Computation of Macaulay constants and degree bounds for Gröbner bases

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Abstract. In this paper, following the approach by Dubé [6] and by applying the Hilbert series method, we provide an efficient algorithm to compute the Macaulay constants of a monomial ideal *without computing any exact cone decomposition of the corresponding quotient ring*. Then, based on this construction and the method proposed by Mayr-Ritscher [18], a new upper bound for the maximum degree of the elements of any reduced Gröbner basis of an ideal generated by a set of homogeneous polynomials is given. The new bound depends on the Krull dimension and the maximum degree of the generating set of the ideal. Finally, we show that the presented upper bound is sharper than the bounds proposed by Dubé [6] and Mayr-Ritscher [18].

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1. Introduction

Gröbner bases (as an efficient tool in working with polynomial ideals) together with the first algorithm to compute them (known as Buchberger's algorithm) were introduced in 1965, by Buchberger in his PhD thesis, see [3, 4]. Due to the fact that these bases can be computed efficiently, they have many applications in various domains, including mathematics, science, and engineering. For example, some of these applications are in the ideal membership problem, computing the dimension of an ideal, solving polynomial systems and so on. Presenting upper bounds for the degrees of the elements of a reduced Gröbner basis is a challenging problem in the computer algebra community, because finding an effective bound is applicable for predicting the practical feasibility of the computations as well as for the complexity analysis of Gröbner bases computations, see [15].

Let us review some of the existing results in literature about degree upper bounds for Gröbner bases. Let denote by $\mathcal{R}$ the polynomial ring $\mathcal{K}[x_1, \dots, x_n]$ where $\mathcal{K}$ is a field of characteristic zero and by $\mathcal{I} \subset \mathcal{R}$ an ideal generated by *homogeneous* polynomials of degree at most d. The first doubly-exponential upper bounds for Gröbner bases were studied by Mayr, Meyer, Bayer, Möller, Mora and Giusti, see [20, Chapter 38] for a comprehensive review of the subject. In 1982, Mayr and Meyer [17] showed that the ideal membership problem has doubly exponential complexity. In 1984, Möller and Mora [19] established the degree upper bound $(2d)^{(2n+2)^{n+1}}$ for any Gröbner basis of $\mathcal{I}$. Then, Giusti [8] gave the upper bound $(2d)^{2^{n-2}}$ for the degree of the reduced Gröbner basis of $\mathcal{I}$ with respect to the degree reverse lexicographic ordering provided that $\mathcal{I}$ is in *generic position*. In 1990, Dubé [6] by applying a constructive combinatorial argument proved the degree bound $2(d^2/2+d)^{2^{n-2}}$. In 2013, Mayr and Ritscher [18] improved Dubé's bound to the dimension-dependent upper bound $2(1/2(d^{n-D}+d))^{2^{D-1}}$ for every reduced Gröbner basis of $\mathcal{I}$ where D stands for $\dim(\mathcal{I})$. Hashemi and Seiler [12] provided dimension and depth depending upper bounds for the degrees of the elements of the reduced Gröbner basis of an ideal in generic position. Finally, by giving a deeper analysis of the method due to Dubé, the authors improved (and correct) Dubé's bound to $(d+1)^{2^{n-2}}$, see [11].

One of the main ingredients in proving Dubé's upper bound [6] is the construction of the Macaulay constants of a given monomial ideal and then bounding this constants by using combinatorial arguments. This construction relies on an exact cone decomposition of the corresponding quotient ring. In this paper, by using only the Hilbert series of a given monomial ideal, we describe an effective method to calculate the Macaulay constants of the ideal *without computing any exact cone decomposition of the corresponding quotient ring*. Then, by applying this construction and the method proposed by Mayr-Ritscher [18] to bound the Macaulay constants, we give a new upper bound for the maximum degree of the elements of any reduced Gröbner basis of an ideal generated by a set of homogeneous polynomials (see Theorem 4.6). We conclude the paper by showing that our new bound is sharper than the bounds proposed by Dubé [6] and Mayr-Ritscher [18].

The structure of the paper is as follows. Section 2 reviews the basic notations and terminologies used throughout this paper. In Section 3, we describe our new method to compute the Macaulay constants of a given monomial ideal by using only the Hilbert series of the ideal. Section 4 is devoted to present our new degree upper bounds for Gröbner bases by applying the relations between the Macaulay constants and Hilbert series developed in the previous section.

2. Preliminaries

In this section, we will briefly review some basic notations and background materials which are used throughout the paper. Let $\mathcal{R}=\mathcal{K}[X]$ be the polynomial ring over an infinite field $\mathcal{K}$ where $X=\{x_1,\dots,x_n\}$ is a an ordered set of variables. Furthermore, let $\mathcal{M}$ be the set of all monomials in $\mathcal{R}$ (a monomial is a power product of the variables and is denoted by X^α with $\alpha\in\mathbb{Z}_{\geq 0}$). We consider a finite set $F=\{f_1,\dots,f_k\}\subset\mathcal{R}$ of *homogeneous* polynomials and the ideal $\mathcal{I}=\langle F\rangle$ generated by F. We denote the total degree of a polynomial $f\in\mathcal{R}$ by $\deg(f)$. The maximum degree of the polynomials in F is denoted by $d=\deg(F)$. The Krull dimension of the factor ring $\mathcal{R}/\mathcal{I}$, denoted by $D=\dim(\mathcal{I})$, is the number of elements of any maximal set $S\subseteq\{X\}$ such that $\mathcal{I}\cap\mathcal{K}[S]=\emptyset$. A total ordering on $\mathcal{M}$ is called a *monomial ordering* if,

1. for monomials $X^\alpha, X^\beta, X^\gamma\in\mathcal{M}$, $X^\alpha\prec X^\beta$ implies that $X^\alpha X^\gamma\prec X^\beta X^\gamma$.
2. for each monomial $X^\alpha\in\mathcal{M}$ it holds $1\prec X^\alpha$.

Let us fix a monomial ordering $\prec$ on $\mathcal{R}$. The *leading monomial* of a polynomial $0\neq f\in\mathcal{R}$, denoted by $\mathrm{LM}(f)$, is the greatest monomial appearing in f with respect to $\prec$ and its coefficient is called the *leading coefficient* of f, denoted by $\mathrm{LC}(f)$. The *leading term* of f is the product $\mathrm{LT}(f)=\mathrm{LC}(f)\mathrm{LM}(f)$. For $F\subset\mathcal{R}$, $\mathrm{LM}(F)$ stands for $\{\mathrm{LM}(f)\mid f\in F\}$. The *leading monomial ideal* of $\mathcal{I}$ is the monomial ideal $\mathrm{LM}(\mathcal{I})=\langle\mathrm{LM}(f)\mid 0\neq f\in\mathcal{I}\rangle$. A finite set $G\subset\mathcal{I}$ is called a *Gröbner basis* for $\mathcal{I}$ with respect to $\prec$, if $\mathrm{LM}(\mathcal{I})=\langle\mathrm{LM}(G)\rangle$. The *remainder* of division f by Gröbner basis G with respect to $\prec$, is denoted by $\mathrm{NF}_G(f)$. For a Gröbner basis G, we let $N_{\mathcal{I}}=\{\mathrm{NF}_G(f)\mid f\in\mathcal{R}\}$. We refer to [5] for more details on the theory of Gröbner bases.

Let us recall definitions of Hilbert function, Hilbert polynomial, and Hilbert series of a homogeneous ideal. If f is an arbitrary polynomial in $\mathcal{R}$, then f can be written as a finite sum of homogeneous polynomials. These homogeneous polynomials are called the *homogeneous components* of f. A subset $S\subset\mathcal{R}$ is called *homogeneous* if it is a $\mathcal{K}$-vector space and for every $f\in S$, each homogeneous component of f lies in S as well. Indeed, one sees that an ideal of $\mathcal{R}$ is homogeneous if it is a homogeneous set. Moreover, for an arbitrary ideal $\mathcal{I}\subset\mathcal{R}$, $N_{\mathcal{I}}$ is a homogeneous set. For a homogeneous set S, the set of all homogeneous polynomials in S of degree i (including zero for being a vector space) is denoted by S_i. The *Hilbert function* of S at i is defined to be $\mathrm{HF}_S(i) = \dim_{\mathcal{K}}(S_i)$; the dimension of S_i as a $\mathcal{K}$-vector space. For a homogeneous ideal $\mathcal{I}$, Hilbert function $\mathrm{HF}_{\mathcal{I}}(i)$ (resp. $\mathrm{HF}_{N_{\mathcal{I}}}(i)$) from a certain value coincides with a unique polynomial in i which is called the *Hilbert polynomial* of $\mathcal{I}$ (resp. $N_{\mathcal{I}}$) and is denoted by $\mathrm{HP}_{\mathcal{I}}$ (resp. $\mathrm{HP}_{N_{\mathcal{I}}}$). We have the equality $\dim(\mathcal{I}) = \deg(\mathrm{HP}_{N_{\mathcal{I}}}) + 1$, see [5, Theorem 12, page 494] and by Macaulay's theorem $\mathrm{HF}_{N_{\mathcal{I}}} = \mathrm{HF}_{N_{\mathrm{LT}(\mathcal{I})}}$, [5, Proposition 9, page 492]. Note that in the case that $\mathrm{HP}_{N_{\mathcal{I}}}$ is the zero polynomial, its degree is defined to be -1. The *Hilbert series* of S is the power series $\mathrm{HS}_S(t) = \sum_{i=0}^{\infty} \mathrm{HF}_S(i)t^i$. This series can be expressed as the quotient $\mathrm{HS}_{N_{\mathcal{I}}}(t) = N(t)/(1-t)^D$ with a polynomial $N \in Q[t]$ satisfying $N(1) \neq 0$ (see [7, Theorem 4.27, page 74]). In the next section, we will provide a new representation of this series using the Macaulay constants. Recall that a sequence of homogeneous polynomials $f_1, \ldots, f_k \in \mathcal{R}$ is called *regular* if $\langle f_1, \ldots, f_k \rangle \neq \mathcal{R}$ and f_i is a non-zero divisor on the ring $\mathcal{R}/\langle f_1, \ldots, f_{i-1} \rangle$ for $i = 2, \ldots, k$. It can be shown that the sequence $f_1, \ldots, f_k$ is regular iff the Hilbert series of $N_{\langle f_1, \ldots, f_k \rangle}$ is equal to $\prod_{i=1}^{k}(1 - t^{\deg(f_i)})/(1-t)^n$ (see e.g. [14, Corollary 5.2.17, page 203]).

Let us now give a short review of Dubé's method [6] and some results from [11, 18] which entail degree upper bounds for Gröbner bases. For this, we first recall some basic definitions from [6]. The notion $\mathrm{indet}(u)$ stands for the set of all variables appearing in the monomial $u \in \mathcal{M}$.

Definition 2.1. *For a given set $T \subset \mathcal{R}$, the sequence $S_1, \ldots, S_t$ (possibly infinite) of subsets of T is called a* direct decomposition *of T if every $p \in T$ can be* uniquely *expressed of the form $p = \sum_{i=j_1}^{j_r} p_i$ where $p_i \in S_i$ and $r \leq t$. This property is shown by $T = S_1 \oplus S_2 \oplus \cdots \oplus S_t$.*

Example 2.2. *As a simple example, the ideal $\mathcal{I} = \langle x_1x_2, x_2x_3 \rangle \subset \mathcal{K}[x_1, x_2, x_3]$ has a direct decomposition $x_1x_2 \cdot \mathcal{K}[x_1, x_2, x_3] \oplus x_2x_3 \cdot \mathcal{K}[x_2, x_3]$.*

Dubé in [6, Example 3] suggested the following construction to find a direct decomposition for a given ideal: Let $F = \{f_1, \ldots, f_k\}$ be a homogeneous generating set for the ideal $\mathcal{I} \subset \mathcal{R}$. Then, there exists the following decomposition for $\mathcal{I}$:

$$\mathcal{I} = \langle f_1 \rangle \oplus \bigoplus_{i=2}^{k} f_i \cdot N_{\langle f_1, \ldots, f_{i-1} \rangle : f_i}. \tag{1}$$

Note that the decomposition $\mathcal{R} = \mathcal{I} \oplus N_{\mathcal{I}}$ is a direct decomposition for the whole ring $\mathcal{R}$. For a homogeneous polynomial h and the subset $u \subseteq X$, the set $C(h, u) = \{ah \mid a \in \mathcal{R} \text{ and } \mathrm{indet}(a) \subseteq u\}$ is called the *cone* generated by h and u.

Definition 2.3. *Let $h_1, \ldots, h_t$ be a sequence of homogeneous polynomials in $\mathcal{R}$, and $u_1, \ldots, u_t$ a sequence of subsets of X. A finite set $P = \{C(h_1, u_1), \ldots, C(h_t, u_t)\}$ is a* cone decomposition *of $T \subset \mathcal{R}$ if these cones form a direct decomposition for T.*

For a cone decomposition P, the notion P^+ signifies $\{C(h, u) \in P \mid u \neq \emptyset\}$. Furthermore, we let denote by $\deg(P)$ the maximum of $\deg(h)$ with $C(h, u) \in P$.

Definition 2.4. *Let k be a non-negative integer and P a cone decomposition. Then, P is called* k-standard *if the following conditions hold:*

1. *there is no cone $C(h, u) \in P^+$ with $\deg(h) < k$,*
2. *for each $C(g, v) \in P^+$ and $k \leq d \leq \deg(g)$, there exists $C(h, u) \in P^+$ with $\deg(h) = d$ and $|u| \geq |v|$.*

By convention, if P^+ is the empty set, then P is k-standard for all k.

Example 2.5. *Let $\mathcal{I} = \langle x_1^3, x_1x_2x_3, x_1^2x_2 \rangle \subset \mathcal{K}[x_1, x_2, x_3]$. Then $C(1, \{x_2, x_3\}) \oplus C(x_1, \{x_3\}) \oplus C(x_1x_2, \{x_2\}) \oplus C(x_1^2, \{x_3\})$ is a 0-standard cone decomposition for $N_{\mathcal{I}}$.*

Dubé [6] described the SPLIT algorithm to compute standard cone decompositions for a given monomial ideal $\mathcal{I}$ as well as for $N_{\mathcal{I}}$.

Definition 2.6. *A cone decomposition P is* exact *if it is k-standard for some k, and in addition for each d, there exists at most one $C(h,u) \in P^+$ with $\deg(h) = d$.*

For example, one observes that the 0-standard cone decomposition in Example 2.5 is not exact. We shall notice that if $\mathcal{I}$ is a homogeneous ideal generated by homogeneous polynomials of degree at most d then by [6, Lemma 5.1], $\mathcal{I}$ possesses a d-standard cone decomposition. Furthermore, for any given d, $N_{\mathcal{I}}$ has a d-standard cone decomposition. Dubé in [6, page 766] proposed the SHIFT algorithm to convert any d-standard cone decomposition into a d-exact one.

Example 2.7. *The decomposition $C(1,\{x_2,x_3\}) \oplus C(x_1,\{x_3\}) \oplus C(x_1x_2^2,\{x_2\}) \oplus C(x_1x_2,\{\}) \oplus C(x_1^2,\{x_3\})$ is a 0-exact cone decomposition for $N_{\mathcal{I}}$ presented in Example 2.5.*

Definition 2.8. *The* Macaulay constants *of a d-exact cone decomposition P is defined to be:*

$$b_i = \min\{\ell \geq d \mid \forall C(h,u) \in P; |u| \geq i \Longrightarrow \deg(h) < \ell\}, \qquad i = 0,\dots,n+1.$$

We note as a simple observation that $b_0 \geq b_1 \geq \cdots \geq b_{n+1} = d$. In Example 2.7, the Macaulay constants for $N_{\mathcal{I}}$ are $b_0 = 4, b_1 = 4, b_2 = 1, b_3 = 0$ and $b_4 = 0$. In [6, Lemma 7.1], it was proved that the Macaulay constants $b_0,\dots,b_n$ for S are uniquely determined provided that b_{n+1} is already fixed. As an application of this theory, once we have found an exact cone decomposition and the Macaulay constants of a homogeneous set then we are able to compute its Hilbert polynomial, see below for more details.

Lemma 2.9 ([6, Lemma 6.1])**.** *Let P be an exact cone decomposition, and $b_0,\dots,b_{n+1}$ the Macaulay constants of P. Then for each $i = 1,\dots,n$ and each number z with $b_{i+1} \leq z < b_i$, there is exactly one cone $C(h,u) \in P^+$ such that $\deg(h) = z$ and $|u| = i$.*

Let P be an exact cone decomposition for a homogeneous set S and $b_0,\dots,b_{n+1}$ the Macaulay constants of P. Based on Lemma 2.9, it is shown that (see [6, page 768]) the Hilbert polynomial of S is equal to

$$\mathrm{HP}_S(z) = \binom{z - b_{n+1} + n}{n} - 1 - \sum_{i=1}^{n} \binom{z - b_i + i - 1}{i}. \tag{2}$$

As another application of introducing the Macaulay constants of $N_{\mathcal{I}}$ (by fixing $b_{n+1} = 0$), Dubé [6, Lemma 7.2] proved that b_0 is an upper bound for the degree of polynomials in any reduced Gröbner basis of $\mathcal{I}$. Then, using the above construction of the Hilbert polynomial and applying the decomposition (1), he found a new upper bound for the maximum degree of the elements of any reduced Gröbner basis of an ideal.

Theorem 2.10 ([6, Theorem 8.2])**.** *Let $\mathcal{I} \subset \mathcal{R}$ be a homogeneous ideal that generated by a set of homogeneous polynomials of degree at most d. Then, the degree of polynomials in any reduced Gröbner basis for $\mathcal{I}$ is bounded above by $2(d^2/2+d)^{2^{n-2}}$.*

In 2013, by improving the Dubé construction, Mayr and Ritscher [18] presented a dimension-depending degree upper bound for Gröbner bases. To review briefly their method, let us quickly recall some well-known facts. Let $\mathcal{I}$ be generated by the homogeneous polynomials $f_1,\dots,f_k \in \mathcal{R}$ with $\deg(f_1) \geq \cdots \geq \deg(f_k)$ and $D = \dim(\mathcal{I})$. One of the main topics discussed in [18] is to embed a homogeneous regular sequence in $\mathcal{I}$. For this end, they employed a result due to Schmid [21, Lemma 2.2] (see also [18, Lemma 9]) according to which one is able to find a homogeneous regular sequence $g_1,\dots,g_{n-D}$ in $\mathcal{I}$ such that $\deg(g_i) = \deg(f_i)$ for $1 \leq i \leq n-D$. In the next step, Mayr and Ritscher provided the following decomposition (see [18, Lemma 21])

$$\mathcal{I} = \langle g_1,\dots,g_{n-D}\rangle \oplus \bigoplus_{i=1}^{k} f_i \cdot N_{\mathcal{J}_{i-1}:f_i} \tag{3}$$

where $\mathcal{J}_i = \{g_1, \ldots, g_{n-D}, f_1, \ldots, f_i\}$. They applied this decomposition [18, Lemma 22] to show that any 0-exact cone decomposition P of $N_{\mathcal{I}}$ may be completed to a $\deg(f_1)$-exact cone decomposition Q of $N_{\mathcal{J}}$ where $\mathcal{J} = \langle g_1, \ldots, g_{n-D}\rangle$ such that $\deg(P) \leq \deg(Q)$. So, the Macaulay constant $a_0 = \deg(Q) + 1$ of Q is an upper bound for the maximum degree of the polynomials in any reduced Gröbner basis of $\mathcal{I}$. Based on this observation, they proved the next dimension-depending upper bound.

Theorem 2.11 ([18, Theorem 33]). *Let $\mathcal{I} \subset \mathcal{R}$ be an ideal of dimension D generated by homogeneous polynomials $f_1, \ldots, f_k$ of degrees respectively $d_1 \geq \cdots \geq d_k$. Then, the maximum degree of the polynomials in any reduced Gröbner basis of $\mathcal{I}$ is bounded by $2\Big(\frac{1}{2}(d_1 \cdots d_{n-D} + d_1)\Big)^{2^{D-1}}$.*

By giving a deeper analysis of the method due to Dubé, we improved in [11] Dubé's bound to $O(1)d^{2^{n-2}}$. In addition, we pointed out and fixed a flaw in the proof of his main result [6, Lemma 8.1]. Below, we state the main result of [11].

Theorem 2.12 ([11, Theorem 4.7]). *Let $\mathcal{I} \subset \mathcal{R}$ be an ideal generated by a set of homogeneous polynomials of degree at most d. Then, the maximum degree of the polynomials in any reduced Gröbner basis of $\mathcal{I}$ is bounded above by $(d+1)^{2^{n-2}}$ if $n > 2$. If $n = 1, 2$, then the upper bound becomes $d, 2d$, respectively.*

3. Efficient computation of Macaulay constants

In this section, by studying the Hilbert polynomial and Hilbert series of a homogeneous set, we describe an efficient algorithm to compute the Macaulay constants of the quotient of a monomial ideal *without* computing any exact cone decomposition of the corresponding quotient ring.

Proposition 3.1. *Let S be a homogeneous set, P an exact cone decomposition for S and $b_0, \ldots, b_{n+1}$ the Macaulay constants of P. Then, the Hilbert series of S can be written as*

$$\mathrm{HS}_S(t) = \frac{\sum_{i=1}^{n}(1-t)^{n-i}(t^{b_{i+1}} + \cdots + t^{b_i - 1}) + (1-t)^n B(t)}{(1-t)^n} \tag{4}$$

for some $B(t) \in \mathbb{Z}[t]$.

Proof. From the assumption, it is clear that the Hilbert series of S (or equivalently the one of P) is equal to the sum of the Hilbert series of the cones in P, i.e.

$$\mathrm{HS}_S(t) = \sum_{C(h,u)\in P} \mathrm{HS}_{C(h,u)}(t).$$

On the other hand, the Hilbert series of $C(h, u) \in P$ is represented in terms of the quantities $\deg(h)$ and $\dim(C(h, u)) := |u|$. So, we can write

$$\mathrm{HS}_{C(h,u)}(t) = \frac{t^{\deg(h)}}{(1-t)^{|u|}}.$$

By applying Lemma 2.9, we can partition the set P into $P = \bigcup_{i=0}^{n} A_i$, where for $1 \leq i \leq n$, $A_i = \{C(h, u) \in P \mid b_{i+1} \leq \deg(h) < b_i, |u| = i\}$ and $A_0 = \{C(h, u) \in P \mid |u| = 0\}$. Therefore, the Hilbert series of S can be reformulated as follows

$$\mathrm{HS}_S(t) = \sum_{i=0}^{n} \sum_{C(h,u)\in A_i} \mathrm{HS}_{C(h,u)}(t). \tag{5}$$

In particular, we know that for each $i = 1, \dots, n$ and degree z such that $b_{i+1} \le z < b_i$, there is exactly one cone $C(h,u) \in P$ with $\deg(h) = z$ and $|u| = i$. Therefore, we get

$$\sum_{C(h,u)\in A_i} \mathrm{HS}_{C(h,u)}(t) = \frac{t^{b_{i+1}} + \cdots + t^{b_i - 1}}{(1-t)^i} = \frac{(1-t)^{n-i}(t^{b_{i+1}} + \cdots + t^{b_i-1})}{(1-t)^n}. \tag{6}$$

For A_0, we can write

$$\sum_{C(h,u)\in A_0} \mathrm{HS}_{C(h,u)}(t) = \sum_{C(h,u)\in A_0} t^{\deg(h)} = \frac{(1-t)^n B(t)}{(1-t)^n} \tag{7}$$

where $B(t) = \sum_{C(h,u)\in A_0} t^{\deg(h)}$. By replacing the equalities (6) and (7) in (5), we obtain the desired equality (4), ending the proof. $\square$

Dubé in [6, Lemma 7.1] applied the Hilbert polynomial to prove the uniqueness of the Macaulay constants. Below, we give an alternative proof of this fact by using Proposition 3.1.

Proposition 3.2. *Let S be a homogeneous set and P a d-exact cone decomposition for S. If the Macaulay constant $b_{n+1} := d$ is fixed, then the Macaulay constants $b_0, \dots, b_{n+1}$ are unique.*

Proof. Suppose that there is another exact cone decomposition P' for S such that $a_0, \dots, a_{n+1} = d$ are the Macaulay constants of P'. We claim that $b_\ell = a_\ell$ for $0 \le \ell \le n$. According to Proposition 3.1, the Hilbert series of S can be written of the forms

$$\begin{aligned}\mathrm{HS}_S(t) &= \frac{\sum_{i=1}^n (1-t)^{n-i}(t^{b_{i+1}} + \cdots + t^{b_i-1}) + (1-t)^n B_1(t)}{(1-t)^n} \\ &= \frac{\sum_{i=1}^n (1-t)^{n-i}(t^{a_{i+1}} + \cdots + t^{a_i-1}) + (1-t)^n B_2(t)}{(1-t)^n}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\sum_{i=1}^n (1-t)^{n-i}(t^{b_{i+1}} + \cdots + t^{b_i-1}) + (1-t)^n B_1(t) = \\ &\quad \sum_{i=1}^n (1-t)^{n-i}(t^{a_{i+1}} + \cdots + t^{a_i-1}) + (1-t)^n B_2(t).\end{aligned} \tag{8}$$

The proof, to show that $b_\ell = a_\ell$ for $1 \le \ell \le n$, proceeds by induction on ℓ. We start with $\ell = n$. If in Equality (8), we put $t = 1$, then we get $b_n - b_{n+1} = a_n - a_{n+1}$ and in consequence $b_n = a_n$. Now, assume that the claim is true for any i with $\ell < i \le n$. We want to prove it for ℓ. By replacing the equalities $b_i = a_i$ for $\ell + 1 \le i \le n$ in Equality (8) and by dividing both sides of the new equality by $(1-t)^{n-\ell}$, we obtain $\sum_{i=1}^{\ell} (1-t)^{\ell-i}(t^{b_{i+1}} + \cdots + t^{b_i-1}) + (1-t)^\ell B_1(t) = \sum_{i=1}^{\ell} (1-t)^{\ell-i}(t^{a_{i+1}} + \cdots + t^{a_i-1}) + (1-t)^\ell B_2(t)$. Again, if in the above equality, we put $t = 1$, we get $b_\ell - b_{\ell+1} = a_\ell - a_{\ell+1}$. By using induction hypothesis, it holds $b_\ell = a_\ell$. Now, it remains to show that $b_0 = a_0$. If in Equality (8), we replace $b_i = a_i$ for each $i > 0$ and simplify it, we have $B_1(t) = B_2(t)$. On the other hand, it follows from the proof of Proposition 3.1 that $B_1(t) = \sum_{C(h,u)\in A_0} \deg(h)$ and $B_2(t) = \sum_{C(h,u)\in A_0'} \deg(h)$, where $A_0 = \{C(h,u) \in P \mid |u| = 0\}$ and $A_0' = \{C(h,u) \in P' \mid |u| = 0\}$. Thus, from Definition 2.8, we have $b_0 = 1 + \max\{b_1 - 1, \deg(B_1)\}$ and $a_0 = 1 + \max\{a_1 - 1, \deg(B_2)\}$ and this shows $b_0 = a_0$, completing the proof. $\square$

Based on this proposition, we are able to give the next definition.

Definition 3.3. *Let S be a homogeneous set and $b_0, \dots, b_{n+1}$ a sequence of Macaulay constants of S such that $d := b_{n+1}$. Then $b_0, \dots, b_{n+1}$ are called the* d-Macaulay constants of S.

Using the fact that the Hilbert function of $\mathcal{I}$ is the same as that of $\mathrm{LM}(\mathcal{I})$, [1] (see also [9, Algorithm 5.2.4]) described an effective method to compute the Hilbert series of $N_\mathcal{I}$ using Gröbner bases. On the other hand, If $\mathcal{I}$ is a homogeneous ideal, then $\mathrm{HS}_\mathcal{I}(t) = \mathrm{HS}_\mathcal{R}(t) - \mathrm{HS}_{N_\mathcal{I}}(t)$, because $\mathcal{R} = \mathcal{I} \oplus N_\mathcal{I}$. In the following, based on Propositions 3.1 and 3.2, we present an algorithm to compute the d-Macaulay constants of $N_\mathcal{I}$ for a given d.

Algorithm 1 COMPUTINGMACAULAYCONSTANTS

Input: The Hilbert series $\mathrm{HS}_{N_\mathcal{I}}(t) = Q(t)/(1-t)^n$ for the homogeneous set $N_\mathcal{I} \subset \mathcal{R}$ and a non-negative integer d
Output: The d-Macaulay constants of $N_\mathcal{I}$
$b_{n+1} := d$, $b_n := b_{n+1} + Q(1)$
for i **from** n **downto** 2 **do**
  $P(t) := (Q(t) - (t^{b_{i+1}} + \cdots + t^{b_i - 1}))/(1-t)$
  $b_{i-1} := b_i + P(1)$
  $Q(t) := P(t)$
end for
$P(t) := (Q(t) - (t^{b_2} + \cdots + t^{b_1 - 1}))/(1-t)$
$b_0 := 1 + \max\{b_1 - 1, \deg(P)\}$
return $(b_0, \ldots, b_{n+1})$

Theorem 3.4. *The* COMPUTINGMACAULAYCONSTANTS *algorithm terminates in finitely many steps and is correct.*

Proof. The termination of the algorithm is ensured by the for-loop in the algorithm. To prove its correctness, it follows from Proposition 3.1 that

$$Q(t) = \sum_{i=1}^{n} (1-t)^{n-i}(t^{b_{i+1}} + \cdots + t^{b_i - 1}) + (1-t)^n B(t).$$

Therefore, $Q(1) = b_n - b_{n+1}$ and in turn $b_n = b_{n+1} + Q(1)$. On the other hand, one observes that $Q(t) - (t^{b_{n+1}} + \cdots + t^{b_n - 1})$ is divisible by $(1-t)$ and we can write $Q(t) - (t^{b_{n+1}} + \cdots + t^{b_n - 1}) = (1-t)P(t)$ with $P(t) = \sum_{i=1}^{n-1} (1-t)^{n-i-1}(t^{b_{i+1}} + \cdots + t^{b_i - 1}) + (1-t)^{n-1}B(t)$. Now, we have $P(1) = b_{n-1} - b_n$, and $b_{n-1} = b_n + P(1)$. By setting $Q(t) := P(t)$ and repeating this process (in the for-loop), we can compute $b_{n-2}, \ldots, b_1$. Finally, for b_0, from Definition 2.8, we know that $b_0 := 1 + \max\{b_1 - 1, \deg(B)\}$. At the end of the for-loop, we have $Q(t) = (t^{b_2} + \cdots + t^{b_1 - 1}) + (1-t)B(t)$. Therefore, by setting $P(t) := (Q(t) - (t^{b_2} + \cdots + t^{b_1 - 1}))/(1-t)$, we have $P(t) = B(t)$ and we get the correct value for b_0, as desired. □

Remark 3.5. *We shall notice that the termination and correctness of this algorithm hold for any homogeneous set which can be represented by an exact cone decomposition.*

We illustrate the steps of this algorithm through a simple example.

Example 3.6. *Let $\mathcal{I} = \langle x_1^2, x_1x_2x_3 \rangle \subset \mathcal{K}[x_1, x_2, x_3]$ and $d = 2$. The ideal $\mathcal{I}$ is a homogeneous set and by using the function* HILBERTSERIES *of* MAPLE, *we have $\mathrm{HS}_\mathcal{I}(t) = -t^2(t^2 - t - 1)/(1-t)^3$. Then, $Q(t) = -t^2(t^2 - t - 1)$, $b_4 = 2$, and $b_3 = b_4 + Q(1) = 2 + 1 = 3$. Next, we have $Q(t) - t^2 = -t^4 + t^3 = (1-t)t^3$ and in turn we obtain $P(t) = t^3$ and $Q(t) = t^3$. Furthermore, we have $b_2 = b_3 + Q(1) = 3 + 1 = 4$. Since $Q(t) - t^3 = 0$ then we set $P(t) = 0$, $Q(t) = 0$ and $b_1 = b_2 + Q(1) = 4 + 0 = 4$. Finally, $P(t) = 0$ and $b_0 = 4$. Note that we the degree of the zero polynomial is defined to be -1.*

Below, based on the next simple observation, we give more explicit formulas for the Hilbert polynomial and Hilbert series of the quotient ring of an ideal in terms of its Macaulay constants.

Lemma 3.7. *Let $\mathcal{I} \subset \mathcal{R}$ be an arbitrary ideal with $D = \dim(\mathcal{I})$. Further, let P be a d-exact cone decomposition for $N_{\mathcal{I}}$ and $b_0, \ldots, b_{n+1}$ the Macaulay constants of P. Then for $i = D+1, \ldots, n+1$ we have $b_i = d$.*

Proof. We know that if P an exact cone decomposition for $N_{\mathcal{I}}$, then $D = \dim(\mathcal{I}) = \max\{|u| \mid C(h,u) \in P\}$. Therefore according to the definition of the Macaulay constants, $b_i = d$ for $i = D+1, \ldots, n+1$. $\square$

Theorem 3.8. *Let $\mathcal{I} \subset \mathcal{R}$ be an arbitrary ideal with $D = \dim(\mathcal{I})$. Let P be a d-exact cone decomposition for $N_{\mathcal{I}}$ and $b_0, \ldots, b_{n+1}$ the Macaulay constants of P. Then*

1. $\mathrm{HP}_{N_{\mathcal{I}}}(z) = \binom{z-d+D}{D} - 1 - \sum_{i=1}^{D} \binom{z-b_i+i-1}{i}$,
2. $\mathrm{HS}_{N_{\mathcal{I}}}(t) = \dfrac{\sum_{i=1}^{D} (1-t)^{D-i}(t^{b_{i+1}} + \cdots + t^{b_i - 1}) + (1-t)^D B(t)}{(1-t)^D}$ *where $B(t) \in \mathbb{Z}[t]$.*

Proof. (1) By Lemma 3.7, we have $b_i = d$ for $i = D+1, \ldots, n+1$. By replacing these values in Equality (2), we deduce that

$$\begin{aligned}
\mathrm{HP}_{N_{\mathcal{I}}}(z) &= \binom{z-d+n}{n} - 1 - \sum_{i=1}^{D} \binom{z-b_i+i-1}{i} - \sum_{i=D+1}^{n} \binom{z-d+i-1}{i} \\
&= \binom{z-d+n}{n} - \sum_{i=D+1}^{n} \binom{z-d+i-1}{i} - 1 - \sum_{i=1}^{D} \binom{z-b_i+i-1}{i}.
\end{aligned}$$

By using the combinatorial identity $\binom{n-1}{t-1} = \binom{n}{t} - \binom{n-1}{t}$, we have

$$\binom{z-d+n}{n} - \sum_{i=D+1}^{n} \binom{z-d+i-1}{i} = \binom{z-d+D}{D}.$$

and the Hilbert polynomial of $N_{\mathcal{I}}$ can be formulated in the form

$$\mathrm{HP}_{N_{\mathcal{I}}}(z) = \binom{z-d+D}{D} - 1 - \sum_{i=1}^{D} \binom{z-b_i+i-1}{i}.$$

To prove (2), we can write

$$\begin{aligned}
\sum_{i=1}^{n} (1-t)^{n-i}(t^{b_{i+1}} + \cdots + t^{b_i-1}) &= \sum_{i=1}^{D} (1-t)^{n-i}(t^{b_{i+1}} + \cdots + t^{b_i-1}) \\
&+ \sum_{i=D+1}^{n} (1-t)^{n-i}(t^{b_{i+1}} + \cdots + t^{b_i-1})
\end{aligned}$$

Since $b_{D+1} = \cdots = b_{n+1} = d$, then P does not contain any cone $C(h,u)$ with $|u| = i$ and $i > D$. Thus, it yields that $\sum_{i=D+1}^{n} (1-t)^{n-i}(t^{b_{i+1}} + \cdots + t^{b_i-1}) = 0$ and we conclude that

$$\begin{aligned}
\mathrm{HS}_{N_{\mathcal{I}}}(t) &= \frac{\sum_{i=1}^{D} (1-t)^{n-i}(t^{b_{i+1}} + \cdots + t^{b_i-1}) + (1-t)^n B(t)}{(1-t)^n} \\
&= \frac{\sum_{i=1}^{D} (1-t)^{D-i}(t^{b_{i+1}} + \cdots + t^{b_i-1}) + (1-t)^D B(t)}{(1-t)^D}.
\end{aligned}$$

$\square$

We conclude this section by discussing two applications of this theorem. As a direct consequence of this theorem, we prove first that the Macaulay constants of a regular sequence depend only on the degree of elements of this sequence.

Corollary 3.9. *Let $g_1, \dots, g_k$ and $g'_1, \dots, g'_k$ be two regular sequences such that $\deg(g_i) = \deg(g'_i)$ for any i. Let $\mathcal{J} = \langle g_1, \dots, g_k \rangle$ and $\mathcal{J}' = \langle g'_1, \dots, g'_k \rangle$. Then, for any given d, the d-Macaulay constants of $N_{\mathcal{J}}$ and of $N_{\mathcal{J}'}$ coincide. In particular, we can choose $g'_1 = x_1^{\deg(g_1)}, \dots, g'_k = x_k^{\deg(g_k)}$.*

Proof. We know that $\mathrm{HS}_{N_{\mathcal{J}}}(t) = \prod_{i=1}^k (1 - t^{\deg(g_i)})/(1-t)^n$ and $\mathrm{HS}_{N_{\mathcal{J}'}}(t) = \prod_{i=1}^k (1 - t^{\deg(g'_i)})/(1-t)^n$. From the equality $\mathrm{HS}_{N_{\mathcal{J}}} = \mathrm{HS}_{N_{\mathcal{J}'}}$ and Proposition 3.2, it follows that the sequence of the Macaulay constants of $N_{\mathcal{J}}$ and $N_{\mathcal{J}'}$ are equal. □

Finally, we give an alternative proof of the result about the dimension-depending upper bound for the degree of a homogeneous ideal, see [2, Theorem 4.5]. Let us recall the definition of the degree of an ideal.

Definition 3.10. [10, page 52] *Let $\mathcal{I} \subset \mathcal{R}$ be a homogeneous ideal with $D = \dim(\mathcal{I})$. If $D > 0$, then the* degree *of $\mathcal{I}$, denoted by $\deg(\mathcal{I})$, is $(D-1)!$ times the leading coefficient of the Hilbert polynomial of $N_{\mathcal{I}}$. If $D = 0$, then $\deg(\mathcal{I})$ is defined to be the sum of the coefficients of $\mathrm{HS}_{N_{\mathcal{I}}}(t)$.*

As already mentioned, one of the main issues in the approach by Mayr and Ritscher [18] is to embed a homogeneous regular sequence in a given homogeneous ideal. However, Lazard in [16, Proposition 21] proved the next proposition which gives a stronger version of the result due to Schmid [21, Lemma 2.2] (see also [18, Lemma 9]) and it can sharpen the bounds presented in [18].

Proposition 3.11. *Let $\mathcal{I} \subset \mathcal{R}$ be an ideal of dimension D generated by homogeneous polynomials $f_1, \dots, f_k$ of degrees $d_1, \dots, d_k$ such that $d_2 \geq \cdots \geq d_k \geq d_1$. Then, there is a regular sequence $g_1, \dots, g_{n-D} \in \mathcal{I}$ such that $\deg(g_i) = d_i$.*

In order to apply this proposition, from now on we consider the next assumption on the sorting of the f_i's.

Notation 3.12. *Let $\mathcal{I} \subset \mathcal{R}$ be an ideal of dimension D generated by the homogeneous polynomials $f_1, \dots, f_k \in \mathcal{R}$ of degrees $d_1, \dots, d_k$ with $d_1 \geq \cdots \geq d_{n-D-1} \geq d_{n-D+1} \geq \cdots \geq d_k \geq d_{n-D}$.*

Under this assumption, we can conclude that there exists a regular sequence $g_1, \dots, g_{n-D} \in \mathcal{I}$ of homogeneous polynomials of degrees $d_1 \geq \cdots \geq d_{n-D}$.

Theorem 3.13 (Dimension-depending Bézout bound). *With the above notations we have $\deg(\mathcal{I}) \leq d_1 \cdots d_{n-D}$.*

Proof. From [13, page 173], we know that $\deg(\mathcal{I}) = N(1)$ where $\mathrm{HS}_{N_{\mathcal{I}}}(t) = N(t)/(1-t)^D$. Let us consider the decomposition (3) for the ideal $\mathcal{I}$. Then, it follows that $\mathrm{HS}_{\mathcal{I}}(t) = \mathrm{HS}_{\mathcal{J}}(t) + \sum_{i=1}^k \mathrm{HS}_{f_i \cdot N_{\mathcal{J}_{i-1}:f_i}}(t)$ where $\mathcal{J} = \langle g_1, \dots, g_{n-D} \rangle$, $\mathcal{J}_i = \{g_1, \dots, g_{n-D}, f_1, \dots, f_i\}$, and the homogeneous polynomials $g_1, \dots, g_{n-D}$ is a regular sequence lying in $\mathcal{I}$ with $\deg(g_i) = d_i$ for $1 \leq i \leq n-D$. By subtracting $\mathrm{HS}_{\mathcal{R}}(t)$ by both sides of this equality of Hilbert series, we get easily

$$\mathrm{HS}_{N_{\mathcal{J}}}(t) = \mathrm{HS}_{N_{\mathcal{I}}}(t) + \sum_{i=1}^k \mathrm{HS}_{f_i \cdot N_{\mathcal{J}_{i-1}:f_i}}(t). \tag{9}$$

Now, by applying the algorithms Split and Shift we can construct a 0-exact cone decomposition Q_i for $N_{\mathcal{J}_{i-1}:f_i}$ for any i. It is easy to see that $f_i \cdot Q_i$ is a d_i-exact cone decomposition of $f_i \cdot N_{\mathcal{J}_{i-1}:f_i}$ for each i. On the other hand, due to the fact that $\mathcal{J} \subset \mathcal{J}_i : f_i$, we have $D_i := \dim(\mathcal{J}_i : f_i) \leq \dim(\mathcal{J}) = D$ for any i. Since $\mathcal{J}$ is generated by a regular sequence, $\mathrm{HS}_{N_{\mathcal{J}}}(t) = \prod_{i=1}^{n-D}(1 + \cdots + t^{d_i - 1})/(1-t)^D$. It follows from Theorem 3.8 and the equality (9) that

$$\frac{\prod_{i=1}^{n-D}(1 + \cdots + t^{d_i-1})}{(1-t)^D} = \frac{N(t)}{(1-t)^D} + \sum_{i=1}^k \frac{\sum_{j=1}^{D_i}(1-t)^{D_i - j}(t^{b_{i,j+1}} + \cdots + t^{b_{i,j}-1}) + (1-t)^{D_i} B_i(t)}{(1-t)^{D_i}}$$

where $b_{i,0}, \dots, b_{i,D_i}, b_{i,D_i+1} = \cdots = b_{i,n+1}$ are the Macaulay constants of $f_i \cdot N_{\mathcal{J}_{i-1}:f_i}$ for each i. Since $D_i \leq D$, then $D - D_i \geq 0$ and by multiplying the numerator and denominator of the latter fraction by $(1-t)^{D-D_i}$ we obtain

$$\frac{\prod_{i=1}^{n-D}(1+\cdots+t^{d_i-1})}{(1-t)^D} = \frac{N(t)}{(1-t)^D} + \sum_{i=1}^{k} \frac{\sum_{j=1}^{D_i}(1-t)^{D-j}(t^{b_{i,j+1}}+\cdots+t^{b_{i,j}-1})+(1-t)^D B_i(t)}{(1-t)^D}.$$

By multiplying both sides of this equality by $(1-t)^D$, we have

$$\prod_{i=1}^{n-D}(1+\cdots+t^{d_i-1}) = N(t) + \sum_{i=1}^{k}\sum_{j=1}^{D_i}(1-t)^{D-j}(t^{b_{i,j+1}}+\cdots+t^{b_{i,j}-1})+(1-t)^D B_i(t). \tag{10}$$

Now, for each i, two cases may arise: If $D_i = D$, then the value of $\sum_{j=1}^{D_i}(1-t)^{D-j}(t^{b_{i,j+1}}+\cdots+t^{b_{i,j}-1})+(1-t)^D B_i(t)$ at $t=1$ is positive. Otherwise the value of $\sum_{j=1}^{D_i}(1-t)^{D-j}(t^{b_{i,j+1}}+\cdots+t^{b_{i,j}-1})+(1-t)^D B_i(t)$ at $t=1$ is zero. Hence, by evaluating both sides of Equality (10) at $t=1$, we get the desired inequality. □

Remark 3.14. *With the notations of Theorem 3.13 and from the proof of this theorem, one sees that in the case that $f_1 \notin \mathcal{J}$ (or equivalently there is some f_i with $f_i \notin \mathcal{J}$) then $D_1 := \dim(\mathcal{J} : f_1) = \dim(\mathcal{J})$ and in consequence the value of $\sum_{j=1}^{D_1}(1-t)^{D-j}(t^{b_{1,j+1}}+\cdots+t^{b_{1,j}-1})+(1-t)^D B_1(t)$ at $t=1$ is positive. Thus, we have $\deg(\mathcal{I}) < d_1 \cdots d_{n-D}$.*

4. New upper bound for degree Gröbner bases

In this section, we use the results discussed in the previous section as well as the method presented in [18] to give a new degree upper bound for Gröbner bases. Assume that $\mathcal{I} = \langle f_1, \dots, f_k \rangle \subset \mathcal{R}$, for each i, f_i is a homogeneous polynomial of total degree d_i and $D = \dim(\mathcal{I}) \neq n$. In whole this section, we follow the sorting on the f_i's presented in Notation 3.12. Note that in the case that the generating set of $\mathcal{I}$ contains a linear polynomial f with the leading monomial x_i, then to compute a Gröbner basis for $\mathcal{I}$, we can eliminate f from the generating set of $\mathcal{I}$ and x_i from the variables provided that for each j, we replace f_j by its reduction with respect to f. Thus, without loss of generality, we may assume that $d_i \geq 2$ for each i and let $d = \max\{d_1, \dots, d_k\}$. Also, let $\prec$ be a monomial ordering on $\mathcal{R}$.

The key idea of Dubé's approach [6, Theorem 4.11] is that for any reduced Gröbner basis G of $\mathcal{I}$ and for any given 0-standard cone decomposition P for $N_{\mathcal{I}}$ we have $\deg(G) \leq \deg(P)+1$ where $\deg(G)$ denotes the maximum degree of the elements of G. Now, let $a_0, \dots, a_{n+1}$ be the Macaulay constants of P, then $\deg(G) \leq a_0$. So, the main goal of [6] is to exploit combinatorial arguments to find an upper bound for a_0.

Now, let $g_1, \dots, g_{n-D} \in \mathcal{I}$ be a regular sequence of degrees $d_1, \dots, d_{n-D}$. As already discussed, we are sure about the existence of such a regular sequence. Let $\mathcal{J} = \langle g_1, \dots, g_{n-D} \rangle \subset \mathcal{R}$ and $b_0, \dots, b_{n+1}$ be the Macaulay constants of $N_{\mathcal{J}}$. Then, Mayr and Ritscher [18] proved that an upper bound for $\max\{b_1, d_1 + \cdots + d_{n-D} - n\}$ remains a upper bound for the maximum degree of the elements of any reduced Gröbner basis of $\mathcal{I}$. In addition, they proved that, for this purpose, instead of the ideal $\mathcal{J}$, we can consider the simpler ideal $\mathcal{L} = \langle x_1^{d_1}, \dots, x_{n-D}^{d_{n-D}} \rangle$. However, we shall note that, a drawback of their approach is that b_0 has not been intervened in their bounds, see [18, Theorem 25] and its proof. By applying our investigation on Hilbert series in the previous section, we are able to involve b_0 and give an explicit formula for computing the b_i's. These allow us to improve the Mayr and Ritscher upper bound. Let us start with the next lemma from [18].

Lemma 4.1 ([18, Lemma 22]). *Keeping the above notations, any* 0*-standard cone decomposition* P *of* $N_{\mathcal{I}}$ *may be completed to a* d*-exact cone decomposition* Q *of* $N_{\mathcal{J}}$ *with* $\deg(P) \leq \deg(Q)$.

Proposition 4.2. *With the above notations, let* P *is a* 0*-standard cone decomposition of* $N_{\mathcal{I}}$*. Then,* $\deg(P) \leq \deg(S)$ *where* S *is a* d*-exact cone decomposition of* $N_{\mathcal{L}}$.

Proof. From Lemma 4.1, we can complete P into a d-exact cone decomposition Q of $N_{\mathcal{J}}$ with $\deg(P) \leq \deg(Q)$. On the other hand, from Corollary 3.9, it follows that the Macaulay constants of Q and S are equal. Let $b_0, \ldots, b_{n+1}$ be the common Macaulay constants of Q and S. Thus, we have $\deg(S) = b_0 - 1 = \deg(Q)$ and in turn $\deg(P) \leq \deg(S)$. □

Remark 4.3. *If we compare Proposition 4.2 with* [18, Theorem 25] *then we see that our upper bound for* $\deg(P)$ *depends only on* b_0*. Beside this, Mayr and Ritscher in the proof of* [18, Theorem 25]*, are not able to show that the* d*-Macaulay constants of* $N_{\mathcal{J}}$ *and* $N_{\mathcal{L}}$ *coincide* completely.

Lemma 4.4. *If* $b_0, \ldots, b_{n+1}$ *are the* d*-Macaulay constants of* $N_{\mathcal{L}}$ *and* $D > 0$ *then it holds* $b_0 = b_1$.

Proof. Let us consider the following cone decomposition for $N_{\mathcal{L}}$

$$T = \{C(h, u) \mid u = \{x_{n-D+1}, \ldots, x_n\}, h = x_1^{\alpha_1} \cdots x_{n-D}^{\alpha_{n-D}} \text{ s.t. } 0 \leq \alpha_i < d_i\}. \tag{11}$$

It is easy to see that T is a 0-standard cone decomposition. Using [6, Lemma 3.1], for any $d \in \mathbb{N}$ we can construct a d-standard cone decomposition T_d for $N_{\mathcal{L}}$ and according to the proof of this lemma and using the assumption $D > 0$ we have $\deg(T_{d,0}) < \deg(T_d)$ where $T_{d,0} = \{C(h, u) \in T_d \mid u = \emptyset\}$. Furthermore, by applying the SHIFT algorithm [6, page 766], we can convert T_d into a d-exact cone decomposition S_d and by the structure of this algorithm $\deg(S_{d,0}) < \deg(S_d)$ where $S_{d,0} = \{C(h, u) \in S_d \mid u = \emptyset\}$. Let $d = b_{n+1}$. From the uniqueness property of Macaulay constants (Proposition 3.2), we know that $b_0, \ldots, b_{n+1}$ are the Macaulay constants of S_d and it follows that $b_0 = b_1$. □

In the following, we present a recursive formula for calculating the d-Macaulay constants of $N_{\mathcal{L}}$ for a given d.

Theorem 4.5. *Let* $b_0, \ldots, b_{n+1}$ *be the* d*-Macaulay constants of* $N_{\mathcal{L}}$*. Then* $b_\ell = d$ *for* $\ell \geq D+1$ *and*

$$\begin{aligned} b_D &= d_1 \cdots d_{n-D} + d \\ b_{D-1} &= \frac{d_1 \cdots d_{n-D}}{2} \big(d_1 \cdots d_{n-D} + 2d + n + 1 - (d_1 + \cdots + d_{n-D} + D)\big) + d \\ b_{D-k} &= \\ b_{D-k+1} &+ \frac{(-1)^k}{k!} \Big(f^{(k)}(1) - \sum_{i=D-k+1}^{D} \Big[\frac{(-1)^{D-i} k!}{(k-D+i)!} \times \sum_{j=b_{i+1}}^{b_i - 1} \frac{j!}{(j-k+D-i)!} \Big] \Big) \end{aligned} \tag{12}$$

where $k = 0, \ldots, D-1$, $f(t) = \prod_{i=1}^{n-D}(1 + \cdots + t^{d_i - 1})$ *and* $f^{(k)}(t)$ *is the* k*-th derivative of* f.

Proof. Let P be a d-exact cone decomposition of $N_{\mathcal{L}}$. Since $\dim(\mathcal{L}) = D$, from Lemma 3.7 and Theorem 3.8, we conclude that $b_\ell = d$ for $D + 1 \leq \ell \leq n + 1$ and in addition

$$\mathrm{HS}_{N_{\mathcal{L}}}(t) = \frac{\sum_{i=1}^{D} (1-t)^{D-i} (t^{b_{i+1}} + \cdots + t^{b_i - 1}) + (1-t)^D B(t)}{(1-t)^D}$$

for some polynomial $B(t)$. Since $x_1^{d_1},\dots,x_{n-D}^{d_{n-D}}$ forms a regular sequence, we have $\mathrm{HS}_{N_{\mathcal{L}}}(t) = \prod_{i=1}^{n-D}(1-t^{d_i})/(1-t)^n = \prod_{i=1}^{n-D}(1+\cdots+t^{d_i-1})/(1-t)^D$. From the two latter equalities, it results

$$\prod_{i=1}^{n-D}(1+\cdots+t^{d_i-1}) = \sum_{i=1}^{D}(1-t)^{D-i}(t^{b_{i+1}}+\cdots+t^{b_i-1}) + (1-t)^D B(t). \tag{13}$$

If in Equality (13), we put $t=1$, then we get $f(1) = d_1\cdots d_{n-D} = b_D - d$ and in turn $b_D = d_1\cdots d_{n-D}+d$. For simplicity, we denote $f_i(t) = t^{b_{i+1}}+\cdots+t^{b_i-1}$ and $h_i(t) = (1-t)^{D-i}$. To calculate b_{D-1}, it is enough to take the derivative with respect to t of both sides of (13) and to put $t=1$. More precisely, some elementary calculus shows that the derivative of the left hand side is $f^{(1)}(1) = \dfrac{d_1\cdots d_{n-D}}{2}\big(d_1+\cdots+d_{n-D}-(n-D)\big)$ and the one of the other side is $f_D^{(1)}(1) + b_D - b_{D-1}$. By replacing $f_D^{(1)}(1) = \dfrac{(b_D-d)(b_D+d-1)}{2}$ in this equality, we get

$$\frac{d_1\cdots d_{n-D}}{2}\big(d_1+\cdots+d_{n-D}-(n-D)\big) = \frac{(b_D-d)(b_D+d-1)}{2} + b_D - b_{D-1}. \tag{14}$$

Replacing b_D by $d_1\cdots d_{n-D}+d$ in (14) and making some simplifications leads to

$$b_{D-1} = \frac{d_1\cdots d_{n-D}}{2}\big(d_1\cdots d_{n-D}+2d+n+1-(d_1+\cdots+d_{n-D}+D)\big)+d. \tag{15}$$

Similarly, to calculate b_{D-k} we shall proceed by taking k times derivative of both sides of (13), i.e. $f(t) = \sum_{i=1}^{D} h_i f_i + (1-t)^D B(t)$ and putting $t=1$ in it. Note that, in this computation, $(1-t)^D B(t)$ can be ignored. Using elementary calculus, it is not hard to show that

$$\begin{aligned}
f^{(k)}(1) &= \sum_{i=D-k}^{D} (h_i f_i)^{(k)}(1) \\
(h_i f_i)^{(k)}(1) &= \sum_{j=0}^{k}\binom{k}{j} h_i^{(j)}(1) f_i^{(k-j)}(1) = \binom{k}{D-i} h_i^{(D-i)}(1) f_i^{(k-D+i)}(1) \\
h_i^{(D-i)}(1) &= (-1)^{D-i}(D-i)!\,,\ f_i^{(k-D+i)}(1) = \sum_{j=b_{i+1}}^{b_i-1}\frac{j!}{(j-k+D-i)!}.
\end{aligned}$$

It is worth noting that to calculate $(h_i f_i)^{(k)}(1)$, we use the well-known Leibniz formula and the fact that if $i \neq D-i$ then $h_i^{(j)}(1) = 0$. Therefore, we can write $f^{(k)}(1)$ as

$$\begin{aligned}
&= \sum_{i=D-k}^{D}\Big[\binom{k}{D-i}(-1)^{D-i}(D-i)! \times \sum_{j=b_{i+1}}^{b_i-1}\frac{j!}{(j-k+D-i)!}\Big] \\
&= \sum_{i=D-k+1}^{D}\Big[\frac{(-1)^{D-i}k!}{(k-D+i)!} \times \sum_{j=b_{i+1}}^{b_i-1}\frac{j!}{(j-k+D-i)!}\Big] + (-1)^k k!(b_{D-k}-b_{D-k+1}).
\end{aligned}$$

From this equality, we can find the value of b_{D-k} as follows

$$b_{D-k+1} + \frac{(-1)^k}{k!}\Big(f^{(k)}(1) - \sum_{i=D-k+1}^{D}\Big[\frac{(-1)^{D-i}k!}{(k-D+i)!} \times \sum_{j=b_{i+1}}^{b_i-1}\frac{j!}{(j-k+D-i)!}\Big]\Big).$$

□

We state now the main theorem of this paper.

Theorem 4.6. *Let $\mathcal{I} \subset \mathcal{R}$ be an ideal generated by homogeneous polynomials $f_1, \ldots, f_k$ of degrees $d_1, \ldots, d_k$ (satisfying Notation 3.12) and $D = \dim(\mathcal{I})$. Then, the maximum degree of the polynomials in any reduced Gröbner basis G of $\mathcal{I}$ is bounded by*

$$2\Big(\frac{1}{4}d_1 \cdots d_{n-D}\big(d_1 \cdots d_{n-D} + 2d_1 + n + 1 - (d_1 + \cdots + d_{n-D} + D)\big) + \frac{d_1}{2}\Big)^{2^{D-2}} \tag{16}$$

whenever $D \geq 2$. In the case that $D = 0, 1$, the upper bound becomes $d_1 + \cdots + d_n - n + 1$ and $d_1 \cdots d_{n-1} + d_1$, respectively.

Proof. Suppose that $D \geq 2$. Let $b_0, \ldots, b_{n+1}$ be the d_1-Macaulay constants of $N_{\mathcal{L}}$ with $\mathcal{L} = \langle x_1^{d_1}, \ldots, x_{n-D}^{d_{n-D}} \rangle$. By Proposition 4.2, $\deg(G) \leq b_0$. On the other hand, according to Theorem 4.4 b_0 equals b_1. To prove the desired bound we follow the approach due to Mayr and Ritscher in [18, Lemma 31] by using the inequality $b_{s-1} \leq b_s^2/2$. It was proved by proceeding an induction on $s = D, \ldots, 2$. However, unfortunately, they have not proved the base step of the induction, i.e. $b_{D-1} \leq b_D^2/2$. Note that to give the proof of this step, one needs the values of b_{D-1} and b_D as given in Theorem 4.5. Let us first complete the proof of this step. By applying Theorem 4.5 for $d = d_1$ and Equality (14), we know that $b_{D-1} = b_D + (b_D - d)(b_D + d - 1)/2 - d_1 \cdots d_{n-D}\big(d_1 + \cdots + d_{n-D} - (n - D)\big)/2$. Thus, by replacing b_D by $d_1 \cdots d_{n-D} + d$ in this expression (only where b_D has degree one), we can write

$$b_{D-1} = \frac{b_D^2}{2} + \frac{-d^2 + 2d}{2} - \frac{d_1 \cdots d_{n-D}}{2}\big(d_1 + \cdots + d_{n-D} - (n - D + 1)\big). \tag{17}$$

Since $d \geq 2$ then $(-d^2 + 2d)/2 \leq 0$. In addition, from $d_i \geq 2$ for each i and $n > D$, it follows that $d_1 + \cdots + d_{n-D} - (n - D + 1) \geq 0$ and in consequence $b_{D-1} \leq b_D^2/2$, proving the base step. Now, we can use the inequality $b_{s-1} \leq b_s^2/2$ for $2 \leq s \leq D$. Therefore, if $D \geq 2$ then $b_s \leq 2\big(b_{D-1}/2\big)^{2^{D-s-1}}$ for $1 \leq s \leq D - 1$. From Theorem 4.4, we have

$$b_{D-1} = \frac{1}{2}d_1 \cdots d_{n-D}\big(d_1 \cdots d_{n-D} + 2d_1 + n + 1 - (d_1 + \cdots + d_{n-D} + D)\big) + d_1$$

and for $D \geq 2$, we deduce that

$$\deg(G) \leq b_0 = b_1 \leq 2\Big(\frac{1}{4}d_1 \cdots d_{n-D}\big(d_1 \cdots d_{n-D} + 2d_1 + n + 1 - (d_1 + \cdots + d_{n-D} + D)\big) + \frac{d_1}{2}\Big)^{2^{D-2}}.$$

Let us now discus the remaining cases $D = 0, 1$. If $D = 1$, then $b_1 = b_D$ and $\deg(G) \leq b_0 = b_1 = b_D = d_1 \cdots d_{n-1} + d_1$. For the case $D = 0$, any standard cone decomposition of $N_{\mathcal{L}}$ is exact. So, the set $\{C(h, \{\}) \mid h = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ s.t. } 0 \leq \alpha_i < d_i\}$ forms a d_1-exact cone decomposition for $N_{\mathcal{L}}$. From the definition of Macaulay constants, b_0 is the maximum of d_1 and one plus the maximum degree of the cones in the above decomposition i.e. $b_0 = \max\{d_1, (d_1 - 1) + \cdots + (d_n - 1) + 1\}$. From the fact that $d_i \geq 2$, it yields that $b_0 = d_1 + \cdots + d_n - n + 1$ and $\deg(G) \leq d_1 + \cdots + d_n - n + 1$, completing the proof. □

Remark 4.7. *To compare our new bound with the one presented in* [18, Theorem 33]*, let us present these bounds by $2(\frac{1}{4}A)^{2^{D-2}}$ and $2(\frac{1}{4}B)^{2^{D-2}}$, respectively where $A = d_1 \cdots d_{n-D}\big(d_1 \cdots d_{n-D} + 2d_1 + n + 1 - (d_1 + \cdots + d_{n-D} + D)\big) + 2d_1$ and $B = (d_1 \cdots d_{n-D} + d_1)^2$. We claim that $B - A \geq 2^{n-D}(n - D - 1)$. Since $n > D$ and $d_i \geq 2$ for $1 \leq i \leq n - D$, we can write*

$$\begin{aligned} B - A &= d_1^2 - 2d_1 + d_1 \cdots d_{n-D}\big(d_1 + \cdots + d_{n-D} - (n - D + 1)\big) \\ &\geq d_1 \cdots d_{n-D}(n - D - 1) \geq 2^{n-D}(n - D - 1) \end{aligned}$$

and this proves that our bound is sharper than the Mayr and Ritscher one. Since our bound depends on the dimension of the ideal, then it is clear that our new bound is sharper than that of Dubé [6].

Corollary 4.8. *Let $\mathcal{I} \subset \mathcal{R}$ be an ideal generated by a set of homogeneous polynomials of degree at most d and $\dim(\mathcal{I}) = D$. Then, the maximum degree of the polynomials in any reduced Gröbner basis of $\mathcal{I}$ is bounded by*

$$2\Big(\frac{1}{4}\big((d^{n-D}+d)^2 - d^2 + 2d\big)\Big)^{2^{D-2}}$$

if $D \geq 2$. If $D = 0, 1$, then upper bound becomes $nd - n + 1$ and $d^{n-1} + d$, respectively.

We end this section by extending Theorem 4.6 to non-necessary homogeneous ideals. It is well-known that by considering a new variable x_{n+1}, we can transform a given non-necessary homogeneous polynomial f into a homogeneous one, denoted by f^h. Let $\mathcal{I} = \langle f_1, \ldots, f_k \rangle \subset \mathcal{R}$ and $d_i = \deg(f_i)$ and $\dim(\mathcal{I}) = D$. We consider the sorting of the f_i's as presented in Notation 3.12. Mayr and Ritscher applied [18, Lemma 35] to prove the upper bound

$$2\Big(\frac{1}{2}\big((d_1 \cdots d_{n-D})^{2(n-D)} + d_1\big)\Big)^{2^D}$$

for the degrees of the elements of any reduced Gröbner basis of $\mathcal{I}$. In doing so, they exploited the fact that the homogenization of $\mathcal{I}$ contains a homogeneous regular sequence of degree at most $(d_1 \cdots d_{n-D})^2$. The proof of [2, Theorem 4.19] entails that there are polynomials $g_1, \ldots, g_{n-D} \in \mathcal{I}$ such that $g_1^h, \ldots, g_{n-D}^h$ is a regular sequence of degree at most $d_1 \cdots d_{n-D}$. Based on this observation and Corollary 4.8, we can state the next theorem.

Theorem 4.9. *Let $\mathcal{I} \subset \mathcal{R}$ be an ideal generated by a set of non-necessary homogeneous polynomials $f_1, \ldots, f_k$ of degrees $d_1, \ldots, d_k$. Then the maximum degree of the polynomials in the reduced Gröbner basis is bounded by*

$$2\Big(\frac{1}{4}\big((d_1^{(n-D)^2} + d_1)^2 - d_1^2 + 2d_1\big)\Big)^{2^{D-1}}$$

provided that $D \geq 1$. If $D = 0$, then upper bound becomes $(d_1 \cdots d_{n-1})^{n-1} + d_1$.

References

[1] BIGATTI, A. M., CONTI, P., ROBBIANO, L., AND TRAVERSO, C. A "divide and conquer" algorithm for Hilbert-Poincaré series, multiplicity and dimension of monomial ideals. In *Applied algebra, algebraic algorithms and error-correcting codes. 10th international symposium, AAECC-10, San Juan de Puerto Rico, Puerto Rico, May 10-14, 1993. Proceedings*. Berlin: Springer-Verlag, 1993, pp. 76–88.

[2] BINAEI, B., HASHEMI, A., AND SEILER, W. M. A Pommaret bases approach to the degree of a polynomial ideal. *Appl. Algebra Eng. Commun. Comput. 29*, 4 (2018), 283–301.

[3] BUCHBERGER, B. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*. PhD thesis, Universität Innsbruck, 1965.

[4] BUCHBERGER, B. Bruno Buchberger's PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. Translation from the German. *J. Symb. Comput. 41*, 3-4 (2006), 475–511.

[5] COX, D. A., LITTLE, J., AND O'SHEA, D. *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra. 4th revised ed.* Cham: Springer, 2015.

[6] DUBÉ, T. W. The structure of polynomial ideals and Gröbner bases. *SIAM J. Comput. 19*, 4 (1990), 750–773.

[7] ENE, V., AND HERZOG, J. *Gröbner bases in commutative algebra.*, vol. 130. Providence, RI: American Mathematical Society (AMS), 2012.

[8] GIUSTI, M. Some effectivity problems in polynomial ideal theory. EUROSAM 84, Symbolic and algebraic computation, Proc. int. Symp., Cambridge/Engl. 1984, Lect. Notes Comput. Sci. 174, 159-171 (1984)., 1984.

[9] GREUEL, G.-M., AND PFISTER, G. *A Singular introduction to commutative algebra. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann. 2nd extended ed*, 2nd extended ed. ed. Berlin: Springer, 2007.

[10] HARTSHORNE, R. *Algebraic geometry. Corr. 8rd printing*, vol. 52. Springer, New York, NY, 1997.

[11] HASHEMI, A., PARNIAN, H., AND SEILER, W. M. Degree upper bounds for involutive bases. *Mathematics in Computer Science, to appear* (2020).

[12] HASHEMI, A., AND SEILER, W. M. Dimension-dependent upper bounds for Gröbner bases. In *Proceedings of the 42th international symposium on symbolic and algebraic computation, ISSAC 2017, Kaiserslautern, Germany, July 25–28, 2017*. New York, NY: Association for Computing Machinery (ACM), 2017, pp. 189–196.

[13] KEMPER, G. *A course in commutative algebra*, vol. 256. Heidelberg: Springer, 2011.

[14] KREUZER, M., AND ROBBIANO, L. *Computational commutative algebra. II.* Berlin: Springer, 2005.

[15] LAZARD, D. Gröbner bases, Gaussian elimination and resolution of systems of algebraic equations. Computer algebra, EUROCAL'83, Proc. Conf., London 1983, Lect. Notes Comput. Sci. 162, pp. 146-156., 1983.

[16] LAZARD, D. Degree of a polynomial ideal and Bézout inequalities. *hal-02897419 (preprint)* (2020).

[17] MAYR, E. W., AND MEYER, A. R. The complexity of the word problems for commutative semigroups and polynomial ideals. *Adv. Math. 46* (1982), 305–329.

[18] MAYR, E. W., AND RITSCHER, S. Dimension-dependent bounds for Gröbner bases of polynomial ideals. *J. Symb. Comput. 49* (2013), 78–94.

[19] MÖLLER, H., AND MORA, F. Upper and lower bounds for the degree of Gröbner bases. EUROSAM 84, Symbolic and algebraic computation, Proc. int. Symp., Cambridge/Engl. 1984, Lect. Notes Comput. Sci. 174, 172-183 (1984)., 1984.

[20] MORA, T. *Solving polynomial equation systems. II. Macaulay's paradigm and Gröbner technology.* Cambridge: Cambridge University Press, 2005.

[21] SCHMID, J. On the affine Bezout inequality. *Manuscr. Math. 88*, 2 (1995), 225–232.

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