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Factorization of fourth-order differential equations for perturbed classical orthogonal polynomials

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Abstract

We factorize the fourth-order differential equations satisfied by the Laguerre–Hahn orthogonal polynomials obtained from some perturbations of classical orthogonal polynomials such as: the rth associated (for generic r), the general co-recursive, the general co-recursive associated, the general co-dilated and the general co-modified classical orthogonal polynomials. Moreover, we find four linearly independent solutions of the fourth-order differential equations, and show that the factorization obtained for modifications of classical orthogonal polynomials is still valid, with some minor changes when the polynomial family modified is semi-classical. Finally, we extend the validity of the results obtained for the associated classical orthogonal polynomials with integer order of association from integers to reals.

MSC: primary 33C45; secondary 33C47

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1. Introduction

Let $\mathscr{U}$ be a regular linear functional [8] on the linear space $\mathscr{P}$ of polynomials with real coefficients and $(P_n)_n$ a sequence of monic polynomials, orthogonal with respect to $\mathscr{U}$, i.e.,

(i) $P_n(x) = x^n +$ lower degree terms,

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(ii) $\langle \mathscr{U}, P_n\ P_m\rangle = k_n\delta_{n,m}, k_n \neq 0,\ n\in\mathbb{N}$,

where $\mathbb{N}=\{0,1,\ldots\}$ denotes the set of nonnegative integers. Here, $\langle\cdot,\cdot\rangle$ means the duality bracket and $\delta_{n,m}$ the Kronecker symbol.

$(P_n)_n$ satisfies a three-term recurrence equation

$$P_{n+1}(x)=(x-\beta_n)P_n(x)-\gamma_nP_{n-1}(x),\quad n\geqslant 1 \tag{1}$$

with the initial conditions

$$P_{-1}(x)=0,\quad P_0(x)=1, \tag{2}$$

where β_n and γ_n are real numbers with $\gamma_n\neq 0$, $\forall n\in\mathbb{N}_{>0}$, and $\mathbb{N}_{>0}$ denotes the set $\mathbb{N}_{>0}=\{1,2,\ldots\}$.

When the polynomial sequence $(P_n)_n$ is classical [28], i.e., orthogonal with respect to a positive weight function ρ (defined on the interval (a,b)) satisfying the first-order differential equation called Pearson equation:

$$(\sigma\rho)'=\tau\rho \tag{3}$$

with

$$x^n\sigma(x)\rho(x)\big|_{x=a}^{x=b}=0,\quad \forall n\in\mathbb{N}, \tag{4}$$

where σ is a polynomial of degree at most two and τ a first-degree polynomial, each P_n satisfies the differential equation

$$\mathbb{L}_n(y(x))=\sigma(x)y''(x)+\tau(x)y'(x)+\lambda_n y(x)=0 \tag{5}$$

and the orthogonality condition (ii) reads as

$$\int_a^b \rho(x)P_n(x)P_m(x)\,\mathrm{d}x=k_n\delta_{n,m},\quad k_n\neq 0.$$

The coefficients β_n, γ_n and λ_n are given by [18,19]

$$\lambda_n=-\frac{n}{2}((n-1)\sigma''+2\tau')=-n((n-1)\sigma_2+\tau_1),$$

$$\beta_n=\frac{-2\sigma_2n^2\sigma_1-2\tau_1\sigma_1n+2\sigma_2\sigma_1n+2\sigma_2\tau_0-\tau_1\tau_0}{(2\sigma_2n-2\sigma_2+\tau_1)(2\sigma_2n+\tau_1)},$$

$$\begin{aligned}\gamma_n={}&-n(\tau_1-2\sigma_2+\sigma_2n)(4\sigma_2^2\sigma_0n^2-8\sigma_2^2\sigma_0n+4\sigma_2^2\sigma_0-\sigma_1^2n^2\sigma_2+2\sigma_1^2n\sigma_2\\&+4\sigma_2\sigma_0n\tau_1-4\sigma_2\sigma_0\tau_1+\sigma_2\tau_0^2-\sigma_2\sigma_1^2-\sigma_1^2n\tau_1-\sigma_1\tau_1\tau_0+\sigma_0\tau_1^2+\sigma_1^2\tau_1)/\\&(2\sigma_2n-2\sigma_2+\tau_1)^2(2\sigma_2n-\sigma_2+\tau_1)(2\sigma_2n-3\sigma_2+\tau_1),\end{aligned} \tag{6}$$

where

$$\sigma(x)=\sigma_2x^2+\sigma_1x+\sigma_0,\quad \tau(x)=\tau_1x+\tau_0$$

with

$$|\tau_1|(|\sigma_2|+|\sigma_1|+|\sigma_0|)\neq 0.$$

The classical families are Jacobi, Laguerre and Hermite orthogonal polynomials [28].

Some modifications of Eq. (1) lead to new families of orthogonal polynomials such as the rth associated (for generic r), the general co-recursive, the general co-recursive associated, the general co-dilated and the general co-modified classical orthogonal polynomials [23]. These new families of orthogonal polynomials satisfy a common fourth-order linear homogeneous differential equation with polynomial coefficients of bounded degree. In general, they cannot satisfy a common second-order linear homogeneous differential equation with polynomial coefficients of bounded degree. Therefore, these new polynomials are not semi-classical but belong to the Laguerre–Hahn class (see Section 2). Many works have been devoted to the derivation of these fourth-order differential equations. Their polynomial coefficients have been given explicitly in [4,5,14,15,29,36,38] for the rth associated classical orthogonal polynomials.

In 1994, using symbolic computation, the coefficients of the fourth-order differential equation for the co-recursive associated Laguerre and Jacobi orthogonal polynomials were given [20]. Also, in [32], general fourth-order differential equation for the generalized co-recursive of all classical orthogonal polynomials was given for any (but fixed) level of recursivity using symbolic computation software.

Despite the fact that apart from the rth associated orthogonal polynomials, the coefficients of the fourth-order differential equation satisfied by the perturbed classical orthogonal polynomials require heavy computations for being very large, we have succeeded in factorizing these fourth-order differential equations and also finding a basis of four linearly independent solutions of all the perturbed systems of classical orthogonal polynomials considered. In Ref. [13], we succeeded also to factorize the equivalent fourth-order difference equation corresponding to the discrete case for which the basic Eqs. (3) and (5) are difference equations of the same order, instead of differential equations. Moreover, we have found interesting relations between the perturbed polynomials, the starting ones and the functions of the second kind (see Section 2.2 for the definition).

In Section 2, we recall definitions and known results needed for this work. Section 3 is devoted to the derivation and the factorization of the fourth-order differential equation. In Section 4, we solve differential equations and represent perturbed classical orthogonal polynomials in terms of solutions of second-order differential equations. In Section 5, we first give asymptotic representation of solutions of the fourth-order differential equation for the rth associated classical orthogonal polynomials; secondly, we extend the results obtained for the rth associated orthogonal polynomials with integer order of association from integers to reals. Finally, we solve a family of second-order differential equations and prove that the factorization obtained for modifications of classical orthogonal polynomials is still valid with some minor changes, when the polynomial family modified is semi-classical (see the next section for the definition).

2. Preliminaries and notations

In this section, we first define the semi-classical and the Laguerre–Hahn class of a given family of orthogonal polynomials. Next, we present the families of rth associated, generalized co-recursive, generalized co-dilated and generalized co-modified orthogonal polynomials, and give relations between new sequences and the starting ones.

Each regular linear functional $\mathscr{U}$ generates a so-called Stieltjes function S of $\mathscr{U}$ defined by

$$S(z) = -\sum_{n\geqslant 0} \frac{\langle \mathscr{U}, x^n\rangle}{z^{n+1}}, \tag{7}$$

where $\langle \mathscr{U}, x^n\rangle$ are the moments of the functional $\mathscr{U}$. The linear functional $\mathscr{U}$ satisfies in general a simple functional equation living in $\mathscr{P}'$, the dual space of $\mathscr{P}$. Appropriate definitions of $(\mathrm{d}/\mathrm{d}x)(\mathscr{U})$ and $P\mathscr{U}$, where P is a polynomial allow to build a simple differential equation for the functional which generalize in some way the Pearson equation for the weight ρ [17] (see also [25,26]).

If the Stieltjes function $S(x)$ satisfies a first-order linear differential equation of the form

$$\phi(x)\, S'(x) = C(x)S(x) + D(x), \tag{8}$$

where ϕ, C and D are polynomials, the functional $\mathscr{U}$ satisfies in $\mathscr{P}'$ a first-order differential equation with polynomial coefficients. In this case, the functional $\mathscr{U}$ and the corresponding orthogonal polynomial sequence $(P_n)_n$ belong to the semi-classical class (and are therefore called semi-classical) which includes the classical families [3,17,25,26].

Each semi-classical orthogonal polynomial sequence $(P_n)_n$ satisfies a common second-order differential equation [17] (see also [25]).

$$\mathbb{M}_n(y(x)) = I_2(x,n)y''(x) + I_1(x,n)y'(x) + I_0(x,n)y(x) = 0, \tag{9}$$

where $I_i(x,n)$ are polynomials in x of degree not depending on n.

An important class, larger than the semi-classical one, appears when the Stieltjes function satisfies a Riccati differential equation [9,11,22]

$$\phi S' = BS^2 + CS + D, \tag{10}$$

where $\phi \neq 0, B, C$ and D are polynomials. The corresponding functional $\mathscr{U}$ satisfies then a complicated quadratic differential equation in $\mathscr{P}'$. $\mathscr{U}$ and the corresponding orthogonal polynomials families are said to belong to the Laguerre–Hahn class [9,11,22].

It is well known that any Laguerre–Hahn orthogonal polynomial sequence satisfies a common fourth-order differential equation of the form [9,11,22]

$$J_4(x,n)y^{(4)}(x) + J_3(x,n)y'''(x) + J_2(x,n)y''(x) + J_1(x,n)y'(x) + J_0(x,n)y(x) = 0,$$

where $J_i(x,n)$ are polynomials of degree not depending on n.

It is shown, from several works [9–11,23,24] that finite perturbations of the recurrence coefficients of any semi-classical family generate orthogonal polynomials belonging to the Laguerre–Hahn and therefore satisfy a fourth-order differential equation.

2.1. Perturbation of recurrence coefficients

Now we consider a sequence of polynomials $(P_n)_n$, orthogonal with respect to a regular linear functional $\mathscr{U}$, satisfying (1). Perturbations we will deal with are the following.

2.1.1. The rth associated orthogonal polynomials $(P_n^{(r)})_n$

Given $r \in \mathbb{N}$, the rth associated of the polynomials $(P_n)_n$, is a polynomial sequence denoted by $(P_n^{(r)})_n$ and defined by the recurrence equation (1) in which β_n and γ_n are replaced by β_{n+r} and γ_{n+r}, respectively,

$$P_{n+1}^{(r)}(x) = (x - \beta_{n+r})P_n^{(r)}(x) - \gamma_{n+r}P_{n-1}^{(r)}(x), \quad n \geqslant 1 \tag{11}$$

with the initial conditions

$$P_0^{(r)}(x) = 1, \quad P_1^{(r)}(x) = x - \beta_r. \tag{12}$$

The family $(P_n^{(r)})_n$, thanks to Favard's theorem [12], is orthogonal. It is related to the starting polynomials and its first associated by the relation [9]

$$P_n^{(r)}(x) = \frac{P_{r-1}(x)}{\Gamma_{r-1}} P_{n+r-1}^{(1)}(x) - \frac{P_{r-2}^{(1)}(x)}{\Gamma_{r-1}} P_{n+r}(x), \quad n \geqslant 0, \ r \geqslant 2, \tag{13}$$

where the sequence $(\Gamma_n)_n$ is defined by

$$\Gamma_n = \prod_{i=1}^{n} \gamma_i, \quad n \geqslant 1, \ \Gamma_0 \equiv 1. \tag{14}$$

2.1.2. The co-recursive $(P_n^{[\mu]})_n$ and the generalized co-recursive orthogonal polynomials $(P_n^{[k,\mu]})_n$

The co-recursive of the orthogonal polynomial sequence $(P_n)_n$, denoted by $(P_n^{[\mu]})_n$, was introduced for the first time by Chihara [7], as the family of polynomials generated by the recursion formula (1) in which β_0 is replaced by $\beta_0 + \mu$

$$P_{n+1}^{[\mu]}(x) = (x - \beta_n)P_n^{[\mu]}(x) - \gamma_n P_{n-1}^{[\mu]}(x), \quad n \geqslant 1 \tag{15}$$

with the initial conditions

$$P_0^{[\mu]}(x) = 1, \quad P_1^{[\mu]}(x) = x - \beta_0 - \mu, \tag{16}$$

where μ denotes a real number.

This notion was extended to the generalized co-recursive orthogonal polynomials in [9,10,30] by modifying the sequence $(\beta_n)_n$ at the level k. This yields an orthogonal polynomial sequence denoted by $(P_n^{[k,\mu]})_n$ and generated by the recursion formula

$$P_{n+1}^{[k,\mu]}(x) = (x - \beta_n^*)P_n^{[k,\mu]}(x) - \gamma_n P_{n-1}^{[k,\mu]}(x), \quad n \geqslant 1 \tag{17}$$

with the initial conditions

$$P_0^{[k,\mu]}(x) = 1, \quad P_1^{[k,\mu]}(x) = x - \beta_0^*, \tag{18}$$

where $\beta_n^* = \beta_n$ for $n \neq k$ and $\beta_k^* = \beta_k + \mu$.

The orthogonal polynomial sequence $(P_n^{[k,\mu]})_n$ is related to $(P_n)_n$ and it is associated by [23]

$$P_n^{[k,\mu]}(x) = P_n(x) - \mu P_k(x) P_{n-(k+1)}^{(k+1)}(x), \quad n \geqslant k+1,$$
$$P_n^{[k,\mu]}(x) = P_n(x), \quad n \leqslant k. \tag{19}$$

Use of (13) transforms the previous equations in

$$P_n^{[k,\mu]}(x) = -\frac{\mu P_k^2(x)}{\Gamma_k} P_{n-1}^{(1)}(x) + \left(1 + \frac{\mu P_k(x) P_{k-1}^{(1)}}{\Gamma_k}\right) P_n(x), \quad n \geqslant k+1,$$
$$P_n^{[k,\mu]}(x) = P_n(x), \quad n \leqslant k. \tag{20}$$

Obviously, we have the relations $P_n^{[0,\mu]} = P_n^{[\mu]}$, and $P_n^{[0]} = P_n$.

2.1.3. *The co-recursive associated $(P_n^{\{r,\mu\}})_n$ and the generalized co-recursive associated orthogonal polynomials $(P_n^{\{r,k,\mu\}})_n$*

The co-recursive associated as well as the generalized co-recursive associated of the orthogonal polynomial sequence $(P_n)_n$, denoted by $(P_n^{\{r,\mu\}})_n$ and $(P_n^{\{r,k,\mu\}})_n$, respectively, are, the co-recursive and the generalized co-recursive (with modification on β_k) of the associated $(P_n^{(r)})_n$ of $(P_n)_n$, respectively. Thanks to (19), they are related with $(P_n)_n$ and it is associated by

$$P_n^{\{r,0,\mu\}} = P_n^{\{r,\mu\}}$$

and

$$P_n^{\{r,k,\mu\}}(x) = P_n^{(r)}(x) - \mu P_k^{(r)}(x) P_{n-(k+1)}^{(r+k+1)}(x), \quad n \geqslant k+1,$$
$$P_n^{\{r,k,\mu\}}(x) = P_n^{(r)}(x), \quad n \leqslant k. \tag{21}$$

The generalized co-recursive associated orthogonal polynomials can also be expressed using (13) and (21) by

$$\begin{aligned} P_n^{\{r,k,\mu\}}(x) &= \left(\frac{P_{r-1}(x)}{\Gamma_{r-1}} - \frac{\mu P_{k+r}(x) P_k^{(r)}(x)}{\Gamma_{r+k}}\right) P_{n+r-1}^{(1)}(x) \\ &\quad - \left(\frac{P_{r-2}^{(1)}(x)}{\Gamma_{r-1}} - \frac{\mu P_{k+r-1}^{(1)}(x) P_k^{(r)}(x)}{\Gamma_{r+k}}\right) P_{n+r}(x), \quad n \geqslant k+1, \end{aligned}$$
$$P_n^{\{r,k,\mu\}}(x) = P_n(x)^{(r)}, \quad n \leqslant k. \tag{22}$$

2.1.4. *The co-dilated $(P_n^{|\lambda|})_n$ and the generalized co-dilated orthogonal polynomials $(P_n^{|k,\lambda|})_n$*

The co-dilated of the orthogonal polynomial sequence $(P_n)_n$, denoted by $(P_n^{|\lambda|})_n$, was introduced by Dini [9], as the family of polynomials generated by the recursion formula (1) in which γ_1, is replaced by $\lambda\gamma_1$, i.e.,

$$P_{n+1}^{|\lambda|}(x) = (x-\beta_n)P_n^{|\lambda|}(x) - \gamma_n P_{n-1}^{|\lambda|}(x), \quad n \geqslant 2 \tag{23}$$

with the initial conditions

$$P_0^{|\lambda|}(x) = 1, \quad P_1^{|\lambda|}(x) = x-\beta_0, \quad P_2^{|\lambda|}(x) = (x-\beta_0)(x-\beta_1) - \lambda\gamma_1, \tag{24}$$

where λ is a nonzero real number.

This notion was extended to the generalized co-dilated orthogonal polynomials in [10,30] by modifying the sequence $(\gamma_n)_n$ at the level k. This yields an orthogonal polynomial sequence denoted by $(P_n^{|k,\lambda|})_n$ and generated by the recurrence equation

$$P_{n+1}^{|k,\lambda|}(x) = (x-\beta_n)P_n^{|k,\lambda|}(x) - \gamma_n^* P_{n-1}^{|k,\lambda|}(x), \quad n \geqslant 1 \tag{25}$$

with the initial conditions

$$P_0^{|k,\lambda|}(x) = 1, \quad P_1^{|k,\lambda|}(x) = x-\beta_0, \tag{26}$$

where $\gamma_n^* = \gamma_n$ for $n \neq k$ and $\gamma_k^* = \lambda\gamma_k$.

The orthogonal polynomial sequence $(P_n^{|k,\lambda|})_n$ is related to $(P_n)_n$ and its associated by [23]

$$\begin{aligned} P_n^{|k,\lambda|}(x) &= P_n(x) + (1-\lambda)\gamma_k P_{k-1}(x) P_{n-(k+1)}^{(k+1)}(x), \quad n \geqslant k+1, \\ P_n^{|k,\lambda|}(x) &= P_n(x), \quad n \leqslant k. \end{aligned} \tag{27}$$

Use of (13) transforms the previous equation in

$$\begin{aligned} P_n^{|k,\lambda|}(x) &= \left(1 - \frac{(1-\lambda)P_{k-1}(x)P_{k-1}^{(1)}(x)}{\Gamma_{k-1}}\right) P_n(x) + \frac{(1-\lambda)P_{k-1}(x)P_k(x)}{\Gamma_{k-1}} P_{n-1}^{(1)}(x), \quad n \geqslant k+1, \\ P_n^{|k,\lambda|}(x) &= P_n(x), \quad n \leqslant k. \end{aligned} \tag{28}$$

For $k=1$ or $\lambda = 1$, we have

$$P_n^{|1,\lambda|} = P_n^{|\lambda|}, \quad P_n^{|k,1|} = P_n.$$

2.1.5. *The generalized co-modified orthogonal polynomials* $(P_n^{[k,\mu,\lambda]})_n$

New families of orthogonal polynomials can also be generated by modifying at the same time the sequences $(\beta_n)_n$ and $(\gamma_n)_n$ at the levels k and k', respectively. When $k = k'$, the new family obtained [23], denoted by $(P_n^{[k,\mu,\lambda]})_n$ is generated by the three-term recurrence relation

$$P_{n+1}^{[k,\mu,\lambda]}(x) = (x-\beta_n^*)P_n^{[k,\mu,\lambda]}(x) - \gamma_n^* P_{n-1}^{[k,\mu,\lambda]}(x), \quad n \geqslant 1 \tag{29}$$

with the initial conditions

$$P_0^{[k,\mu,\lambda]}(x) = 1, \quad P_1^{[k,\mu,\lambda]}(x) = x - \beta_0^*, \tag{30}$$

where $\beta_n^* = \beta_n$, $\gamma_n^* = \gamma_n$ for $n \neq k$ and $\beta_k^* = \beta_k + \mu$, $\gamma_k^* = \lambda\gamma_k$. This family is represented in terms of the starting polynomials and their associated ones by [23]

$$\begin{aligned} P_n^{[k,\mu,\lambda]}(x) &= P_n(x) + ((1-\lambda)\gamma_k P_{k-1}(x) - \mu P_k(x))P_{n-(k+1)}^{(k+1)}(x), \quad n \geqslant k+1, \\ P_n^{|k,\lambda|}(x) &= P_n(x), \quad n \leqslant k. \end{aligned} \tag{31}$$

The latter relation can also be written as

$$\begin{aligned} P_n^{[k,\mu,\lambda]}(x) = {} & \left(1 - \frac{(1-\lambda)P_{k-1}(x)P_{k-1}^{(1)}(x)}{\Gamma_{k-1}} + \frac{\mu P_k(x)P_{k-1}^{(1)}(x)}{\Gamma_k}\right)P_n(x) \\ & + \left(\frac{(1-\lambda)P_{k-1}(x)P_k(x)}{\Gamma_{k-1}} - \frac{\mu P_k^2(x)}{\Gamma_k}\right)P_{n-1}^{(1)}(x), \quad n \geqslant k+1, \\ P_n^{[k,\lambda]}(x) = {} & P_n(x), \quad n \leqslant k. \end{aligned} \tag{32}$$

2.2. Results on classical orthogonal polynomials

Next, we state the following lemmas which are essential for this work.

Lemma 1 (Ronveaux [29]). *Given a classical orthogonal polynomial sequence $(P_n)_n$ satisfying (5), the following relation holds*:

$$\mathbb{L}_n^*(P_{n-1}^{(1)}(x)) = (\sigma'' - 2\tau')P_n'(x), \tag{33}$$

where $\mathbb{L}_n^*$, which is the adjoint of $\mathbb{L}_n$ is given by

$$\mathbb{L}_n^* = \sigma\mathscr{D}^2 + (2\sigma' - \tau)\mathscr{D} + (\lambda_n + \sigma'' - \tau'). \tag{34}$$

It should be noticed that $\mathbb{L}_n$ and $\mathbb{L}_n^*$ are related by

$$\rho\mathbb{L}_n(y) = \mathbb{L}_n^*(\rho y), \quad \forall y, \tag{35}$$

where ρ is the weight function satisfying Eqs. (3) and (4).

Lemma 2 (Nikiforov and Uvarov [28]). (1) *Two linearly independent solutions of the differential equation*

$$\mathbb{L}_n(y(x)) = \sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0,$$

are P_n and Q_n, where $(P_n)_n$ is the polynomial sequence, orthogonal with respect to the weight function ρ defined on the interval (a,b), satisfying Eqs. (3) *and* (4). *The constant λ_n*

is given by

$$\lambda_n = -\frac{n}{2}((n-1)\sigma'' + 2\tau').$$

Q_n is the function of the second kind, defined by

$$Q_n(x) = \frac{1}{\rho(x)} \int_a^b \frac{\rho(s)P_n(s)}{s-x}\, \mathrm{d}s. \tag{36}$$

(2) *The polynomials P_n and the function Q_n are two linearly independent solutions of the recurrence equation* (1).

Notice that the representation of Q_n given above is valid for $x \notin [a,b]$. But this representation is still valid for $x \in [a,b]$ by analytic continuation [28] or by taking Cauchy's principal part in the integral of (36) [14]. P_n and Q_n are given for each classical situation in terms of hypergeometric functions (see Section 5).

3. Factorization of fourth-order differential operators

Given $(P_n)_n$ a classical orthogonal polynomial sequence, we consider in general all transformations which lead to new families of orthogonal polynomials denoted by $(\bar{P}_n)_n$ and are related to the starting sequence by

$$\bar{P}_n(x) = A_n(x)P^{(1)}_{n+k-1} + B_n(x)P_{n+k}, \quad n \geqslant k', \tag{37}$$

where A_n and B_n are polynomials of degree not depending on n, and $k, k' \in \mathbb{N}$. We have the following:

Theorem 1. (1) *The orthogonal polynomials $(\bar{P}_n)_{n\geqslant k'}$ satisfy a common fourth-order linear differential equation*

$$\begin{aligned}\mathbb{F}_n(y(x)) = {}& J_4(x,n)y''''(x) + J_3(x,n)y'''(x) + J_2(x,n)y''(x)\\ &+ J_1(x,n)y'(x) + J_0(x,n)y(x) = 0,\end{aligned} \tag{38}$$

where the coefficients J_i are polynomials in x, with degree not depending on n.

(2) *The operator $\mathbb{F}_n$ can be factored as product of two second-order linear differential operators $\mathbb{S}_n$ and $\mathbb{T}_n$*

$$\mathbb{F}_n = \mathbb{S}_n \mathbb{T}_n, \tag{39}$$

where the coefficients in $\mathbb{S}_n$ and $\mathbb{T}_n$ are polynomials of degree not depending on n.

Proof. In the first step, we solve Eq. (37) in terms of $P^{(1)}_{n+k-1}$

$$P^{(1)}_{n+k-1}(x) = \frac{\bar{P}_n(x) - B_n(x)P_{n+k}(x)}{A_n(x)} \tag{40}$$

and substitute the previous relation in Eq. (33) in which n is replaced by $n+k$. Then we use (5) (for P_{n+k}) to eliminate the term $P''_{n+k}(x)$ and get

$$\mathbb{M}_{n+k}(\bar{P}_n) = b_1 P'_{n+k} + b_0 P_{n+k}, \tag{41}$$

where b_i are rational functions and $\mathbb{M}_{n+k}$ a second-order linear operator given in terms of operator $\mathbb{L}^*_{n+k}$ (see (33)) by

$$\mathbb{M}_{n+k}(y) = A_n^3 \mathbb{L}^*_{n+k}\left(\frac{y}{A_n}\right). \tag{42}$$

Next, we take derivative in (41) and use again (5) to eliminate $P''_{n+k}(x)$, and get

$$[\mathbb{M}_{n+k}(\bar{P}_n)]' = c_1 P'_{n+k} + c_0 P_{n+k}. \tag{43}$$

We reiterate the same process using the previous equation and get

$$[\mathbb{M}_{n+k}(\bar{P}_n)]'' = d_1 P'_{n+k} + d_0 P_{n+k}, \tag{44}$$

where c_i and d_i are again rational functions.

The fourth-order differential equation is given in determinantal form from (41), (43) and (44)

$$\mathbb{F}_n(\bar{P}_n) = \begin{vmatrix} b_1 & b_0 & \mathbb{M}_{n+k}(\bar{P}_n) \\ c_1 & c_0 & [\mathbb{M}_{n+k}(\bar{P}_n)]' \\ d_1 & d_0 & [\mathbb{M}_{n+k}(\bar{P}_n)]'' \end{vmatrix} = 0. \tag{45}$$

The previous equation can be written as

$$\mathbb{F}_n(\bar{P}_n) = e_2[\mathbb{M}_{n+k}(\bar{P}_n)]'' + e_1[\mathbb{M}_{n+k}(\bar{P}_n)]' + e_0\mathbb{M}_{n+k}(\bar{P}_n) = [\mathbb{S}_n\mathbb{T}_n](\bar{P}_n) = 0, \tag{46}$$

where the second-order differential operators $\mathbb{S}_n$ and $\mathbb{T}_n$ are given by

$$\mathbb{S}_n = e_2\mathscr{D}^2 + e_1\mathscr{D} + e_0, \quad \mathbb{T}_n = \mathbb{M}_{n+k}. \tag{47}$$

We conclude the proof by noticing that after cancellation of the denominator in (45), the coefficients e_i are polynomials of degree not depending on n. □

It should be mentioned that the previous method was first developed in [4]. The more general situation considered in [23] gives the fourth-order differential equation for the orthogonal polynomial sequence in the form

$$\tilde{P}_n = P_n + QP^{(k+1)}_{n-(k+1)},$$

where Q is a polynomial of degree k and $(P_n)_n$ a semi-classical orthogonal polynomial sequence. The previous theorem (also valid for semi-classical orthogonal polynomials (see Section 5.5)) therefore extends the results given in [23]. In addition, we would like to mention that the factorization pointed out in the previous theorem (except some particular cases listed below) seems to be a new result and has many applications as will be shown later. This factorization was known for some particular cases: the case of the first associated classical orthogonal polynomials (see [29]), which is obviously a consequence of (33) and the case of co-recursive orthogonal polynomials given explicitly in [31].

In what follows, we will denote by $\mathbb{F}_n^{(r)}$, $\mathbb{F}_n^{[k,\mu]}$, $\mathbb{F}_n^{\{r,k,\mu\}}$ $\mathbb{F}_n^{|k,\lambda|}$ and $\mathbb{F}_n^{[k,\mu,\lambda]}$ the fourth-order differential operators (obtained after cancellation of common factors) for the rth associated, the generalized co-recursive, the generalized co-recursive associated, the generalized co-dilated, and the generalized co-modified orthogonal polynomials.

3.1. Some consequences

For the rth associated classical orthogonal polynomials $(P_n^{(r)})_n$, we have used the previous theorem and the representation given in (13) to compute the operators $\mathbb{S}_n$ and $\mathbb{T}_n$ using Maple 7 [27].

Proposition 1. *The two differential operator factors of the fourth-order differential operator for the rth associated classical orthogonal polynomials are*

$$\mathbb{S}_n^{(r)} = \sigma P_{r-1}\mathscr{D}^2 + [(\tau+\sigma')P_{r-1} - 2\sigma P'_{r-1}]\mathscr{D} + [(\tau' + \lambda_{n+r} - \lambda_{r-1})P_{r-1} - 2\tau P'_{r-1}], \tag{48}$$

$$\begin{aligned}\mathbb{T}_n^{(r)} = {}& \sigma P_{r-1}^2\mathscr{D}^2 - P_{r-1}[(\tau - 2\sigma')P_{r-1} + 2\sigma P'_{r-1}]\mathscr{D} \\ & + [2(\tau-\sigma')P_{r-1}P'_{r-1} + 2\sigma P'_{r-1}P'_{r-1} + (\sigma'' - \tau' + \lambda_{n+r} + \lambda_{r-1})P_{r-1}^2],\end{aligned} \tag{49}$$

where $r \in \mathbb{N}_{>0}$ and $(P_n)_n$ is the sequence of classical orthogonal polynomials satisfying (5).

Moreover, we have

$$\mathbb{S}_n^{(r)}\mathbb{T}_n^{(r)} = P_{r-1}^3\mathbb{F}_n^{(r)}, \tag{50}$$

where

$$\begin{aligned}\mathbb{F}_n^{(r)} = {}& \sigma^2\mathscr{D}^4 + 5\sigma\sigma'\mathscr{D}^3 + (6\sigma\sigma'' - 2\tau'\sigma + 2\tau\sigma' + 2\lambda_{n+r}\sigma + 2\lambda_{r-1}\sigma - \tau^2 + 3\sigma'^2)\mathscr{D}^2 \\ & + 3(\lambda_{r-1}\sigma' + \lambda_{n+r}\sigma' - \tau\tau' + \tau\sigma'' + \sigma'\sigma'')\mathscr{D} \\ & + [(\lambda_{n+r} - \lambda_{r-1})^2 + (\lambda_{n+r} + \lambda_{r-1})\sigma'' + \tau'\sigma'' - \tau'^2].\end{aligned} \tag{51}$$

Remark 1. It should be mentioned that the factorization pointed out in (50) for generic r was already known for the first associated ($r = 1$) [29]:

$$\mathbb{F}_n^{(1)} = (\sigma\mathscr{D}^2 + (\tau+\sigma')\mathscr{D} + \tau' + \lambda_{n+1})(\sigma\mathscr{D}^2 + (2\sigma' - \tau)\mathscr{D} + \sigma'' - \tau' + \lambda_{n+1}). \tag{52}$$

Eq. (51) gives a new representation of the fourth-order differential equation for the rth associated classical orthogonal polynomials in terms of σ, τ and λ_n and of course can be brought in the form of known results [36,5,38]. From this representation, we recover easily the simple form of the fourth-order differential equation for rth associated classical orthogonal polynomials, given in [21]

$$\mathbb{F}_n^{(r)} = \mathbb{F}_n^{(1)} + (1-r)[(n+r-2)\sigma'' + 2\tau'][2\sigma\mathscr{D}^2 + 3\sigma'\mathscr{D} - (n^2-1)\sigma''],$$

where $\mathbb{F}_n^{(1)}$ is given by (52).

Corollary 1. *The fourth-order differential operator can also be factorized as*

$$\tilde{\mathbb{S}}_n^{(r)}\tilde{\mathbb{T}}_n^{(r)} = Q_{r-1}^3\mathbb{F}_n^{(r)}, \tag{53}$$

where the operators $\tilde{\mathbb{S}}_n^{(r)}$ *and* $\tilde{\mathbb{T}}_n^{(r)}$ *are obtained from the operators* $\mathbb{S}_n^{(r)}$ *and* $\mathbb{T}_n^{(r)}$, *respectively by replacing the polynomials* P_{r-1} *with the function* Q_{r-1}, *i.e.*,

$$\tilde{\mathbb{S}}_n^{(r)} = \sigma Q_{r-1}\mathscr{D}^2 + [(\tau+\sigma')Q_{r-1} - 2\sigma Q'_{r-1}]\mathscr{D} + [(\tau' + \lambda_{n+r} - \lambda_{r-1})Q_{r-1} - 2\tau Q'_{r-1}], \tag{54}$$

$$\begin{aligned}\tilde{\mathbb{T}}_n^{(r)} = {} & \sigma Q_{r-1}^2\mathscr{D}^2 - Q_{r-1}[(\tau - 2\sigma')Q_{r-1} + 2\sigma Q'_{r-1}]\mathscr{D} \\ & + [2(\tau-\sigma')Q_{r-1}Q'_{r-1} + 2\sigma Q'_{r-1}Q'_{r-1} + (\sigma'' - \tau' + \lambda_{n+r} + \lambda_{r-1})Q_{r-1}^2].\end{aligned} \tag{55}$$

The proof is obtained by computation utilizing the fact that Q_n satisfies (5).

Proposition 2. *The operator* $\mathbb{T}_n$ *for the generalized co-recursive and co-dilated classical orthogonal polynomials* $(P_n^{[k,\mu]})_n$ *and* $(P_n^{|k,\lambda|})_n$ (*with* $k \geqslant 1$), *denoted*, *respectively*, *by* $\mathbb{T}_n^{[k,\mu]}$, $\mathbb{T}_n^{|k,\lambda|}$ *are obtained in the same way*:

$$\begin{aligned}\mathbb{T}_n^{[k,\mu]} = {} & \sigma P_k^2\mathscr{D}^2 - P_k[(\tau - 2\sigma')P_k + 4\sigma P'_k]\mathscr{D} \\ & + [4(\tau-\sigma')P_kP'_k + 6\sigma P'_kP'_k + (\lambda_n + 2\lambda_k + \sigma'' - \tau')P_k^2],\end{aligned} \tag{56}$$

$$\begin{aligned}\mathbb{T}_n^{|k,\lambda|} = {} & \sigma P_{k-1}^2P_k^2 - P_{k-1}P_k[2\sigma(P_{k-1}P_k)' + (\tau - 2\sigma')P_{k-1}P_k]\mathscr{D} \\ & + [(\lambda_{k-1} + \lambda_k + \lambda_n + \sigma'' - \tau')P_{k-1}^2P_k^2 + (\tau - \sigma')(P_{k-1}^2P_k^2)' \\ & + 2\sigma P'_{k-1}P'_{k-1}P_k^2 + 2\sigma P_{k-1}^2P'_kP'_k + 2\sigma P'_{k-1}P'_kP_{k-1}P_k].\end{aligned} \tag{57}$$

The operators $\mathbb{S}_n$ for the generalized co-recursive and co-dilated classical orthogonal polynomials are very large expressions; however, they can be obtained using the previous theorem and Eqs. (22) and (32). The same remark applies for the factors $\mathbb{S}_n$ and $\mathbb{T}_n$ of the fourth-order differential equation satisfied by the generalized co-recursive associated and co-modified classical orthogonal polynomials.

4. Solutions of the fourth-order differential equations

In the following, we solve the fourth-order differential equation satisfied by the five perturbations listed in Section 2 and represent the new families of orthogonal polynomials in terms of solutions of second-order differential equations.

Theorem 2. *Let* $(P_n)_n$ *be a classical orthogonal polynomial sequence*, $r \in \mathbb{N}_{>0}$ *and* $(P_n^{(r)})_n$ *the rth associated of* $(P_n)_n$. *Four linearly independent solutions of the differential equation*

$$\mathbb{F}_n^{(r)}(y) = 0 \tag{58}$$

satisfied by $(P_n^{(r)})_n$*, where* $\mathbb{F}_n^{(r)}$ *is given by* (51)*, are*:

$$\begin{aligned} A_n^{(r)}(x) &= \rho(x)P_{r-1}(x)P_{n+r}(x), \\ B_n^{(r)}(x) &= \rho(x)P_{r-1}(x)Q_{n+r}(x), \\ C_n^{(r)}(x) &= \rho(x)Q_{r-1}(x)P_{n+r}(x), \\ D_n^{(r)}(x) &= \rho(x)Q_{r-1}(x)Q_{n+r}(x), \end{aligned} \tag{59}$$

Q_n *denoting the function of second kind associated to* $(P_n)_n$ *which is defined by* (36).

Moreover, $P_n^{(r)}$ *is related to these solutions by*

$$\begin{aligned} P_n^{(r)}(x) &= \frac{B_n^{(r)}(x) - C_n^{(r)}(x)}{\gamma_0 \Gamma_{r-1}} \\ &= \frac{\rho(x)(P_{r-1}(x)Q_{n+r}(x) - Q_{r-1}(x)P_{n+r}(x))}{\gamma_0 \Gamma_{r-1}}, \quad \forall n \in \mathbb{N},\ \forall r \in \mathbb{N}_{>0}, \end{aligned} \tag{60}$$

where Γ_k *is given by* (14) *and* γ_0 *defined as*

$$\gamma_0 = \int_a^b \rho(x)\, \mathrm{d}x. \tag{61}$$

Proof. In the first step, we solve the differential equation

$$\mathbb{T}_n^{(r)}(y) = 0.$$

To do this, we use (35), (42) and (47) to get

$$\begin{aligned} \mathbb{T}_n^{(r)}(y) &= \mathbb{M}_{n+r}(y) \\ &= P_{r-1}^3 \mathbb{L}_{n+r}^* \left(\frac{y}{P_{r-1}} \right) \\ &= P_{r-1}^3 \rho \mathbb{L}_{n+r}(z), \end{aligned} \tag{62}$$

where $y = z\rho P_{r-1}$. Since the two linearly independent solutions of $\mathbb{L}_{n+r}(z) = 0$ are P_{n+r} and Q_{n+r} (see Lemma 2), the two linearly independent solutions of $\mathbb{T}_n^{(r)}(y) = 0$ (which are also solutions of (58) thanks to (50)) are

$$\begin{aligned} A_n^{(r)}(x) &= \rho(x)P_{r-1}(x)P_{n+r}(x), \\ B_n^{(r)}(x) &= \rho(x)P_{r-1}(x)Q_{n+r}(x). \end{aligned} \tag{63}$$

Use of (55) utilizing the fact that the weight function ρ and the function Q_n satisfy (3) and (5), respectively, leads to

$$\tilde{\mathbb{T}}_n^{(r)}(y) = Q_{r-1}^3 \rho \mathbb{L}_{n+r}(z), \tag{64}$$

where $y = z\rho Q_{r-1}$. Eq. (64) permits us to conclude that the two independent solutions of $\tilde{\mathbb{T}}_n^{(r)}(y) = 0$ (which are also solutions of (58) thanks to (53)) are given by

$$C_n^{(r)}(x) = \rho(x)Q_{r-1}(x)P_{n+r}(x),$$

$$D_n^{(r)}(x) = \rho(x)Q_{r-1}(x)Q_{n+r}(x).$$

The four solutions of (58) obtained are linearly independent since P_n and Q_n are two linearly independent solutions of (5) and have different asymptotic behaviour (see Section 5.1).

It should be mentioned that computations with Maple 7 using the fact that P_n and Q_n satisfy (5), confirm that the functions $A_n^{(r)}$, $B_n^{(r)}$, $C_n^{(r)}$ and $D_n^{(r)}$ satisfy (58). Also, notice that the structure of the solutions of Eq. (58) given by (59) was suggested by Hahn [16].

To prove (60) one has to remark that since $(P_n)_n$ and $(Q_n)_n$ satisfy (1), each solution given in (59) satisfies the recurrence equation

$$X_{n+1} = (x - \beta_{n+r})X_n - \gamma_{n+r}X_{n-1}, \quad n \geqslant 1. \tag{65}$$

Therefore, the function $X_n^{(r)}$ defined by

$$X_n^{(r)}(x) = \frac{\rho(x)(P_{r-1}(x)Q_{n+r}(x) - Q_{r-1}(x)P_{n+r}(x))}{\gamma_0\Gamma_{r-1}}, \quad r \geqslant 1,$$

fulfills (65). It remains to prove that the initial values are $X_0^{(r)} = 1, X_1^{(r)} = x - \beta_r$. We have

$$\begin{aligned}
X_0^{(r+1)} &= \frac{\rho(x)(P_r(x)Q_{r+1}(x) - Q_r(x)P_{r+1}(x))}{\gamma_0\Gamma_r} \\
&= \frac{\rho(x)(P_r(x)[(x-\beta_r)Q_r(x) - \gamma_r Q_{r-1}(x)] - Q_r(x)[(x-\beta_r)P_r(x) - \gamma_r P_{r-1}(x)])}{\gamma_0\Gamma_r} \\
&= \frac{\rho(x)(P_{r-1}(x)Q_r(x) - Q_{r-1}(x)P_r(x))}{\gamma_0\Gamma_{r-1}} \\
&= X_0^{(r)}.
\end{aligned}$$

We deduce that

$$X_0^{(r)} = X_0^{(1)}, \quad r \geqslant 1.$$

A computation using (36) and (61) gives

$$\begin{aligned}
X_0^{(1)} &= \frac{\rho(x)(P_0(x)Q_1(x) - Q_0(x)P_1(x))}{\gamma_0} \\
&= \frac{1}{\gamma_0}\int_a^b \frac{(s-\beta_0)\rho(s)\,\mathrm{d}s}{s-x} - \frac{(x-\beta_0)}{\gamma_0}\int_a^b \frac{\rho(s)\,\mathrm{d}s}{s-x} \\
&= 1.
\end{aligned}$$

Therefore,

$$X_0^{(r)} = 1, \quad r \geqslant 1.$$

Use of (1) for P_{r+1} and Q_{r+1} and the previous equation gives

$$X_1^{(r)} = (x - \beta_r)X_0^{(r)} = x - \beta_r.$$

We therefore conclude that

$$X_n^{(r)} = P_n^{(r)}, \quad r \geqslant 1, \ n \geqslant 0,$$

since $(P_n^{(r)})_n$ is the unique solution of recurrence equation (65) with the initial conditions

$$P_0^{(r)} = 1, \quad P_1^{(r)} = x - \beta_r. \qquad \square$$

Remark 2. The results of the previous theorem are still valid if we replace P_n and Q_n in Eqs. (59) and (60) by two other linearly independent solutions of Eqs. (1) and (5). In fact, the structure of the solutions given in (59) remains the same. The same remark applies for (60) except that the denominator in (60) may be a different constant (with respect to x) factor.

Theorem 3. *Let $(P_n)_n$ be a classical orthogonal polynomial sequence, $k \in \mathbb{N}$ and $(P_n^{[k,\mu]})_n$ the generalized co-recursive of $(P_n)_n$. Four linearly independent solutions of the differential equation*

$$\mathbb{F}_n^{[k,\mu]}(y) = 0, \quad n \geqslant k+1 \tag{66}$$

satisfied by $(P_n^{[k,\mu]})_n$, are (with $n \geqslant k+1$)

$$\begin{aligned} A_n^{[k,\mu]}(x) &= \rho(x)P_k^2(x)P_n(x), \\ B_n^{[k,\mu]}(x) &= \rho(x)P_k^2(x)Q_n(x), \\ C_n^{[k,\mu]}(x) &= [\gamma_0\Gamma_k + \mu\rho(x)P_k(x)Q_k(x)]P_n(x), \\ D_n^{[k,\mu]}(x) &= [\gamma_0\Gamma_k + \mu\rho(x)P_k(x)Q_k(x)]Q_n(x), \end{aligned} \tag{67}$$

where Q_n is the function of second kind associated to $(P_n)_n$ defined by (36).

Moreover, $P_n^{[k,\mu]}$ is related to these solutions by

$$P_n^{[k,\mu]} = \frac{[\gamma_0\Gamma_k + \mu\rho(x)P_k(x)Q_k(x)]P_n(x) - \mu\rho(x)P_k^2(x)Q_n(x)}{\gamma_0\Gamma_k}, \quad k \geqslant 0, \ n \geqslant k+1. \tag{68}$$

Proof. By analogy with the proof of Theorem 2, we show using (20), (42) and (47) that

$$\mathbb{T}_n^{[k,\mu]}(y) = \rho P_k^6 \mathbb{L}_n(z),$$

where $\mathbb{T}_n^{[k,\mu]}$ is given by (56) and $y = z\rho P_k^2$. Therefore, $A_n^{[k,\mu]}$ and $B_n^{[k,\mu]}$ given by

$$A_n^{[k,\mu]}(x) = \rho(x)P_k^2(x)P_n(x), \quad B_n^{[k,\mu]}(x) = \rho(x)P_k^2(x)Q_n(x),$$

are two linearly independent solutions of

$$\mathbb{T}_n^{[k,\mu]}(y) = 0.$$

Next, we use (19) and (60) and get

$$P_n^{[k,\mu]} = \frac{C_n^{[k,\mu]} - \mu B_n^{[k,\mu]}}{\gamma_0 \Gamma_k}, \quad n \geqslant k+1. \tag{69}$$

Since the generalized co-dilated polynomials $P_n^{[k,\mu]}$ and the function $B_n^{[k,\mu]}$ given by (67), are both solutions of the linear homogeneous differential equation

$$\mathbb{F}_n^{[k,\mu]}(y) = 0, \quad n \geqslant k+1,$$

it follows from (69) that the function $C_n^{[k,\mu]}$, given by (67), is also solution of the previous equation. Computations show that the function $D_n^{[k,\mu]}$, given by (67) is also solution of the previous differential equation. One can also prove that $D_n^{[k,\mu]}$ is solution of the previous differential equation by following the proof given in [13] for the discrete case.

To complete the proof, we notice that $A_n^{[k,\mu]}$, $B_n^{[k,\mu]}$, $C_n^{[k,\mu]}$ and $C_n^{[k,\mu]}$ are four linearly independent solutions of $\mathbb{F}_n^{[k,\mu]}(y) = 0$ since P_n and Q_n are two linearly independent solutions of (5) enjoying different asymptotic properties. □

In the following, we give the equivalent of the previous theorem for the co-dilated classical orthogonal polynomials. The proof is similar to the one of the previous theorem by using relations (27), (28), (42), (47), and (60).

Theorem 4. *Let $(P_n)_n$ be a classical orthogonal polynomial sequence, $k \in \mathbb{N}$ and $(P_n^{|k,\lambda|})_n$ the generalized co-dilated of $(P_n)_n$. Four linearly independent solutions of the differential equation*

$$\mathbb{F}_n^{|k,\lambda|}(y) = 0, \quad n \geqslant k+1, \tag{70}$$

satisfied by $(P_n^{|k,\lambda|})_n$ are (with $n \geqslant k+1$)

$$\begin{aligned}
A_n^{|k,\lambda|}(x) &= \rho(x) P_{k-1}(x) P_k(x) P_n(x),\\
B_n^{|k,\lambda|}(x) &= \rho(x) P_{k-1}(x) P_k(x) Q_n(x),\\
C_n^{|k,\lambda|}(x) &= [\gamma_0 \Gamma_k + (\lambda - 1)\gamma_k \rho(x) P_{k-1}(x) Q_k(x)] P_n(x),\\
D_n^{|k,\lambda|}(x) &= [\gamma_0 \Gamma_k + (\lambda - 1)\gamma_k \rho(x) P_{k-1}(x) Q_k(x)] Q_n(x).
\end{aligned} \tag{71}$$

The co-dilated $P_n^{|k,\lambda|}$ is related to these solutions by

$$P_n^{|k,\lambda|} = \frac{[\gamma_0 \Gamma_k + (\lambda-1)\gamma_k \rho(x) P_{k-1}(x) Q_k(x)] P_n(x) - (\lambda-1)\gamma_k \rho(x) P_{k-1}(x) P_k(x) Q_n(x)}{\gamma_0 \Gamma_k},$$
$$n \geqslant k+1. \tag{72}$$

We furthermore give the solutions for the generalized co-recursive associated and the generalized co-modified classical orthogonal polynomials. The proofs are similar to the previous ones.

Theorem 5. *Let $(P_n)_n$ be a classical orthogonal polynomial sequence, $k \in \mathbb{N}$, $r \in \mathbb{N}_{>0}$ and $(P_n^{\{r,k,\mu\}})_n$ the generalized co-recursive associated of $(P_n)_n$. Four linearly independent solutions of the differential equation*

$$\mathbb{F}_n^{\{r,k,\mu\}}(y) = 0, \quad n \geqslant k+1, \tag{73}$$

satisfied by $(P_n^{\{r,k,\mu\}})_n$ are (with $n \geqslant k+1$)

$$A_n^{\{r,k,\mu\}}(x) = (\gamma_0 \Gamma_{k+r} P_{r-1}(x) - \mu\rho(x) P_{k+r}(x)[P_{r-1}(x)Q_{k+r}(x) - Q_{r-1}(x)P_{k+r}(x)])\rho(x), P_{n+r}(x),$$

$$B_n^{\{r,k,\mu\}}(x) = (\gamma_0 \Gamma_{k+r} P_{r-1}(x) - \mu\rho(x) P_{k+r}(x)[P_{r-1}(x)Q_{k+r}(x) - Q_{r-1}(x)P_{k+r}(x)])\rho(x) Q_{n+r}(x),$$

$$C_n^{\{r,k,\mu\}}(x) = (\gamma_0 \Gamma_{k+r} Q_{r-1}(x) - \mu\rho(x) Q_{k+r}(x)[P_{r-1}(x)Q_{k+r}(x) - Q_{r-1}(x)P_{k+r}(x)])\rho(x) P_{n+r}(x),$$

$$D_n^{\{r,k,\mu\}}(x) = (\gamma_0 \Gamma_{k+r} Q_{r-1}(x) - \mu\rho(x) Q_{k+r}(x)[P_{r-1}(x)Q_{k+r}(x) - Q_{r-1}(x)P_{k+r}(x)])\rho(x) Q_{n+r}(x).$$

Moreover, $P_n^{\{r,k,\mu\}}$ is related to these solutions by

$$\begin{aligned} P_n^{\{r,k,\mu\}} &= \left(\frac{P_{r-1}(x)}{\gamma_0 \Gamma_{r-1}} - \frac{\mu\rho(x) P_{k+r}(x)[P_{r-1}(x)Q_{k+r}(x) - Q_{r-1}(x)P_{k+r}(x)]}{\gamma_0^2 \Gamma_{r-1} \Gamma_{k+r}} \right) \rho(x) Q_{n+r}(x) \\ &\quad - \left(\frac{Q_{r-1}(x)}{\gamma_0 \Gamma_{r-1}} - \frac{\mu\rho(x) Q_{k+r}(x)[P_{r-1}(x)Q_{k+r}(x) - Q_{r-1}(x)P_{k+r}(x)]}{\gamma_0^2 \Gamma_{r-1} \Gamma_{k+r}} \right) \rho(x) P_{n+r}(x), \\ &\quad r \geqslant 1, \; n \geqslant k+1. \end{aligned} \tag{74}$$

Theorem 6. *Let $(P_n)_n$ be a classical orthogonal polynomial sequence, $k \in \mathbb{N}$ and $(P_n^{[k,\mu,\lambda]})_n$ the generalized co-modified of $(P_n)_n$. Four linearly independent solutions of the differential equation*

$$\mathbb{F}_n^{[k,\mu,\lambda]}(y) = 0, \quad n \geqslant k+1, \tag{75}$$

satisfied by $(P_n^{[k,\mu,\lambda]})_n$ are (with $n \geqslant k+1$)

$$\begin{aligned} A_n^{[k,\mu,\lambda]}(x) &= [(\lambda-1)\gamma_k P_{k-1}(x) P_k(x) + \mu P_k^2(x)]\rho(x) P_n(x), \\ B_n^{[k,\mu,\lambda]}(x) &= [(\lambda-1)\gamma_k P_{k-1}(x) P_k(x) + \mu P_k^2(x)]\rho(x) Q_n(x), \\ C_n^{[k,\mu,\lambda]}(x) &= [\gamma_0 \Gamma_k + (\lambda-1)\gamma_k \rho(x) P_{k-1}(x) Q_k(x) + \mu\rho(x) P_k(x) Q_k(x)] P_n(x), \\ D_n^{[k,\mu,\lambda]}(x) &= [\gamma_0 \Gamma_k + (\lambda-1)\gamma_k \rho(x) P_{k-1}(x) Q_k(x) + \mu\rho(x) P_k(x) Q_k(x)] Q_n(x). \end{aligned} \tag{76}$$

The co-dilated $P_n^{[k,\mu,\lambda]}$ is related to these solutions by

$$\begin{aligned} P_n^{[k,\mu,\lambda]} &= \left(1 + \frac{(\lambda-1)\gamma_k \rho(x) P_{k-1}(x) Q_k(x) + \mu\rho(x) P_k(x) Q_k(x)}{\gamma_0 \Gamma_k} \right) P_n(x) \\ &\quad - \frac{(\lambda-1)\gamma_k \rho(x) P_{k-1}(x) P_k(x) + \mu\rho(x) P_k^2(x)}{\gamma_0 \Gamma_k} Q_n(x), \quad n \geqslant k+1. \end{aligned} \tag{77}$$

5. Applications

5.1. Asymptotic formulas for the four solutions

We use results given in Theorem 2 and the asymptotic formula for the function of the second kind (see [28, p. 98])

$$Q_n(x) = -\frac{\prod_{i=0}^{n} \gamma_i}{\rho(x) x^{n+1}} \left(1 + \mathrm{O}\left(\frac{1}{x}\right)\right)$$

to get the following formulas for the solutions given in relation (59):

Theorem 7.

$$\begin{aligned}
A_n^{(r)}(x) &= x^{n+2r-1}\rho(x)\left(1 + \mathrm{O}\left(\frac{1}{x}\right)\right),\\
B_n^{(r)}(x) &= -\frac{\prod_{i=0}^{n+r} \gamma_i}{x^{n+2}}\left(1 + \mathrm{O}\left(\frac{1}{x}\right)\right),\\
C_n^{(r)}(x) &= -x^n \prod_{i=0}^{r-1} \gamma_i \left(1 + \mathrm{O}\left(\frac{1}{x}\right)\right),\\
D_n^{(r)}(x) &= \frac{\prod_{i=0}^{r-1} \gamma_i \prod_{i=0}^{n+r} \gamma_i}{\rho(x) x^{n+2r+1}}\left(1 + \mathrm{O}\left(\frac{1}{x}\right)\right).
\end{aligned} \tag{78}$$

5.2. Hypergeometric representation of the solutions

We give for each classical situation a hypergeometric representation of the polynomials P_n, the function of the second kind Q_n, four linearly independent solutions of the differential equation for $P_n^{(r)}$ and relations between these solutions and the associated polynomials. In what follows, $(a)_k$ and ${}_pF_q$ denote the Pochhammer symbol and the generalized hypergeometric function, respectively, and are defined by

$$(a)_k = a(a+1)\cdots(a+k-1), \quad k \in \mathbb{N}, \ (a)_0 \equiv 1,$$

$${}_pF_q\left(\left.\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right| x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where p and q belong to $\mathbb{N}$, and x, a, a_i and b_i are complex numbers. The ${}_pF_q(x)$ is well defined if no b_i, $1 \leqslant i \leqslant q$ is a negative integer or zero and it constitutes a convergent series for all x if $p \leqslant q$, or if $p = q+1$ and $|x| < 1$.

5.2.1. Monic Jacobi polynomials $p_n^{(\alpha,\beta)}$

We denote by $p_n^{(\alpha,\beta)}$ the monic Jacobi polynomials and $q_n^{(\alpha,\beta)}$ the corresponding function of the second kind. The data are [28, p. 286]:

$$\sigma(x) = 1 - x^2, \quad \tau(x) = -(\alpha+\beta+2)x + \beta - \alpha, \quad \alpha > -1,\ \beta > -1,$$

$$\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad I = [-1, 1],$$

$$\lambda_n = n(n+\alpha+\beta+1),$$

$$\beta_n = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

$$\gamma_n = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)^2(2n+\alpha+\beta-1)},$$

$$p_n^{(\alpha,\beta)} = \frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\, {}_2F_1\left(\left.\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}\right| \frac{1-x}{2}\right),$$

$$q_n^{(\alpha,\beta)} = \frac{(-1)^n 2^{2n+\alpha+\beta+1} n!\,\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(1-x)^{n+\alpha+1}(1+x)^{\beta}(n+\alpha+\beta+1)_n\Gamma(2n+\alpha+\beta+2)}$$
$$\times {}_2F_1\left(\left.\begin{matrix} n+1, n+\alpha+1 \\ 2n+\alpha+\beta+2 \end{matrix}\right| \frac{2}{1-x}\right).$$

It should be mentioned that for $n = 0$ and $\alpha+\beta+1 = 0$, $q_0^{(\alpha,-1-\alpha)}$ is constant with respect to x. In this case, the nonconstant solution of Eq. (5) is given in [34, p. 75].

From Theorem 2 and the previous data, four linearly independent solutions of the fourth-order differential equation

$$\begin{aligned}
&(1-x^2)^2 y''''(x) - 10x(1-x^2)y'''(x) + (-8 + 2x\beta^2 - 2x\alpha^2 - \alpha^2 + 2n^2 - 2n\alpha x^2 - 2n\beta x^2 \\
&\quad - 2x^2\alpha\beta - 4nrx^2 - 4r\alpha x^2 - 4r\beta x^2 - 2nx^2 - x^2\alpha^2 - 4r^2x^2 - 2n^2x^2 \\
&\quad - x^2\beta^2 + 2n + 4\beta r + 4\alpha r + 24x^2 + 2n\alpha + 4nr + 2n\beta + 2\alpha\beta - \beta^2 + 4r^2)y''(x) \\
&\quad + (-12xr^2 - 3x\beta^2 - 12xnr - 6xn^2 - 6xn - 3x\alpha^2 - 12xr\beta - 6x\alpha\beta \\
&\quad - 6xn\beta - 6xn\alpha - 12xr\alpha + 12x - 3\alpha^2 + 3\beta^2)y'(x) \\
&\quad + n(2+n)(n+1+\alpha+\beta+2r)(n-1+\alpha+\beta+2r)y(x) = 0 \qquad (79)
\end{aligned}$$

satisfied by the rth associated Jacobi orthogonal polynomials [36] (see also [5]) are:

$$\begin{aligned}
A_{n,J}^{(r)} &= (1-x)^{\alpha}(1+x)^{\beta}{}_2F_1\left(\begin{matrix}1-r, r+\alpha+\beta\\ \alpha+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right)\\
&\quad\times{}_2F_1\left(\begin{matrix}-n-r, n+r+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right),\\
B_{n,J}^{(r)} &= (1-x)^{-n-r-1}{}_2F_1\left(\begin{matrix}1-r, r+\alpha+\beta\\ \alpha+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right){}_2F_1\left(\begin{matrix}n+r+1, n+r+\alpha+1\\ 2n+2r+\alpha+\beta+2\end{matrix}\,\middle|\,\frac{2}{1-x}\right),\\
C_{n,J}^{(r)} &= (1-x)^{-r}{}_2F_1\left(\begin{matrix}r, r+\alpha\\ 2r+\alpha+\beta\end{matrix}\,\middle|\,\frac{2}{1-x}\right){}_2F_1\left(\begin{matrix}-n-r, n+r+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right),\\
D_{n,J}^{(r)} &= \frac{(1-x)^{-n-2r-\alpha-1}}{(1+x)^{\beta}}{}_2F_1\left(\begin{matrix}r, r+\alpha\\ 2r+\alpha+\beta\end{matrix}\,\middle|\,\frac{2}{1-x}\right)\\
&\quad\times{}_2F_1\left(\begin{matrix}n+r+1, n+r+\alpha+1\\ 2n+2r+\alpha+\beta+2\end{matrix}\,\middle|\,\frac{2}{1-x}\right).
\end{aligned}\tag{80}$$

The use of (60), the previous data and the fact that

$$\prod_{i=0}^{r-1}\gamma_i = \frac{2^{2r+\alpha+\beta-1}(2r+\alpha+\beta-1)\Gamma(r)\Gamma(r+\alpha)\Gamma(r+\beta)\Gamma(r+\alpha+\beta)}{\Gamma(2r+\alpha+\beta)^2}, \quad r\in\mathbb{N}_{>0},$$

allow us to represent the associated Jacobi polynomials $P_n^{(r)}$ ($r\in\mathbb{N}$) in terms of the hypergeometric functions (see also [36]).

$$P_n^{(r)}(x) = \frac{(1-x)^{\alpha}(1+x)^{\beta}(p_{r-1}^{(\alpha,\beta)}(x)q_{n+r}^{(\alpha,\beta)}(x) - q_{r-1}^{(\alpha,\beta)}(x)p_{n+r}^{(\alpha,\beta)}(x))}{\prod_{i=0}^{r-1}\gamma_i}, \quad r\in\mathbb{N}_{>0}.$$

Remark 3. Note that $A_{n,J}^{(r)}$, $B_{n,J}^{(r)}$, $C_{n,J}^{(r)}$ and $D_{n,J}^{(r)}$ are multiples of the functions (59) in order to be as simple as possible. This applies also for the Laguerre and Hermite case below.

Since ${}_2F_1$ constitutes a convergent series for $|x|<1$, functional relation (see [28, p. 270])

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}\alpha, \beta\\ \gamma\end{matrix}\,\middle|\,x\right) &= \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-x)^{-\alpha}{}_2F_1\left(\begin{matrix}\alpha, 1+\alpha-\gamma\\ 1+\alpha-\beta\end{matrix}\,\middle|\,\frac{1}{x}\right)\\
&\quad+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-x)^{-\beta}{}_2F_1\left(\begin{matrix}\beta, 1+\beta-\gamma\\ 1+\beta-\alpha\end{matrix}\,\middle|\,\frac{1}{x}\right)
\end{aligned}\tag{81}$$

can be used in order to get for the functions $q_n^{\alpha,\beta}$, $A_{n,J}^{(r)}$, $B_{n,J}^{(r)}$, $C_{n,J}^{(r)}$ and $D_{n,J}^{(r)}$ given above a representation with convergent series expansion when $|x| > 1$.

5.2.2. Monic Laguerre polynomials l_n^α

We denote by l_n^α the monic Laguerre polynomials and q_n^α the corresponding function of the second kind. The data are [28, p. 286]:

$$\sigma(x) = x, \quad \tau(x) = \alpha + 1 - x, \quad \alpha > -1,$$

$$\rho(x) = x^\alpha e^{-x}, \quad I = [0, \infty),$$

$$\lambda_n = n, \quad \beta_n = 2n + 1 + \alpha, \quad \gamma_n = n(n + \alpha),$$

$$l_n^\alpha(x) = (-1)^n (\alpha + 1)_n \quad {}_1F_1\left(\left.\begin{matrix} -n \\ \alpha + 1 \end{matrix}\right| x\right),$$

$$q_n^\alpha(x) = (-1)^n n! e^{i\pi\alpha} \Gamma(n + \alpha + 1) e^x G(n + \alpha + 1, \alpha + 1, -x),$$

where $G(a, b, x)$ is the confluent hypergeometric function of the second kind (see [28, p. 272]) and related to the confluent hypergeometric function ${}_1F_1$ by

$$G(\alpha, \gamma, x) = \frac{\Gamma(1 - \gamma)}{\Gamma(\alpha - \gamma + 1)} {}_1F_1\left(\left.\begin{matrix} \alpha \\ \gamma \end{matrix}\right| x\right) + \frac{\Gamma(\gamma - 1)}{\Gamma(\alpha)} x^{1-\gamma} {}_1F_1\left(\left.\begin{matrix} \alpha - \gamma + 1 \\ 2 - \gamma \end{matrix}\right| x\right). \tag{82}$$

In case of convergence, the function $G(\alpha, \gamma, x)$ can also be represented by the hypergeometric function ${}_2F_0$ (see [1, Chapter 13]) as

$$G(\alpha, \gamma, x) = x^{-\alpha} {}_2F_0\left(\left.\begin{matrix} \alpha, 1 + \alpha - \gamma \\ - \end{matrix}\right| -\frac{1}{x}\right). \tag{83}$$

From Theorem 2 and the previous data, four linearly independent solutions of the differential equation [5] (see also [2])

$$\begin{aligned} &x^2 y''''(x) + 5x y'''(x) + (4 + 2xn + 4xr - x^2 + 2x\alpha - \alpha^2) y''(x) \\ &\quad + (6r - 3x + 3\alpha + 3n) y'(x) + n(2 + n) y(x) = 0 \end{aligned} \tag{84}$$

satisfied by the rth associated Laguerre orthogonal polynomials are:

$$A_{n,L}^{(r)} = x^\alpha e^{-x} {}_1F_1\left(\left.\begin{matrix} 1 - r \\ \alpha + 1 \end{matrix}\right| x\right) {}_1F_1\left(\left.\begin{matrix} -n - r \\ \alpha + 1 \end{matrix}\right| x\right),$$

$$B_{n,L}^{(r)} = x^\alpha {}_1F_1\left(\left.\begin{matrix} 1 - r \\ \alpha + 1 \end{matrix}\right| x\right) G(n + r + \alpha + 1, \alpha + 1, -x),$$

$$C_{n,L}^{(r)} = x^{\alpha} G(r+\alpha, \alpha+1, -x)\,{}_1F_1\left(\left.\begin{array}{c} -n-r \\ \alpha+1 \end{array}\right| x\right),$$

$$D_{n,L}^{(r)} = x^{\alpha} \mathrm{e}^{x} G(r+\alpha, \alpha+1, -x) G(n+r+\alpha+1, \alpha+1, -x). \tag{85}$$

The use of (60), the previous data and the fact that

$$\prod_{i=0}^{r-1} \gamma_i = \Gamma(r)\Gamma(r+\alpha),$$

allow us to represent the associated Laguerre polynomials $P_n^{(r)}$ $(r \in \mathbb{N}_{>0})$ in terms of the hypergeometric functions

$$\begin{aligned} P_n^{(r)}(x) = {} & \frac{(-1)^{n-1} \mathrm{e}^{\mathrm{i}\pi\alpha} \Gamma(n+r+\alpha+1) x^{\alpha}}{\Gamma(\alpha+1)} \\ & \times \left(\frac{\Gamma(n+r+1)}{\Gamma(r)}\,{}_1F_1\left(\left.\begin{array}{c} 1-r \\ \alpha+1 \end{array}\right| x\right) G(n+r+\alpha+1, \alpha+1, -x) \right. \\ & \left. - \; G(r+\alpha, \alpha+1, -x)\,{}_1F_1\left(\left.\begin{array}{c} -n-r \\ \alpha+1 \end{array}\right| x\right) \right), \quad r \in \mathbb{N}_{>0}. \end{aligned}$$

5.2.3. Monic Hermite polynomials h_n

We denote by h_n the monic Hermite polynomials and by q_n the corresponding function of the second kind. The data are [28, p. 286]:

$$\sigma(x) = 1, \quad \tau(x) = -2x,$$

$$\rho(x) = \mathrm{e}^{-x^2}, \quad I = (-\infty, +\infty),$$

$$\lambda_n = 2n, \quad \beta_n = 0, \quad \gamma_n = \frac{n}{2},$$

$$h_n(x) = G\left(-\frac{n}{2}, \frac{1}{2}, x^2\right) = x^n\,{}_2F_0\left(\left.\begin{array}{c} -\frac{n}{2}, \frac{1-n}{2} \\ - \end{array}\right| -\frac{1}{x^2}\right),$$

$$\begin{aligned} q_n(x) &= \sqrt{\pi} n! 2^{-n} \mathrm{e}^{x^2 + \mathrm{i}\pi(n-1)/2} h_{-n-1}(\mathrm{i}x) \\ &= \sqrt{\pi} n! 2^{-n} \mathrm{e}^{x^2 + \mathrm{i}\pi(n-1)/2} G\left(\frac{n+1}{2}, \frac{1}{2}, -x^2\right). \end{aligned}$$

From Theorem 2 and the previous data, the four linearly independent solutions of the differential equation [5] (see also [2])

$$y^{(\mathrm{iv})}(x) + (4n + 8r - 4x^2) y''(x) - 12x y'(x) + 4n(2+n) y(x) = 0 \tag{86}$$

satisfied by the rth associated Hermite orthogonal polynomials are:

$$
\begin{aligned}
A_{n,H}^{(r)} &= \mathrm{e}^{-x^2} G\left(\frac{1-r}{2}, \frac{1}{2}, x^2\right) G\left(-\frac{n+r}{2}, \frac{1}{2}, x^2\right), \\
B_{n,H}^{(r)} &= G\left(\frac{1-r}{2}, \frac{1}{2}, x^2\right) G\left(\frac{n+r+1}{2}, \frac{1}{2}, -x^2\right), \\
C_{n,H}^{(r)} &= G\left(\frac{r}{2}, \frac{1}{2}, -x^2\right) G\left(-\frac{n+r}{2}, \frac{1}{2}, x^2\right), \\
D_{n,H}^{(r)} &= \mathrm{e}^{x^2} G\left(\frac{r}{2}, \frac{1}{2}, -x^2\right) G\left(\frac{n+r+1}{2}, \frac{1}{2}, -x^2\right).
\end{aligned}
\tag{87}
$$

By using (60), the previous data and the fact that

$$\prod_{i=0}^{r-1} \gamma_i = \sqrt{\pi}\Gamma(r)2^{1-r},$$

we represent the associated Hermite orthogonal polynomials $P_n^{(r)}$ $(r \in \mathbb{N}_{>0})$ as

$$
\begin{aligned}
P_n^{(r)}(x) = \mathrm{e}^{\mathrm{i}\pi(r-2)/2} \Bigg(& 2^{-n-1} \mathrm{e}^{\mathrm{i}\pi(n+1)/2} \frac{\Gamma(n+r)}{\Gamma(r)} G\left(\frac{1-r}{2}, \frac{1}{2}, x^2\right) G\left(\frac{n+r+1}{2}, \frac{1}{2}, -x^2\right) \\
& - G\left(\frac{r}{2}, \frac{1}{2}, -x^2\right) G\left(-\frac{n+r}{2}, \frac{1}{2}, x^2\right)\Bigg), \quad r \in \mathbb{N}_{>0}.
\end{aligned}
$$

5.3. Extension of results to real order of association

Let v be a real number with $v \geqslant 0$ and $(P_n^{(v)})_n$ the family of polynomials defined by

$$P_{n+1}^{(v)}(x) = (x - \beta_{n+v}) P_n^{(v)}(x) - \gamma_{n+v} P_{n-1}^{(v)}(x), \quad n \geqslant 1 \tag{88}$$

with the initial conditions

$$P_0^{(v)}(x) = 1, \quad P_1^{(v)}(x) = x - \beta_v, \tag{89}$$

where β_{n+v} and γ_{n+v} are the coefficients β_n and γ_n of Eq. (1) with n replaced by $n+v$.

We assume that the starting family $(P_n)_n$ defined in (1) is classical. The coefficients β_n and γ_n are therefore rational functions in the variable n [18,19,28] and the coefficients β_{n+v} and γ_{n+v} are well defined. When $\gamma_{n+v} \neq 0$, $\forall n \geqslant 1$, the family $(P_n^{(r)})_n$, thanks to Favard's theorem [12,8], is orthogonal and represents the associated of the family $(P_n)_n$ with real order of association. The notion of associated orthogonal polynomials with real order of association has been investigated by several authors (see for example [2,6,20,36]).

Theorem 8. *Let $(P_n)_n$ be a family of classical orthogonal polynomial, $v \geqslant 0$ a real number and $(P_n^{(v)})_n$ the v-associated of $(P_n)_n$. We have:*

(1) *$(P_n^{(v)})_n$ satisfies*

$$\mathbb{F}_n^{(v)}(y) = 0, \tag{90}$$

where $\mathbb{F}_n^{(v)}$ is the operator given in (51) with r replaced by v.

(2) *The differential operator $\mathbb{F}_n^{(v)}$ factorizes as*

$$\begin{aligned} &\mathbb{S}_n^{(v)}\mathbb{T}_n^{(v)} = U_{v-1}^3\mathbb{F}_n^{(v)}, \\ &\tilde{\mathbb{S}}_n^{(v)}\tilde{\mathbb{T}}_n^{(v)} = V_{v-1}^3\mathbb{F}_n^{(v)}, \end{aligned} \tag{91}$$

where the operators $\mathbb{S}_n^{(v)}$, $\mathbb{T}_n^{(v)}$, $\tilde{\mathbb{S}}_n^{(v)}$, $\tilde{\mathbb{T}}_n^{(v)}$ are those given in Eqs. (48)–(55) with r replaced by v, P_r and Q_r are replaced by U_v and V_v, respectively. U_v and V_v are the two linearly independent solutions of the differential equation (see [28])

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_v y(x) = 0 \tag{92}$$

with $U_r = P_r$, $V_r = Q_r$ for $v = r \in \mathbb{N}$ and

$$\lambda_v = -\frac{v}{2}((v-1)\sigma'' + 2\tau'). \tag{93}$$

Four linearly independent solutions of the differential equation (90) are given by

$$\begin{aligned} A_n^{(v)}(x) &= \rho(x)U_{v-1}(x)U_{n+v}(x), \\ B_n^{(v)}(x) &= \rho(x)U_{v-1}(x)V_{n+v}(x), \\ C_n^{(v)}(x) &= \rho(x)V_{v-1}(x)U_{n+v}(x), \\ D_n^{(v)}(x) &= \rho(x)V_{v-1}(x)V_{n+v}(x), \end{aligned} \tag{94}$$

where $\rho(x)$ is the weight function given by (3).

Proof. (1) Let n be a fixed integer number and define the function Φ by

$$\begin{aligned} \Phi:\quad & \mathbb{R}_+ \to \mathbb{R}, \\ & v \to \mathbb{F}_n^{(v)}(P_n^{(v)}(x)), \end{aligned}$$

where $\mathbb{R}_+$ is the set of positive real numbers. Using relation (88) for fixed x, $\Phi(v)$ can be written as rational function in v. In fact, for the classical orthogonal polynomials, the three-term recurrence relation coefficients β_n and γ_n are rational functions in the variable n. Using Eq. (58) we get

$$\Phi(r) = \mathbb{F}_n^{(r)}(P_n^{(r)}(x)) = 0, \quad \forall r \in \mathbb{N}. \tag{95}$$

We then conclude that $\Phi(v)$ is a rational function with an infinite number of zeros. Therefore, $\Phi(v) = 0$, $\forall v \in \mathbb{R}_+$, and $(P_n^{(v)})_n$ satisfies (90).

(2) Eq. (91) are proved by a straightforward computation using that U_v and U_v satisfy (92).

(3) The functions given in (94) are represented as products of functions satisfying homogeneous differential equation of order 1 (for ρ) and 2 (for U and V). These functions therefore satisfy

a differential equation of order $4(=1\times 2\times 2)$ which is identical to (90). Notice that by linear algebra one can deduce the differential equation of the product (94), given the differential equations of the factors, since they have polynomial coefficients. This can be done, e.g., by the Maple command `diffeq*diffeq` [33] of the gfun package.

We conclude the proof by noticing that the results of the previous theorem can be used to extend Theorem 5 to the generalized co-recursive associated of classical orthogonal polynomials with real order of association as was done for the classical discrete case in [13]. □

5.4. Solutions of some second-order differential equations

Proposition 3. *The two linearly independent solutions of the differential equation*

$$\mathbb{S}^{(r)}(y)=0,$$

where the operator $\mathbb{S}^{(r)}$ *(see (48)) is given by*

$$\mathbb{S}_n^{(r)}=\sigma P_{r-1}\mathscr{D}^2+[(\tau+\sigma')P_{r-1}-2\sigma P'_{r-1}]\mathscr{D}+[(\tau'+\lambda_{n+r}-\lambda_{r-1})P_{r-1}-2\tau P'_{r-1}]$$

are:

$$E_n^{(r)}(x)=\sigma(x)\rho(x)(Q_{r-1}(x)P'_{r-1}(x)-P_{r-1}(x)Q'_{r-1}(x))(P_{n+r}(x)P'_{r-1}(x)-P_{r-1}(x)P'_{n+r}(x)),$$

$$F_n^{(r)}(x)=\sigma(x)\rho(x)(Q_{r-1}(x)P'_{r-1}(x)-P_{r-1}(x)Q'_{r-1}(x))(Q_{n+r}(x)P'_{r-1}(x)-P_{r-1}(x)Q'_{n+r}(x)).$$

Proposition 4. *The two linearly independent solutions of the differential equation*

$$\mathbb{S}^{(v)}(y)=0,$$

where the operator $\mathbb{S}^{(v)}$ *(see (94)) is given by*

$$\tilde{\mathbb{S}}_n^{(v)}=\sigma U_{v-1}\mathscr{D}^2+[(\tau+\sigma')U_{v-1}-2\sigma U'_{v-1}]\mathscr{D}+[(\tau'+\lambda_{n+v}-\lambda_{v-1})U_{v-1}-2\tau U'_{v-1}]$$

are:

$$E_n^{(v)}(x)=\sigma(x)\rho(x)(V_{v-1}(x)U'_{v-1}(x)-U_{v-1}(x)V'_{v-1}(x))(U_{n+v}(x)U'_{v-1}(x)-U_{v-1}(x)U'_{n+v}(x)),$$

$$F_n^{(v)}(x)=\sigma(x)\rho(x)(V_{v-1}(x)U'_{v-1}(x)-U_{v-1}(x)V'_{v-1}(x))(V_{n+v}(x)U'_{v-1}(x)-U_{v-1}(x)V'_{n+v}(x)).$$

Here, U_v *and* V_v *are solutions of* (92).

Proof. Since the functions $C_n^{(r)}$ and $D_n^{(r)}$ are solutions of equation $\mathbb{F}_n^{(r)}(y)=0$ (see Theorem 2), we use the factorization given by (50) and get

$$\mathbb{S}_n^{(r)}(\mathbb{T}_n^{(r)}(y))=P_{r-1}^3\mathbb{F}_n^{(r)}(y)=0$$

for $y\in\{C_n^{(r)},D_n^{(r)}\}$. We therefore conclude that the functions $E_n^{(r)}$ and $F_n^{(r)}$ defined by

$$E_n^{(r)}=\mathbb{T}_n^{(r)}(C_n^{(r)}),\quad F_n^{(r)}=\mathbb{T}_n^{(r)}(D_n^{(r)}),$$

satisfy $\mathbb{S}_n^{(r)}(y)=0$. Computations using the fact that P_n and Q_n satisfy (5) lead to the expressions given in Proposition 3. The proof of Proposition 4 is similar to the one of Proposition 3 by using Theorem 7. □

Remark 4. (1) The previous propositions give solutions to families of second-order differential equations. In particular, Proposition 3 solves a family of second-order differential equations with polynomial coefficients.

(2) For fixed integer r, and for different classical situations, we have tried without success to solve the differential equation $\mathbb{S}_n^{(r)}(y)=0$ using Maple 7 or Mathematica 4.1 [37].

(3) Mark van Hoeij [35] was able to solve differential equation $\mathbb{S}_n^{(r)}(y)=0$ for fixed integers r using an algorithm he is currently developing and which extends the capabilities of algorithms aimed at solving second-order linear homogeneous differential equation with polynomial coefficients.

5.5. Extension of results to semi-classical cases

The proof of Theorem 1, which is the starting point of this paper, uses merely the second-order differential equation (5) and relation (33). Now, we suppose that the family $(P_n)_n$ is semi-classical [3,17,25,26]. This implies that $(P_n)_n$ is orthogonal satisfying a second-order differential equation of the form

$$\bar{\mathbb{M}}_n(y(x)) = I_2(x,n)y''(x) + I_1(x,n)y'(x) + I_0(x,n)y(x) = 0, \tag{96}$$

where the coefficients $I_i(x,n)$ are polynomials in x of degree not depending on n. For semi-classical orthogonal polynomials an equation of type (33) is known and can be stated as [4,5]

$$\tilde{\mathbb{M}}_n(P_{n-1}^{(1)}(x)) = a_1(x)P_n'(x) + a_0(x)P_n(x), \tag{97}$$

where a_i are polynomials and $\tilde{\mathbb{M}}_n$ a second-order linear differential operator with polynomial coefficients. Use of the two previous equations leads to the following extension.

Theorem 9. *Given $(P_n)_n$ a sequence of semi-classical orthogonal polynomials satisfying (96) and $(\bar{P}_n)_n$ a family of orthogonal polynomials obtained by modifying $(P_n)_n$ and satisfying*

$$\bar{P}_n(x) = A_n(x)P_{n+k-1}^{(1)} + B_n(x)P_{n+k}, \quad n \geqslant k', \tag{98}$$

where A_n and B_n are polynomials of degree not depending on n, and $k,k' \in \mathbb{N}$, we have the following:

(1) *The orthogonal polynomials $(\bar{P}_n)_{n\geqslant k'}$ satisfy a common fourth-order linear differential equation*

$$\begin{aligned}\bar{\mathbb{F}}_n(y(x)) = {} & K_4(x,n)y''''(x) + K_3(x,n)y'''(x) + K_2(x,n)y''(x) \\ & + K_1(x,n)y'(x) + K_0(x,n)y(x) = 0,\end{aligned} \tag{99}$$

where the coefficients K_i are polynomials in x, with degree not depending on n.

(2) *The operator $\bar{\mathbb{F}}_n$ can be factored as product of two second-order linear differential operators*

$$\bar{X}_n\bar{\mathbb{F}}_n = \bar{\mathbb{S}}_n\bar{\mathbb{T}}_n, \tag{100}$$

where $\bar{X}_n$ and the coefficients of $\bar{\mathbb{S}}_n$ and $\bar{\mathbb{T}}_n$ are polynomials of degree not depending on n.

The proof is similar to the one of Theorem 1 but with (96) and (97) playing the role of (5) and (33), respectively.

The previous theorem covers the modifications such as the rth associated, the general co-recursive, the general co-dilated, the general co-recursive associated and the general co-modified semi-classical orthogonal polynomials.

When the orthogonal polynomial sequence $(P_n)_n$ is semi-classical, in general it is difficult to represent the coefficients of the differential operators $\bar{\mathbb{M}}_n$, $\tilde{\mathbb{M}}_n$, $\bar{\mathbb{F}}_n$, $\bar{\mathbb{S}}_n$ and $\bar{\mathbb{T}}_n$ in terms of the polynomials ϕ and ψ, the coefficients of the functional equation (see [26, p. 37]) satisfied by the regular functional with respect to which $(\bar{P}_n)_n$ is orthogonal.

However, for particular cases (for example if the degrees of polynomials ϕ and ψ are small), it is possible after huge computations to give the coefficients of the differential operators $\bar{\mathbb{M}}_n$, $\tilde{\mathbb{M}}_n$, $\bar{\mathbb{F}}_n$, $\bar{\mathbb{S}}_n$ and $\bar{\mathbb{T}}_n$ explicitly, and therefore look for functions annihilating these differential operators.

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References

[1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series, Vol. 55, US Government, Washington, DC, 1964.
[2] R. Askey, J. Wimp, Associated Laguerre and Hermite polynomials, Proc. Roy. Soc. Edinburgh 96A (1984) 15–37.
[3] S. Belmehdi, On semi-classical linear functionals of class $s=1$ Classification and integral representation, Idag. Math. (N.S.) 3 (3) (1992) 253–276.
[4] S. Belmehdi, A. Ronveaux, Polynômes associés des polynômes orthogonaux classiques. Construction via “REDUCE”, in: L. Arias, et al. (Eds.), Orthogonal Polynomials and their Applications, Universidad de Oviedo, 1989, pp. 72–83.
[5] S. Belmehdi, A. Ronveaux, The fourth-order differential equation satisfied by the associated orthogonal polynomials, Rend. Mat. Ser. VII 11 (1991) 313–326.
[6] J. Bustoz, M.E.H. Ismail, The associated ultraspherical polynomials and their q-analogues, Canad. J. Math. XXXIV (3) (1982) 718–736.
[7] T.S. Chihara, On co-recursive orthogonal polynomials, Proc. Amer. Math. Soc. 8 (1957) 899–905.
[8] T.S. Chihara, Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[9] J. Dini, Sur les formes linéaires et polynômes orthogonaux de Laguerre–Hahn, Thèse de Doctorat, Université Pierre et Marie Curie, Paris VI, 1988.
[10] J. Dini, P. Maroni, A. Ronveaux, Sur une perturbation de la récurrence vérifiée par une suite de polynômes orthogonaux, Portugal. Math. 6 (3) (1989) 269–282.
[11] J. Dzoumba, Sur les Polynômes de Laguerre–Hahn, Thèse de 3ème cycle, Université Pierre et Marie Curie, Paris VI, 1985.
[12] J. Favard, Sur les polynômes de Tchebicheff, C. R. Acad. Sci. Paris 200 (1935) 2052–2053.
[13] M. Foupouagnigni, W. Koepf, A. Ronveaux, On fourth-order difference equations for orthogonal polynomials of a discrete variable: derivation, factorization and solutions, J. Difference Equations Appl. 9 (2003) 777–804.

[14] C.C. Grosjean, Theory of recursive generation of systems of orthogonal polynomials: an illustrative example, J. Comput. Appl. Math. 12& 13 (1985) 299–318.
[15] C.C. Grosjean, The weight function, generating functions and miscellaneous properties of the sequences of orthogonal polynomials of the second kind associated with the Jacobi and the Gegenbauer polynomials, J. Comput. Appl. Math. 16 (1986) 259–307.
[16] W. Hahn, On differential equations for orthogonal polynomials, Funkcial. Ekvac. 21 (1978) 1–9.
[17] E. Hendriksen, H. van Rossum, Semi-classical orthogonal polynomials, in: C. Brezinski et al. (Eds.), Polynômes Orthogonaux et Applications, Lecture Notes in Mathematics, Vol. 1171, Springer, Berlin, 1985, pp. 355–361.
[18] W. Koepf, D. Schmersau, Representations of orthogonal polynomials, J. Comput. Appl. Math. 90 (1998) 57–94.
[19] P. Lesky, Über Polynomlösungen von Differentialgleichungen und Differenzengleichungen zweiter Ordnung, Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. 121 (1985) 29–33.
[20] J. Letessier, Some results on co-recursive associated Laguerre and Jacobi polynomials, SIAM J. Math. Anal. 25 (2) (1994) 528–548.
[21] S. Lewanowicz, Results on the associated classical orthogonal polynomials, J. Comput. Appl. Math. 65 (1995) 215–231.
[22] A. Magnus, Riccati acceleration of Jacobi continued fractions and Laguerre–Hahn orthogonal polynomials, Lecture Notes in Mathematics, Vol. 1071, Springer, Berlin, 1984, pp. 213–230.
[23] F. Marcellán, J.S. Dehesa, A. Ronveaux, On orthogonal polynomials with perturbed recurrence relations, J. Comput. Appl. Math. 30 (2) (1990) 203–212.
[24] F. Marcellán, E. Prianes, Perturbations of Laguerre–Hahn functionals, J. Comput. Appl. Math. 105 (1999) 109–128.
[25] P. Maroni, Prolégomènes à l'étude des polynômes orthogonaux semi-classiques, Ann. Mat. Pura Appl. 4 (1987) 165–184.
[26] P. Maroni, Une théorie algébrique des polynômes orthogonaux: Applications aux polynômes orthogonaux semi-classiques, in: C. Brezinski, et al., (Eds.), Orthogonal Polynomials and Applications, Annals on Computing and Applications of Mathematics, Vol. 9, J.C. Baltzer AG, Basel, 1991, pp. 98–130.
[27] M.B. Monagan, K.O. Geddes, K.M. Heal, G. Labahn, S.M. Vorkoetter, J. McCarron, P. DeMarco, Maple 7, Waterloo Maple, Inc., 2001.
[28] A.F. Nikiforov, V.B. Uvarov, Special Functions of Mathematical Physics, Birkhäuser, Basel, Boston, 1988.
[29] A. Ronveaux, Fourth-order differential equation for numerator polynomials, J. Phys. A.: Math. Gen. 21 (1988) 749–753.
[30] A. Ronveaux, S. Belmehdi, J. Dini, P. Maroni, Fourth-order differential equation for the co-modified semi-classical orthogonal polynomials, J. Comput. Appl. Math. 29 (2) (1990) 225–231.
[31] A. Ronveaux, F. Marcellán, Co-recursive orthogonal polynomials and fourth-order differential equation, J. Comput. Appl. Math. 25 (1989) 295–328.
[32] A. Ronveaux, A. Zarzo, E. Godoy, Fourth-order differential equation satisfied by the generalized co-recursive of all classical orthogonal polynomials. A study of their distribution of zeros, J. Comput. Appl. Math. 59 (1995) 105–109.
[33] B. Salvy, P. Zimmermann, GFUN: A Maple package for the manipulation of generating and holonomic functions in one variable, ACM Trans. Math. Software 20 (1994) 163–177.
[34] G. Szegö, Orthogonal Polynomials, Vol. XXIII, American Mathematical Society, Colloquium Publications, 1939.
[35] M. van Hoeij, Private communication, May 15, 2002.
[36] J. Wimp, Explicit formulas for the associated Jacobi polynomials and applications, Canad. J. Math. XXXIX (4) (1987) 983–1000.
[37] S. Wolfram, The Mathematica Book, 4th Edition, Wolfram Media, Cambridge University Press, Cambridge 1999.
[38] A. Zarzo, A. Ronveaux, E. Godoy, Fourth-order differential equation satisfied by the associated of any order of all classical orthogonal polynomials. A study of their distribution of zeros, J. Comput. Appl. Math. 49 (1993) 349–359.