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Irrationality of certain infinite series II

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Summary: In a recent paper a new direct proof for the irrationality of Euler's number

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

and on the same lines a simple criterion for some fast converging series representing irrational numbers was given. In the present paper, we give some generalizations of our previous results.

1 Irrationality criterion

Our considerations in [3] lead us to the following criterion for irrationality, where

$$\lfloor x \rfloor := \max\{n \in \mathbb{N} \mid n \leqq x\}$$

denotes the floor function (Gauss bracket).

Theorem 1.1 *Let $\sum_{j=1}^{\infty} b_j$ be a convergent series with $b_j \geqq 0$ for all natural numbers $j \in \mathbb{N} := \{1, 2, 3, \dots\}$ and $b_j > 0$ for infinitely many $j \in \mathbb{N}$. Let $a := \sum_{j=1}^{\infty} b_j \in \mathbb{R}$ denote its limit and $s_n := \sum_{j=1}^{n} b_j$ denote the corresponding partial sums. If*

$$\lfloor n\, s_n \rfloor = \lfloor n\, a \rfloor$$

for almost all $n \in \mathbb{N}$, i. e. for all but finitely many $n \in \mathbb{N}$, then a is irrational.

Proof: From the given assumptions on b_j it follows that $s_n < a$ for all $n \in \mathbb{N}$ and therefore

$$n\, s_n < n\, a \quad (n \in \mathbb{N}).$$

We write $n\, s_n = \lfloor n\, s_n \rfloor + R_n$ with $0 \leqq R_n < 1$ and $n\, a = \lfloor n\, a \rfloor + \widetilde{R}_n$ with $0 \leqq \widetilde{R}_n < 1$. Therefore we have for all $n \in \mathbb{N}$

$$\lfloor n\, s_n \rfloor + R_n < \lfloor n\, a \rfloor + \widetilde{R}_n$$

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from which it follows by assumption that for almost all $n \in \mathbb{N}$

$$\lfloor n\, s_n \rfloor + R_n < \lfloor n\, s_n \rfloor + \widetilde{R}_n$$

and hence $\widetilde{R}_n > R_n$. Since $R_n \geqq 0$ for all $n \in \mathbb{N}$, we therefore get $\widetilde{R}_n > 0$ for almost all $n \in \mathbb{N}$. Using [3, Lemma 4.4], we deduce that a is irrational. □

For convenience, we cite here Lemma 4.4 from [3]:

Lemma 4.4 [3] *Let $c \in \mathbb{R}_{>0}$ be arbitrary, and let the remainder $0 \leqq R_n < 1$ be defined by the division algorithm as*

$$n\, c = \lfloor n\, c \rfloor + R_n .$$

If $R_n > 0$ for almost all $n \in \mathbb{N}$, then c is irrational.

Our goal is to find rather general families of series satisfying the assumptions of Theorem 1.1. To identify such families we will emphasize on series for which $b_j \in \mathbb{Q}$.

2 Irrational series of rational numbers

In this section we consider converging infinite series $\sum_{j=1}^{\infty} b_j$ with $b_j \in \mathbb{Q}$, $b_j \geqq 0$. As in § 1, we still use the notations $a = \sum_{j=1}^{\infty} b_j$ and $s_n = \sum_{j=1}^{n} b_j$, and we write $n\, s_n = \lfloor n\, s_n \rfloor + R_n$ with $0 \leqq R_n < 1$ and $n\, a = \lfloor n\, a \rfloor + \widetilde{R}_n$ with $0 \leqq \widetilde{R}_n < 1$. Finally we use the abbreviations $M_n := \lfloor n\, s_n \rfloor \in \mathbb{N}_{\geqq 0}$ and $\widetilde{M}_n := \lfloor n\, a \rfloor \in \mathbb{N}_{\geqq 0}$. To apply Theorem 1.1, we need to find sufficient conditions for the property

$$\lfloor n\, s_n \rfloor = \lfloor n\, a \rfloor \quad \text{for almost all } n \in \mathbb{N}.$$

For that purpose, we use a constructive approach. Assume $(p_n)_{n \in \mathbb{N}}$ is a sequence of integers $p_n \in \mathbb{N}_{\geqq 0}$ with

$$n\, p_n\, s_n \in \mathbb{N}_{\geqq 0} \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

If $b_j = \frac{a_j}{c_j}$ with $a_j \in \mathbb{N}_{\geqq 0}$, $c_j \in \mathbb{N}$ and $\gcd(a_j, c_j) = 1$, then $p_n = \operatorname{lcm}(c_1, \ldots, c_n)$ is such a choice.

We would like to note a crucial consequence of (2.1). Since $R_n = n\, s_n - M_n$, we have $p_n\, R_n = n\, p_n\, s_n - p_n\, M_n$. (2.1) implies $n\, p_n\, s_n - p_n\, M_n \in \mathbb{Z}$, and since $R_n \geqq 0$, we get

$$p_n\, R_n \in \mathbb{N}_{\geqq 0} \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

Now we consider the remainder sequence $\sum_{j=n+1}^{\infty} b_j = \sum_{j=1}^{\infty} b_{n+j}$. If we demand furthermore that the remainder tends to zero in a certain specific way, namely

$$\sum_{j=1}^{\infty} b_{n+j} < \frac{1}{n\, p_n}, \tag{2.3}$$

then it follows from $a = s_n + \sum_{j=1}^{\infty} b_{n+j}$ that

$$a < s_n + \frac{1}{n\,p_n} \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

This finally yields together with $s_n \leqq a$

$$0 \leqq a - s_n < \frac{1}{n\,p_n} \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

At this point, we would like to interrupt our general considerations with an instructive example.

Example 2.1 Consider the series $a = \sum_{j=1}^{\infty} \frac{1}{j!}$ converging to $a = e - 1$. Then $s_n = \sum_{j=1}^{n} \frac{1}{j!}$. Here obviously $n!\, s_n \in \mathbb{N}$, hence $p_n := (n-1)!$ is a suitable candidate sequence satisfying (2.1). The question is whether p_n does also satisfy (2.3). Since

$$\sum_{j=1}^{\infty} \frac{1}{(n+j)!} < \sum_{j=1}^{\infty} \frac{n+j-1}{(n+j)!} = \sum_{j=1}^{\infty} \left(\frac{1}{(n+j-1)!} - \frac{1}{(n+j)!} \right),$$

(i. e., $\frac{n+j-1}{(n+j)!}$ is Gosper-summable [2] w. r. t. j), we set

$$S_m := \sum_{j=1}^{m} \left(\frac{1}{(n+j-1)!} - \frac{1}{(n+j)!} \right),$$

and we get by telescoping $S_m = \frac{1}{n!} - \frac{1}{(n+m)!}$. Therefore $\lim\limits_{m\to\infty} S_m = \frac{1}{n!}$, hence

$$\sum_{j=1}^{\infty} \frac{1}{(n+j)!} < \frac{1}{n!} = \frac{1}{n\,p_n}$$

so that p_n satisfies (2.3).

Our considerations result in the following

Theorem 2.2 *Assume the series $\sum_{j=1}^{\infty} b_j$ converges to a and the summands b_j satisfy $b_j \in \mathbb{Q}$ and $b_j \geqq 0$ for all $j \in \mathbb{N}$. If there is a sequence $(p_n)_{n\in\mathbb{N}}$ satisfying (2.1) and (2.3), then $\lfloor n\, s_n \rfloor = \lfloor n\, a \rfloor$ is valid for all $n \in \mathbb{N}$.*

Proof: By (2.5), we have

$$0 \leqq a - s_n < \frac{1}{n\,p_n}$$

for all $n \in \mathbb{N}$, and therefore $0 \leqq p_n\, n\, a - p_n\, n\, s_n < 1$ or

$$0 \leqq p_n\,(\widetilde{M}_n + \widetilde{R}_n) - p_n\,(M_n + R_n) < 1.$$

Hence

$$-p_n\,\widetilde{R}_n \leqq p_n\,\widetilde{M}_n - p_n\,M_n - p_n\,R_n < 1 - p_n\,\widetilde{R}_n.$$

Because $0 \leqq \widetilde{R}_n < 1$, we get

$$-p_n < p_n\,\widetilde{M}_n - p_n\,M_n - p_n\,R_n < 1.$$

Since by (2.2) $p_n\,\widetilde{M}_n - p_n\,M_n - p_n\,R_n \in \mathbb{Z}$, this yields furthermore

$$p_n\,R_n - p_n < p_n\,\widetilde{M}_n - p_n\,M_n \leqq p_n\,R_n.$$

Because $0 \leqq R_n < 1$, this leads to

$$-p_n < p_n\,\widetilde{M}_n - p_n\,M_n < p_n,$$

and dividing by p_n finally gives

$$-1 < \widetilde{M}_n - M_n < 1\,.$$

This relation, however, shows that $M_n = \widetilde{M}_n$ for all $n \in \mathbb{N}$ as announced because $\widetilde{M}_n - M_n \in \mathbb{Z}$. □

Combining Theorem 2.2 with Theorem 1.1 yields

Theorem 2.3 *Assume the series* $\sum_{j=1}^{\infty} b_j$ *converges to a and the summands* b_j *satisfy* $b_j \in \mathbb{Q}$, $b_j \geqq 0$ *for all* $j \in \mathbb{N}$ *and* $b_j > 0$ *for infinitely many* $n \in \mathbb{N}$. *If there is a sequence* $(p_n)_{n\in\mathbb{N}}$ *satisfying (2.1) and (2.3), then a is irrational.*

If we apply Theorem 2.3 to Example 2.1, then we deduce the well-known result that e is irrational.

3 Example types of irrational series

In this section, we give some rather general example types for which the above criterion is applicable.

Definition 3.1 (Cantor series) *A sequence* $(g_j)_{j\in\mathbb{N}}$ *of positive integers* $g_j \in \mathbb{N}_{\geqq 2}$ *is called a Cantor basis. We use the abbreviation* $G_n := g_1 \cdots g_n$. *A series of the form*

$$z_0 + \sum_{j=1}^{\infty} \frac{z_j}{G_j}$$

with $z_j \in \mathbb{N}_{\geqq 0}$, $z_j \leqq g_j - 1$ *for all* $j \in \mathbb{N}$ *and* $z_j < g_j - 1$ *for infinitely many* $j \in \mathbb{N}$ *is called Cantor series with basis* $(g_j)_{j\in\mathbb{N}}$*, see [4, pp. 69].*

Example 3.2 (Cantor series) If we set $g_j = j + 1$, we have $G_n = g_1 \cdots g_n = (n+1)!$. Hence the exponential series of Example 2.1 is clearly a Cantor series.

W. l. o. g. we assume $z_0 = 0$. Therefore let $a = \sum_{j=1}^{\infty} \frac{z_j}{(j+1)!}$, $z_j \in \mathbb{N}_{\geqq 0}$, $z_j \leqq j$ for all j and $z_j < j$ for infinitely many j. We would like to prove that every such Cantor series converges towards an irrational number a if the series does not terminate.

Again we set $s_n = \sum_{j=1}^{n} \frac{z_j}{(j+1)!}$. Of course $(n+1)!\, s_n \in \mathbb{N}$ and therefore $p_n = (n-1)!\,(n+1)$ satisfies (2.1). Since $z_j \leqq j$ for all j and $z_j < j$ for infinitely many j, we get for the residues

$$a - s_n = \sum_{j=1}^{\infty} \frac{z_{n+j}}{(n+j+1)!} < \sum_{j=1}^{\infty} \frac{n+j}{(n+j+1)!} = \frac{1}{(n+1)!} = \frac{1}{n\, p_n}$$

by the computation of Example 2.1. Therefore p_n satisfies (2.3), too. Therefore, by Theorem 2.3, if the series does not terminate, a is irrational.

We remark that this criterion is sharp in the sense that if we assume $z_j \leqq j$ without having $z_j < j$ for infinitely many j, then irrationality is not guaranteed which is shown by the non-terminating series $\sum_{j=1}^{\infty} \frac{j}{(j+1)!} = 1$. Note that the summand $\frac{j}{(j+1)!}$ again is Gosper-summable [2], hence the sum is telescoping, so that this sum can be computed automatically.

We would like to note that for these very special Cantor series, [4] shows moreover that every $a \in \mathbb{R}_{\geqq 0}$ has exactly one representation of the form

$$a = z_0 + \sum_{j=1}^{\infty} \frac{z_j}{(j+1)!} \tag{3.1}$$

with $z_j \in \mathbb{N}$, $z_j \leqq j$ for all j and $z_j < j$ for infinitely many j.

As another comment we mention that as soon as one knows the representation (3.1) of a positive real number a, its irrationality status in clear. This would be of particular interest for the Euler–Mascheroni constant

$$\gamma = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{j} - \ln n \right),$$

whose irrationality status is still unknown, see e. g. [6].

We give another example.

Example 3.3 In this example we assume that $(q_j)_{j\in\mathbb{N}}$ is an arbitrary sequence of positive integers, and $Q_j = q_1 \cdots q_j$. We consider the series $a = \sum_{j=1}^{\infty} \frac{z_j}{j!\, Q_j}$ with $z_j \in \mathbb{N}_{\geqq 0}$, $z_j \leqq j\, q_j - 1$ for all j and $z_j < j\, q_j - 1$ for infinitely many j, and set $s_n := \sum_{j=1}^{n} \frac{z_j}{j!\, Q_j}$.

Then obviously $s_n \leqq a$ for all $n \in \mathbb{N}$, and $n!\, Q_n\, s_n \in \mathbb{N}_{\geqq 0}$. Therefore the numbers $p_n := (n-1)!\, Q_n$ satisfy (2.1). We will show that (2.3) is also valid so that Theorem 2.3 applies.

We get again a telescoping series

$$a - s_n = \sum_{j=1}^{\infty} \frac{z_{n+j}}{(n+j)!\,Q_{n+j}} < \sum_{j=+1}^{\infty} \frac{(n+j)\,q_{n+j}-1}{(n+j)!\,Q_{n+j}}$$
$$= \sum_{j=1}^{\infty} \left(\frac{1}{(n+j-1)!\,Q_{n+j-1}} - \frac{1}{(n+j)!\,Q_{n+j}} \right),$$

so that we conclude that

$$a - s_n = \sum_{j=1}^{\infty} \frac{z_{n+j}}{(n+j)!\,Q_{n+j}} < \frac{1}{n!\,Q_n} = \frac{1}{n\,p_n}$$

so that by Theorem 2.3 a is irrational if additionally $z_j > 0$ for infinitely many j, i. e., if the series does not terminate.

We apply this criterion to certain specific cases:

1. $q_j = j$ leads to

$$a = \sum_{j=1}^{\infty} \frac{z_j}{j!^2}$$

with $z_j \in \mathbb{N}_{\geqq 0}$, $z_j \leqq j^2 - 1$ and $z_j < j^2 - 1$ for infinitely many j. In particular one deduces that $a = \sum_{j=1}^{\infty} \frac{1}{j!^2}$ is irrational. Note that a is related to the modified Bessel function of the first kind:

$$a = I_0(2) - 1.$$

2. $q_j \geqq 2$ for infinitely many j. Then $j\,q_j - 1 \geqq 2j - 1$ for infinitely many j. If we set $z_j = j - 1$, then we get $z_j \leqq j\,q_j - 1$ for all j and $z_j = j - 1 < 2j - 1$ for infinitely many j. Therefore $\sum_{j=1}^{\infty} \frac{j-1}{j!\,Q_j}$ is irrational.

Next, we consider a similar example.

Example 3.4 In this example we assume again that $(q_j)_{j\in\mathbb{N}}$ is an arbitrary sequence of positive integers, and $Q_j = q_1 \cdots q_j$. We consider the series $a = \sum_{j=1}^{\infty} \frac{z_j}{j!\,Q_j}$ with $z_j \in \mathbb{N}_{\geqq 0}$, $z_j > 0$ for infinitely many j, and $z_j \leqq K$ for some $K \in \mathbb{N}$ and all $j \in \mathbb{N}$. We set $s_n := \sum_{j=1}^{n} \frac{z_j}{j!\,Q_j}$ and claim that from these assumptions it follows that $a := \lim\limits_{n\to\infty} s_n$ is irrational.

We have obviously $s_n < a$ for all $n \in \mathbb{N}$ since $z_j > 0$ for infinitely many j. Furthermore $n!\,Q_n\,s_n \in \mathbb{N}_{\geqq 0}$. Therefore $p_n = (n-1)!\,Q_n$ is a suitable "candidate" satisfying (2.1).

Inspecting the residues for this series, we get

$$\sum_{j=n+1}^{\infty} \frac{z_j}{j!\,Q_j} = \sum_{j=1}^{\infty} \frac{z_{n+j}}{(n+j)!\,Q_{n+j}} \leqq K \cdot \sum_{j=1}^{\infty} \frac{1}{(n+j)!\,Q_{n+j}}.$$

For the latter series we can write

$$\sum_{j=1}^{\infty} \frac{1}{(n+j)!\,Q_{n+j}} = \frac{1}{n!\,Q_n} \sum_{j=1}^{\infty} \frac{n!\,Q_n}{(n+j)!\,Q_{n+j}}.$$

Since

$$\frac{Q_n}{Q_{n+j}} \leqq 1 \quad \text{and} \quad \frac{n!}{(n+j)!} \leqq \frac{1}{(n+1)^j}$$

for all j and for $j > 1$ further

$$\frac{n!}{(n+j)!} < \frac{1}{(n+1)^j}$$

(see e. g. [1]), we get

$$\sum_{j=1}^{\infty} \frac{n!\,Q_n}{(n+j)!\,Q_{n+j}} < \sum_{j=1}^{\infty} \frac{1}{(n+1)^j} = \frac{1}{n}\,.$$

Collecting the above inequalities, we deduce for the residues

$$\sum_{j=n+1}^{\infty} \frac{z_j}{j!\,Q_j} < \frac{1}{n!\,Q_n} \cdot \frac{K}{n},$$

and therefore

$$s_n < a < s_n + \frac{1}{n!\,Q_n} \cdot \frac{K}{n}.$$

This yields—as usual—the desired result.

Finally, we will discuss hypergeometric series that in many instances are of one of the types of Examples 3.3–3.4. By

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| \, x \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \cdot \frac{x^j}{j!}$$

we denote as usual the generalized hypergeometric series, where

$$(a)_j := a(a+1)\cdots(a+j-1)$$

is the Pochhammer symbol or shifted factorial. Let ${}_pF_q^\star := {}_pF_q - 1$. This series starts with $j = 1$ (like the series considered in our article). Obviously a series ${}_pF_q^\star$ is irrational if and only if the corresponding ${}_pF_q$ is.

Example 3.5 In this example, we consider hypergeometric series representing irrational values.

1. Let $m \in \mathbb{N}$. We consider
$$ {}_0F_1^\star\left(\begin{array}{c|c} - & \\ & 1 \\ m & \end{array}\right) = \sum_{j=1}^{\infty} \frac{1}{(m)_j\, j!}. $$
Example 3.3 with $q_j := m + j - 1$ and therefore $Q_j := (m)_j$ shows that ${}_0F_1^\star(m; 1)$ and therefore ${}_0F_1(m; 1)$ is irrational. The modified Bessel function is of this type. Analogously ${}_0F_1(m; \frac{1}{k})$ is irrational for every $k \in \mathbb{N}$. Furthermore, a similar argument shows that ${}_0F_q(m; \frac{1}{k})$ is irrational for every $m, q, k \in \mathbb{N}$.

2. The hyperbolic functions
$$ \cosh x = \sum_{j=0}^{\infty} \frac{x^{2k}}{(2k)!} \quad \text{and} \quad \sinh x = \sum_{j=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} $$
are given as power series whose every second Taylor coefficient vanishes. Here Example 3.4 applies, and the values $\cosh(\frac{1}{k})$ and $\sinh(\frac{1}{k})$ are irrational whenever $k \in \mathbb{N}$.

References

[1] K. Knopp. *Theorie und Anwendung der unendlichen Reihen*. Springer, fifth edition, 1964.

[2] W. Koepf. *Hypergeometric Summation*. Vieweg, 1998.

[3] W. Koepf, D. Schmersau. Irrationality of certain infinite series. *Analysis* 30, 2010, 27–34.

[4] W. Rautenberg. *Elementare Grundlagen der Analysis*. BI Wissenschaftsverlag, 1993.

[5] D. Schmersau, W. Koepf. *Die reellen Zahlen als Fundament und Baustein der Analysis*. Oldenbourg, 2000.

[6] J. Sondow. Criteria for irrationality of Euler's constant. *Proc. Amer. Math. Soc.* 131, 2003, 3335–3344.

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