A new algorithm for the development of algebraic functions in Puiseux series

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Abstract:

There are several well-known algorithms to calculate the Puiseux series developments of the branches of an algebraic function. None of them, however, produces the formal series, even in those cases where such a formal result is available. They produce, instead, truncated series, and give information that can be used to handle the series as streams.

Here we give a new approach to the given problem. We show an algorithmic procedure which enables us to produce the closed form result if it is of hypergeometric type (see [6] and [7]). Furthermore in each case the algorithm produces a homogeneous and linear recurrence equation with polynomial coefficients for the series coefficients. Other techniques were used in ([2]–[3]) to arrive at the same differential and recurrence equations.

A finite linear recurrence equation is optimal for a representation by streams. In ([2]–[3]) it is pointed out that this algorithm requires only $O(N)$ operations if N is the order of the number of series terms considered. However, the complexity of the resulting recurrence equation — as well as of the differential equation — can be extremely high, a fact, which supersedes the complexity order. We give an example to illustrate that point.

It turns out, however, that many algebraic functions of low order with a sparse representation are of hypergeometric type, and so closed form representations for the corresponding series can be given.

Keywords: formal power series, formal Puiseux series, Laurent-Puiseux series, linear differential equations, linear recurrence equations, hypergeometric functions, functions of hypergeometric type.

1 Introduction

We consider algebraic functions $y(x)$ which are given by some bivariate polynomial equation

$$F(x, y(x)) = \sum_{k=0}^{N} p_k(x) y(x)^k = 0 \tag{1}$$

with coefficient functions $p_k \in \mathbb{K}(x)$ where $\mathbb{K}$ is one of $\mathbb{Z}, \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.

Locally the branches of an algebraic function in a neighborhood of the origin $x = 0$ can be represented by Laurent-Puiseux type series

$$y(x) = \sum_{k=k_0}^{\infty} a_k x^{k/n} \tag{2}$$

for some $k_0 \in \mathbb{N}$ and $n \in \mathbb{N}$, with coefficients $a_k \in \mathbb{C}$ $(k \in \mathbb{N}_0)$. If there is a power series representation, i.e. if $k_0 = 0$ and $n = 1$, the origin is called a *regular*, otherwise a *singular* point of the algebraic function.

There are well-known algorithms to find the coefficients of these representations iteratively, see e.g. [10]. Roughly, they consist of both a method to find initial values which, in general, are algebraic quantities, and of an iteration procedure to generate higher coefficients.

These algorithms are implemented in certain Computer Algebra Systems, e.g. in Axiom [1] (previously Scratchpad), and Maple [8], see [11].

However, none of these algorithms leads to a formula for a_k. For that reason Axiom internally works with streams, and lazy evaluation, i.e. the series objects are given by a finite number of initial terms, and an (internally used) formula to calculate further coefficients, see e.g. [9]. Infinite series representations, however, are not supported.

We will not emphasize on the question how to find suitable initial values. However, we present an algorithm which generates a homogeneous linear recurrence equation for the coefficients searched for. In the special case of hypergeometric type, i.e. if the generated recurrence equation possesses only two summands, the formal solution can explicitly be given. Many algebraic functions of low order are examples of that type.

2 The Algorithm

In ([6], see also [7]) we published an algorithm which generates the formal Laurent-Puiseux type series representation $\sum\limits_{k=k_0}^{\infty} a_k x^{k/n}$ of an explicitly given function $f(x)$ if certain conditions are satisfied. We will use a modified version of the same algorithm which constructs the coefficients of the local Laurent-Puiseux representations (2) for the branches of an algebraic function given by (1).

In ([2]–[3]) a different algorithm was introduced generating the same differential equation for $y(x)$, and recurrence equation for its Puisieux coefficients. These authors gave an elegant algebraic argument that each algebraic function of order N, given by (1), satisfies a homogeneous, linear differential equation

$$\sum_{k=0}^{N} a_k(x) y^{(k)}(x) = 0 \tag{3}$$

with polynomial coefficients $a_k \in \mathbb{K}(x)$ of order at most N: Equation (3) states that the $N+1$ distinct functions $y^{(k)}$ $(k = 0, \ldots, N)$ are linearly dependent members of the (at most) N-dimensional linear space over $\mathbb{K}(x)$ which is generated by the extension of $\mathbb{K}(x)$ according to (1).

Algorithm

1. Let be given equation (1) with some bivariate polynomial F for an algebraic function $y(x)$ of order N. Solving (1) for $y(x)^N$ produces then an equation of the form

$$y(x)^N = R_0(x, y(x)), \tag{4}$$

where R_0 is rational in the first variable x, and a polynomial of degree smaller than N with respect to the second variable y.

2. **Find a homogeneous linear differential equation with polynomial coefficients of degree at most N satisfied by $y(x)$**

   (a) **Search for a differential equation of first order**
   Differentiate (1) with respect to x taking into account that $y(x)$ is a function of y. This leads to a relation

   $$F_1(x, y(x), y'(x)) = 0 \tag{5}$$

   with a polynomial F_1 that is linear in its third argument. Solving this equation for $y'(x)$ thus produces

   $$y'(x) = R_1(x, y(x)) \tag{6}$$

   with rational R_1. Simplify the expression

   $$\frac{y'(x)}{y(x)} = \frac{R_1(x, y(x))}{y(x)}$$

   using (4) if $y(x)^N$ occurs. If the resulting expression is independent of $y(x)$, then the equation

   $$y'(x) - R_1(x)y(x) = 0$$

   is a valid differential equation of first degree for $y(x)$ which by a multiplication by the denominator of R_1 yields the desired form.

   (b) **Search for a differential equation of second order**
   If (a) was not successful, differentiation of (5) produces an equation of the form

   $$F_2(x, y(x), y'(x), y''(x)) = 0 \tag{7}$$

   with a further polynomial F_2, linear in its fourth argument. Solving (7) for $y''(x)$ gives

   $$y''(x) = R_2(x, y(x), y'(x)) \tag{8}$$

   with rational R_2. Introduce the setting

   $$y''(x) + A_1 y'(x) + A_0 y(x) = 0 \ , \tag{9}$$

   a second order differential equation for $y(x)$ with (still unknown) functions A_0, A_1 which are supposed to be rational in x. Substitute (8) in (9) (which eliminates $y''(x)$), then multiply the equation by the denominator of the resulting expression (to work with polynomials), substitute (6) in the resulting equation (which eliminates $y'(x)$), and multiply again by the denominator. The resulting polyonomial may contain high powers of $y(x)$. Eliminate all powers greater than $N-1$ by a recursive application of (4) beginning with the highest power. Finally we get a formula

   $$P(x, y(x), A_0, A_1) = 0 \tag{10}$$

with a polynomial P whose degree with respect to its second argument y is smaller than N.

Equate the coefficients of P with respect to the second variable, and try to solve the corresponding linear system for the 2 unknowns A_0, A_1. If solution functions A_0, A_1 exist, then they are rational functions in x. Substituting these solutions in the setting (9), and multiplying by the common denominator, generates a second order differential equation of the desired form for $y(x)$.

(c) **Search for a differential equation of higher order**

If (b) was not successful, continue with the same procedure by iterative differentiation of (7) and corresponding settings

$$y^{(m)}(x) + \sum_{k=0}^{m-1} A_k y^{(k)}(x) = 0 \,, \tag{11}$$

a differential equation of degree m for $y(x)$. The successive substitution of the derivatives, multiplication by the denominators, and substitution of the high powers of $y(x)$ results in a polynomial equation

$$P(x, y(x), A_0, A_1, \ldots, A_{m-1}) = 0$$

of degree at most $N-1$ with respect to the second variable y. Equate the coefficients of P with respect to y, and solve the corresponding linear system for the m unknowns $A_0, A_1, \ldots, A_{m-1}$. If a solution exists, then the resulting expressions are rational functions in x that we substitute in the setting (11), and multiply with the common denominator, generating a differential equation of order m of the desired form for $y(x)$.

Continue with this procedure until a differential equation is found. By (3) this procedure stops at least for $m = N$.

3. **Find the corresponding recurrence equation** (for details, see ([6], Section 6))

Transfer the differential equation that was found in step 2 into a recurrence equation for the coefficients a_k. This is done by the formal substitution

$$x^l y^{(j)}(x) \mapsto (k+1-l)_j \cdot a_{k+j-l} \tag{12}$$

into the differential equation. The resulting recurrence equation is then of the special type

$$\sum_{j=0}^{M} P_j a_{k+j} = 0 \,, \tag{13}$$

where P_j $(j = 0, \ldots, M)$ are polynomials in k, and $M \in \mathbb{N}$.

4. **Type of recurrence equation** (for details, see ([6], Section 7))

If the recurrence equation (13) contains only two summands then $y(x)$ is of hypergeometric type, and an explicit formula for the coefficients can be found by the hypergeometric coefficient formula (see [6], Equation (2.2)), and some initial conditions.

The following is a MATHEMATICA[1] [12] implementation of part 2 of the above algorithm. The procedure **AlgebraicDE[F,x,y]** results in the differential equation that is satisfied by the algebraic function $y(x)$ given by the equation $F(x,y) = 0$.

```
DEout[de_,y_,x_]:=Module[{X,DE,delist,k,n},
        DE=(Expand[de] /. {y[x]->1,Derivative[n_][y][x]->X^n});
        delist=CoefficientList[DE,X];
        Sum[Factor[delist[[k]]] Derivative[k-1][y][x],
        {k,2,Length[delist]}]+Factor[delist[[1]]] y[x]==0
];

AlgebraicDE[F_,x_,y_]:=Module[
        {n,k,j,z,eq,order,y0,eqlist,ylist,term,setting,intermediate,solution,list,values},
        plugin[expr_,form_,order_]:=Module[{level},
                level=Length[CoefficientList[expr,y[x]]]-1;
                If[level>=order,plugin[Expand[expr /.
                y[x]^level->y[x]^(level-order)*form[[1,1,2]]],form,order],
                expr]
        ];
        If[FreeQ[F,y[x]],eq=(F /. y->y[x]),eq=F];
        If[SameQ[Head[F],Equal],eq=Expand[eq[[1]]-eq[[2]]]==0,eq=Expand[eq]==0];
        order=Length[CoefficientList[eq[[1]],y[x]]]-1;
        y0=(Solve[eq /. y[x]^order->z,z] /. z->y[x]^order);
        eqlist={};
        ylist={};
        (* normalization done *)
        (* first order *)
        AppendTo[eqlist,D[eq,x]];
        AppendTo[ylist,Solve[eqlist[[1]],y'[x]][[1,1]]];
        term=Cancel[Together[ExpandAll[ylist[[1,2]]/y[x]] /.
                {y0[[1,1]],y[x]^(-order)->1/y0[[1,1,2]]}]];
        If[FreeQ[term,y[x]],
        Return[Denominator[term] y'[x]-Numerator[term] y[x]==0]];
        Print["No differential equation of order ",1," found"];
        (* higher order *)
        Do[
        AppendTo[eqlist,D[eqlist[[n-1]],x]];
        AppendTo[ylist,Solve[eqlist[[n]],Derivative[n][y][x]][[1,1]]];
        setting=Sum[A[k] Derivative[k][y][x] /. A[n]->1,{k,0,n}];
        intermediate=setting;
        Do[intermediate=Numerator[Together[intermediate /. ylist[[k]]]],{k,n,1,-1}];
        solution=plugin[intermediate,y0,order];
        Clear[intermediate];
        list=CoefficientList[solution,y[x]];
        Clear[solution];
        values=Solve[list==Table[0,{k,1,Length[list]}],Table[A[k],{k,0,n-1}]];
        If[Not[values=={}],Return[DEout[Numerator[Together[setting /. values[[1]]]],y,x]]],
        {n,2,order}]
]
```

[1]MATHEMATICA is a registered trademark of Wolfram Research, Inc.

3 Examples

In this section we want to give some examples for the use of the algorithm.

1. First we consider the algebraic equation

$$y(x)^2 + x^2 - r^2 = 0 \ , \tag{14}$$

i.e.

$$y(x)^2 = r^2 - x^2 \ . \tag{15}$$

Differentiation of (14) leads to

$$2x + 2y(x)y'(x) = 0 \tag{16}$$

with the explicit form

$$y'(x) = -\frac{x}{y(x)} \ . \tag{17}$$

The expression $y'(x)/y(x)$ can be simplified using (15)

$$\frac{y'(x)}{y(x)} = -\frac{x}{y(x)^2} = -\frac{x}{r^2 - x^2} \ ,$$

which is independent of $y(x)$ so that the first order differential equation

$$(r^2 - x^2)y'(x) + xy(x) = 0 \tag{18}$$

holds for $y(x)$. By (12) the differential equation (18) is converted to the recurrence equation

$$(-2 + k)\ a_{k-1} - (1 + k)\ r^2\, a_{k+1} = 0$$

which is of hypergeometric type, and produces the two formal series

$$y(x) = \pm \sum_{k=0}^{\infty} \frac{|r|\ (2\,k)!}{4^k\, r^{2k}\ (2\,k - 1)\ k!^2} x^{2\,k} \ .$$

2. Next we consider the similar equation

$$(y(x) - r)^2 + x^2 - r^2 = 0 \tag{19}$$

with the explicit form

$$y(x)^2 = 2ry(x) - x^2 \ . \tag{20}$$

Differentiation of (19) leads to

$$2x + 2(-r + y(x))y'(x) = 0 \ , \tag{21}$$

or

$$y'(x) = -\frac{x}{y(x) - r} \ . \tag{22}$$

In this case a simplification of expression $y'(x)/y(x)$ using (20) yields

$$\frac{y'(x)}{y(x)} = \frac{x}{x^2 - r\,y(x)}$$

which is **not** independent of $y(x)$ so that no first order differential equation for $y(x)$ is found. Thus we continue our search, and differentiate (21) again, getting

$$2 + 2\,y'(x)^2 + 2\,(-r + y(x))\;y''(x) = 0\;, \tag{23}$$

and

$$y''(x) = -\frac{1 + y'(x)^2}{-r + y(x)}\;. \tag{24}$$

In the left hand side of the setting (9) we substitute $y''(x)$ by (24), and multiply by the common denominator of the resulting rational expression to get

$$-1 - r\,A_0\,y(x) + A_0\,y(x)^2 - r\,A_1\,y'(x) + A_1\,y(x)\,y'(x) - y'(x)^2\;.$$

Then we substitute $y'(x)$ by (22), and multiply by the common denominator again with the result

$$-r^2 - x^2 - r^2 x A_1 + \left(2r - r^3 A_0 + 2rxA_1\right) y(x) + \left(-1 + 3r^2 A_0 - xA_1\right) y(x)^2 - 3rA_0 y(x)^3 + A_0 y(x)^4.$$

This expression contains powers of $y(x)$ up to order 4. We reduce the degree to 1 by iteratively using (20), and get finally

$$-\,r^2 + 7\,r^2\,x^2\,A_0 - x^4\,A_0 - r^2\,x\,A_1 + x^3\,A_1 + \left(-15\,r^3\,A_0 + 7\,r\,x^2\,A_0\right) y(x)\;. \tag{25}$$

Now we set the coefficients of the powers of $y(x)$ in this expression zero, and solve the resulting linear system for the unknowns A_0 and A_1, which gives

$$A_0 = 0\;, \qquad A_1 = \frac{r^2}{-r^2\,x + x^3}\;. \tag{26}$$

If we substitute these values in the setting (9), and multiply by the common denominator, again, we arrive at the differential equation for $y(x)$

$$-\,r^2\,y'(x) + r^2\,x\,y''(x) - x^3\,y''(x) = 0\;. \tag{27}$$

By (12) we get the equivalent recurrence equation for the Puisieux coefficients a_k

$$(1-k)\;(-2+k)\;a_{k-1} + (-1+k)\;(1+k)\;r^2\,a_{k+1} = 0 \tag{28}$$

which, again, is of hypergeometric type, producing the two formal series

$$y(x) = \sum_{k=0}^{\infty} \frac{(2\,k)!}{2\cdot 4^k\,r^{2k}\;(1+k)\;r\,k!^2} x^{2+2\,k}\;, \quad \text{and} \quad y(x) = 2r - \sum_{k=0}^{\infty} \frac{(2\,k)!}{2\cdot 4^k\,r^{2k}\;(1+k)\;r\,k!^2} x^{2+2\,k}\;.$$

3. Now we consider the algebraic equation

$$y(x)^3 + xy(x)^2 + x^2 = 0 \tag{29}$$

of the explicit form

$$y(x)^3 = -\left(xy(x)^2 + x^2\right) . \tag{30}$$

Differentiation of (29) leads to

$$2x + y(x)^2 + (2xy(x) + 3y(x)^2)y'(x) = 0 , \tag{31}$$

and thus to

$$y'(x) = -\frac{2x - y(x)^2}{2xy(x) + 3y(x)^2} . \tag{32}$$

In this case the first step of considering the expression $y'(x)/y(x)$ is not successful, and similarly no second order differential equation is found. We present the development for the differential equation of egree 3.

Therefore we differentiate (31) again twice producing

$$2 + 4\,y(x)\,y'(x) + 2\,x\,y'(x)^2 + 6\,y(x)\,y'(x)^2 + 2\,x\,y(x)\,y''(x) + 3\,y(x)^2\,y''(x) = 0$$

and

$$6\,y'(x)^2 + 6\,y'(x)^3 + \left(6\,y(x) + 6\,x\,y'(x) + 18\,y(x)\,y'(x)\right)y''(x) + \left(2\,x\,y(x) + 3\,y(x)^2\right)y'''(x) = 0$$

corresponding to

$$y''(x) = -\frac{2 + 4\,y(x)\,y'(x) + 2\,x\,y'(x)^2 + 6\,y(x)\,y'(x)^2}{2\,x\,y(x) + 3\,y(x)^2} \tag{33}$$

and

$$y'''(x) = -\frac{6\,y'(x)^2 + 6\,y'(x)^3 + 6\,y(x)\,y''(x) + 6\,x\,y'(x)\,y''(x) + 18\,y(x)\,y'(x)\,y''(x)}{2\,x\,y(x) + 3\,y(x)^2} . \tag{34}$$

We introduce the setting

$$y'''(x) + A_2 y''(x) + A_1 y'(x) + A_0 y(x) = 0 . \tag{35}$$

After substituting (34), multiplying by the denominator, substituting (33), multiplying by the denominator, substituting (32), and multiplying by the denominator again, we get the following polynomial result (where we suppress the argument x of $y(x)$)

$$\begin{aligned}
&-96\,x^5 - 480\,x^4\,y + \left(-720\,x^3 - 48\,x^4 - 32\,x^5\,A_2\right)\,y^2 + \left(-288\,x^3 - 192\,x^4\,A_2\right)\,y^3 \\
&+ \left(-1080\,x^2 - 24\,x^3 - 360\,x^3\,A_2 - 32\,x^5\,A_1\right)\,y^4 \\
&+ \left(-648\,x + 216\,x^2 - 216\,x^2\,A_2 - 96\,x^3\,A_2 - 192\,x^4\,A_1\right)\,y^5 \\
&+ \left(216\,x - 60\,x^2 - 360\,x^2\,A_2 + 24\,x^3\,A_2 - 432\,x^3\,A_1 - 16\,x^4\,A_1 + 32\,x^5\,A_0\right)\,y^6 \\
&+ \left(-96\,x - 432\,x\,A_2 + 96\,x^2\,A_2 - 432\,x^2\,A_1 - 96\,x^3\,A_1 + 240\,x^4\,A_0\right)\,y^7 \\
&+ \left(-36 - 162\,A_2 + 126\,x\,A_2 - 162\,x\,A_1 - 216\,x^2\,A_1 + 720\,x^3\,A_0\right)\,y^8 \\
&+ \left(54\,A_2 - 216\,x\,A_1 + 1080\,x^2\,A_0\right)\,y^9 + (-81\,A_1 + 810\,x\,A_0)\,y^{10} + 243\,A_0\,y^{11}
\end{aligned}$$

which after an elimination of the powers of y larger than 2 using (30) recursively yields

$$\begin{aligned}
&-24\,x^5+12\,x^6\,A_2-81\,x^7\,A_1-15\,x^8\,A_1-x^9\,A_1-108\,x^8\,A_0-16\,x^9\,A_0-x^{10}\,A_0\\
&+\left(-48\,x^4+36\,x^5\,A_2-27\,x^6\,A_1+14\,x^7\,A_1+x^8\,A_1-81\,x^7\,A_0+15\,x^8\,A_0+x^9\,A_0\right)\,y\\
&+\Big(-72\,x^3-12\,x^4+54\,x^4\,A_2+4\,x^5\,A_2-162\,x^5\,A_1-78\,x^6\,A_1-16\,x^7\,A_1-x^8\,A_1\\
&\quad-243\,x^6\,A_0-90\,x^7\,A_0-17\,x^8\,A_0-x^9\,A_0\Big)\,y^2\;.
\end{aligned}$$

If we set the coefficients of this expression with respect to y zero, and solve the corresponding linear system with respect to A_0, A_1, and A_2, we get the rational solutions

$$\begin{aligned}
A_0 &= \frac{-24}{x^3\,(27+4\,x)}\\
A_1 &= \frac{24}{x^2\,(27+4\,x)}\\
A_2 &= \frac{6}{27+4\,x}
\end{aligned}\;,$$

and after a multiplication with the common denominator we arrive at the differential equation for $y(x)$

$$-24\,y(x)+24\,x\,y'(x)+6\,x^3\,y''(x)+\left(27\,x^3+4\,x^4\right)y'''(x)=0\;.$$

By an application of (12) this differential equation is converted to the recurrence equation for the coefficients

$$2\,(-2+k)\,(-1+k)\,(-3+2\,k)\,a_{k-1}+3\,(-1+k)\,(-4+3\,k)\,(-2+3\,k)\,a_k=0$$

which is of hypergeometric type.

If we use the initialization data, we get the three closed term series solutions

$$y_1(x)=\frac{-x}{3}+\sum_{k=0}^{\infty}\frac{(-1)^{\frac{5}{3}}\,(-1)^k\,\left(-\frac{2}{3}\right)_{2k}}{27^k\,k!\,\left(\frac{1}{3}\right)_k}x^{\frac{2}{3}+k}+\sum_{k=0}^{\infty}\frac{(-1)^{\frac{1}{3}}\,(-1)^k\,\left(\frac{2}{3}\right)_{2k}}{9\cdot 27^k\,k!\,\left(\frac{5}{3}\right)_k}x^{\frac{4}{3}+k}\;,$$

$$y_2(x)=\frac{-x}{3}+\sum_{k=0}^{\infty}\frac{(-1)^{\frac{7}{6}}(-1)^k(-i+\sqrt{3})\left(-\frac{2}{3}\right)_{2k}}{2\cdot 27^k\,k!\,\left(\frac{1}{3}\right)_k}x^{\frac{2}{3}+k}+\sum_{k=0}^{\infty}\frac{(-1)^{\frac{5}{6}}(-1)^k(i+\sqrt{3})\left(\frac{2}{3}\right)_{2k}}{18\cdot 27^k\,k!\,\left(\frac{5}{3}\right)_k}x^{\frac{4}{3}+k},$$

and

$$y_3(x)=\frac{-x}{3}+\sum_{k=0}^{\infty}\frac{(-1)^{\frac{1}{6}}(-1)^k(i+\sqrt{3})\left(-\frac{2}{3}\right)_{2k}}{2\cdot 27^k\,k!\,\left(\frac{1}{3}\right)_k}x^{\frac{2}{3}+k}+\sum_{k=0}^{\infty}\frac{(-1)^{\frac{11}{6}}(-1)^k(-i+\sqrt{3})\left(\frac{2}{3}\right)_{2k}}{18\cdot 27^k\,k!\,\left(\frac{5}{3}\right)_k}x^{\frac{4}{3}+k},$$

where $(a)_k$ denotes the *Pochhammer symbol* (or *shifted factorial*) defined by

$$(a)_k:=\left\{\begin{array}{cl} 1 & \text{if } k=0\\ a\cdot(a+1)\cdots(a+k-1) & \text{if } k\in\mathbb{N}\end{array}\right.\;.$$

4. Finally we consider the equation of fifth order (see e.g. [11], Example 1)

$$y(x)^5 + 2\,x\,y(x)^4 - x\,y(x)^2 - 2\,x^2\,y(x) + x^4 - x^3 = 0\ .$$

It turns out that no differential equation of order smaller than 5 is found. The algorithm then produces the highly complicated differential equation of order 5 for $y(x)$

$$\begin{aligned}
0 \;=\; & 120\,\Big(54432 - 1134108\,x + 37203408\,x^2 + 281967786\,x^3 + 1230374520\,x^4 \\
& +11657644290\,x^5 + 12450677400\,x^6 + 21363256476\,x^7 - 161323223184\,x^8 \\
& +191964729654\,x^9 + 650433460248\,x^{10} + 2279414773653\,x^{11} - 4536683337810\,x^{12} \\
& -1667119660824\,x^{13} + 2238685852584\,x^{14} + 17601207569391\,x^{15} + 9634391651640\,x^{16} \\
& -51665406476124\,x^{17} + 11346782526764\,x^{18} - 31807391940894\,x^{19} + 1929550976364\,x^{20} \\
& +52687275029734\,x^{21} + 1543734656904\,x^{22} - 16908484455900\,x^{23} - 15461359024160\,x^{24} \\
& +6966972049632\,x^{25} - 288513300096\,x^{26} - 197485208704\,x^{27} + 28588643328\,x^{28} \\
& -579962880\,x^{29} - 7225344\,x^{30}\Big)\,\Big(y(x) - xy'(x)\Big) \\
& +30x^2\,\Big(120960 - 2069424\,x + 56432844\,x^2 + 400213278\,x^3 + 1182018600\,x^4 \\
& +18113119074\,x^5 - 138332777886\,x^6 - 412899747018\,x^7 - 176460847236\,x^8 \\
& +1810751206488\,x^9 + 6932166985338\,x^{10} + 30112459757565\,x^{11} - 142261156307412\,x^{12} \\
& -41799613173558\,x^{13} + 4992317227050\,x^{14} + 998106311239713\,x^{15} \\
& -841181120323800\,x^{16} - 497453297897334\,x^{17} - 661731936759476\,x^{18} \\
& +2146290039043276\,x^{19} - 1848127081491960\,x^{20} + 618680925247376\,x^{21} \\
& +604952379151376\,x^{22} + 730462307204592\,x^{23} - 1275500751217600\,x^{24} \\
& +329313306725248\,x^{25} + 16592169798144\,x^{26} - 2172758109440\,x^{27} - 5639438529536\,x^{28} \\
& +159425753088\,x^{29} + 5876097024\,x^{30} + 12845056\,x^{31}\Big)\;y''(x) \\
& +10x^3\,\Big(-145152 - 190944\,x + 3437964\,x^2 - 769768434\,x^3 + 3487469904\,x^4 \\
& -29768539302\,x^5 - 662986405314\,x^6 - 2965998057066\,x^7 - 4492564719768\,x^8 \\
& -25026922213788\,x^9 + 11253928822218\,x^{10} + 89916578926269\,x^{11} - 94148948385972\,x^{12} \\
& -289277304749562\,x^{13} - 430857781924080\,x^{14} + 1777988237992521\,x^{15} \\
& -442086792624120\,x^{16} - 423610968754578\,x^{17} - 4733089717358374\,x^{18} \\
& +11692972472188356\,x^{19} - 10091792544402636\,x^{20} - 191982280714744\,x^{21} \\
& +1757061267466128\,x^{22} + 6037508014604256\,x^{23} - 6142357758832608\,x^{24} \\
& +1370459170644480\,x^{25} + 58525075983936\,x^{26} + 41726406595840\,x^{27} \\
& -40779892978944\,x^{28} + 1237224640512\,x^{29} + 52568965120\,x^{30} + 143130624\,x^{31}\Big)\;y'''(x) \\
& +5x^4\,\Big(-1306368 + 6905088\,x + 5846580\,x^2 - 835380540\,x^3 + 13312685826\,x^4 \\
& +88564809906\,x^5 - 249799299786\,x^6 - 1029878798427\,x^7 - 6402220009146\,x^8 \\
& -26405493601773\,x^9 - 995942403738\,x^{10} + 88139507176443\,x^{11} + 129185204574438\,x^{12} \\
& -147685369152168\,x^{13} - 570160264560330\,x^{14} + 361007472690882\,x^{15}
\end{aligned}$$

$$
\begin{aligned}
&+1458452822920470\,x^{16}-1322236450284177\,x^{17}-3829698595215556\,x^{18}\\
&+7915652686624152\,x^{19}-4395830268317756\,x^{20}-1939579125489742\,x^{21}\\
&-124551120304104\,x^{22}+5343794312610332\,x^{23}-4025593530875232\,x^{24}\\
&+777670186871232\,x^{25}-244909732864\,x^{26}+69733357360384\,x^{27}-35221389934080\,x^{28}\\
&++1104965650432\,x^{29}+55056719872\,x^{30}+176160768\,x^{31}\Big)\;y^{(iv)}(x)\\
&+x^5\left(-108-108\,x+1080\,x^2-1863\,x^3-12836\,x^4+4270\,x^5+14076\,x^6-21451\,x^7+8192\,x^8\right)\\
&\Big(24192-309888\,x+7154010\,x^2+43413102\,x^3+128874429\,x^4+921434754\,x^5\\
&+641396625\,x^6-785077536\,x^7-9150010194\,x^8-1382801547\,x^9+17592751371\,x^{10}\\
&+30927169413\,x^{11}-75484940452\,x^{12}+64747317160\,x^{13}+60583653282\,x^{14}\\
&-5107979484\,x^{15}-101167000076\,x^{16}+38111990992\,x^{17}+1908260032\,x^{18}\\
&+590340992\,x^{19}-1933935360\,x^{20}+76462848\,x^{21}+3971072\,x^{22}+14336\,x^{23}\Big)\;y^{(v)}(x)
\end{aligned}
$$

to be converted to the recurrence equation

$$
\begin{aligned}
0\;=\;&3670016\;(-32+k)\;(-31+k)\;(-63+2\,k)\;(-127+4\,k)\;(-125+4\,k)\;a_{k-31}\\
&+14336\;(-31+k)\;\Big(1847584169040-245474685292\,k+12229317488\,k^2\\
&\quad-270755297\,k^3+2247733\,k^4\Big)\;a_{k-30}\\
&+71680\;(-30+k)\;\Big(5106340398832-712934084634\,k+37304640535\,k^2\\
&\quad-867053838\,k^3+7553041\,k^4\Big)\;a_{k-29}\\
&+1792\;(-29+k)\;\Big(-5733035987328000+828032758801088\,k-44832058741478\,k^2\\
&\quad+1078440566413\,k^3-9724913503\,k^4\Big)\;a_{k-28}\\
&+256\;(-28+k)\;\Big(104159014914036720-15228984809713838\,k+834446273409049\,k^2\\
&\quad-20307692886592\,k^3+185210876261\,k^4\Big)\;a_{k-27}\\
&+128\;(-27+k)\;\Big(-115213994699295920+16581492512727702\,k-890793330408059\,k^2\\
&\quad+21158083696953\,k^3-187324340561\,k^4\Big)\;a_{k-26}\\
&+320\;(-26+k)\;\Big(261523489660826088-44478219897618088\,k+2831030436466297\,k^2\\
&\quad-79934222131760\,k^3+844819438547\,k^4\Big)\;a_{k-25}\\
&+48\;(-25+k)\;(-9362885878701552592+1627753310296951180\,k\\
&\quad-105989435137460860\,k^2+3063677307904995\,k^3-33171523498773\,k^4\Big)\;a_{k-24}\\
&+4\;(-24+k)\;(178027620674797339280-31360093098029528886\,k\\
&\quad+2070363509913533231\,k^2-60712730831598066\,k^3+667252215072881\,k^4\Big)\;a_{k-23}\\
&+4\;(-23+k)\;(-54336072942087552000+9246137431966092272\,k\\
&\quad-586371145952587046\,k^2+16409668072287133\,k^3-170776876006879\,k^4\Big)\;a_{k-22}
\end{aligned}
$$

$$
\begin{aligned}
&+2\,(-22+k)\,(-2001976100400556972800+36772496390714994748\,k\\
&\quad-2529609061143644621\,k^2+77222406585725912\,k^3-882487681714519\,k^4\Big)\;a_{k-21}\\
&+4\,(-21+k)\,(48030754028234174320-7266357224497754322\,k\\
&\quad+371587223129803393\,k^2-6684720091603977\,k^3+13370025350638\,k^4\Big)\;a_{k-20}\\
&+4\,(-20+k)\,(92964173086601713938-20367858196521329885\,k\\
&\quad+1672814255784693545\,k^2-61029211282013905\,k^3+834397378121107\,k^4\Big)\;a_{k-19}\\
&+5\,(-19+k)\,(-47134872307190169408+10371375732712393708\,k\\
&\quad-857803740661958792\,k^2+31600868198734667\,k^3-437414404863583\,k^4\Big)\;a_{k-18}\\
&+3\,(-18+k)\,(-41228087445218516160+9266251586762053626\,k\\
&\quad-779191321811937023\,k^2+29067781184443494\,k^3-406123488332257\,k^4\Big)\;a_{k-17}\\
&+(-17+k)\,(97077035030491596800-23486019628240708324\,k\\
&\quad+2120748804586146328\,k^2-84659168158617701\,k^3+1259608661276987\,k^4\Big)\;a_{k-16}\\
&+6\,(-16+k)\,(711758856796006200-201419678538192560\,k\\
&\quad+3678814686285739\,k^2-309641096342211\,k^3+8847545752434\,k^4\Big)\;a_{k-15}\\
&+3\,(-15+k)\,(-7924587028888143672+2131827489759700260\,k\\
&\quad-215420923842410090\,k^2+9690285768386445\,k^3-163700813476723\,k^4\Big)\;a_{k-14}\\
&+12\,(-14+k)\,\Big(-162804033545303100+10163157396963119\,k-1798786291196759\,k^2\\
&\quad+125511521736614\,k^3-3066345774044\,k^4\Big)\;a_{k-13}\\
&+9\,(-13+k)\,\Big(393319820385627520-124089824864656012\,k+14596866130117436\,k^2\\
&\quad-756656795626553\,k^3+14533795679359\,k^4\Big)\;a_{k-12}\\
&+15\,(-12+k)\,\Big(16520649136330852-7939611929676696\,k+1299577291158435\,k^2\\
&\quad-89658804052518\,k^3+2246012071283\,k^4\Big)\;a_{k-11}\\
&+18\,(-11+k)\,\Big(-15350964221844580+5205607425117416\,k-657415970944339\,k^2\\
&\quad+36547228152226\,k^3-751507731019\,k^4\Big)\;a_{k-10}\\
&+15\,(-10+k)\,\Big(-3303883425538368+1727821099727602\,k-325306212813695\,k^2\\
&\quad+26380432162394\,k^3-781828074733\,k^4\Big)\;a_{k-9}\\
&+18\,(-9+k)\,\Big(598893763169360-142404489157962\,k+3440503510327\,k^2\\
&\quad+1210614678738\,k^3-72902661603\,k^4\Big)\;a_{k-8}\\
&+9\,(-8+k)\,\Big(346209600203900-158105415207988\,k+25096693312979\,k^2\\
&\quad-1554669870312\,k^3+26554458981\,k^4\Big)\;a_{k-7}
\end{aligned}
$$

$$+18\ (-7+k)\ \Big(-35193377957700 + 17382891058784\,k - 3335152431823\,k^2$$
$$+303312539586\,k^3 - 11293976787\,k^4\Big)\ a_{k-6}$$
$$+18\ (-6+k)\ \Big(-18307046703640 + 9594956615268\,k - 1818119107317\,k^2$$
$$+146714117238\,k^3 - 4210785557\,k^4\Big)\ a_{k-5}$$
$$+54\ (-5+k)\ \Big(-461116148696 + 290897986010\,k - 65516804275\,k^2 + 6146512445\,k^3$$
$$-196554254\,k^4\Big)\ a_{k-4}$$
$$+108\ (-4+k)\ \Big(-32307660700 + 26963077044\,k - 8237236449\,k^2 + 1097074359\,k^3$$
$$-54083304\,k^4\Big)\ a_{k-3}$$
$$+108\ (-3+k)\ \Big(-1665497540 + 1793441596\,k - 689288103\,k^2 + 112508109\,k^3$$
$$-6602202\,k^4\Big)\ a_{k-2}$$
$$+864\ (-2+k)\ (-4+3\,k)\ \Big(-471485 + 427626\,k - 125560\,k^2 + 11904\,k^3\Big)\ a_{k-1}$$
$$+145152\ (3-k)\ (-1+k)\ (-3+2\,k)\ (-5+3\,k)\ (-1+3\,k)\ a_k$$

This example shows that even for small degree of F with respect to y the complexity of the resulting differential and recurrence equations can be extremely high.

It turns out, however, that algebraic functions of low degree (smaller than 5 or 6) with sparse representations often are of hypergeometric type. The following is a list of examples of that type which we produced by an experimental implementation of our algorithm in MATHEMATICA [12].

$F(x,y)$	differential equation
$x^3 + x\,y - y^2$	$2\ (1+3\,x)\ y - 2\,x\ (1+3\,x)\ y' + x^2\ (1+4\,x)\ y'' = 0$
$x + y^2 + x\,y^3$	$4y + 2x\left(-4+81x^3\right) y' + 6x^2\,(1+3x)\left(1-3x+9x^2\right) y'' + x^3\left(4+27x^3\right) y''' = 0$
$x^2 + y^2 + x\,y^3$	$3\left(4-7\,x^4\right)\ y + 3\,x\ \left(-4+47\,x^4\right)\ y' + 162\,x^6\,y'' + x^3\ \left(4+27\,x^4\right)\ y''' = 0$
$1 + y^2 + x\,y^3$	$15\,y + 177\,x\,y' + 6\ \left(2+27\,x^2\right)\ y'' + x\ \left(4+27\,x^2\right)\ y''' = 0$
$-x + y + x\,y^3$	$-2\,y + 2\,x\ (1+3\,x)\ \left(1-3\,x+9\,x^2\right)\ y' + x^2\ \left(4+27\,x^3\right)\ y'' = 0$
$-y^2 + x^2\,y^4 - x$	$\left(-2+7\,x^3\right)\ y + 6\,x\ (1+2\,x)\ \left(1-2\,x+4\,x^2\right)\ y' + 4\,x^2\ \left(1+4\,x^3\right)\ y'' = 0$
$-1 + x^2\,y^2 + x\,y^3$	$-3\ \left(7+4\,x^4\right)\ y + 3\,x\ \left(7+4\,x^4\right)\ y' + 24\,x^6\,y'' + x^3\ \left(-27+4\,x^4\right)\ y''' = 0$
$-1 + x^2\,y + x\,y^2$	$2\ (1+x)\ \left(-1+x-x^2\right)\ y + 2\,x\ (1+x)\ \left(1-x+x^2\right)\ y' + x^2\ \left(4+x^3\right)\ y'' = 0$
$(x^2+y^2)^3 - 4x^2y^2$	$3\ \left(8+x^2\right)\ y - x\ \left(28+3\,x^2\right)\ y' + 2\,x^2\ \left(8+27\,x^2\right)\ y'' + x^3\ \left(-16+27\,x^2\right)\ y''' = 0$
$(x^2+y^2)^4 - Ax^2y^2$	$-45\,A\,y + 69\,A\,x\,y' - 2\,x^2\ \left(27\,A+512\,x^4\right)\ y'' + x^3\ \left(27\,A-256\,x^4\right)\ y''' = 0$

corresponding to the recurrence equations of hypergeometric type

$F(x,y)$	recurrence equation
$x^3 + x\,y - y^2$	$2\,(-2+k)\,(-5+2\,k)\,a_{k-1} + (-2+k)\,(-1+k)\,a_k = 0$
$x + y^2 + x\,y^3$	$27\,(-3+k)\,(-2+k)\,(-1+k)\,a_{k-3} + 2\,(-2+k)\,(1+k)\,(-1+2\,k)\,a_k = 0$
$x^2 + y^2 + x\,y^3$	$3\,(-3+k)\,(-13+3\,k)\,(-5+3\,k)\,a_{k-4} + 4\,(-3+k)\,(-1+k)\,(1+k)\,a_k = 0$
$1 + y^2 + x\,y^3$	$3\,(1+k)\,(1+3\,k)\,(5+3\,k)\,a_k + 4\,(1+k)\,(2+k)\,(3+k)\,a_{k+2} = 0$
$-x + y + x\,y^3$	$27\,(-3+k)\,(-2+k)\,a_{k-3} + 2\,(-1+k)\,(1+2\,k)\,a_k = 0$
$-y^2 + x^2\,y^4 - x$	$(-11+4\,k)\,(-5+4\,k)\,a_{k-3} + 2\,(1+k)\,(-1+2\,k)\,a_k = 0$
$-1 + x^2\,y^2 + x\,y^3$	$4\,(-5+k)\,(-3+k)\,(-1+k)\,a_{k-4} + 3\,(1-k)\,(-7+3\,k)\,(1+3\,k)\,a_k = 0$
$-1 + x^2\,y + x\,y^2$	$(-4+k)\,(-1+k)\,a_{k-3} + 2\,(-1+k)\,(1+2\,k)\,a_k = 0$
$(x^2+y^2)^3 - 4x^2y^2$	$3\,(-3+k)\,(-7+3\,k)\,(-5+3\,k)\,a_{k-2} + 4\,(2-k)\,(-3+2\,k)\,(-1+2\,k)\,a_k = 0$
$(x^2+y^2)^4 - Ax^2y^2$	$256\,(2-k)\,(-5+k)\,(-4+k)\,a_{k-4} + 3\,A\,(-3+k)\,(-5+3\,k)\,(-1+3\,k)\,a_k = 0$

The above examples were connected with differential equations of order at most 3. Here are final examples corresponding to fourth order differential equations. The algebraic functions defined by

$$-y^3 + x^2\,y^4 - x^2 = 0\ , \qquad \text{and} \qquad x^2 + x\,y(x)^3 + y(x)^5\ ,$$

respectively, satisfy the differential equations

$$-720y+15x(69+1792x^8)y'+3x^2(-97+7424x^8)y''+18x^3(-3+256x^8)y'''+x^4(27+256x^8)y^{(iv)}=0\ ,$$

and

$$720y + 6x\,(-125+8x)\,y' + 6x^2\,(625+97x)\,y'' + 3x^3\,(3125+198x)\,y''' + x^4\,(3125+108x)\,y^{(iv)} = 0\ ,$$

respectively, which correspond to the recurrence equations

$$256\,(-8+k)\,(-6+k)\,(-4+k)\,(-2+k)\,a_{k-8} + 3\,(-6+k)\,(2+k)\,(-10+3k)\,(-2+3k)\,a_k = 0\ .$$

and

$$6\,(-1+k)\,(-3+2k)\,(-4+3k)\,(-2+3k)\,a_{k-1} + 5\,(2-5k)\,(3-5k)\,(4-5k)\,(6-5k)\,a_k = 0\ ,$$

respectively.

4 First order differential equations

The algorithm given in § 2 is self-explanatory, and it is clear how it produces a homogeneous linear differential equation with polynomial coefficients for an algebraic function of order N. The fact that there always exists such a differential equation of order N corresponds to the fact that the linear system of N equations in the N variables $A_0, A_1, \ldots, A_{N-1}$ which has to be solved in the final step of the algorithm is regular if there does not exist a differential equation of smaller order.

In this section we give a complete description of algebraic functions satisfying a first order differential equation of the type considered.

Theorem An algebraic function $y(x)$ satisfies a first order homogeneus, linear differential equation with polynomial coefficients in $\mathbb{C}(x)$ if and only if

$$y(x) = \sqrt[n]{r(x)} = \sqrt[n]{\frac{p(x)}{q(x)}} \qquad (p, q \quad \text{polynomials})$$

for some $n \in \mathbb{N}$ and some rational function r, i.e. if

$$F(x,y) = q(x)y^n - p(x) = 0\ . \tag{36}$$

Proof: Assume firstly, $y(x)$ is of the given type (36). Then by taking the logarithmic derivative we get

$$\frac{q'(x)}{q(x)} + n\frac{y'(x)}{y(x)} = \frac{p'(x)}{p(x)}\ ,$$

and thus we arrive at the first order differential equation

$$\Big(- q(x)\,p'(x) + p(x)\,q'(x)\Big)\,y(x) + n\,p(x)\,q(x)\,y'(x) = 0$$

for $y(x)$.

Assume now, on the other hand, that y is an algebraic function satisfying a first order homogeneus, linear differential equation with polynomial coefficients. Then there is a rational function $R(x)$ such that

$$\frac{y'(x)}{y(x)} = R(x)\ . \tag{37}$$

If we integrate (37) using a complex partial fraction decomposition of $R(x)$ we get a representation of $\ln y(x)$ as a sum

$$\ln y(x) = R_0(x) + \sum_{k=1}^{d} \alpha_k \ln R_k(x)$$

with rational functions $R_k(x)$ $(k = 0, \ldots, d)$ and $\alpha_k \in \mathbb{C}$ $(k = 1, \ldots, d)$. Thus

$$y(x) = e^{R_0(x)} \cdot \prod_{k=1}^{d} \Big(R_k(x)\Big)^{\alpha_k}\ .$$

Now, as $y(x)$ is algebraic, it turns out that the first factor $e^{R_0(x)}$ must be constant as otherwise clearly $y(x)$ is transcendental. This can be seen by a consideration of a neighborhood of $x = \infty$. Considering a neighborhood of the zeros and poles of the functions R_k $(k = 1, \ldots, d)$ shows that the same is true if one of the exponents α_k is irrational or complex, so that for all $k = 1, \ldots, d$ the numbers $\alpha_k \in \mathbb{Q}$, i.e. $\alpha_k = \frac{p_k}{q_k}$ $(p_k \in \mathbb{Z},\ q_k \in \mathbb{N})$. Then, if n denotes the least common multiple of the numbers q_k $(k = 1, \ldots, d)$, then $S_k := R_k^{np_k/q_k}$ $(k = 1, \ldots, d)$ are rational functions, and we get

$$y(x) = \prod_{k=1}^{d} S_k^{1/n} = \sqrt[n]{S}$$

with rational $S := \prod_{k=1}^{d} S_k$ which finishes the result. $\square$

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