Mehrdimensionale Integration

In[5]:= `RechtsRiemannSumme[f_, {x_, y_}, {{a_, b_}, {c_, d_}}] :=`

$$\frac{(b-a)\,(d-c)}{n^2}\sum_{k=1}^{n}\sum_{j=1}^{n}\left(f\ /.\ \left\{x\to a+\frac{k}{n}\,(b-a),\ y\to c+\frac{j}{n}\,(d-c)\right\}\right)$$

In[6]:= `riemann = RechtsRiemannSumme[(x + y)^3, {x, y}, {{0, 1}, {0, 1}}]`

Out[6]= $$\frac{3\,n^3+7\,n^2+5\,n+1}{2\,n^3}$$

In[7]:= `Limit[riemann, n → ∞]`

Out[7]= $$\frac{3}{2}$$

In[8]:= `riemann = RechtsRiemannSumme[(x + y)^3, {x, y}, {{a, b}, {c, d}}]`

Out[8]= $$\frac{1}{4\,n^3}$$

$$(b-a)\,(d-c)\left(a^3 n^3-2\,a^3 n^2+a^3 n+a^2 b\,n^3-a^2 b\,n+2\,a^2 c\,n^3-5\,a^2 c\,n^2+4\,a^2 c\,n-a^2 c+2\,a^2 d\,n^3-a^2 d\,n^2-2\,a^2 d\,n+\right.$$
$$a^2 d+a\,b^2 n^3-a\,b^2 n+2\,a\,b\,c\,n^3-2\,a\,b\,c\,n^2-2\,a\,b\,c\,n+2\,a\,b\,c+2\,a\,b\,d\,n^3+2\,a\,b\,d\,n^2-2\,a\,b\,d\,n-$$
$$2\,a\,b\,d+2\,a\,c^2 n^3-5\,a\,c^2 n^2+4\,a\,c^2 n-a\,c^2+2\,a\,c\,d\,n^3-2\,a\,c\,d\,n^2-2\,a\,c\,d\,n+2\,a\,c\,d+2\,a\,d^2 n^3+$$
$$a\,d^2 n^2-2\,a\,d^2 n-a\,d^2+b^3 n^3+2\,b^3 n^2+b^3 n+2\,b^2 c\,n^3+b^2 c\,n^2-2\,b^2 c\,n-b^2 c+2\,b^2 d\,n^3+5\,b^2 d\,n^2+$$
$$4\,b^2 d\,n+b^2 d+2\,b\,c^2 n^3-b\,c^2 n^2-2\,b\,c^2 n+b\,c^2+2\,b\,c\,d\,n^3+2\,b\,c\,d\,n^2-2\,b\,c\,d\,n-2\,b\,c\,d+2\,b\,d^2 n^3+$$
$$\left.5\,b\,d^2 n^2+4\,b\,d^2 n+b\,d^2+c^3 n^3-2\,c^3 n^2+c^3 n+c^2 d\,n^3-c^2 d\,n+c\,d^2 n^3-c\,d^2 n+d^3 n^3+2\,d^3 n^2+d^3 n\right)$$

In[9]:= `Limit[riemann, n → ∞] // Factor`

Out[9]= $$\frac{1}{4}\,(a-b)\,(c-d)\,(a+b+c+d)\left(a^2+a\,c+a\,d+b^2+b\,c+b\,d+c^2+d^2\right)$$

■ Vergleich

In[10]:= $$\int_0^1\int_0^1 (x+y)^3\,dx\,dy$$

Out[10]= $$\frac{3}{2}$$

In[11]:= $$\int_c^d\int_a^b (x+y)^3\,dx\,dy$$

Out[11]= $$\frac{1}{4}\left(a^4\,(c-d)+2\,a^3\left(c^2-d^2\right)+2\,a^2\left(c^3-d^3\right)+a\left(c^4-d^4\right)+b\left(b^3\,(d-c)-2\,b^2\left(c^2-d^2\right)-2\,b\left(c^3-d^3\right)-c^4+d^4\right)\right)$$

In[12]:= $$\int_a^b\int_c^d (x+y)^3\,dx\,dy\ \text{// Factor}$$

Out[12]= $$\frac{1}{4}\,(a-b)\,(c-d)\,(a+b+c+d)\left(a^2+a\,c+a\,d+b^2+b\,c+b\,d+c^2+d^2\right)$$

■ Beispiel 9.2

In[13]:= $$\int_{-1}^1\int_{-2}^2 x_1^2\,x_2^4\,dx_2\,dx_1$$

Out[13]= $$\frac{128}{15}$$

Beispiel 9.3

In[14]:= $$\int_0^2\left(\int_{-1}^1\int_0^1 x_1^2\, x_2^2\, x_3^2\, dx_1\, dx_2\right) dx_3$$

Out[14]= $$\frac{16}{27}$$

Beispiel 9.4

In[15]:= $$\int_{-1}^1\int_0^{\sqrt{1-x_1^2}} 1\, dx_2\, dx_1$$

Out[15]= $$\frac{\pi}{2}$$

In[16]:= $$\int\left(\int_0^{\sqrt{1-x_1^2}} 1\, dx_2\right) dx_1$$

Out[16]= $$\frac{1}{2}\left(x_1\sqrt{1-x_1^2}+\sin^{-1}(x_1)\right)$$

Beispiel 9.5

In[17]:= $$\pi\int_0^H \frac{R^2}{H^2}\,(H-x_3)^2\, dx_3$$

Out[17]= $$\frac{1}{3}\pi H R^2$$

Beispiel 9.6

In[18]:= $$V = \text{Factor}\left[\int_0^{b-\frac{b}{c}x_3}\int_0^{a-\frac{a}{b}x_2-\frac{a}{c}x_3} 1\, dx_1\, dx_2\right]$$

Out[18]= $$\frac{a\, b\,(c-x_3)^2}{2\, c^2}$$

In[19]:= $$\int_0^c V\, dx_3$$

Out[19]= $$\frac{a\, b\, c}{6}$$

Beispiel 9.7

In[20]:= $$\int_0^a\int_0^{\sqrt{b^2-\frac{b^2}{a^2}x_1^2}} x_2\, dx_2\, dx_1$$

Out[20]= $$\frac{a\, b^2}{3}$$

In[21]:= $$\int_0^b\int_0^{\sqrt{a^2-\frac{a^2}{b^2}x_2^2}} x_2\, dx_1\, dx_2$$

Out[21]= $$\frac{1}{3}\sqrt{a^2}\; b^2$$

Beispiel 9.8

In[22]:= $$\int_0^1\int_{x^2}^{-(x-1)^2+1} x\, dy\, dx$$

Out[22]= $$\frac{1}{6}$$

In[23]:= $\int_0^1 \int_{-\sqrt{1-y}+1}^{\sqrt{y}} x \, dx \, dy$

Out[23]= $\frac{1}{6}$

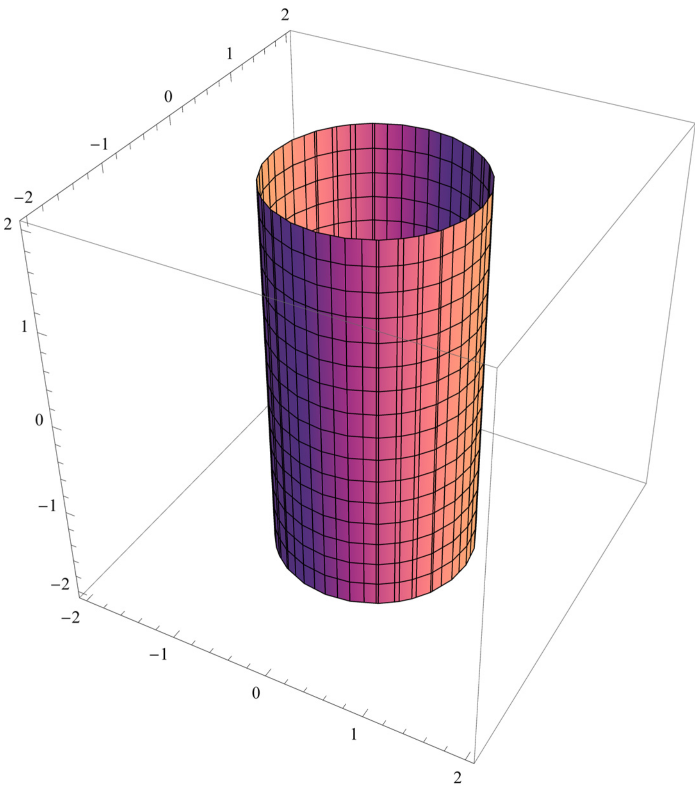
- **Beispiel 9.10**

In[24]:= $8 \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} 1 \, dz \, dy \right) dx$

Out[24]= $\frac{16}{3}$

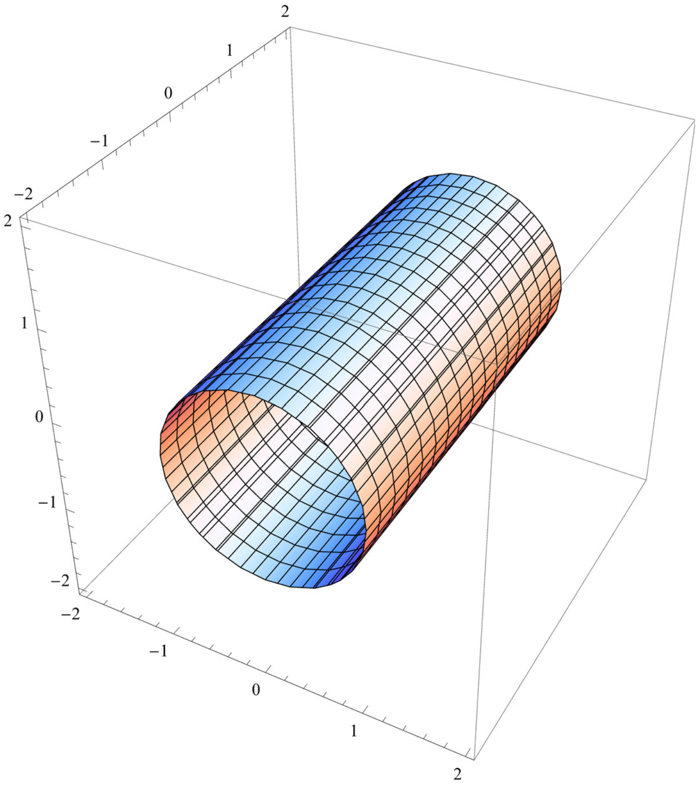
In[25]:= `plot1 = ContourPlot3D[x^2 + y^2 == 1, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]`

Out[25]=



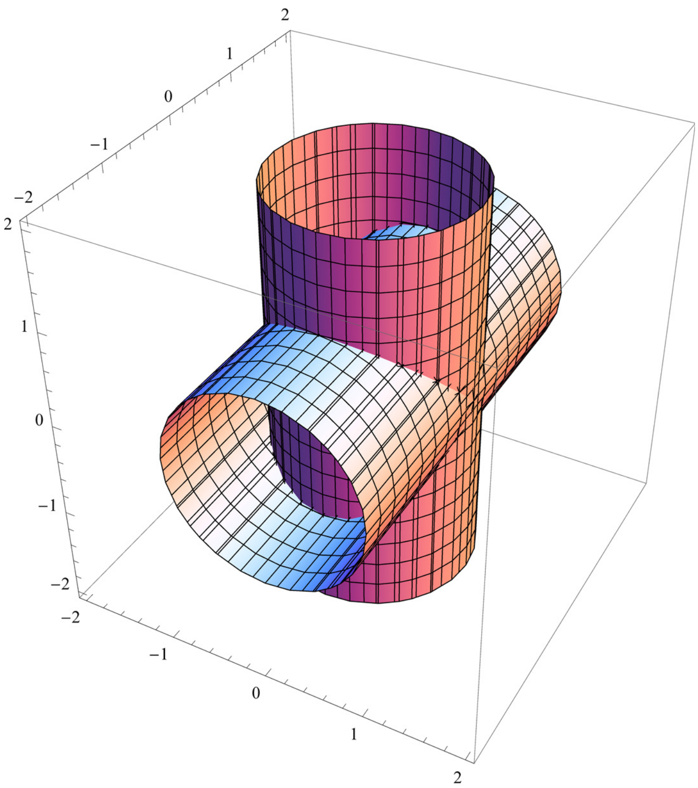
In[26]:= `plot2 = ContourPlot3D[x^2 + z^2 == 1, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]`

Out[26]=



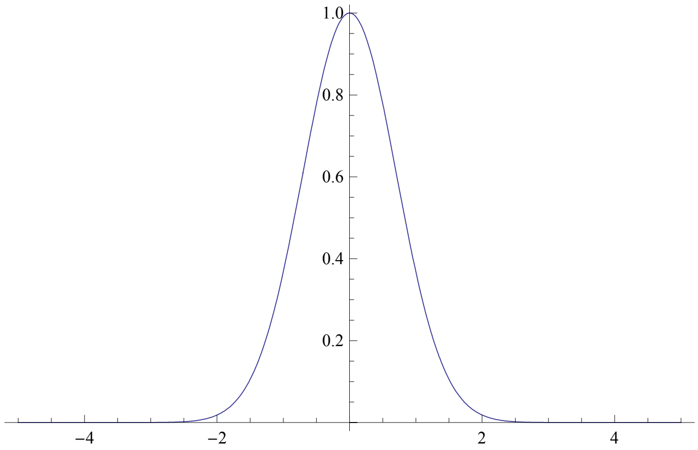
In[27]:= `Show[plot1, plot2]`

Out[27]=



In[28]:= `Plot[Exp[-x^2], {x, -5, 5}]`

Out[28]=



In[29]:= $\int \text{Exp}[-x^2]\, dx$

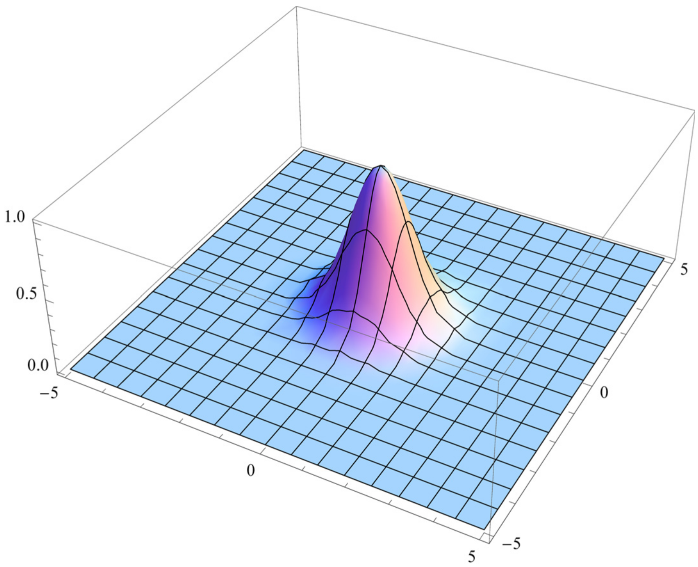
Out[29]= $\frac{1}{2}\sqrt{\pi}\ \text{erf}(x)$

In[30]:= $\int_{-\infty}^{\infty} \text{Exp}[-x^2]\, dx$

Out[30]= $\sqrt{\pi}$

In[31]:= `Plot3D[Exp[-x^2 - y^2], {x, -5, 5}, {y, -5, 5}, PlotRange → All]`

Out[31]=



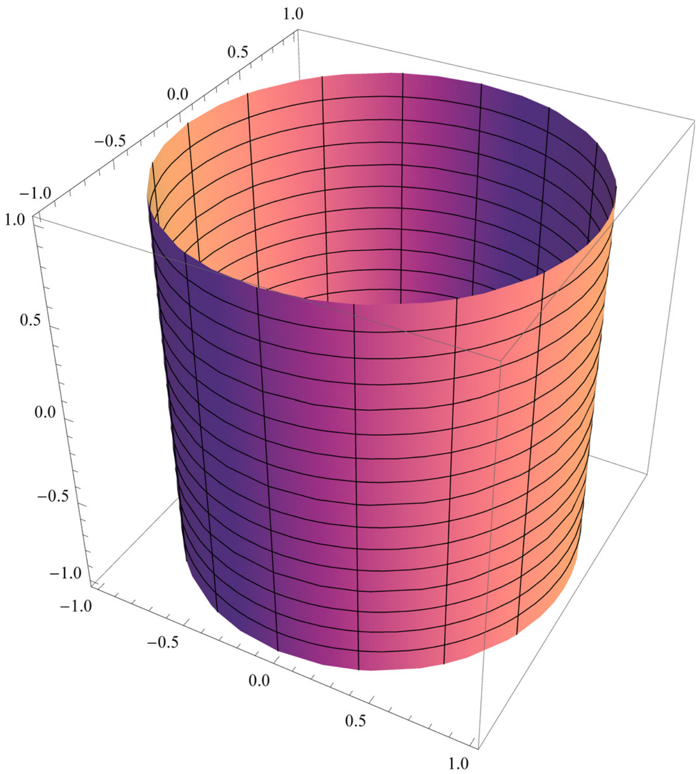
■ Jacobimatrix

In[32]:= `JacobiMatrix[f_, var_] := Table[D[f[[k]], var[[j]]], {k, Length[f]}, {j, Length[var]}]`

Zylinderkoordinaten

In[33]:= `zylinder = ParametricPlot3D[{Cos[ϕ], Sin[ϕ], z}, {ϕ, 0, 2 π}, {z, -1, 1}]`

Out[33]=



In[34]:= `jacobi = JacobiMatrix[{r Cos[ϕ], r Sin[ϕ], z}, {r, ϕ, z}]`

Out[34]= $\begin{pmatrix} \cos(\phi) & -r\sin(\phi) & 0 \\ \sin(\phi) & r\cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In[35]:= `Det[jacobi] // Simplify`

Out[35]= r

Beispiel 9.11

In[36]:= $\int_0^{2\pi}\int_0^{r} r \, dr \, d\phi$

Out[36]= πr^2

Beispiel 9.12

In[37]:= int1 = $\int_0^{2\pi}\int_0^{R} r \, \text{Exp}[-r^2] \, dr \, d\phi$

Out[37]= $\pi - \pi e^{-R^2}$

In[38]:= `Limit[int1, R → ∞]`

Out[38]= π

In[39]:= int2 = $\int_{-S}^{S}\int_{-S}^{S} \text{Exp}[-x^2 - y^2] \, dx \, dy$

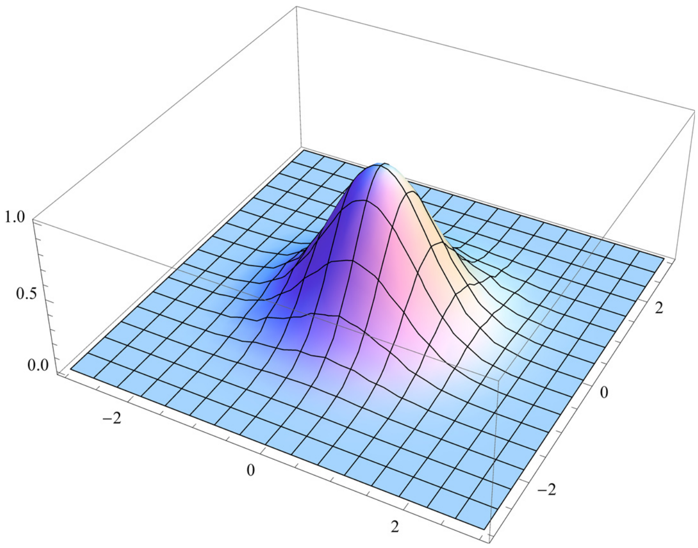
Out[39]= $\pi \, \text{erf}(S)^2$

In[40]:= `Limit[int2, S → ∞]`

Out[40]= π

In[41]:= `Plot3D[Exp[-x^2 - y^2], {x, -3, 3}, {y, -3, 3}, PlotRange → All]`

Out[41]=



■ **Beispiel 9.13**

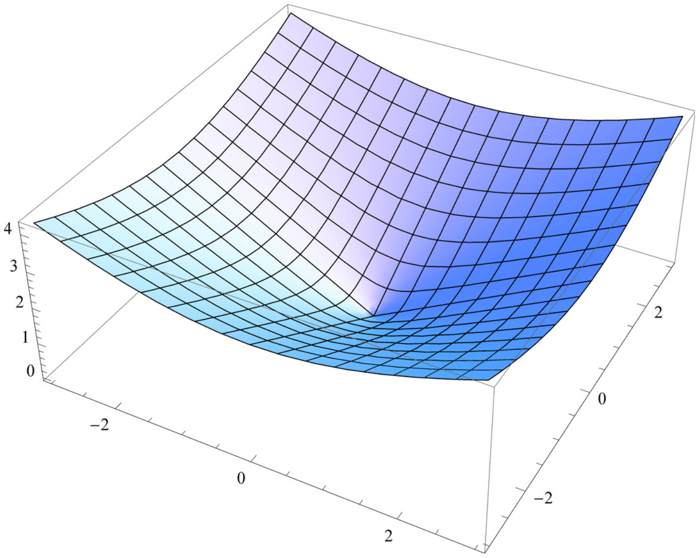
In[42]:= $\text{int} = \int_0^{2\pi}\int_0^{H}\int_0^{\frac{R}{H}z} r^3 \, dr \, dz \, d\phi$

Out[42]= $\frac{1}{10}\pi H R^4$

■ **Beispiel 9.14**

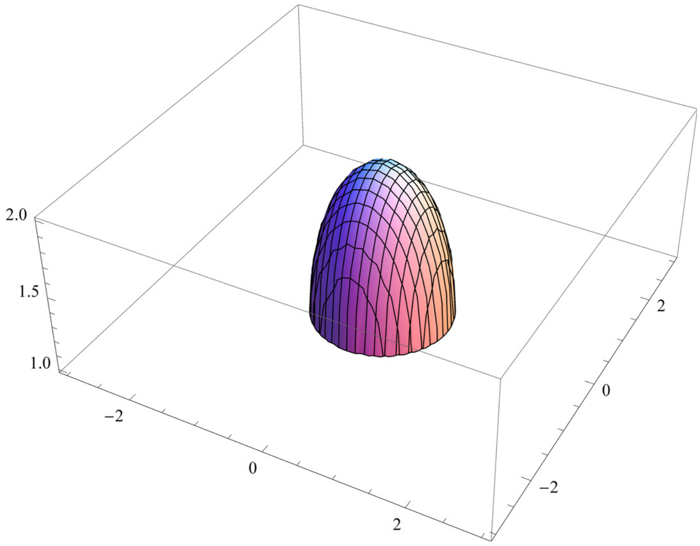
In[43]:= `Plot3D[Sqrt[x^2 + y^2], {x, -3, 3}, {y, -3, 3}]`

Out[43]=



In[44]:= `Plot3D[1 + Sqrt[1 - (x^2 + y^2)], {x, -3, 3}, {y, -3, 3}]`

Out[44]=



In[45]:= $\texttt{int} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1+\sqrt{1-(x^2+y^2)}} dz\,dy\,dx$

Out[45]= π

In[46]:= $\texttt{inner1} = \int_{\sqrt{x^2+y^2}}^{1+\sqrt{1-(x^2+y^2)}} dz$

Out[46]= $\sqrt{-x^2-y^2+1} - \sqrt{x^2+y^2} + 1$

In[47]:= `inner2 = Integrate[1, {y, -Sqrt[1 - x^2], Sqrt[1 - x^2]},`
`{z, Sqrt[x^2 + y^2], 1 + Sqrt[1 - (x^2 + y^2)]}, GenerateConditions → False]`

Out[47]= $\frac{1}{2}\left(\pi - \pi x^2\right) + \sqrt{1-x^2} + x^2\left(-\tanh^{-1}\left(\sqrt{1-x^2}\right)\right)$

In[48]:= $\texttt{int} = \int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{1+\sqrt{1-r^2}} r\,dz\,dr\,d\phi$

Out[48]= π

In[49]:= $\texttt{inner1} = \int_{r}^{1+\sqrt{1-r^2}} r\,dz$

Out[49]= $r\left(\sqrt{1-r^2} - r + 1\right)$

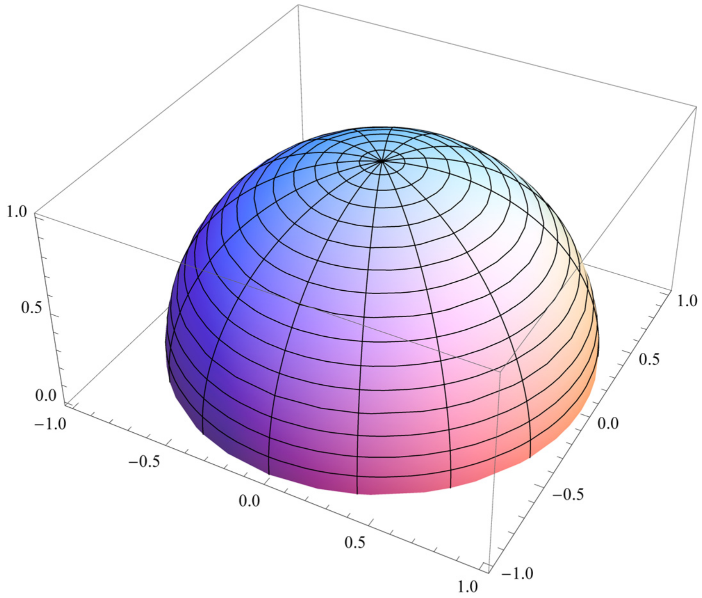
In[50]:= $\texttt{inner2} = \int_{0}^{1}\int_{r}^{1+\sqrt{1-r^2}} r\,dz\,dr$

Out[50]= $\frac{1}{2}$

▪ Kugelkoordinaten

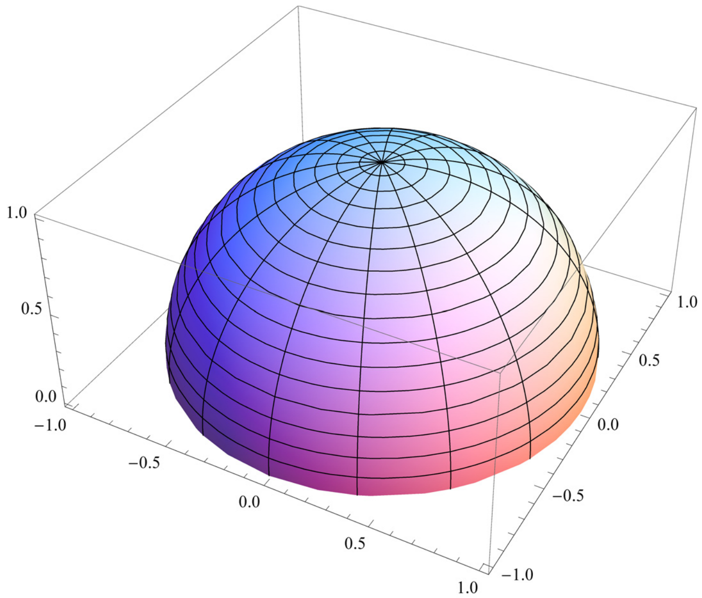
In[51]:=
```
halbkugel =
  ParametricPlot3D[{Cos[ϕ] Sin[θ], Sin[ϕ] Sin[θ], Cos[θ]}, {ϕ, 0, 2 π}, {θ, 0, π/2}]
```

Out[51]=



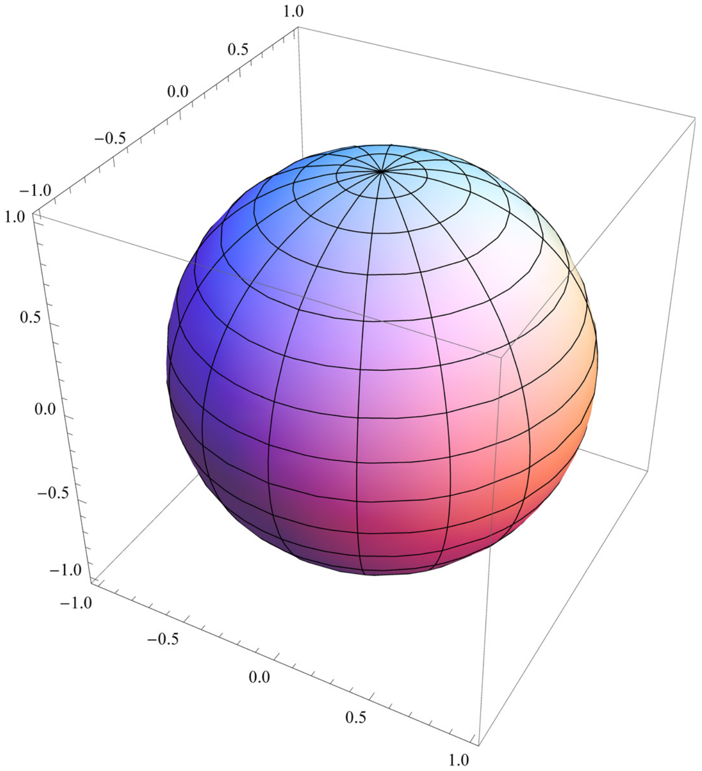
In[52]:=
```
halbkugel =
  ParametricPlot3D[{Cos[ϕ] Cos[θ], Sin[ϕ] Cos[θ], Sin[θ]}, {ϕ, 0, 2 π}, {θ, 0, π/2}]
```

Out[52]=



In[53]:= `kugel = ParametricPlot3D[{Cos[ϕ] Cos[θ], Sin[ϕ] Cos[θ], Sin[θ]}, {ϕ, 0, 2 π}, {θ, -π/2, π/2}]`

Out[53]=



In[54]:= `jacobi = JacobiMatrix[{r Cos[ϕ] Cos[θ], r Sin[ϕ] Cos[θ], r Sin[θ]}, {r, ϕ, θ}]`

Out[54]=
$$\begin{pmatrix} \cos(\theta)\cos(\phi) & -r\cos(\theta)\sin(\phi) & -r\cos(\phi)\sin(\theta) \\ \cos(\theta)\sin(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ \sin(\theta) & 0 & r\cos(\theta) \end{pmatrix}$$

In[55]:= `Det[jacobi] // Simplify`

Out[55]= $r^2 \cos(\theta)$

- **Beispiel 9.15**

In[56]:= `int =` $\int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2 \,\mathrm{Sin}[\theta]\, dr\, d\phi\, d\theta$

Out[56]=
$$\frac{4\pi R^3}{3}$$

$$\frac{4\pi R^3}{3}$$

- **Beispiel 9.16**

In[57]:= `int =` $\int_{R0}^{R}\int_0^{2\pi}\int_0^{\mathrm{ArcCos}\left[\frac{R0}{r}\right]} r^4 \,\mathrm{Sin}[\theta]\, d\theta\, d\phi\, dr$

Out[57]=
$$\frac{1}{10}\pi\left(4R^5 - 5R^4\,\mathrm{R0} + \mathrm{R0}^5\right)$$

$$\frac{1}{10}\pi\left(4R^5 - 5R^4\,\mathrm{R0} + \mathrm{R0}^5\right)$$

▪ Beispiel 9.17

In[58]:= `int = c` $\int_0^1 \int_0^{\frac{\pi}{2}} \sqrt{1 - r^2}\ a\,b\,r\, d\phi\, dr$

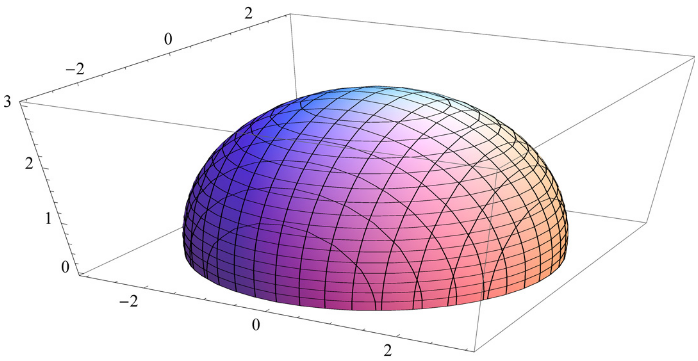
Out[58]= $\frac{1}{6} \pi a b c$

$\frac{1}{6} \pi a b c$

▪ Übung 9.10

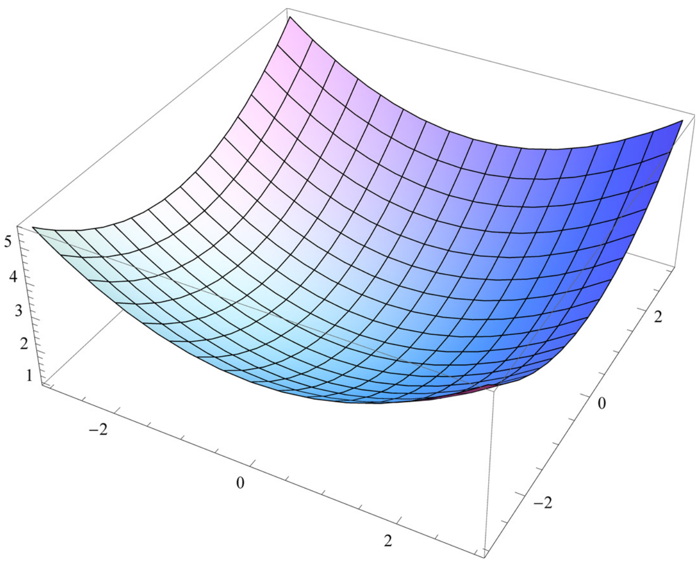
In[59]:= `plot1 = ContourPlot3D[`$x^2 + y^2 + z^2 == 8$`, {x, -3, 3}, {y, -3, 3}, {z, 0, 3}, AspectRatio →` $\frac{1}{2}$`]`

Out[59]=



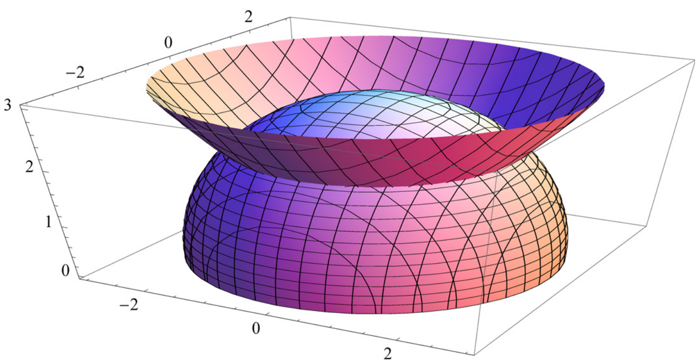
In[60]:= `plot2 = Plot3D[`$\frac{x^2 + y^2 + 4}{4}$`, {x, -3, 3}, {y, -3, 3}]`

Out[60]=



In[61]:= `Show[plot1, plot2]`

Out[61]=



In[62]:= `int =` $\int_0^{2\pi}\int_0^2\int_{\frac{r^2}{4}+1}^{\sqrt{8-r^2}} r\,dz\,dr\,d\phi$

Out[62]= $\frac{2}{3}\left(16\sqrt{2}-17\right)\pi$

$\frac{2}{3}\left(16\sqrt{2}-17\right)\pi$

■ Mittlerer Abstand vom Ursprung im Einheitswürfel

In[63]:= `int =` $\int_0^1\int_0^1\int_0^1 \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$

Out[63]= $\frac{1}{72}\left(2\left(9\sqrt{3}+\log\left(9\,863\,382\,151+5\,694\,626\,340\sqrt{3}\right)\right)-3\pi\right)$

In[64]:= `FullSimplify[int]`

Out[64]= $\frac{1}{72}\left(2\left(9\sqrt{3}+\log\left(9\,863\,382\,151+5\,694\,626\,340\sqrt{3}\right)\right)-3\pi\right)$

In[65]:= `N[int]`

Out[65]= 0.960592