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Close-to-convex Functions, Univalence Criteria and Quasiconformal Extension

Funkcje prawie wypukłe, kryteria jednolistności i przedłużenie quasikonforemne

Почти выпуклые функции и признаки однолистности и квазиконформное продолжение

Let S denote the family of univalent functions f of the unit disk D, normalized by

$$(1) \qquad f(z) = z + a_2 z^2 + a_3 z^3 + \dots \quad .$$

Let K denote the subset of convex functions, i.e. functions that have a convex range. A function f, normalized by (1), is called close-to-convex of order β, $\beta \geqslant 0$, if there is a convex function φ such that $|\arg(e^{i\alpha} f'(z)/\varphi'(z)| \leqslant \beta \frac{\pi}{2}$ for some $\alpha \in R$. Let $C(\beta)$ denote the family of close-to-convex functions of order β. For $\beta \leqslant 1$ it turns out that a function is close-to-convex of order β, if and only if it maps D univalently onto a domain whose complement E is the union of rays which are pairwise disjoint up to their tips, such that every ray is the bisector of a sector of angle

$(1-\beta)\pi$ which wholly lies in E (see [9], p.176) . Obviously $C(0)$ equals K .

Nehari showed that if a locally univalent function f satisfies a condition of the form

$$(2) \qquad (1-|z|^2)^2 \left| \left(\frac{f''}{f'}\right)'(z) - \left(\frac{f''}{f'}\right)^2(z) \right| \leqslant \sigma$$

with $\sigma = 2$, then f is univalent [7]. Later he showed that each convex function satisfies the univalence criterion [8]. The next Theorem gives a generalized version of this result for close-to-convex functions.

Theorem 1. Let $\beta \geqslant 0$ and $f \in C(\beta)$. Then f satisfies a Nehari type condition (2) with

$$(3) \qquad \sigma = \begin{cases} 2 + 4\beta & \text{if} \quad \beta \leqslant 1 \\ 2\beta^2 + 4\beta & \text{if} \quad \beta \geqslant 1 \end{cases} ,$$

and this result is sharp.

We indicate the idea of the proof, which is in [4]. The first step is to show the result at the origin, i.e.

$$(4) \qquad |a_3 - a_2^2| \leqslant \sigma/6 ,$$

where σ is defined by (3). Therefore one uses the representation

$$f'(z) = e^{-i\alpha}\,\varphi'(z)(\cos\alpha \cdot p(z) + i\sin\alpha)^{\beta}$$

with some $\varphi(z) = z + \varphi_2 z^2 + \varphi_3 z^3 + \ldots \in K$,
$p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ with positive real part and
$\cos\alpha > 0$. With aid of the estimates

$$|\varphi_3 - \varphi_2^2| \leqslant \frac{1}{3}(1 - |\varphi_2|^2)$$

[11] and

$$|p_2 - \tfrac{1}{2}p_1^2| \leqslant 2 - \tfrac{1}{2}|p_1|^2$$

(see e.g. [10], p.166, formula (10)) one gets using the triangle inequality

(5)
$$3|a_3 - a_2^2| \leqslant 1 - |\varphi_2|^2 + \beta\cos\alpha\left(2 - \frac{|p_1|^2}{2}\left(1 - \sqrt{1 - \cos^2\alpha\left(1 - \frac{\beta^2}{4}\right)}\right)\right) + \beta\cos\alpha\,|p_1|\,|\varphi_2| .$$

Now it is well known that $|p_1| \in [0, 2]$, $|\varphi_2| \in [0, 1]$ and $\cos\alpha \in [0, 1]$. A careful analysis shows that the right hand side of (5) is maximized at $(|p_1|, |\varphi_2|, \cos\alpha) = (0, 0, 1)$ if $\beta \in [0, 1]$ and at $(|p_1|, |\varphi_2|, \cos\alpha) = (2, 1, 1)$ if $\beta \geqslant 1$ which gives (4).

Now the linear-invariance of $C(\beta)$ is used (see e.g. [3]). Composing with an automorphism of D the information at the origin is transported to an arbitrary point $z \in D$, which finishes the proof. The sharpness is easily verified.

Becker showed that if a locally univalent function f

satisfies a condition of the form

$$(6) \qquad (1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right| \leqslant \lambda$$

with $\lambda = 1$, then f is univalent [1]. The Becker univalence criterion is often used to construct exotic univalent functions with "bad behaviour". In most cases these examples are gap series. We show conversely that m-fold symmetric close-to-convex functions with large m fulfill the Becker univalence criterion. A function f is called m-fold symmetric if it has the special form

$$f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \ldots \quad .$$

Theorem 2.

(a) Let $f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \ldots \in K$. Then f satisfies a Becker type condition (6) with $\lambda = 4/m$. In particular, if $m \geqslant 4$, then f fulfills the Becker univalence criterion.

(b) Let $\beta \in (0, 1)$ and let f be an m-fold symmetric close-to-convex function of order β. Then f satisfies a Becker type condition (6) with $\lambda = 4/m + 2\beta$. In particular, if $\beta < 1/2$ and $m \geqslant 4/(1-2\beta)$, then f fulfills the Becker univalence criterion. Moreover, the results are sharp.

The proof of Theorem 2 is in [5]. The idea for (a) is to use a subordination result for convex functions of the given form, namely $f' \prec (1-z)^{-2/m}$. The statement $f \prec g$ means that $f = g \circ \omega$, where ω satisfies the hypotheses of Schwarz's Lemma. It implies that

$$(7) \qquad \sup_{z \in D} (1 - |z|^2)|f'(z)| \leqslant \sup_{z \in D} (1 - |z|^2)|g'(z)|$$

(see e.g. [10], p.35, formula (4)), which is used to deduce (a). This result and a further application of (7) leads to (b). The sharpness is again easily verified.

From Theorem 2 one could deduce a condition which guarantees that an m-fold symmetric close-to-convex function of order β has a quasiconformal extension. A more precise result is

Theorem 3. Let $\beta \in [0, 1)$ and f be an m-fold symmetric close-to-convex function of order β .
(a) If $m > 2/(1-\beta)$, then f has a rectifiable boundary;
(b) if $m > 4/(1-\beta)$, then f has a quasiconformal extension.

This result is in [6]. For to prove (a) one uses an estimate on the integral means of the derivative of m-fold symmetric close-to-convex functions of order β ([6], Theorem 5), and for (b) a general condition, which implies the existence of a quasiconformal extension for Bazilevič functions [2], gives the result.

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STRESZCZENIE

Niech $C(\beta)$ będzie rodziną funkcji prawie wypukłych rzędu $\beta \in (0;\ 1)$. Znaleziono dokładną wartość normy szwarcjanu $\|S_f\| = \sigma$ dla $f \in C(\beta)$ (tw. 1). Wykazano również, że m-symetryczna funkcja f klasy $C(\beta)$ spełnia warunek jednolistności Beckera, jeśli $\beta < 1/2$ oraz $m > 4/(1-2\beta)$. (tw. 2). Ponadto, jeśli $m > 4/(1-\beta)$, to f ma przedłużenie quasikonforemne na całą płaszczyznę (tw. 3).

РЕЗЮМЕ

Пусть $C(\beta)$ семейство почти выпуклых функций порядка $\beta \in (0;\ 1)$. Найдена точная оценка нормы шварциана $\|S_f\| = \sigma$ для $f \in C(\beta)$ (теор. 1). Доказано тоже, что m-симметрическая функция $f \in C(\beta)$ исполняет признак Бекера для $\beta < \frac{1}{2}$, m $4/(1-2\beta)$ (теор. 2). Кроме того, если m $4/(1-\beta)$, тогда f квасиконформно продолжима на целую плоскость (теор. 3).