Bieberbach Conjecture and the de Branges and Weinstein Functions

Prof. Dr. Wolfram Koepf
University of Kassel

koepf@mathematik.uni-kassel.de

http://www.mathematik.uni-kassel.de/~koepf

AbiTUMath 2004
International Autumn School *Mathematical Modelling*
TU München, Germany
September 26–30, 2004

UNIKASSEL
VERSITÄT

Bieberbach Conjecture

- Let

$$S = \{ f : \mathbb{D} \to \mathbb{C} \mid f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

$$\text{analytic and univalent } \} \; .$$

- The Riemann mapping theorem states that for every simply connected domain $G \in \mathbb{C}$ there is a univalent function $f \in S$ with range similar to G.

Compactness of S

- Riemann had formulated his mapping theorem already in 1851, but his proof was incomplete. Carathéodory and Koebe proved the mapping theorem around 1909.

- Moreover, Koebe showed that S is *compact* w.r.t. the topology of locally uniform convergence.

- Hence, since a_n is a continuous functional,

$$k_n := \max_{f \in S} |a_n(f)|$$

exists.

Bieberbach Conjecture

- In his 1916 article
  - Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. Semesterberichte der Preussischen Akademie der Wissenschaften 38, 940–955

  Bieberbach proved that $k_2 = 2$. In a footnote he wrote

  - Vielleicht ist überhaupt $k_n = n$.

- This statement is called the Bieberbach conjecture.

Prof. Dr. Hecke

Überreicht vom Verfasser.

SITZUNGSBERICHTE 1916. XXXVIII.

DER

KÖNIGLICH PREUSSISCHEN

AKADEMIE DER WISSENSCHAFTEN.

Gesamtsitzung vom 20. Juli.
Mitteilung vom 6. Juli.

Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln.

Von Prof. Dr. Ludwig Bieberbach
in Frankfurt a. M.

haben damit nicht nur unseren Satz IV bewiesen, sondern darüber hinaus auch erkannt, daß die Zahl r_2 des Satzes IV die 2 ist, und daß für a_2 auch wirklich alle Werte dieses Kreises $|a_2| \leq 2$ vorkommen. Wenn es mir auch nicht gelungen ist, für die andern Koeffizienten ein ähnlich abschließendes Resultat zu erreichen, so möchte ich doch noch zeigen, daß *auch der Wertevorrat jedes andern Koeffizienten gerade einen Kreis erfüllt.* Das folgt einfach daraus, daß für $|k| \leq 1$ mit $f(z)$ stets auch $\frac{1}{k}f(kz)$ für $|z| < 1$ regulär und schlicht ist. Der $n$te Koeffizient dieser Funktion ist aber $a_n k^{n-1}$. Ist also $a_n^{(0)}$ $n$ter Koeffizient einer schlichten Abbildung, so auch alle a_n aus dem Kreis $|a_n| \leq |a_n^{(0)}|$. Darin liegt bekanntermaßen unsere Behauptung[1]. Man muß sich indessen hüten, dies Resultat in zu starkem Maße umzukehren. Es bildet ganz und gar nicht jede Funktion schlicht ab, deren Koeffizienten den gefundenen Kreisen angehören. Z. B. bildet schon die Funktion $w = z + 2z^2$ den Kreis $|z| \leq 1$ nicht auf einen schlichten Bereich ab, denn wir haben gesehen, daß $\sum nz^n$ die einzige schlicht abbildende Funktion mit $a_2 = 2$ ist[2].

Wir ziehen noch eine Folgerung aus diesen Betrachtungen:

Satz V. Wenn $|z| > 1$ durch $w = F(z) = z + \frac{\alpha_1}{z} + \cdots$ schlicht abgebildet wird, so liegen alle Randpunkte des Bildbereiches im Kreise $|w| \leq 2$, und es finden sich auf diesem Kreise nur dann Randpunkte des Bildbereiches, wenn es sich um die durch die Funktion $F(z) = z + \frac{1}{z}$ vermittelte Abbildung von $|z| > 1$ auf die von -1 bis $+1$ aufgeschlitzte Ebene handelt, oder wenn die Abbildungsfunktionen $\frac{1}{e^{i\varphi}}F(e^{i\varphi}z)$ vorliegen, die gleichfalls auf Schlitzbereiche abbilden, welche aus den eben genannten durch Drehung hervorgehen[3,4].

---

[1] Daß $k_n \geq n$ zeigt das Beispiel $\sum nz^n$. Vielleicht ist überhaupt $k_n = n$.

[2] Die hier gefundene Tatsache, daß 2 die genaue Schranke für $|a_2|$ ist, erlaubt es, gewisse Untersuchungen über den Koebeschen Verzerrungssatz zu Ende zu führen, welche schon Hr. Plemelj auf der Wiener Naturforscherversammlung vorgetragen hat, und die unabhängig davon kürzlich Hr. Pick angestellt und (Leipz. Ber. 1916) veröffentlicht hat.

[3] Man vgl. zu diesem Satz einen von Koebe, Göttinger Nachrichten 1908, S. 348. Der hier bewiesene Satz V liefert zugleich den Koebeschen und zeigt, daß der genaue Wert der Konstanten, deren Existenz dort bewiesen ist, die 4 ist, und daß diese Konstante nur bei Schlitzabbildungen erreicht wird.

[4] Der Satz ist ferner nahe verwandt mit dem Satz I, den ich auf S. 153 von Bd. 77 der Math. Ann. aufgestellt habe, besagt aber ersichtlich noch etwas mehr als dieser.

[1] Daß $k_n \geqq n$ zeigt das Beispiel $\sum n z^n$. Vielleicht ist überhaupt $k_n = n$.

[2] Die hier gefundene Tatsache, daß 2 die genaue Schranke für $|a_2|$ ist, erlaubt es, gewisse Untersuchungen über den Koebeschen Verzerrungssatz zu Ende zu führen, welche schon Hr. Plemelj auf der Wiener Naturforscherversammlung vorgetragen hat, und die unabhängig davon kürzlich Hr. Pick angestellt und (Leipz. Ber. 1916) veröffentlicht hat.

[3] Man vgl. zu diesem Satz einen von Koebe, Göttinger Nachrichten 1908, S. 348. Der hier bewiesene Satz V liefert zugleich den Koebeschen und zeigt, daß der genaue Wert der Konstanten, deren Existenz dort bewiesen ist, die 4 ist, und daß diese Konstante nur bei Schlitzabbildungen erreicht wird.

[4] Der Satz ist ferner nahe verwandt mit dem Satz I, den ich auf S. 153 von Bd. 77 der Math. Ann. aufgestellt habe, besagt aber ersichtlich noch etwas mehr als dieser.

Where does the conjecture come from?

- Bieberbach showed moreover that for $n = 2$ essentially only the *Koebe function*

$$K(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=0}^{\infty} n z^n$$

that maps the unit disk onto a radially slit plane, solves the coefficient problem.

Where does the conjecture come from?

- Bieberbach showed moreover that for $n = 2$ essentially only the *Koebe function*

$$K(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=0}^{\infty} n z^n$$

  that maps the unit disk onto a radially slit plane, solves the coefficient problem.

Where does the conjecture come from?

- Bieberbach showed moreover that for $n = 2$ essentially only the *Koebe function*

$$K(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=0}^{\infty} n z^n$$

that maps the unit disk onto a radially slit plane, solves the coefficient problem.

Where does the conjecture come from?

- Bieberbach showed moreover that for $n = 2$ essentially only the *Koebe function*

$$K(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=0}^{\infty} n z^n$$

that maps the unit disk onto a radially slit plane, solves the coefficient problem.

- Hence he conjectured that the same is true for $n > 2$.

Ludwig Bieberbach conjectured $|a_n| \leqq n$ (1916)

UNIKASSEL VERSITÄT

Loewner Method

- In 1923 Loewner proved the Bieberbach conjecture for $n = 3$.
- His method was to embed a univalent function $f(z)$ into a *Loewner chain*, i.e. a family $\{f(z,t) \mid t \geqq 0\}$ of univalent functions of the form

$$f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \quad (z \in \mathbb{D}, t \geqq 0, a_n(t) \in \mathbb{C})$$

which start with f

$$f(z,0) = f(z)\ ,$$

Loewner Differential Equation

and for which the relation

$$\mathrm{Re}\, p(z,t) = \mathrm{Re}\left(\frac{\dot{f}(z,t)}{zf'(z,t)}\right) > 0 \quad (z \in \mathbb{D})$$

is satisfied.

- The above equation is referred to as the *Loewner differential equation*.
- It geometrically states that the image domains of $f(\mathbb{D},t)$ expand as t increases.

Charles Loewner proved $|a_3| \leqq 3$ (1923)

Logarithmic Coefficients

- After some decades one knew that it was very difficult to obtain informations about the coefficients $a_n(f)$ of univalent functions.

- By Loewner's method, it seemed to be simpler to obtain results about the coefficients $d_n(f)$ of the function
$$\varphi(z) = \ln \frac{f(z)}{z} =: \sum_{n=1}^{\infty} d_n z^n \; .$$

- These are called the *logarithmic coefficients* of f.

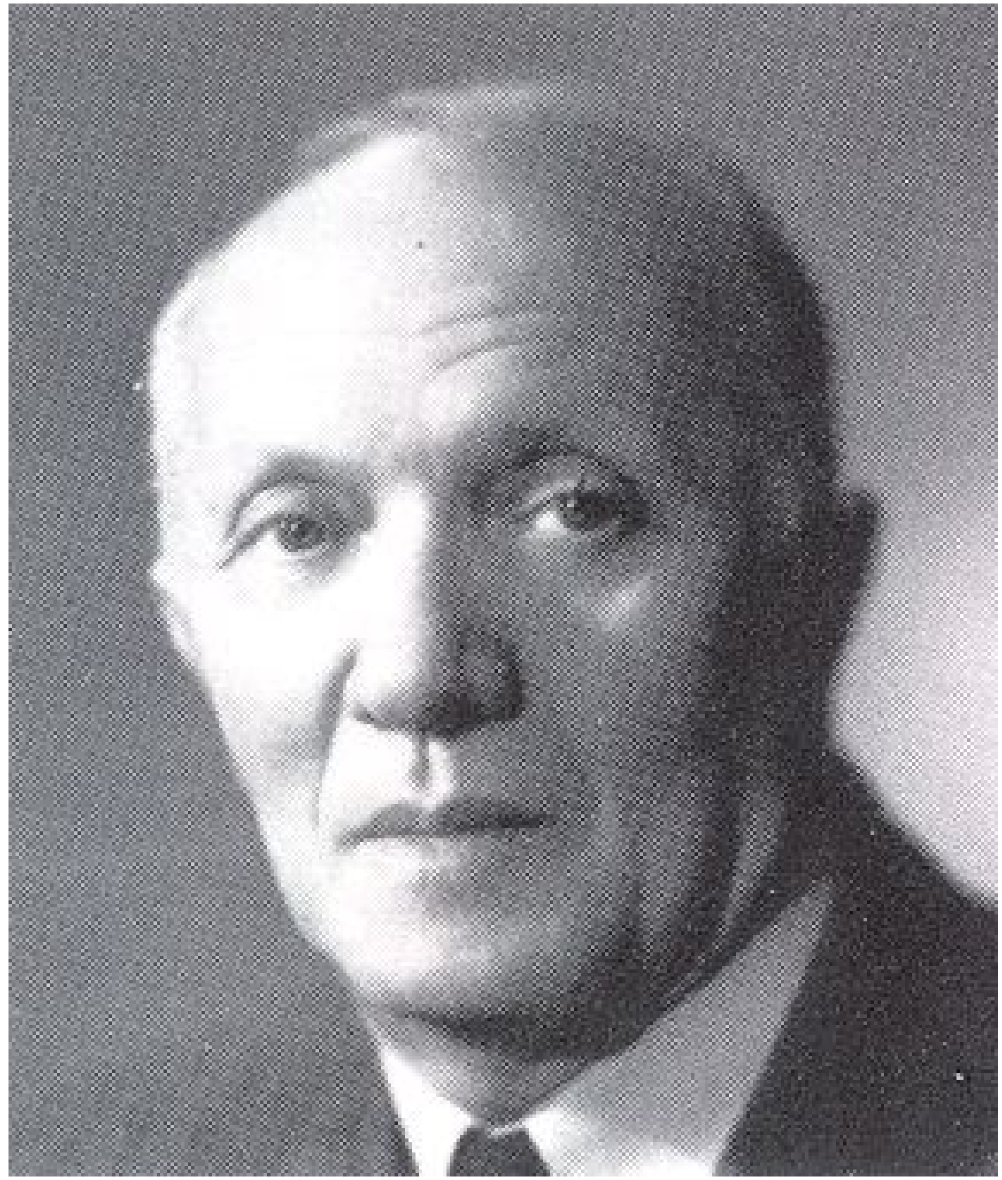
The Milin Conjecture

- In the mid 1960s Lebedev and Milin developed methods to exponentiate informations about d_n.

- They showed in 1965 by an application of the Cauchy-Schwarz inequality, that the inequality

$$\sum_{k=1}^{n}(n+1-k)\left(k|d_k|^2-\frac{4}{k}\right)\leqq 0$$

for some $n\in\mathbb{N}$ implies the Bieberbach conjecture for the index $n+1$. This is called the *Milin conjecture*.

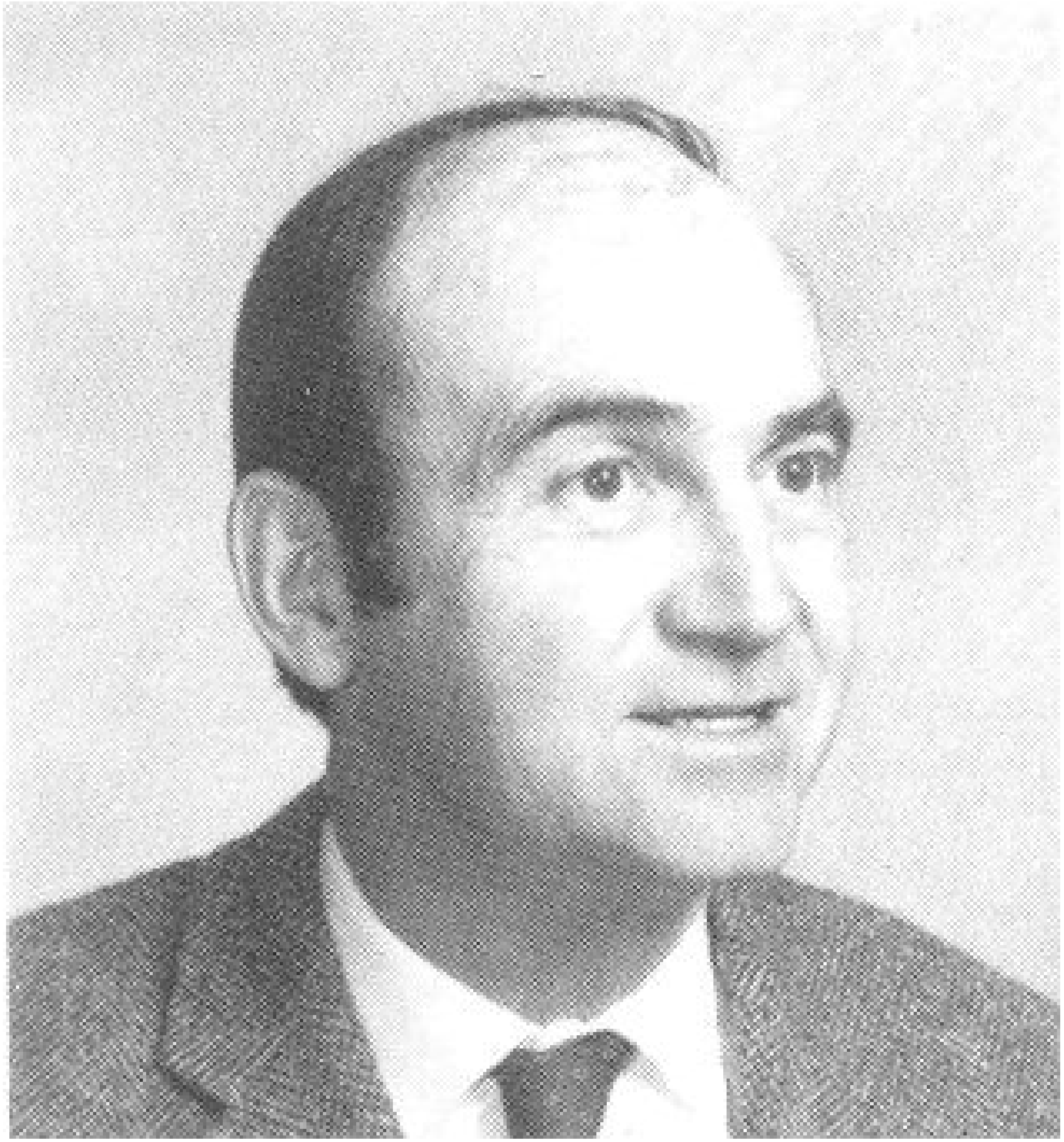
N. A. Lebedev and I. M. Milin

The de Branges Theorem

- In 1984 Louis de Branges proved the Milin and therefore the Bieberbach conjecture for all $n \in \mathbb{N}$.

- In 1991 Lenard Weinstein gave a completely independent proof of the Milin conjecture.

- Nevertheless, it turned out that the two proofs share more than expected.



Louis de Branges proved the Milin conjecture

De Branges' Proof

- De Branges considered the function

$$\psi(t) := \sum_{k=1}^{n} \tau_k^n(t) \left(k|d_k(t)|^2 - \frac{4}{k} \right) .$$

- Applying Loewner's theory he could show that for suitably chosen functions $\tau_k^n(t)$ the relation $\dot{\psi}(t) \geqq 0$ and therefore

$$\psi(0) = -\int_0^\infty \dot{\psi}(t) \mathrm{d}t \leqq 0$$

follows, hence Milin's conjecture is valid.

The de Branges Functions

- The de Branges functions $\tau_k^n(t), k = 1, \ldots, n$ are defined by the coupled system of differential equations

$$\tau_k^n(t) - \tau_{k+1}^n(t) = -\frac{1}{k}\,\dot{\tau}_k^n(t) - \frac{1}{k+1}\,\dot{\tau}_{k+1}^n(t)$$

with the initial values

$$\tau_k^n(0) = n + 1 - k\ .$$

Further Properties of the de Branges Functions

- By these properties the family $\tau_k^n(t)$ is already uniquely determined. For the success of de Branges' proof, however, we need moreover the properties

$$\lim_{t\to\infty} \tau_k^n(t) = 0$$

as well as

$$\dot{\tau}_k^n(t) \leqq 0 \; .$$

The Askey-Gasper Inequality

- Whereas the limit $\lim_{t\to\infty} \tau_k^n(t) = 0$ can be established easily, de Branges could not verify the relation $\dot{\tau}_k^n(t) \leqq 0$.

- By a phone call of Walter Gautschi with Dick Askey, de Branges finally realized that this relation had been proved by Askey and Gasper not long before, namely in 1976. To find this connection, an explicit representation of his functions $\tau_k^n(t)$ was necessary.

Dick Askey and George Gasper

POSITIVE JACOBI POLYNOMIAL SUMS, II.

By Richard Askey* and George Gasper.**

Abstract. Among the positive polynomial sums of Jacobi polynomials, there are two which have been very useful, the Cesàro means of the formal reproducing kernel and the sum $(*)$ $\sum_{k=0}^{n} P_k^{(\alpha,\beta)}(x)/P_k^{(\beta,\alpha)}(1)$, which was first considered by Fejér when $\alpha=1/2$, $\beta=\pm 1/2$ and when $\alpha=\beta=0$. A conjecture is given which connects these two sets of inequalities and this conjecture is proven for many values of (α,β). In particular, it is shown that if $\beta \geqslant 0$, then the sum $(*)$ is nonnegative for $-1 \leqslant x \leqslant 1$ if and only if $\alpha+\beta \geqslant -2$. It is also shown that $\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!}\frac{(\lambda+1)_k}{k!}\frac{\sin(k+1)\theta}{k+1} > 0$, $0<\theta<\pi$, $-1<\lambda\leqslant 1$, and that for real α the function $(1-r)^{-|\alpha|}[1\pm r+(1-2xr+r^2)^{1/2}]^{\alpha}$ is absolutely monotonic for $-1\leqslant x\leqslant 1$, i.e., it has nonnegative power series coefficients when it is expanded in a power series in r. Limiting cases involving Laguerre and Hermite polynomials are also considered.

1. Introduction. Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{k!(\alpha+1)_k}\left(\frac{1-x}{2}\right)^k$$

$$= \frac{(\alpha+1)_n}{n!}\,{}_2F_1[-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2], \qquad (1.1)$$

where $(a)_0=1, (a)_k=a(a+1)\dots(a+k-1), k\geqslant 1$. These polynomials are orthogonal on the interval $[-1,1]$ when $\alpha,\beta>-1$; but we shall also consider them for other values of α,β. Jacobi polynomials are the most general class of functions for which it is now possible to prove many deep results, and they contain as special cases or limiting cases most of the useful classical functions.

Manuscript received December 17, 1973.

*Supported in part by NSF Grant GP-33897X.

**Supported in part by NSF Grant GP-32116 and in part by a fellowship from the Alfred P. Sloan Foundation.

In order to prove further cases of Conjecture 1 we derive the identity

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(x)$$

$$= \sum_{j=0}^{[n/2]} \frac{\left(\frac{1}{2}\right)_j \left(\frac{\alpha+2}{2}\right)_{n-j} \left(\frac{\alpha+3}{2}\right)_{n-2j} (n-2j)!}{j! \left(\frac{\alpha+3}{2}\right)_{n-j} \left(\frac{\alpha+1}{2}\right)_{n-2j} (\alpha+1)_{n-2j}} \left\{ C_{n-2j}^{(\alpha+1)/2}\left(\left(\frac{1+x}{2}\right)^{1/2}\right)\right\}^2, \tag{1.16}$$

where $C_n^\lambda(x)$ is the ultraspherical polynomial of order λ, and use it to obtain

UNIKASSEL VERSITÄT

Askey-Gasper Identity

- The Askey-Gasper inequality was proved by detecting ingeniously the Askey-Gasper identity

$$\frac{(\alpha+2)_n}{n!}\cdot {}_3F_2\left(\left.\begin{matrix}-n, n+2+\alpha, \frac{\alpha+1}{2}\\ \alpha+1, \frac{\alpha+3}{2}\end{matrix}\right| x\right)$$

$$= \sum_{j=0}^{\lfloor\frac{n-k}{2}\rfloor} \frac{\left(\frac{1}{2}\right)_j \left(\frac{\alpha}{2}+1\right)_{n-j} \left(\frac{\alpha+3}{2}\right)_{n-2j} (\alpha+1)_{n-2j}}{j!\,\left(\frac{\alpha+3}{2}\right)_{n-j} \left(\frac{\alpha+1}{2}\right)_{n-2j} (n-2j)!} \cdot$$

$$ {}_3F_2\left(\left.\begin{matrix}2j-n, n-2j+\alpha+1, \frac{\alpha+1}{2}\\ \alpha+1, \frac{\alpha+2}{2}\end{matrix}\right| x\right)$$

Hypergeometric Functions

- The power series

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| \, x\right) = \sum_{k=0}^{\infty} A_k \, x^k = \sum_{k=0}^{\infty} \alpha_k \ , $$

whose coefficients A_k have a rational term ratio

$$ \frac{\alpha_{k+1}}{\alpha_k} = \frac{A_{k+1} \, x^{k+1}}{A_k \, x^k} = \frac{(k + a_1) \cdots (k + a_p)}{(k + b_1) \cdots (k + b_q)} \cdot \frac{x}{k+1} \ , $$

is called the generalized hypergeometric function.

Coefficients of the Hypergeometric Function

- For the coefficients of the hypergeometric function that are called hypergeometric terms, one gets the formula

$$ {}_pF_q\left(\left.\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right| x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!} \,, $$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is called the Pochhammer symbol or shifted factorial.

Examples of Hypergeometric Functions

$$e^z = {}_0F_0(z)$$

$$\sin z = z \cdot {}_0F_1\left(\left.\begin{matrix} - \\ 3/2 \end{matrix}\right| -\frac{z^2}{4}\right)$$

- Further examples: $\cos(z), \arcsin(z), \arctan(z),$ $\ln(1+z), \mathrm{erf}(z), L_n^{(\alpha)}(z), \ldots,$ but for example not $\tan(z)$.

UNIKASSEL VERSITÄT

Computer Algebra Proof of the Askey-Gasper Identity

- Using Zeilberger's algorithm from 1990, computer calculations can prove the Askey-Gasper identity easily:

$$\frac{(\alpha+2)_n}{n!}\cdot {}_3F_2\left(\left.\begin{array}{c}-n,\, n+2+\alpha,\, \frac{\alpha+1}{2}\\ \alpha+1,\, \frac{\alpha+3}{2}\end{array}\right| x\right)$$

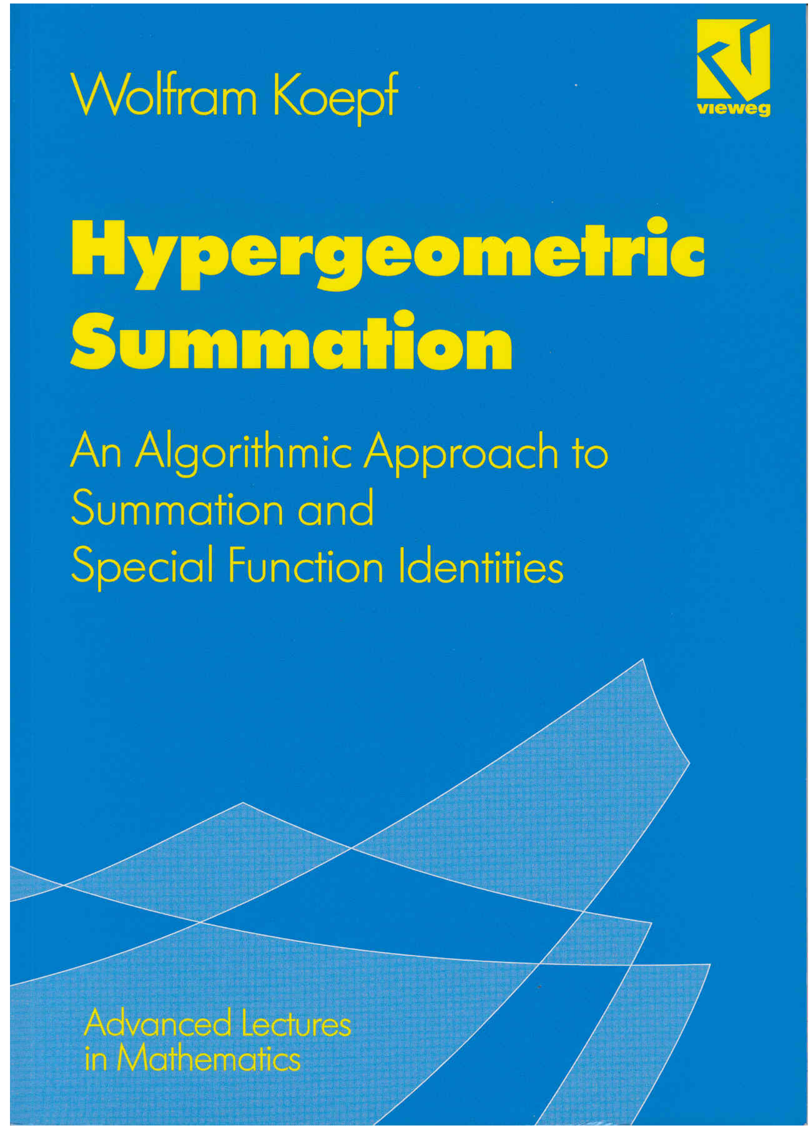
$$=\sum_{j=0}^{\lfloor\frac{n-k}{2}\rfloor}\frac{\left(\frac{1}{2}\right)_j\,\left(\frac{\alpha}{2}+1\right)_{n-j}\,\left(\frac{\alpha+3}{2}\right)_{n-2j}\,(\alpha+1)_{n-2j}}{j!\,\left(\frac{\alpha+3}{2}\right)_{n-j}\,\left(\frac{\alpha+1}{2}\right)_{n-2j}\,(n-2j)!}\cdot {}_3F_2\left(\left.\begin{array}{c}2j-n,\, n-2j+\alpha+1,\, \frac{\alpha+1}{2}\\ \alpha+1,\, \frac{\alpha+2}{2}\end{array}\right| x\right).$$

- This proof goes from right to left, but there is no way to detect the right hand side from the left hand side. This part still needs Askey's and Gasper's ingenuity.

- Computer demonstration using *Maple*

The software used was developed in connection with my book

Hypergeometric Summation, Vieweg, 1998, Braunschweig/Wiesbaden



and can be downloaded from my home page:

http://www.mathematik.uni-kassel.de/~koepf

Clausen's Identity

- The clue of the Askey-Gasper identity is the fact that the hypergeometric function occurring in its right hand summand is a complete square by Clausen's identity

$$
{}_3F_2\left(\left.\begin{array}{c} 2a, 2b, a+b \\ 2a+2b, a+b+1/2 \end{array}\right| x\right) = {}_2F_1\left(\left.\begin{array}{c} a, b \\ a+b+1/2 \end{array}\right| x\right)^2 .
$$

- Clausen's identity can also be proved by Zeilberger's algorithm. For this purpose, we write the right hand side as a Cauchy product.

Weinstein's Proof

- In 1991 Weinstein published a completely different proof of the Milin conjecture.
- Whereas de Branges takes fixed $n \in \mathbb{N}$, Weinstein considers the conjecture for all $n \in \mathbb{N}$ at the same time.
- For his proof Weinstein needs the following Loewner chain of the Koebe function, sometimes called Pick function.

Loewner Chain of the Koebe Function

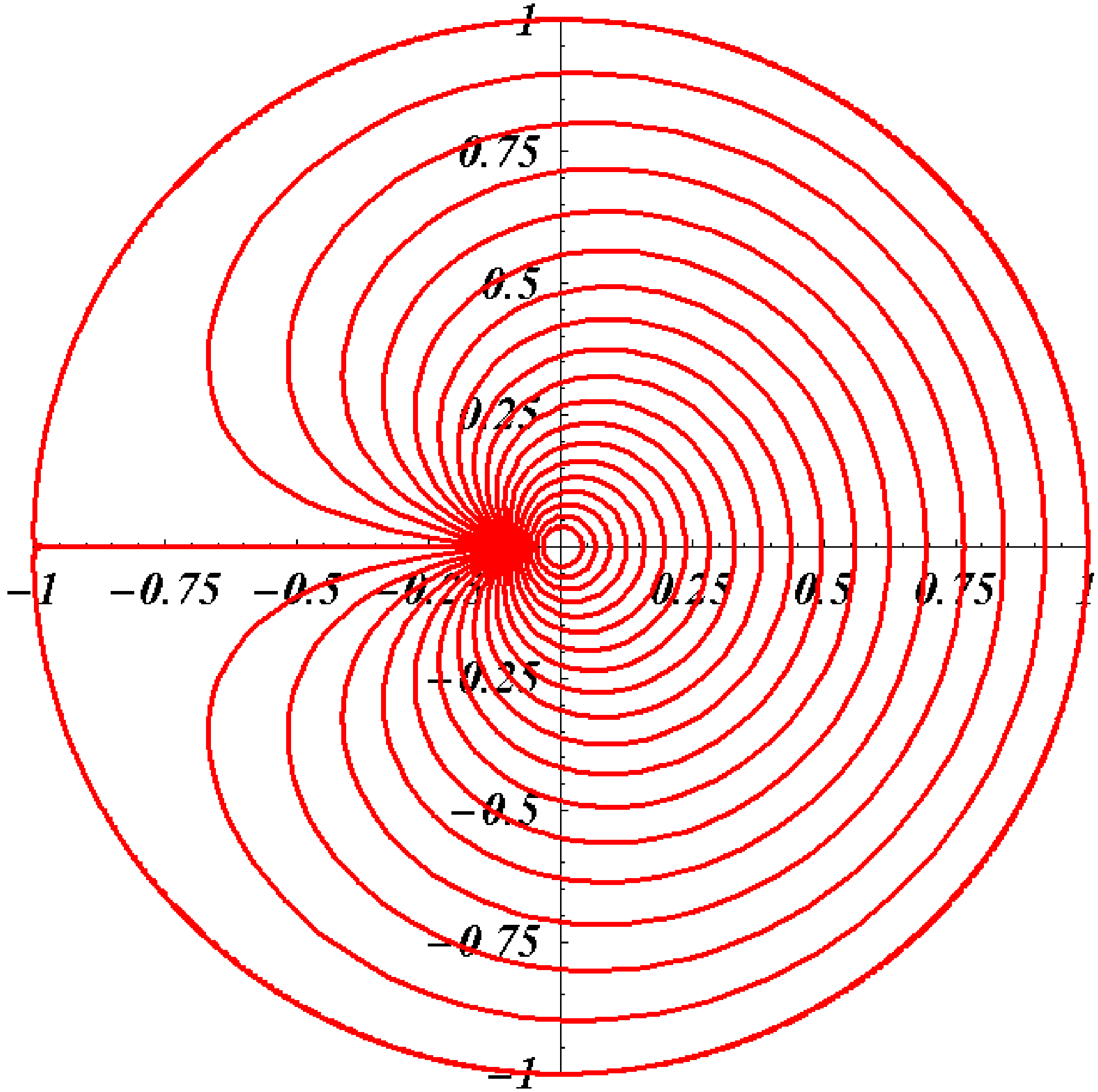
- This function family $W : \mathbb{D} \times \mathbb{R}_{\geqq 0} \to \mathbb{D}$ is given by

$$W(z,t) = K^{-1}\Big(e^{-t}K(z)\Big) \,.$$

- For any particular $t > 0$ the range of this function is the unit disk with a radial slit that grows with growing t.

- A computation shows that

$$W(z,t) = \frac{4e^{-t}z}{\left(1 - z + \sqrt{1 - 2(1 - 2e^{-t})z + z^2}\right)^2} \,.$$



The mapping behavior of $W(z,t)$

International Mathematics Research Notices 1991 No. 5

THE BIEBERBACH CONJECTURE

LENARD WEINSTEIN

1. Introduction. Bieberbach [1] conjectured bounds on the coefficients of functions analytic and univalent in the open unit disk. Schiffer [3], Garabedian [3], Löwner [4], and Bieberbach [1] verified this conjecture for the first several coefficients, and de Branges [2] in February 1984 verified the entire conjecture. Here, we give a short proof of the conjecture of Bieberbach. We show the following theorem.

THEOREM. *Let f be a function, analytic and univalent in the open unit disk* $\mathbf{D}$, *such that* $f(0) = 0$, $f'(0) = 1$; *so* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in \mathbf{D}$. *Then* $|a_n| < n$, $n = 2, 3, \ldots$, *unless* $f(z) = z/(1 - \gamma z)^2$, $|\gamma| = 1$.

2. Lemmas.

LEMMA 1 (Löwner theory). *There exists a family* f_t *of functions,* $f_t\colon \mathbf{D} \to \mathbb{C}$ *for* $t \geqslant 0$, *analytic and univalent in* $\mathbf{D}$ *such that*:

1. $f_0 = f$;

2. $f_t(z) = e^t z + \sum_{k=2}^{\infty} a_k(t) z^k, \qquad a_k(t) \in \mathbb{C},\ z \in \mathbf{D},\ t \geqslant 0$

3. $\log \dfrac{f_t(z)}{e^t z} = \sum_{k=1}^{\infty} c_k(t) z^k, \qquad$ *and* $c_k(\infty) = 2/k$.

4. $\operatorname{Re}\left\{ \dfrac{\dfrac{\partial f_t(z)}{\partial t}}{z \dfrac{\partial f_t(z)}{\partial z}} \right\} > 0 \qquad$ *for all* $z \in \mathbf{D}$.

Proof. This may be found in [4].

LEMMA 2 (Milin conjecture).

$$\sum_{k=1}^{n} \left(\frac{4}{k} - k|c_k(0)|^2\right)(n - k + 1) > 0, \qquad n = 1, 2, \ldots$$

implies the Bieberbach conjecture $|a_n| < n$ *for* $n = 2, 3, \ldots$.

Proof. See [5].

Received 15 August 1991.
Communicated by Peter Sarnak.

3. Proof of theorem. We show that

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \left(\frac{4}{k} - k|c_k(0)|^2\right)(n - k + 1) \right) z^{n+1} = \sum_{n=1}^{\infty} \int_0^{\infty} g_n(t)\, dt\, z^{n+1}$$

where $g_n(t) \geqslant 0$ for $t \geqslant 0$, $n = 1, 2, \ldots$. Fix $z \in \mathbf{D}$; define $w = w_t(z)$ by $z/(1 - z)^2 = e^t w/(1 - w)^2$, $t \geqslant 0$, and let $z_1 = re^{i\theta}$. We have

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \left(\frac{4}{k} - k|c_k(0)|^2\right)(n - k + 1) \right) z^{n+1}$$

$$= \frac{z}{(1 - z)^2} \sum_{k=1}^{\infty} \left(\frac{4}{k} - k|c_k(0)|^2\right) z^k$$

$$= \int_0^{\infty} -\frac{z}{(1 - z)^2} \frac{d}{dt}\left(\sum_{k=1}^{\infty} \left(\frac{4}{k} - k|c_k(t)|^2\right) w^k \right) dt$$

$$= \int_0^{\infty} \frac{e^t w}{1 - w^2} \frac{1 + w}{1 - w} \left(\sum_{k=1}^{\infty} k(c_k(t)\overline{c_k(t)})' w^k + \sum_{k=1}^{\infty} (4 - k^2|c_k(t)|^2) w^k \frac{1 - w}{1 + w} \right) dt$$

$$= \int_0^{\infty} \frac{e^t w}{1 - w^2} \left(\frac{1 + w}{1 - w} \left(1 + \sum_{k=1}^{\infty} \left(\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{\frac{\partial f_t(z_1)}{\partial t}}{f_t(z_1)} k\overline{c_k(t) z_1^k}\, d\theta \right) w^k \right) \right.$$

$$+ \frac{1 + w}{1 - w} \left(1 + \sum_{k=1}^{\infty} \left(\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{\overline{\frac{\partial f_t(z_1)}{\partial t}}}{\overline{f_t(z_1)}} k c_k(t) z_1^k\, d\theta \right) w^k \right)$$

$$\left. - 2\left(\frac{1 + w}{1 - w}\right) + \frac{4w}{1 - w} + \sum_{k=1}^{\infty} -k^2|c_k(t)|^2 w^k \right) dt$$

$$= \int_0^{\infty} \frac{e^t w}{1 - w^2} \left(1 + \sum_{k=1}^{\infty} \left(\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{\frac{\partial f_t(z_1)}{\partial t}}{f_t(z_1)} (2(1 + \cdots + k\overline{c_k(t) z_1^k}) - k\overline{c_k(t) z_1^k})\, d\theta \right) w^k \right.$$

$$+ 1 + \sum_{k=1}^{\infty} \left(\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{\overline{\frac{\partial f_t(z_1)}{\partial t}}}{\overline{f_t(z_1)}} (2(1 + \cdots + k c_k(t) z_1^k) - k c_k(t) z_1^k)\, d\theta \right) w^k$$

$$\left. - 2 + \sum_{k=1}^{\infty} -k^2|c_k(t)|^2 w^k \right) dt$$

$$= \int_0^\infty \frac{e^t w}{1-w^2}\left(\sum_{k=1}^\infty \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\partial f_t(z_1)}{\partial t}\bigg/ \frac{z_1 \frac{\partial f_t(z_1)}{\partial z_1}}{f_t(z_1)}\right\}\left(1+\sum_{l=1}^k l c_l(t) z_1^l\right)\right.$$

$$\times\, (2(1+\cdots+k\overline{c_k(t)z_1^k}) - k\overline{c_k(t)z_1^k})\, d\theta\, w^k$$

$$+ \sum_{k=1}^\infty \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\left\{\overline{\frac{\partial f_t(z_1)}{\partial t}\over f_t(z_1)}\bigg/ \overline{\frac{z_1 \frac{\partial f_t(z_1)}{\partial z_1}}{f_t(z_1)}}\right\}\left(1+\sum_{l=1}^\infty \overline{l c_l(t) z_1^l}\right)$$

$$\times\, (2(1+\cdots+k c_k(t)z_1^k) - k c_k(t)z_1^k)\, d\theta\, w^k$$

$$\left. + \sum_{k=1}^\infty -k^2 |c_k(t)|^2 w^k\right) dt;$$

by Cauchy's theorem,

$$= \int_0^\infty \frac{e^t w}{1-w^2}\sum_{k=1}^\infty \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left\{\frac{\frac{\partial f_t(z_1)}{\partial t}}{z_1\frac{\partial f_t(z_1)}{\partial z_1}}\right\}$$

$$\times\, |2(1+\cdots+k c_k(t)z_1^k) - k c_k(t) z_1^k|^2\, d\theta \quad w^k\, dt$$

$$= \int_0^\infty \frac{e^t w}{1-w^2}\left(\sum_{k=1}^\infty A_k(t) w^k\right) dt, \qquad A_k(t) \geqslant 0, \quad \text{for } t \geqslant 0, \quad k = 1, 2, \ldots,$$

and since we shall show that

$$\frac{e^t w^{k+1}}{1-w^2} = \sum_{n=0}^\infty \Lambda_k^n(t) z^{n+1}$$

with $\Lambda_i^j(t) \geqslant 0$ for $t \geqslant 0$, this implies the Bieberbach conjecture.

We now show that $\Lambda_i^j(t) \geqslant 0$ for $t \geqslant 0$. We have:

$$\frac{z}{1-2z(\cos^2\phi + \sin^2\phi\cos\theta) + z^2} = \frac{e^t w}{1-w^2} + 2\sum_{k=1}^\infty \frac{e^t w^{k+1}}{1-w^2}\cos k\theta,$$

where $\sin\phi = e^{-t/2}$. Now consider

$$\frac{1}{\sqrt{1-2z(\cos^2\phi + \sin^2\phi\cos\theta) + z^2}} = \sum_{n=0}^\infty P_n(\cos^2\phi + \sin^2\phi\cos\theta) z^n$$

where P_n is the nth Legendre polynomial. By the addition theorem for Legendre polynomials

$$P_n(\cos^2\phi + \sin^2\phi\cos\theta) = (P_n(\cos\phi))^2 + 2\sum_{k=1}^n \frac{(n-k)!}{(n+k)!}(P_n^k(\cos\phi))^2 \cos k\theta,$$

where P_n^k is the associated Legendre function. Thus

$$\frac{1}{\sqrt{1-2z(\cos^2\phi + \sin^2\phi\cos\theta) + z^2}}$$

$$= \sum_{n=0}^\infty (P_n(\cos\phi))^2 z^n + 2\sum_{k=1}^\infty \left(\sum_{n=k}^\infty \frac{(n-k)!}{(n+k)!}(P_n^k(\cos\phi))^2 z^n\right)\cos k\theta.$$

The result now follows by squaring the above and multiplying by z.

The case of equality for the Bieberbach conjecture follows immediately from consideration of $A_1(t)$ and Lemma 2, using the fact that $|a_2| = 2$ implies that $f(z) = z/(1-\gamma z)^2$, $|\gamma| = 1$.

Acknowledgement. I wish to thank Professors Smale, Schiffer, Sarnak, Erdös, Duren, and Osgood for their imagination and encouragement, and Professors Schiffer and Smale in particular, Professor Schiffer for checking the correctness of the proof, and Professor Smale for giving his seminar on the Bieberbach conjecture. I would also like to thank all the mathematicians who have contributed toward this paper.

REFERENCES

1. L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, Sitzungsber. Preuss. Akad. Wiss. (1916), 940–955.
2. L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.
3. P. R. Garabedian and M. Schiffer, *A proof of the Bieberbach conjecture for the fourth coefficient*, J. Rational Mech. Anal. **4** (1955), 427–465.
4. C. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitkreises, I*, Math. Ann. **89** (1923), 103–121.
5. I. M. Milin, *Univalent functions and orthonormal systems*, Izdat. "Nauka", Moscow, 1971; Transl. Math. Monographs **49**, Amer. Math. Soc., Providence, 1977.
6. S. Smale, Lecture notes, informal, Univ. of California, Berkeley, 1987.

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Weinstein's Proof

- Weinstein computes the generating function of the (negative) Milin expression using the Pick function W:

$$
\begin{aligned}
M(z) &:= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(n+1-k)\left(\frac{4}{k}-k|d_k(0)|^2\right)\right)z^{n+1} \\
&= \frac{z}{(1-z)^2}\sum_{k=1}^{\infty}\left(\frac{4}{k}-k|d_k(0)|^2\right)z^k \\
&= -\int_0^{\infty}\frac{e^t W}{(1-W)^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=1}^{\infty}\left(\frac{4}{k}-k|d_k(t)|^2\right)W^k\right)\mathrm{d}t\,.
\end{aligned}
$$

Weinstein's Computation

- Using the Loewner differential equation Weinstein shows that for the generating function $M(z)$ one finally gets the equation

$$M(z) = \sum_{n=1}^{\infty} \left(\int_0^{2\pi} \Lambda_k^n(t) A_k(t) \mathrm{d}t \right) z^{n+1}$$

where $A_k(t) \geqq 0$ (by the Loewner differential equation) and

$$\frac{e^t W(z,t)^{k+1}}{1 - W(z,t)^2} =: \sum_{n=1}^{\infty} \Lambda_k^n(t) z^{n+1} \ .$$

End of Proof

- Hence, Milin's conjecture follows if the coefficients $\Lambda_k^n(t)$ of the function

$$L_k(z,t) := \frac{e^t W(z,t)^{k+1}}{1 - W(z,t)^2} = \sum_{n=1}^{\infty} \Lambda_k^n(t) z^{n+1}$$

  satisfy the relation $\Lambda_k^n(t) \geqq 0$.

- Weinstein shows this relation with the aid of the Addition theorem of the Legendre polynomials (1782).

De Branges versus Weinstein

- The question was posed to identify the *Weinstein functions* $\Lambda_k^n(t)$.

- Todorov (1992) and Wilf (1994) independently proved the surprising identity

$$\dot{\tau}_k^n(t) = -k\,\Lambda_k^n(t)\ .$$

- This shows that the t-derivatives of the de Branges functions and the Weinstein functions essentially agree. In particular, the essential inequalities are the same.

Another Generating Function

- The strong relation between the de Branges functions and the Koebe function can be seen by their generating function w.r.t. n [Koepf, Schmersau (1996)]:

$$B_k(z,t) \;:=\; \sum_{n=k}^{\infty} \tau_k^n(t)\, z^{n+1} = K(z)\, W(z,t)^k$$

$$= \; K(z)^{k+1} e^{-kt} {}_2F_1\left(\left.\begin{matrix} k, k+1/2 \\ 2k+1 \end{matrix}\right| -4K(z)e^{-t}\right)$$

$$= \sum_{n=k}^{\infty} e^{-kt} \binom{n+k+1}{2k+1} {}_4F_3\left(\left.\begin{matrix} k+\frac{1}{2}, n+k+2, k, k-n \\ k+1, 2k+1, k+\frac{3}{2} \end{matrix}\right| e^{-t}\right) z^{n+1}.$$

Automatic Computation of Power Series

- Given an expression $f(x)$ in the variable x, one would like to find the Taylor series

$$f(x) = \sum_{k=0}^{\infty} A_k \, x^k \ ,$$

i.e., a formula for the coefficient A_k.

- For example, if $f(x) = e^x$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \, x^k \ ,$$

hence $A_k = \frac{1}{k!}$.

FPS Algorithm

The main idea behind the FPS algorithm [Koepf (1992)] is

- to compute a holonomic differential equation for $f(x)$, i.e., a homogeneous linear differential equation with polynomial coefficients,
- to convert the differential equation to a holonomic recurrence equation for A_k,
- and to solve the recurrence equation for A_k.

The above procedure is successful at least is $f(x)$ is a hypergeometric power series.

FPS Algorithm for $W(z,t)^k$

- As a first application of the FPS algorithm, we compute the Taylor series of $w(z,y)^k = W(z,-\ln y)^k$, considered as function of the variable $y = e^{-t}$,

$$w(z,y)^k = \frac{(4yz)^k}{\left(1 - z + \sqrt{1-2(1-2y)z+z^2}\right)^{2k}}\,.$$

which turns out to be a hypergeometric power series.

- After multiplying by $K(z)$, we apply the FPS procedure a second time, this time w.r.t. the variable z, and get the hypergeometric representation of $\tau_k^n(t)$.

UNIKASSEL VERSITÄT

Closed Form Representation of the Weinstein functions

- An application of the same method automatically generates the hypergeometric representation for the Weinstein functions

$$\Lambda_k^n(z,t) = e^{-kt} \binom{k+n+1}{1+2\,k} \, {}_3F_2\left(\left. \begin{array}{c} \frac{1}{2}+k,\; k-n,\; 2+k+n \\ 1+2\,k,\; \dfrac{3}{2}+k \end{array} \right| e^{-t} \right)$$

directly from their defining generating function

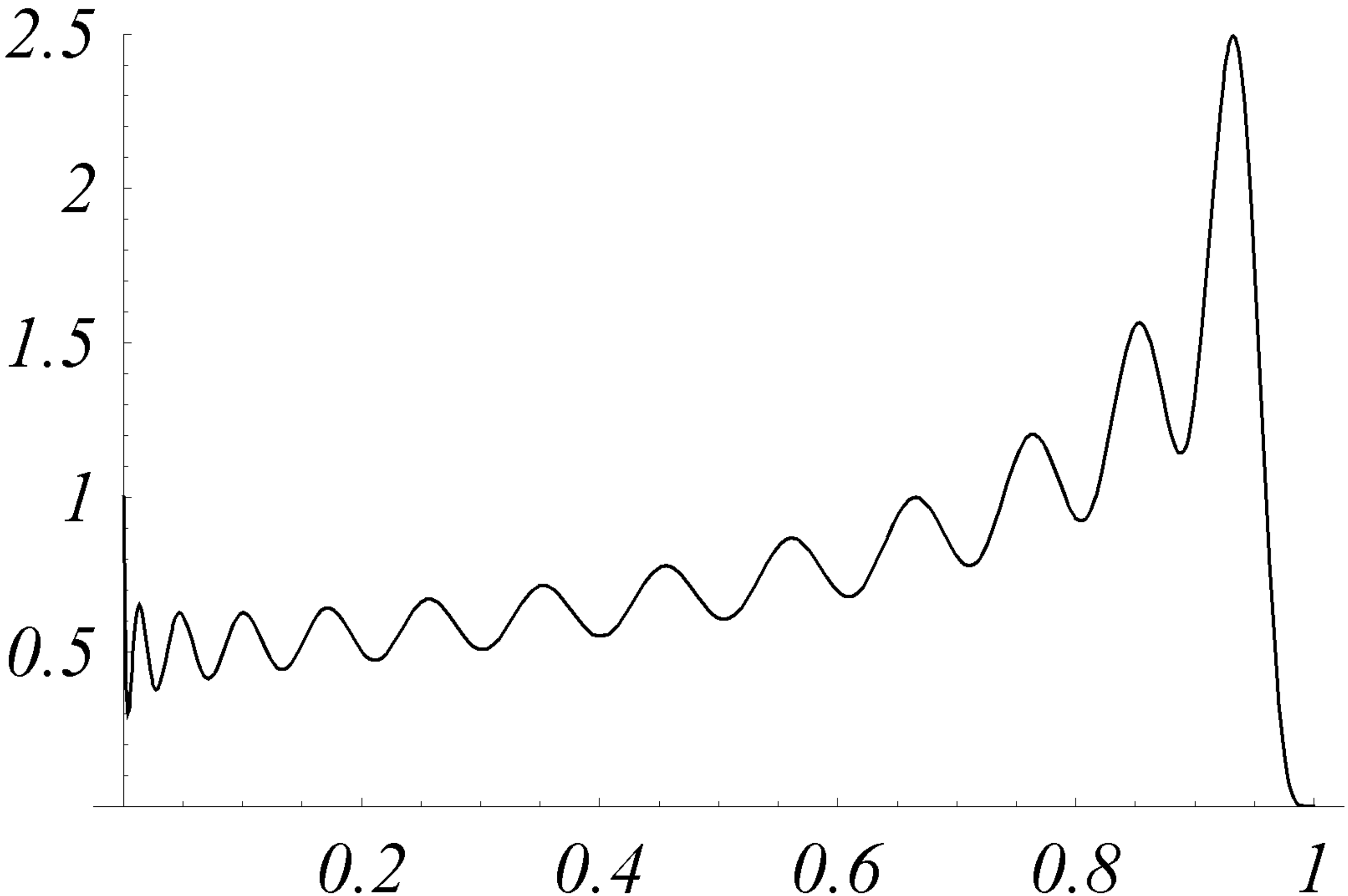
$$\frac{e^t W(z,t)^{k+1}}{1-W(z,t)^2} = \sum_{n=1}^{\infty} \Lambda_k^n(t) z^{n+1} \; .$$

UNIKASSEL VERSITÄT

Positivity for specific n

- When de Branges had found his function system $\tau_k^n(t)$, he was able to check the Bieberbach conjecture by hand computations for $n \leqq 6$—exactly the values for which it was already known. For this purpose he proved the nonpositivity of $\dot{\tau}_k^n(t)$ for $1 \leqq n \leqq 5$, $1 \leqq k \leqq n$, $t \geqq 0$.

- He asked his colleague Walter Gautschi from Purdue University to verify this inequality numerically, which was done for $n \leqq 30$, and de Branges became confident of the validity of the general statement.



Λ_6^{30} shows the highly oscillatory character of the Weinstein functions

Sturm Sequences

- Note, however, although Gautschi (under several methods) used Sturm sequences, because of the oscillatory nature of Λ_k^n his numerical computations had to be very careful to obtain correct results.

- Nowadays, we can apply Sturm sequences in a computer algebra system and count the roots easily by rational arithmetic to obtain correct countings since the input polynomials have rational coefficients.

- In this way Milin's conjecture for $n \leq 30$, e. g., is easily checked within seconds with a PC of these days.