On the de Branges and Weinstein Functions

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KÖNIGLICH PREUSSISCHEN

AKADEMIE DER WISSENSCHAFTEN.

Gesamtsitzung vom 20. Juli.

Mitteilung vom 6. Juli.

Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln.

Von Prof. Dr. Ludwig Bieberbach

in Frankfurt a. M.

haben damit nicht nur unseren Satz IV bewiesen, sondern darüber hinaus auch erkannt, daß die Zahl r_2 des Satzes IV die 2 ist, und daß für a_2 auch wirklich alle Werte dieses Kreises $|a_2| \leq 2$ vorkommen. Wenn es mir auch nicht gelungen ist, für die andern Koeffizienten ein ähnlich abschließendes Resultat zu erreichen, so möchte ich doch noch zeigen, daß *auch der Wertevorrat jedes andern Koeffizienten gerade einen Kreis erfüllt.* Das folgt einfach daraus, daß für $|k| \leq 1$ mit $f(z)$ stets auch $\frac{1}{k}f(kz)$ für $|z| < 1$ regulär und schlicht ist. Der $n$te Koeffizient dieser Funktion ist aber $a_n k^{n-1}$. Ist also $a_n^{(0)}$ $n$ter Koeffizient einer schlichten Abbildung, so auch alle a_n aus dem Kreis $|a_n| \leq |a_n^{(0)}|$. Darin liegt bekanntermaßen unsere Behauptung[1]. Man muß sich indessen hüten, dies Resultat in zu starkem Maße umzukehren. Es bildet ganz und gar nicht jede Funktion schlicht ab, deren Koeffizienten den gefundenen Kreisen angehören. Z. B. bildet schon die Funktion $w = z + 2z^2$ den Kreis $|z| \leq 1$ nicht auf einen schlichten Bereich ab, denn wir haben gesehen, daß $\sum nz^n$ die einzige schlicht abbildende Funktion mit $a_2 = 2$ ist[2].

Wir ziehen noch eine Folgerung aus diesen Betrachtungen:

Satz V. Wenn $|z| > 1$ durch $w = F(z) = z + \frac{\alpha_1}{z} + \cdots$ schlicht abgebildet wird, so liegen alle Randpunkte des Bildbereiches im Kreise $|w| \leq 2$, und es finden sich auf diesem Kreise nur dann Randpunkte des Bildbereiches, wenn es sich um die durch die Funktion $F(z) = z + \frac{1}{z}$ vermittelte Abbildung von $|z| > 1$ auf die von -1 bis $+1$ aufgeschlitzte Ebene handelt, oder wenn die Abbildungsfunktionen $\frac{1}{e^{i\varphi}}F(e^{i\varphi}z)$ vorliegen, die gleichfalls auf Schlitzbereiche abbilden, welche aus den eben genannten durch Drehung hervorgehen[3,4].

[1] Daß $k_n \geq n$ zeigt das Beispiel $\sum nz^n$. Vielleicht ist überhaupt $k_n = n$.

[2] Die hier gefundene Tatsache, daß 2 die genaue Schranke für $|a_2|$ ist, erlaubt es, gewisse Untersuchungen über den Koebeschen Verzerrungssatz zu Ende zu führen, welche schon Hr. Plemelj auf der Wiener Naturforscherversammlung vorgetragen hat, und die unabhängig davon kürzlich Hr. Pick angestellt und (Leipz. Ber. 1916) veröffentlicht hat.

[3] Man vgl. zu diesem Satz einen von Koebe, Göttinger Nachrichten 1908, S. 348. Der hier bewiesene Satz V liefert zugleich den Koebeschen und zeigt, daß der genaue Wert der Konstanten, deren Existenz dort bewiesen ist, die 4 ist, und daß diese Konstante nur bei Schlitzabbildungen erreicht wird.

[4] Der Satz ist ferner nahe verwandt mit dem Satz I, den ich auf S. 153 von Bd. 77 der Math. Ann. aufgestellt habe, besagt aber ersichtlich noch etwas mehr als dieser.

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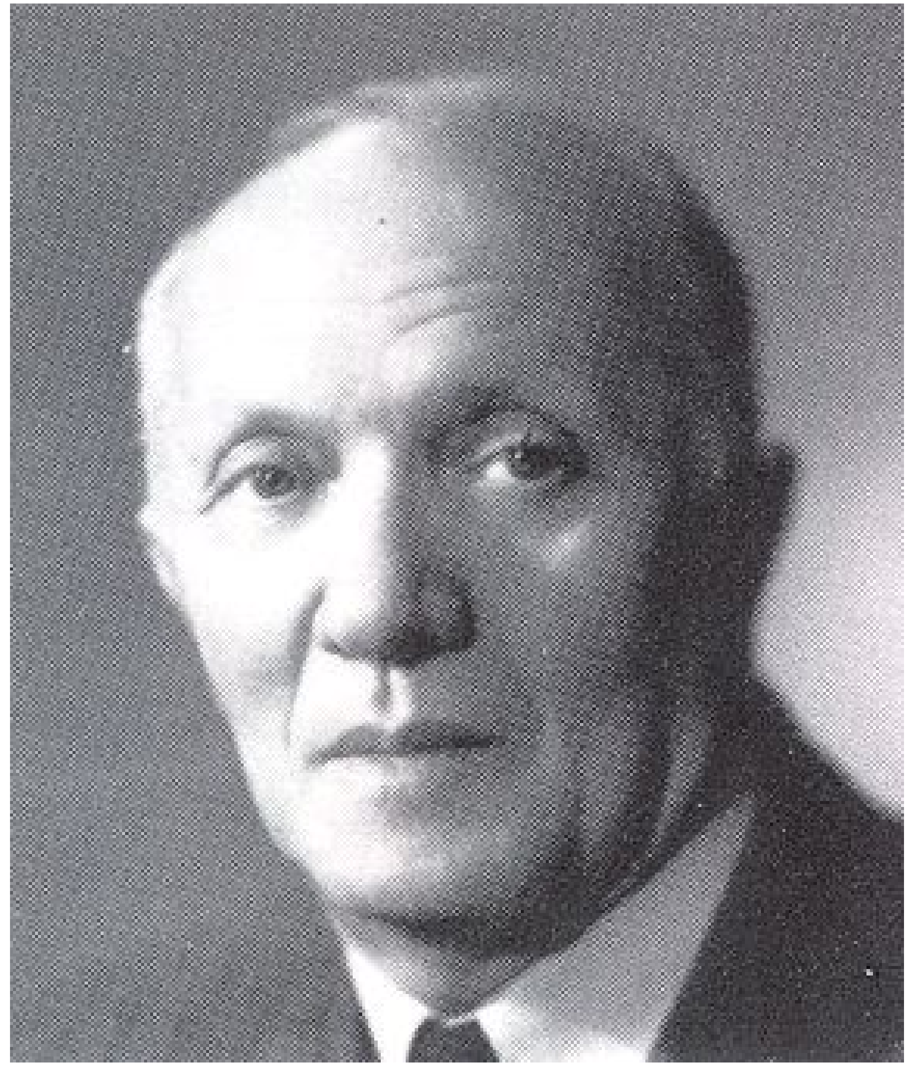
Ludwig Bieberbach conjectured $|a_n| \leq n$ (1916)

Charles Loewner proved $|a_3| \leq 3$ (1923)

N. A. Lebedev and I. M. Milin invented the logarithmic coefficients and deduced the Milin Conjecture (1965)

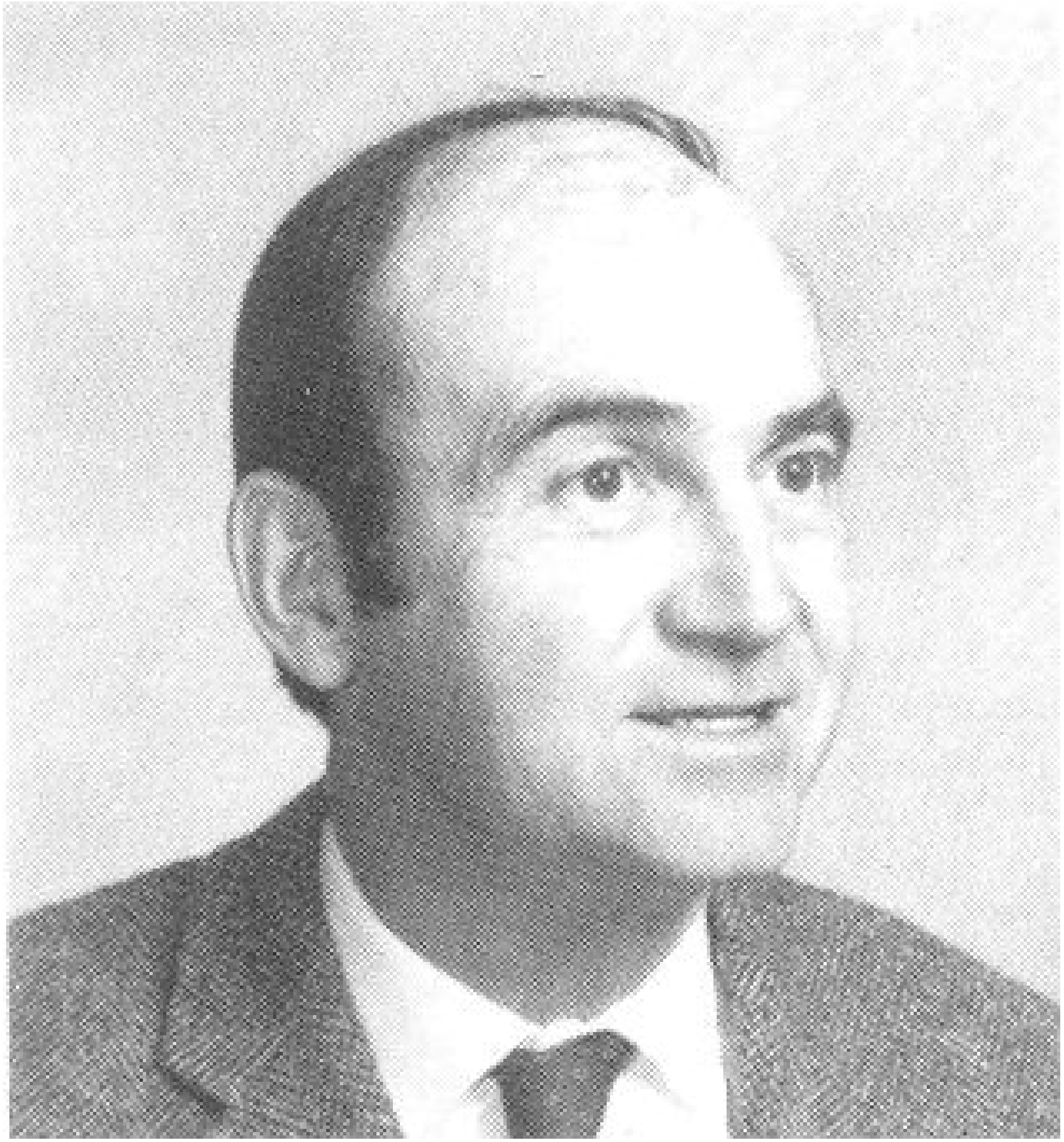
The de Branges Theorem

- In 1984 Louis de Branges proved the Milin and therefore the Bieberbach conjecture for all $n \in \mathbb{N}$.

- In 1991 Lenard Weinstein gave a completely independent proof of the Milin conjecture.

- Nevertheless, it turned out that the two proofs share more than expected.

- Large parts of both proofs can be computerized.



Louis de Branges proved the Milin conjecture (1984)

De Branges' Proof

- De Branges considered the function

$$\psi(t) := \sum_{k=1}^{n} \tau_k^n(t) \left(k|d_k(t)|^2 - \frac{4}{k} \right)$$

  $d_k(t)$ denoting the logarithmic coefficients of $e^{-t} f(z,t)$, where $f(z,t)$ is a Loewner chain of f.

- Applying Loewner's theory he could show that for suitably chosen functions $\tau_k^n(t)$ the relation $\dot{\psi}(t) \geqq 0$ and therefore $\psi(0) = -\int_0^\infty \dot{\psi}(t) \mathrm{d}t \leqq 0$ follows.

- Inequality $\psi(0) \leqq 0$, however, is Milin's conjecture.

The de Branges Functions

- The de Branges functions $\tau_k^n(t), k = 1, \ldots, n$ are defined by the coupled system of differential equations

$$\tau_k^n(t) - \tau_{k+1}^n(t) = -\frac{1}{k}\,\dot{\tau}_k^n(t) - \frac{1}{k+1}\,\dot{\tau}_{k+1}^n(t)$$

with the initial values

$$\tau_k^n(0) = n + 1 - k\ .$$

Further Properties of the de Branges Functions

- By these properties the family $\tau_k^n(t)$ is already uniquely determined. For the success of de Branges' proof, however, one needs moreover the properties

$$\lim_{t\to\infty} \tau_k^n(t) = 0$$

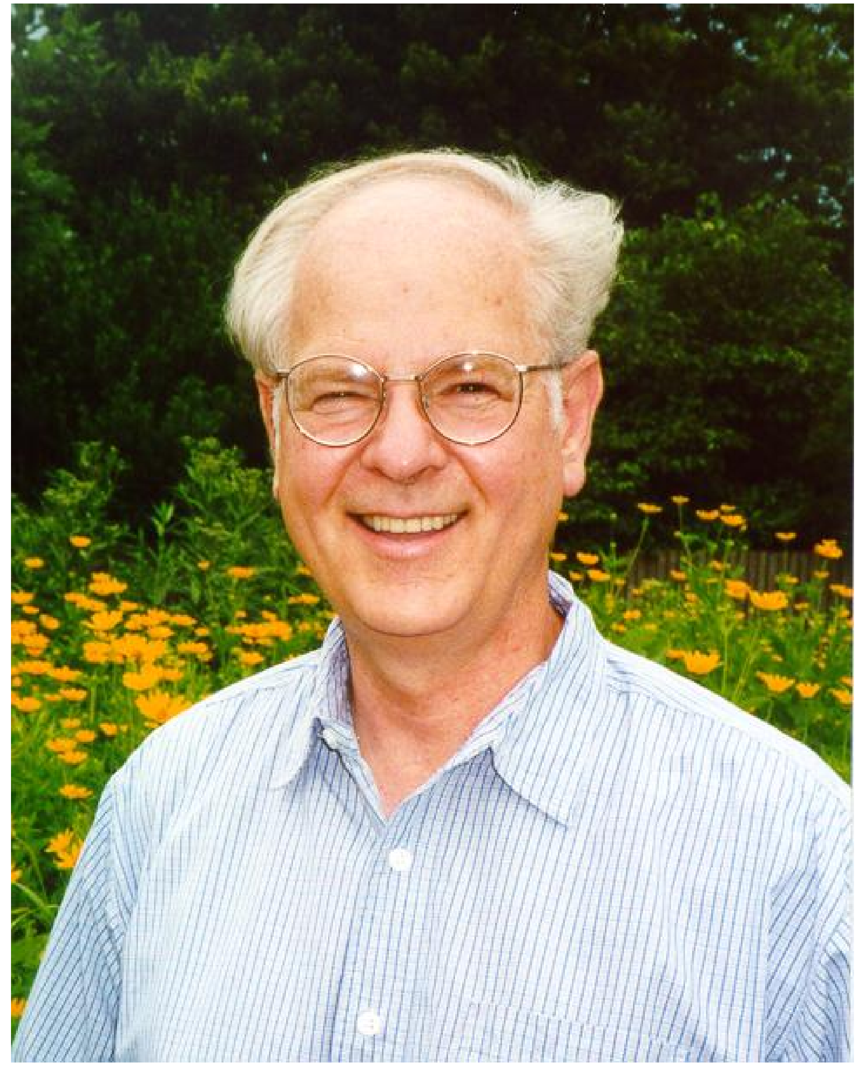
as well as

$$\dot{\tau}_k^n(t) \leqq 0 \quad (t \geqq 0) .$$

The Askey-Gasper Inequality

- Whereas the limit $\lim_{t \to \infty} \tau_k^n(t) = 0$ can be established easily, de Branges could not verify the relation $\dot{\tau}_k^n(t) \leqq 0$.

- By a phone call of Walter Gautschi with Dick Askey, de Branges finally realized that this relation had been proved by Askey and Gasper not long before, namely in 1976. To find this connection, an explicit representation of his functions $\tau_k^n(t)$ was necessary.

Dick Askey and George Gasper

POSITIVE JACOBI POLYNOMIAL SUMS, II.

By Richard Askey* and George Gasper.**

Abstract. Among the positive polynomial sums of Jacobi polynomials, there are two which have been very useful, the Cesàro means of the formal reproducing kernel and the sum (*) $\sum_{k=0}^{n} P_k^{(\alpha,\beta)}(x)/P_k^{(\beta,\alpha)}(1)$, which was first considered by Fejér when $\alpha=1/2$, $\beta=\pm 1/2$ and when $\alpha=\beta=0$. A conjecture is given which connects these two sets of inequalities and this conjecture is proven for many values of (α,β). In particular, it is shown that if $\beta \geqslant 0$, then the sum (*) is nonnegative for $-1 \leqslant x \leqslant 1$ if and only if $\alpha+\beta \geqslant -2$. It is also shown that $\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}}{(n-k)!} \frac{(\lambda+1)_k}{k!} \frac{\sin(k+1)\theta}{k+1} > 0$, $0<\theta<\pi$, $-1<\lambda\leqslant 1$, and that for real α the function $(1-r)^{-|\alpha|}[1 \pm r + (1-2xr+r^2)^{1/2}]^{\alpha}$ is absolutely monotonic for $-1 \leqslant x \leqslant 1$, i.e., it has nonnegative power series coefficients when it is expanded in a power series in r. Limiting cases involving Laguerre and Hermite polynomials are also considered.

1. Introduction. Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (n+\alpha+\beta+1)_k}{k!(\alpha+1)_k} \left(\frac{1-x}{2}\right)^k$$

$$= \frac{(\alpha+1)_n}{n!} {}_2F_1[-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2], \qquad (1.1)$$

where $(a)_0 = 1, (a)_k = a(a+1)\ldots(a+k-1), k \geqslant 1$. These polynomials are orthogonal on the interval $[-1,1]$ when $\alpha, \beta > -1$; but we shall also consider them for other values of α, β. Jacobi polynomials are the most general class of functions for which it is now possible to prove many deep results, and they contain as special cases or limiting cases most of the useful classical functions.

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In order to prove further cases of Conjecture 1 we derive the identity

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(x)$$

$$= \sum_{j=0}^{[n/2]} \frac{\left(\frac{1}{2}\right)_j \left(\frac{\alpha+2}{2}\right)_{n-j} \left(\frac{\alpha+3}{2}\right)_{n-2j} (n-2j)!}{j! \left(\frac{\alpha+3}{2}\right)_{n-j} \left(\frac{\alpha+1}{2}\right)_{n-2j} (\alpha+1)_{n-2j}} \left\{ C_{n-2j}^{(\alpha+1)/2}\left(\left(\frac{1+x}{2}\right)^{1/2}\right)\right\}^2, \tag{1.16}$$

where $C_n^\lambda(x)$ is the ultraspherical polynomial of order λ, and use it to obtain

Hypergeometric Functions

- The power series

$$
{}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| \, x \right) = \sum_{k=0}^{\infty} A_k \, x^k = \sum_{k=0}^{\infty} a_k \ ,
$$

whose coefficients A_k have a rational term ratio

$$
\frac{a_{k+1}}{a_k} = \frac{A_{k+1} \, x^{k+1}}{A_k \, x^k} = \frac{(k+a_1)\cdots(k+a_p)}{(k+b_1)\cdots(k+b_q)} \cdot \frac{x}{k+1} \ ,
$$

is called the generalized hypergeometric function.

Coefficients of the Hypergeometric Function

- For the coefficients of the hypergeometric function that are called hypergeometric terms, one gets the formula

$$ {}_pF_q\left(\left.\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} \,, $$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is called the Pochhammer symbol or shifted factorial.

Askey-Gasper Identity

- The Askey-Gasper inequality was proved by detecting ingeniously the Askey-Gasper identity

$$\frac{(\alpha+2)_n}{n!}\cdot {}_3F_2\left(\left.\begin{matrix}-n, n+2+\alpha, \frac{\alpha+1}{2}\\ \alpha+1, \frac{\alpha+3}{2}\end{matrix}\right| x\right)$$

$$= \sum_{j=0}^{\lfloor n/2\rfloor} \frac{\left(\frac{1}{2}\right)_j \left(\frac{\alpha}{2}+1\right)_{n-j} \left(\frac{\alpha+3}{2}\right)_{n-2j} (\alpha+1)_{n-2j}}{j! \left(\frac{\alpha+3}{2}\right)_{n-j} \left(\frac{\alpha+1}{2}\right)_{n-2j} (n-2j)!}.$$

$${}_3F_2\left(\left.\begin{matrix}2j-n, n-2j+\alpha+1, \frac{\alpha+1}{2}\\ \alpha+1, \frac{\alpha+2}{2}\end{matrix}\right| x\right)$$

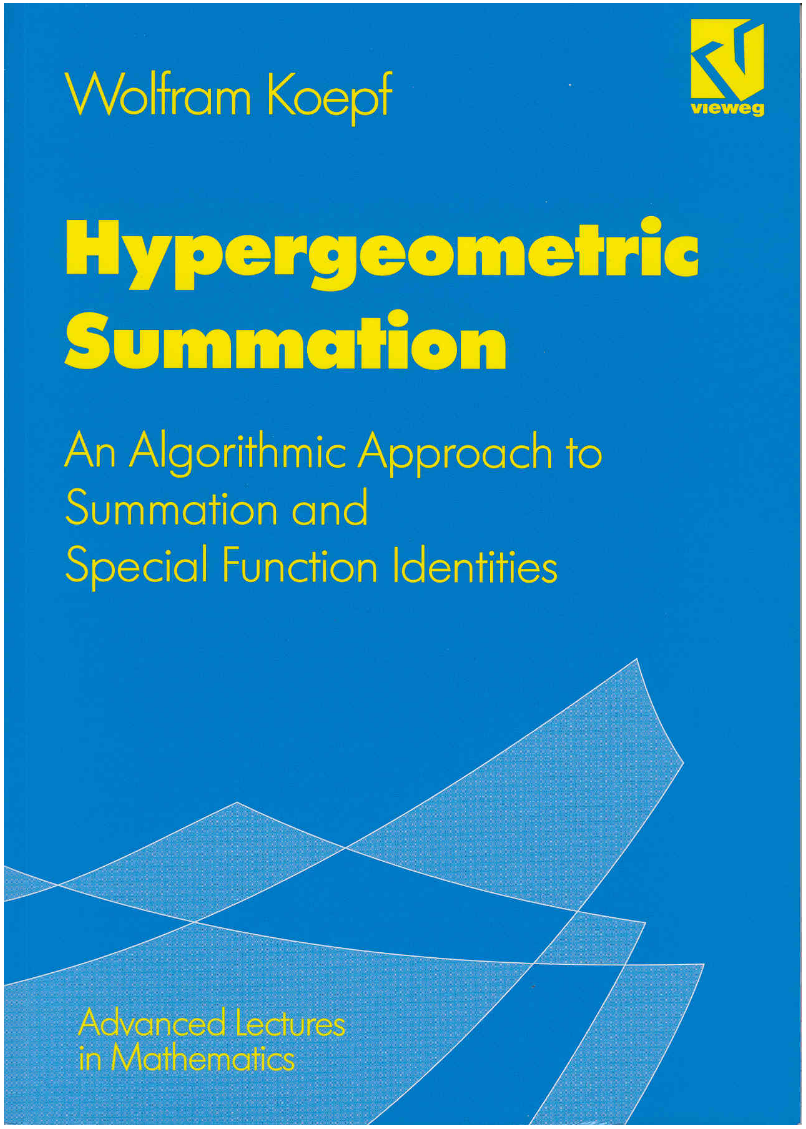
Computer Algebra Proof of the Askey-Gasper Identity

- Using Zeilberger's algorithm which was developed in 1990, computer calculations can prove the Askey-Gasper identity easily.

- This proof goes from right to left, but there is no way to detect the right hand side from the left hand side. This part still needs Askey's and Gasper's ingenuity.

- Computer demonstration using *Maple*

The software used was developed in connection with my book



Hypergeometric Summation, Vieweg, 1998, Braunschweig/Wiesbaden

and can be downloaded from my home page:

http://www.mathematik.uni-kassel.de/~koepf

Clausen's Identity

- The clue of the Askey-Gasper identity is the fact that the hypergeometric function occurring in its right hand summand is a complete square by Clausen's identity

$$
{}_3F_2\left(\left.\begin{array}{c} 2a, 2b, a+b \\ 2a+2b, a+b+1/2 \end{array}\right| x\right) = {}_2F_1\left(\left.\begin{array}{c} a, b \\ a+b+1/2 \end{array}\right| x\right)^2 .
$$

- Clausen's identity can also be proved by Zeilberger's algorithm. For this purpose, we write the right hand side as a Cauchy product.

Weinstein's Proof

- In 1991 Weinstein published a completely different proof of the Milin conjecture.
- Whereas de Branges takes fixed $n \in \mathbb{N}$, Weinstein considers the conjecture for all $n \in \mathbb{N}$ at the same time.
- For his proof Weinstein needs the following Loewner chain of the Koebe function

$$K(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=0}^{\infty} n z^n \,,$$

sometimes called Pick function.

Loewner Chain of the Koebe Function

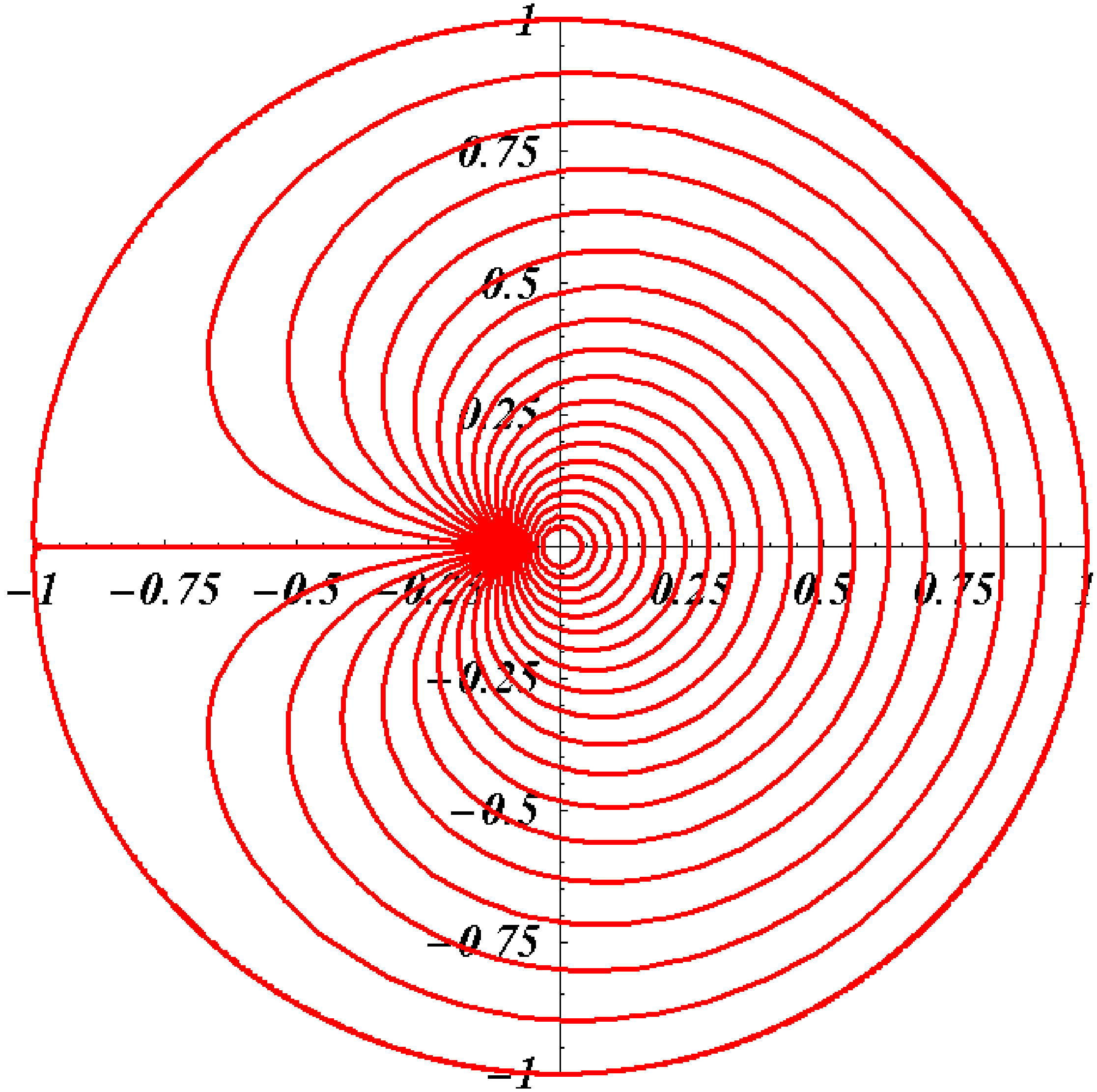
- This function family $W : \mathbb{D} \times \mathbb{R}_{\geqq 0} \to \mathbb{D}$ is given by

$$W(z,t) = K^{-1}\Big(e^{-t}K(z)\Big) \,.$$

- For any particular $t > 0$ the range of this function is the unit disk with a radial slit that grows with growing t.

- A computation shows that

$$W(z,t) = \frac{4e^{-t}z}{\left(1 - z + \sqrt{1 - 2(1 - 2e^{-t})z + z^2}\right)^2} \,.$$



The mapping behavior of $W(z,t)$

Weinstein's Proof

- Weinstein computes the generating function of the (negative) Milin expression using the Pick function W:

$$
\begin{aligned}
M(z) &:= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(n+1-k)\left(\frac{4}{k}-k|d_k(0)|^2\right)\right)z^{n+1} \\
&= \frac{z}{(1-z)^2}\sum_{k=1}^{\infty}\left(\frac{4}{k}-k|d_k(0)|^2\right)z^k \\
&= -\int_0^{\infty}\frac{e^t W}{(1-W)^2}\frac{\mathsf{d}}{\mathsf{d}t}\left(\sum_{k=1}^{\infty}\left(\frac{4}{k}-k|d_k(t)|^2\right)W^k\right)\mathsf{d}t\,.
\end{aligned}
$$

Weinstein's Proof

- Using the Loewner differential equation Weinstein shows that for the generating function $M(z)$ one finally gets the equation

$$M(z) = \sum_{n=1}^{\infty} \left(\int_0^{2\pi} \Lambda_k^n(t) A_k(t) \mathrm{d}t \right) z^{n+1}$$

where $A_k(t) \geqq 0$ (by the Loewner differential equation) and

$$\frac{e^t W(z,t)^{k+1}}{1 - W(z,t)^2} =: \sum_{n=1}^{\infty} \Lambda_k^n(t) z^{n+1} \; .$$

End of Proof

- Hence, Milin's conjecture follows if the coefficients $\Lambda_k^n(t)$ of the function

$$L_k(z,t) := \frac{e^t W(z,t)^{k+1}}{1-W(z,t)^2} = \sum_{n=1}^{\infty} \Lambda_k^n(t) z^{n+1}$$

satisfy the relation $\Lambda_k^n(t) \geqq 0$.

- Weinstein shows this relation with the aid of the Addition theorem of the Legendre polynomials (1782).

De Branges versus Weinstein

- The question was posed to identify the *Weinstein functions* $\Lambda_k^n(t)$.

- Todorov (1992) and Wilf (1994) independently proved the surprising identity

$$\dot{\tau}_k^n(t) = -k\,\Lambda_k^n(t)\;.$$

- This shows that the t-derivatives of the de Branges functions and the Weinstein functions essentially agree. In particular, the essential inequalities are the same.

Another Generating Function

- The strong relation between the de Branges functions and the Koebe function can be seen by their generating function w.r.t. n [Koepf, Schmersau 1996]:

$$
\begin{aligned}
B_k(z,t) \;&:=\; \sum_{n=k}^{\infty} \tau_k^n(t)\, z^{n+1} = K(z)\, W(z,t)^k \\
&=\; K(z)^{k+1} e^{-kt}\, {}_2F_1\left(\left.\begin{array}{c} k,\, k+1/2 \\ 2k+1 \end{array}\right| -4K(z)e^{-t}\right)
\end{aligned}
$$

$$
= \sum_{n=k}^{\infty} e^{-kt} \binom{n+k+1}{2k+1} {}_4F_3\left(\left.\begin{array}{c} k+\frac{1}{2},\, n+k+2,\, k,\, k-n \\ k+1,\, 2k+1,\, k+\frac{3}{2} \end{array}\right| e^{-t}\right) z^{n+1} .
$$

Automatic Computation of Power Series

- Given an expression $f(x)$ in the variable x, one would like to find the Taylor series

$$f(x) = \sum_{k=0}^{\infty} A_k\, x^k \ ,$$

i.e., a formula for the coefficient A_k.

- For example, if $f(x) = e^x$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k \ ,$$

hence $A_k = \frac{1}{k!}$.

FPS Algorithm

The main idea behind the FPS algorithm [Koepf 1992] is

- to compute a holonomic differential equation for $f(x)$, i.e., a homogeneous linear differential equation with polynomial coefficients,

- to convert the differential equation to a holonomic recurrence equation for A_k,

- and to solve the recurrence equation for A_k.

The above procedure is successful at least is $f(x)$ is a hypergeometric power series.

FPS Algorithm for $W(z,t)^k$

- As an application of the FPS algorithm, we compute the Taylor series of $w(z,y)^k = W(z, -\ln y)^k$, considered as function of the variable $y = e^{-t}$,

$$w(z,y)^k = \frac{(4yz)^k}{\left(1 - z + \sqrt{1 - 2(1-2y)z + z^2}\right)^{2k}} .$$

  which turns out to be a hypergeometric power series.
- After multiplying by $K(z)$, we apply the FPS procedure a second time, this time w.r.t. the variable z, and get the hypergeometric representation of $\tau_k^n(t)$.

Further Properties

- The de Branges system of differential equations gives
$$S_k^n(t) = \frac{1}{2}\Big(\tau_k^n(t) + \Lambda_k^n(t)\Big) \geqq 0$$
and
$$S_{k+1}^n(t) = \frac{1}{2}\Big(\tau_k^n(t) - \Lambda_k^n(t)\Big) \geqq 0$$
hence $\tau_k^n(t)$ and $\Lambda_k^n(t)$ are the sum and difference of the same nonnegative ${}_4F_3$ hypergeometric function.
- One can also prove that $\tau_k^n(t)$ increases w.r.t. n and $e^{kt}\,\tau_k^n(t)$ increases w.r.t. t [Koepf, Schmersau 2004].