Cauchyprodukt der Exponentialreihe

```
sum[x^k/k!, {k, 0, ∞}] * sum[y^k/k!, {k, 0, ∞}] == sum[Sum[x^j/j! y^(k-j)/(k-j)!, {j, 0, k}], {k, 0, ∞}]
```

$$\text{sum}\left(\frac{x^k}{k!}, \{k, 0, \infty\}\right) \text{sum}\left(\frac{y^k}{k!}, \{k, 0, \infty\}\right) = \text{sum}\left(\frac{(x+y)^k}{k!}, \{k, 0, \infty\}\right)$$

```
Sum[x^j/j! y^(k-j)/(k-j)!, {j, 0, k}]
```

$$\frac{(x+y)^k}{k!}$$

Potenzreihen: Konvergenzradius

```
Konvergenzradius[a_, k_] := Limit[Abs[a/(a /. k -> k + 1)], k -> ∞]
```

Exponentialreihe

```
Konvergenzradius[1/k!, k]
```

$$\infty$$

Cosinusreihe

```
Konvergenzradius[(-1)^k/(2 k)!, k]
```

$$\infty$$

Geometrische Reihe

```
Konvergenzradius[1, k]
```

$$1$$

Geometrische Reihe

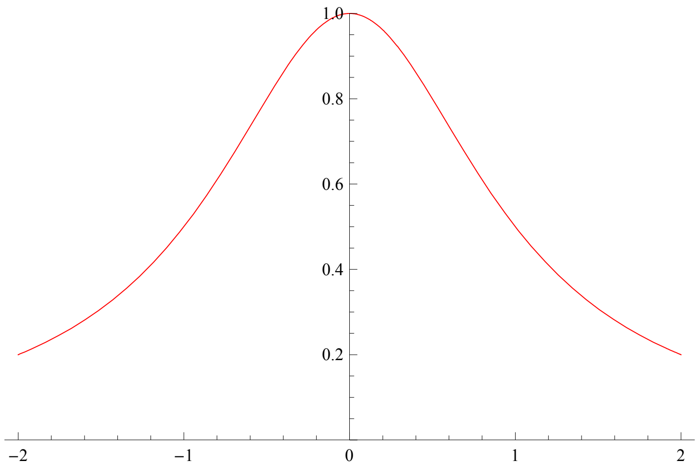
```
Konvergenzradius[(-1)^k, k]
```

$$1$$

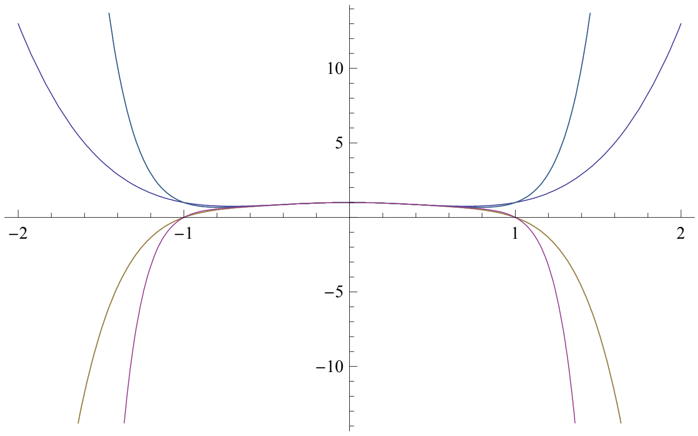
```
f = 1/(1 + x^2)
```

$$\frac{1}{x^2+1}$$

```
plot1 = Plot[f, {x, -2, 2}, PlotRange → {0, 1}, PlotStyle → RGBColor[1, 0, 0]]
```

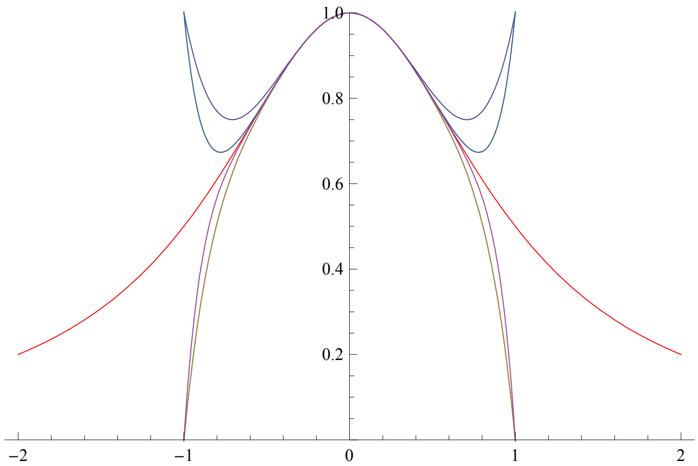



```
plot2 = Plot[Evaluate[Table[Normal[Series[f, {x, 0, k}]], {k, 5, 10}]], {x, -2, 2}]
```



- **Konvergenz nur im Konvergenzkreis**

```
Show[plot1, plot2]
```



- **Der Integralsinus**

- **Wir interessieren uns für eine Stammfunktion der Funktion $\frac{\text{Sin[x]}}{\text{x}}$:**

$$\int \frac{\text{Sin[x]}}{\text{x}}\, dx$$

Si(x)

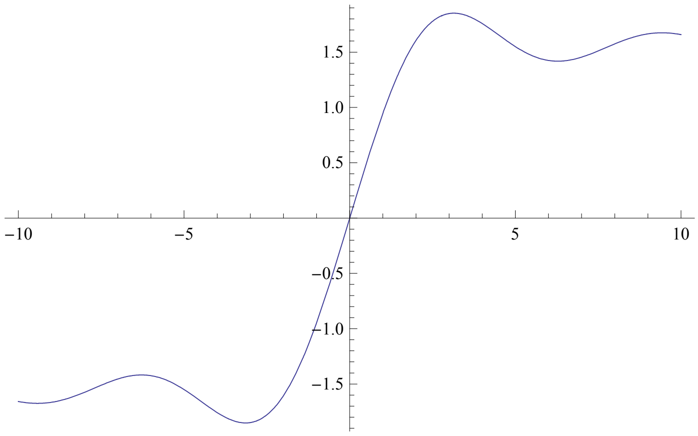
$$\int_0^x \frac{\text{Sin}[t]}{t}\, dt$$

Si(x)

```
FullForm[%]
```

SinIntegral[x]

```
Plot[SinIntegral[x], {x, -10, 10}]
```



- **Man nennt diese Funktion den Integralsinus. Die Sinusfunktion hat die Potenzreihendarstellung**

$$\text{Sin}[x] == \text{sum}\left[\frac{(-1)^k}{(2k+1)!} x^{2k+1}, \{k, 0, \infty\}\right]$$

$$\sin(x) = \text{sum}\left(\frac{(-1)^k x^{2k+1}}{(2k+1)!}, \{k, 0, \infty\}\right)$$

- **also**

$$\frac{\text{Sin}[x]}{x} == \text{sum}\left[\frac{(-1)^k}{(2k+1)!} x^{2k}, \{k, 0, \infty\}\right]$$

$$\frac{\sin(x)}{x} = \text{sum}\left(\frac{(-1)^k x^{2k}}{(2k+1)!}, \{k, 0, \infty\}\right)$$

- **und schließlich**

$$\int \frac{\text{Sin}[x]}{x}\, dx == \text{sum}\left[\int \frac{(-1)^k}{(2k+1)!} x^{2k}\, dx, \{k, 0, \infty\}\right]$$

$$\text{Si}(x) = \text{sum}\left(\frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!}, \{k, 0, \infty\}\right)$$

- **Die sich ergebende Potenzreihe lässt sich zur numerischen Berechnung des Integralsinus nutzen:**

$$\text{Table}\left[\text{Sum}\left[\frac{(-1)^k 1^{2k+1}}{(2k+1)\ (2k+1)!}, \{k, 0, n\}\right], \{n, 1, 10\}\right] \text{ // N}$$

{0.944444, 0.946111, 0.946083, 0.946083, 0.946083, 0.946083, 0.946083, 0.946083, 0.946083, 0.946083}

```
SinIntegral[1.0]
```

0.946083

$$f = \frac{\text{Log}[1 + x]}{1 - x}$$

$$\frac{\log(x+1)}{1-x}$$

```
Series[f, {x, 0, 10}]
```

$$x+\frac{x^2}{2}+\frac{5x^3}{6}+\frac{7x^4}{12}+\frac{47x^5}{60}+\frac{37x^6}{60}+\frac{319x^7}{420}+\frac{533x^8}{840}+\frac{1879x^9}{2520}+\frac{1627x^{10}}{2520}+O(x^{11})$$

- **Konvergenzgeschwindigkeit der alternierenden harmonischen Reihe bei der Approximation von ln(2)**

$$\texttt{s[n\_] :=} \sum_{k=0}^{n} (-1)^k \frac{1}{k+1}$$

```
s[100] - Log[2] // N
```

0.00492599

```
s[1000] - Log[2] // N
```

0.000499251

```
s[10 000] - Log[2] // N
```

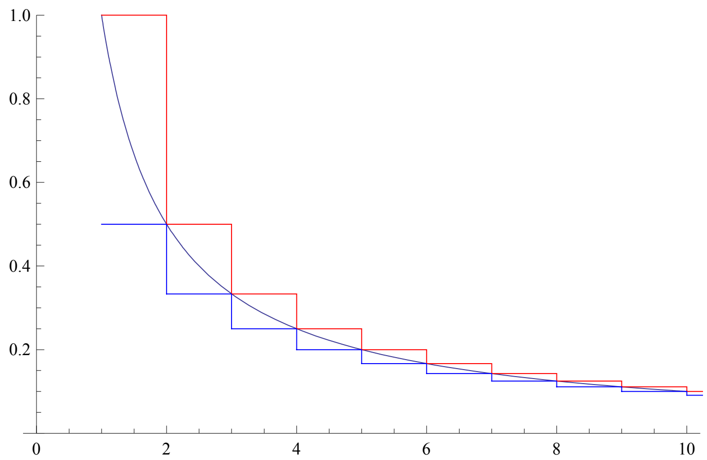
0.0000499925

```
s[100 000] - Log[2] // N
```

4.99993×10^{-6}

- **Graphische Darstellung der harmonischen Reihe und des Integralkriteriums**

```
plot1 = Plot[1/x, {x, 0, 10}, PlotRange → {0, 1}, DisplayFunction → Identity];
plot2 = Table[
   Graphics[{RGBColor[1, 0, 0], Line[{{k, 1/k}, {k + 1, 1/k}, {k + 1, 1/(k + 1)}}]}], {k, 1, 10}];
plot3 = Table[Graphics[
    {RGBColor[0, 0, 1], Line[{{k, 1/(k + 1)}, {k + 1, 1/(k + 1)}, {k + 1, 1/(k + 2)}}]}], {k, 1, 10}];
Show[plot1, plot2, plot3, DisplayFunction → $DisplayFunction]
```



- **Abschätzung der Approximation von $\sum_{k=1}^{\infty} \frac{1}{k^5}$ mit dem Integralkriterium. Zunächst der exakte Wert:**

$$\sum_{k=1}^{\infty} \frac{1}{k^5}$$

$\zeta(5)$

$\sum_{k=1}^{\infty} \frac{1}{k^5}$ // N

1.03693

- **Approximation durch 100 Summanden**

$\sum_{k=1}^{100} \frac{1}{k^5}$

13 665 863 048 356 383 670 978 767 877 205 311 114 172 007 580 136 980 515 111 892 289 714 456 354 157 226 357 505 124 914 ⋱
499 589 679 179 949 790 863 693 285 625 009 456 673 978 234 127 682 400 597 828 733 358 769 121 259 952 785 302 489 985 ⋱
064 354 139 668 191 245 263 /
13 179 185 351 019 348 393 269 383 948 867 758 575 959 215 380 852 037 147 528 985 841 492 344 929 720 350 742 751 601 ⋱
893 318 093 712 781 376 563 994 811 320 764 825 662 981 908 820 197 919 859 781 615 615 970 727 080 743 696 272 335 ⋱
141 540 540 173 453 885 440 000 000

$\sum_{k=1}^{100} \frac{1}{k^5}$ // N

1.03693

- **Fehlerabschätzung**

$\int_{101}^{\infty} \frac{1}{x^5} \, dx$

$\frac{1}{416\,241\,604}$

$\int_{101}^{\infty} \frac{1}{x^5} \, dx$ // N

2.40245×10^{-9}

- **Abschätzung der Approximation von $\sum_{k=1}^{\infty} \frac{1}{k \,\mathrm{Log}[k]}$ mit dem Integralkriterium. Zunächst der exakte Wert:**

$\sum_{k=2}^{\infty} \frac{1}{k \,\mathrm{Log}[k]}$

Sum::div : Sum does not converge. ≫

$$\sum_{k=2}^{\infty} \frac{1}{k \log(k)}$$

- **Den kennt *Mathematica* nicht. Anwendung des Integralkriteriums:**

$\int_{2}^{\infty} \frac{1}{x \,\mathrm{Log}[x]} \, dx$

Integrate::idiv : Integral of $\frac{1}{x \log(x)}$ does not converge on $\{2, \infty\}$. ≫

$$\int_{2}^{\infty} \frac{1}{x \log(x)} \, dx$$

- ***Mathematica* behauptet, das Integral (und daher auch die Summe) divergiere. Nachweis über die Stammfunktion:**

int = $\int \frac{1}{x \,\mathrm{Log}[x]} \, dx$

$\log(\log(x))$

```
Limit[int, x → ∞]
```

∞

■ Ein Beispiel, bei welchem die Ableitungen nicht gleichmäßig konvergieren

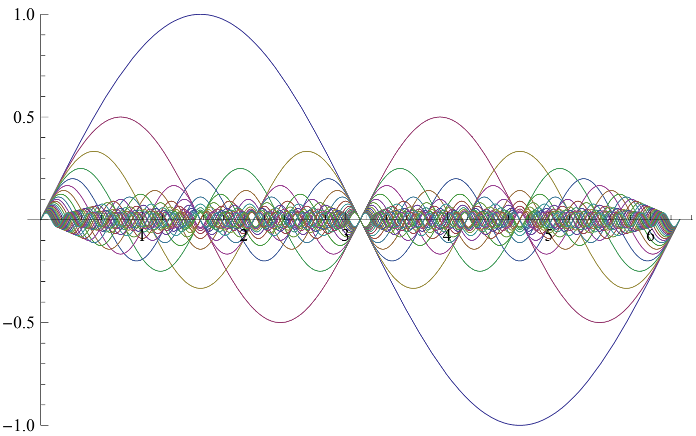
```
f = Sin[n x]/n
```

$$\frac{\sin(n\,x)}{n}$$

```
Limit[f, n → ∞]
```

$$\lim_{n\to\infty} \frac{\sin(n\,x)}{n}$$

```
Plot[Evaluate[Table[f, {n, 1, 30}]], {x, 0, 2 π}, PlotRange → {-1, 1}]
```



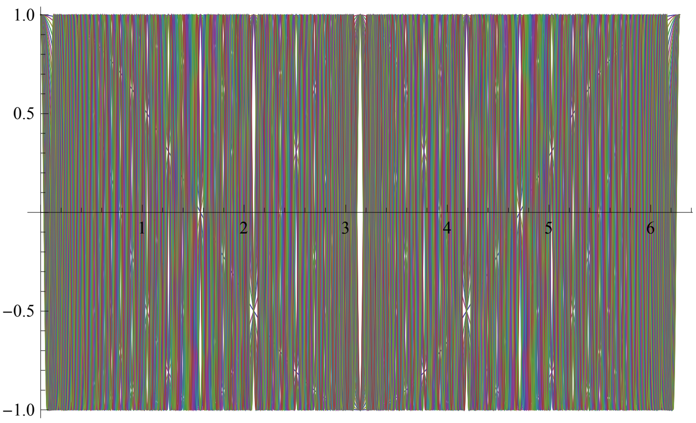

```
g = D[f, x]
```

$\cos(n\,x)$

```
Limit[g, n → ∞]
```

$$\lim_{n\to\infty} \cos(n\,x)$$

```
Plot[Evaluate[Table[g, {n, 1, 50}]], {x, 0, 2 π}]
```



■ Ein Kehrwert

```
reihe1 = Series[f = (E^x - 1)/x, {x, 0, 10}]
```

$$1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+\frac{x^4}{120}+\frac{x^5}{720}+\frac{x^6}{5040}+\frac{x^7}{40\,320}+\frac{x^8}{362\,880}+\frac{x^9}{3\,628\,800}+\frac{x^{10}}{39\,916\,800}+O(x^{11})$$

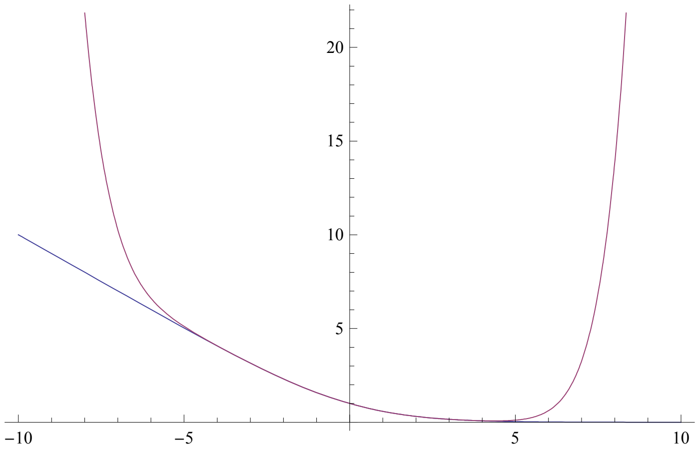
```
reihe2 = Series[1/f, {x, 0, 10}]
```

$$1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30\,240} - \frac{x^8}{1\,209\,600} + \frac{x^{10}}{47\,900\,160} + O(x^{11})$$

```
reihe1 * reihe2
```

$$1 + O(x^{11})$$

```
Plot[Evaluate[{1/f, Normal[reihe2]}], {x, -10, 10}]
```



- **Der Konvergenzradius ist 2π. Man beachte immer: Außerhalb des Konvergenzbereichs wird die Funktion nicht von ihrer Taylorreihe approximiert!**
- **Berechnung der Koeffizienten des Kehrwerts**

```
reihe2 = Sum[a[k] * x^k, {k, 0, 10}]
```

$$a(10)\,x^{10} + a(9)\,x^9 + a(8)\,x^8 + a(7)\,x^7 + a(6)\,x^6 + a(5)\,x^5 + a(4)\,x^4 + a(3)\,x^3 + a(2)\,x^2 + a(1)\,x + a(0)$$

```
produkt = Expand[Normal[reihe1] * reihe2]
```

$$\frac{a(10)\,x^{20}}{39\,916\,800}+\frac{a(9)\,x^{19}}{39\,916\,800}+\frac{a(10)\,x^{19}}{3\,628\,800}+\frac{a(8)\,x^{18}}{39\,916\,800}+\frac{a(9)\,x^{18}}{3\,628\,800}+\frac{a(10)\,x^{18}}{362\,880}+\frac{a(7)\,x^{17}}{39\,916\,800}+\frac{a(8)\,x^{17}}{3\,628\,800}+$$
$$\frac{a(9)\,x^{17}}{362\,880}+\frac{a(10)\,x^{17}}{40\,320}+\frac{a(6)\,x^{16}}{39\,916\,800}+\frac{a(7)\,x^{16}}{3\,628\,800}+\frac{a(8)\,x^{16}}{362\,880}+\frac{a(9)\,x^{16}}{40\,320}+\frac{a(10)\,x^{16}}{5040}+\frac{a(5)\,x^{15}}{39\,916\,800}+\frac{a(6)\,x^{15}}{3\,628\,800}+$$
$$\frac{a(7)\,x^{15}}{362\,880}+\frac{a(8)\,x^{15}}{40\,320}+\frac{a(9)\,x^{15}}{5040}+\frac{1}{720}\,a(10)\,x^{15}+\frac{a(4)\,x^{14}}{39\,916\,800}+\frac{a(5)\,x^{14}}{3\,628\,800}+\frac{a(6)\,x^{14}}{362\,880}+\frac{a(7)\,x^{14}}{40\,320}+\frac{a(8)\,x^{14}}{5040}+$$
$$\frac{1}{720}\,a(9)\,x^{14}+\frac{1}{120}\,a(10)\,x^{14}+\frac{a(3)\,x^{13}}{39\,916\,800}+\frac{a(4)\,x^{13}}{3\,628\,800}+\frac{a(5)\,x^{13}}{362\,880}+\frac{a(6)\,x^{13}}{40\,320}+\frac{a(7)\,x^{13}}{5040}+\frac{1}{720}\,a(8)\,x^{13}+$$
$$\frac{1}{120}\,a(9)\,x^{13}+\frac{1}{24}\,a(10)\,x^{13}+\frac{a(2)\,x^{12}}{39\,916\,800}+\frac{a(3)\,x^{12}}{3\,628\,800}+\frac{a(4)\,x^{12}}{362\,880}+\frac{a(5)\,x^{12}}{40\,320}+\frac{a(6)\,x^{12}}{5040}+\frac{1}{720}\,a(7)\,x^{12}+$$
$$\frac{1}{120}\,a(8)\,x^{12}+\frac{1}{24}\,a(9)\,x^{12}+\frac{1}{6}\,a(10)\,x^{12}+\frac{a(1)\,x^{11}}{39\,916\,800}+\frac{a(2)\,x^{11}}{3\,628\,800}+\frac{a(3)\,x^{11}}{362\,880}+\frac{a(4)\,x^{11}}{40\,320}+\frac{a(5)\,x^{11}}{5040}+$$
$$\frac{1}{720}\,a(6)\,x^{11}+\frac{1}{120}\,a(7)\,x^{11}+\frac{1}{24}\,a(8)\,x^{11}+\frac{1}{6}\,a(9)\,x^{11}+\frac{1}{2}\,a(10)\,x^{11}+\frac{a(0)\,x^{10}}{39\,916\,800}+\frac{a(1)\,x^{10}}{3\,628\,800}+\frac{a(2)\,x^{10}}{362\,880}+$$
$$\frac{a(3)\,x^{10}}{40\,320}+\frac{a(4)\,x^{10}}{5040}+\frac{1}{720}\,a(5)\,x^{10}+\frac{1}{120}\,a(6)\,x^{10}+\frac{1}{24}\,a(7)\,x^{10}+\frac{1}{6}\,a(8)\,x^{10}+\frac{1}{2}\,a(9)\,x^{10}+a(10)\,x^{10}+$$
$$\frac{a(0)\,x^{9}}{3\,628\,800}+\frac{a(1)\,x^{9}}{362\,880}+\frac{a(2)\,x^{9}}{40\,320}+\frac{a(3)\,x^{9}}{5040}+\frac{1}{720}\,a(4)\,x^{9}+\frac{1}{120}\,a(5)\,x^{9}+\frac{1}{24}\,a(6)\,x^{9}+\frac{1}{6}\,a(7)\,x^{9}+\frac{1}{2}\,a(8)\,x^{9}+$$
$$a(9)\,x^{9}+\frac{a(0)\,x^{8}}{362\,880}+\frac{a(1)\,x^{8}}{40\,320}+\frac{a(2)\,x^{8}}{5040}+\frac{1}{720}\,a(3)\,x^{8}+\frac{1}{120}\,a(4)\,x^{8}+\frac{1}{24}\,a(5)\,x^{8}+\frac{1}{6}\,a(6)\,x^{8}+\frac{1}{2}\,a(7)\,x^{8}+$$
$$a(8)\,x^{8}+\frac{a(0)\,x^{7}}{40\,320}+\frac{a(1)\,x^{7}}{5040}+\frac{1}{720}\,a(2)\,x^{7}+\frac{1}{120}\,a(3)\,x^{7}+\frac{1}{24}\,a(4)\,x^{7}+\frac{1}{6}\,a(5)\,x^{7}+\frac{1}{2}\,a(6)\,x^{7}+a(7)\,x^{7}+$$
$$\frac{a(0)\,x^{6}}{5040}+\frac{1}{720}\,a(1)\,x^{6}+\frac{1}{120}\,a(2)\,x^{6}+\frac{1}{24}\,a(3)\,x^{6}+\frac{1}{6}\,a(4)\,x^{6}+\frac{1}{2}\,a(5)\,x^{6}+a(6)\,x^{6}+\frac{1}{720}\,a(0)\,x^{5}+\frac{1}{120}\,a(1)\,x^{5}+$$
$$\frac{1}{24}\,a(2)\,x^{5}+\frac{1}{6}\,a(3)\,x^{5}+\frac{1}{2}\,a(4)\,x^{5}+a(5)\,x^{5}+\frac{1}{120}\,a(0)\,x^{4}+\frac{1}{24}\,a(1)\,x^{4}+\frac{1}{6}\,a(2)\,x^{4}+\frac{1}{2}\,a(3)\,x^{4}+a(4)\,x^{4}+$$
$$\frac{1}{24}\,a(0)\,x^{3}+\frac{1}{6}\,a(1)\,x^{3}+\frac{1}{2}\,a(2)\,x^{3}+a(3)\,x^{3}+\frac{1}{6}\,a(0)\,x^{2}+\frac{1}{2}\,a(1)\,x^{2}+a(2)\,x^{2}+\frac{1}{2}\,a(0)\,x+a(1)\,x+a(0)$$

```
liste = Take[CoefficientList[produkt, x], 11]
```

$$\Big\{a(0),\ \frac{a(0)}{2}+a(1),\ \frac{a(0)}{6}+\frac{a(1)}{2}+a(2),\ \frac{a(0)}{24}+\frac{a(1)}{6}+\frac{a(2)}{2}+a(3),$$
$$\frac{a(0)}{120}+\frac{a(1)}{24}+\frac{a(2)}{6}+\frac{a(3)}{2}+a(4),\ \frac{a(0)}{720}+\frac{a(1)}{120}+\frac{a(2)}{24}+\frac{a(3)}{6}+\frac{a(4)}{2}+a(5),$$
$$\frac{a(0)}{5040}+\frac{a(1)}{720}+\frac{a(2)}{120}+\frac{a(3)}{24}+\frac{a(4)}{6}+\frac{a(5)}{2}+a(6),\ \frac{a(0)}{40\,320}+\frac{a(1)}{5040}+\frac{a(2)}{720}+\frac{a(3)}{120}+\frac{a(4)}{24}+\frac{a(5)}{6}+\frac{a(6)}{2}+a(7),$$
$$\frac{a(0)}{362\,880}+\frac{a(1)}{40\,320}+\frac{a(2)}{5040}+\frac{a(3)}{720}+\frac{a(4)}{120}+\frac{a(5)}{24}+\frac{a(6)}{6}+\frac{a(7)}{2}+a(8),$$
$$\frac{a(0)}{3\,628\,800}+\frac{a(1)}{362\,880}+\frac{a(2)}{40\,320}+\frac{a(3)}{5040}+\frac{a(4)}{720}+\frac{a(5)}{120}+\frac{a(6)}{24}+\frac{a(7)}{6}+\frac{a(8)}{2}+a(9),$$
$$\frac{a(0)}{39\,916\,800}+\frac{a(1)}{3\,628\,800}+\frac{a(2)}{362\,880}+\frac{a(3)}{40\,320}+\frac{a(4)}{5040}+\frac{a(5)}{720}+\frac{a(6)}{120}+\frac{a(7)}{24}+\frac{a(8)}{6}+\frac{a(9)}{2}+a(10)\Big\}$$

```
sol = Solve[liste == {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, Table[a[k], {k, 0, 10}]]
```

$$\Big\{\Big\{a(0)\to 1,\ a(1)\to -\frac{1}{2},\ a(2)\to \frac{1}{12},\ a(3)\to 0,\ a(4)\to -\frac{1}{720},$$
$$a(5)\to 0,\ a(6)\to \frac{1}{30\,240},\ a(7)\to 0,\ a(8)\to -\frac{1}{1\,209\,600},\ a(9)\to 0,\ a(10)\to \frac{1}{47\,900\,160}\Big\}\Big\}$$

```
reihe2 /. sol[[1]]
```

$$\frac{x^{10}}{47\,900\,160} - \frac{x^8}{1\,209\,600} + \frac{x^6}{30\,240} - \frac{x^4}{720} + \frac{x^2}{12} - \frac{x}{2} + 1$$

```
reihe1 * (reihe2 /. sol[[1]])
```

$$1 + O(x^{11})$$