence equation for the square root of $\frac{(n-k)!}{(n+k)!}(1-x^2)^k C_{k,n}(x)$ rather than for the square root of $C_{k,n}(x)$. However, such a recurrence equation does not exist.

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Classification:

- 30C50 Coefficient problems for univalent and multivalent functions
- 33C20 Generalized hypergeometric series