Funktionentheorie

```
Clear[Realteil, Imaginärteil]

Realteil[f_] := Simplify[
  ComplexExpand[Re[f /. z → x + ⅈ y], TargetFunctions → Conjugate]]

Imaginärteil[f_] := Simplify[
  ComplexExpand[Im[f /. z → x + ⅈ y], TargetFunctions → Conjugate]]
```

■ Übungen und Cauchy-Riemannsche Differentialgleichungen

```
Realteil[z^3]
```

$$x^3 - 3\,x\,y^2$$

```
Imaginärteil[z^3]
```

$$3\,x^2\,y - y^3$$

```
u = Realteil[(1 + z)/(1 - z)]
```

$$-\frac{x^2 + y^2 - 1}{x^2 - 2\,x + y^2 + 1}$$

```
v = Imaginärteil[(1 + z)/(1 - z)]
```

$$\frac{2\,y}{x^2 - 2\,x + y^2 + 1}$$

```
{D[u, x], D[v, y]}
```

$$\left\{\frac{(2\,x-2)\left(x^2+y^2-1\right)}{\left(x^2-2\,x+y^2+1\right)^2} - \frac{2\,x}{x^2-2\,x+y^2+1},\ \frac{2}{x^2-2\,x+y^2+1} - \frac{4\,y^2}{\left(x^2-2\,x+y^2+1\right)^2}\right\}$$

```
{D[u, x], D[v, y]} // Together
```

$$\left\{\frac{2\left(x^2-2x-y^2+1\right)}{\left(x^2-2x+y^2+1\right)^2}, -\frac{2\left(-x^2+2x+y^2-1\right)}{\left(x^2-2x+y^2+1\right)^2}\right\}$$

```
D[u, x] - D[v, y] // Together
```

$$0$$

```
D[u, y] + D[v, x] // Together
```

$$0$$

```
ableitung = D[u + i v, x]
```

$$-\frac{2x}{x^2-2x+y^2+1} - \frac{2i(2x-2)y}{\left(x^2-2x+y^2+1\right)^2} + \frac{(2x-2)\left(x^2+y^2-1\right)}{\left(x^2-2x+y^2+1\right)^2}$$

```
Together[ableitung]
```

$$\frac{2}{(x+iy-1)^2}$$

```
D[(1 + z)/(1 - z), z] // Together
```

$$\frac{2}{(z-1)^2}$$

```
u = Realteil[Cos[z]]
```

$$\cos(x)\cosh(y)$$

```
v = Imaginärteil[Cos[z]]
```

$$-\sin(x)\sinh(y)$$

```
{D[u, x], D[v, y]}
```

$$\{\sin(x)\,(-\cosh(y)),\ \sin(x)\,(-\cosh(y))\}$$

```
{D[u, y], D[v, x]}
```

$$\{\cos(x)\sinh(y),\ -\cos(x)\sinh(y)\}$$

```
ableitung = D[u + ⅈ v, x]
```

$$\sin(x)\,(-\cosh(y)) - i\cos(x)\sinh(y)$$

```
TrigReduce[ableitung]
```

$$-\sin(x + i\,y)$$

- **Beispiel 2.5**

```
u = Realteil[Log[z]]
```

$$\frac{1}{2}\log\left(x^2 + y^2\right)$$

```
v = Imaginärteil[Log[z]]
```

$$\frac{1}{2}\,i\log\left(x^2 + y^2\right) - i\log(x + i\,y)$$

```
v = ArcTan[y/x]
```

$$\tan^{-1}\left(\frac{y}{x}\right)$$

```
v2 = 2 ArcTan[y/(x + Sqrt[x^2 + y^2])]
```

$$2\tan^{-1}\left(\frac{y}{\sqrt{x^2+y^2}+x}\right)$$

```
{D[u, x], D[v, y], D[v2, y]} // FullSimplify
```

$$\left\{\frac{x}{x^2+y^2}, \frac{x}{x^2+y^2}, \frac{x}{x^2+y^2}\right\}$$

```
{D[u, y], D[v, x], D[v2, x]} // FullSimplify
```

$$\left\{\frac{y}{x^2+y^2}, -\frac{y}{x^2+y^2}, -\frac{y}{x^2+y^2}\right\}$$

```
ableitung = D[u + i v, x]
```

$$\frac{x}{x^2+y^2} - \frac{i\,y}{x^2\left(\frac{y^2}{x^2}+1\right)}$$

```
Together[ableitung]
```

$$\frac{1}{x+i\,y}$$

- **Beispiel 2.6**

```
ComplexExpand[Conjugate[x + I y]]
```

$$x - i\,y$$

```
u = Realteil[Conjugate[z] * ℯ^z²]
```

$$e^{x^2-y^2}\,(y\sin(2\,x\,y) + x\cos(2\,x\,y))$$

```
v = Imaginärteil[Conjugate[z] * ℯ^z²]
```

$$e^{x^2-y^2}\,(x\sin(2\,x\,y) - y\cos(2\,x\,y))$$

```
D[u, x]
```

$$2\,x\,e^{x^2-y^2}\,(y\sin(2\,x\,y) + x\cos(2\,x\,y)) + e^{x^2-y^2}\left(2\,y^2\cos(2\,x\,y) - 2\,x\,y\sin(2\,x\,y) + \cos(2\,x\,y)\right)$$

```
D[v, y] // Together
```

$$e^{x^2-y^2}\left(2\,x^2 + 2\,y^2 - 1\right)\cos(2\,x\,y)$$

```
D[u, x] - D[v, y] // Together
```

$$2\,e^{x^2-y^2}\cos(2\,x\,y)$$

```
D[u, y]
```

$$e^{x^2-y^2}\left(-2\,x^2\sin(2\,x\,y) + \sin(2\,x\,y) + 2\,x\,y\cos(2\,x\,y)\right) - 2\,y\,e^{x^2-y^2}\,(y\sin(2\,x\,y) + x\cos(2\,x\,y))$$

```
D[v, x] // Together
```

$$e^{x^2-y^2}\left(2\,x^2 + 2\,y^2 + 1\right)\sin(2\,x\,y)$$

```
D[u, y] + D[v, x] // Together
```

$$2\, e^{x^2-y^2} \sin(2\, x\, y)$$

- **Beispiel 2.7**

```
u = Realteil[e^(z^2)]
```

$$e^{x^2-y^2} \cos(2\, x\, y)$$

```
v = Imaginärteil[e^(z^2)]
```

$$e^{x^2-y^2} \sin(2\, x\, y)$$

```
D[u, x] // Factor
```

$$2\, e^{x^2-y^2}\, (x \cos(2\, x\, y) - y \sin(2\, x\, y))$$

```
D[v, y] // Factor
```

$$2\, e^{x^2-y^2}\, (x \cos(2\, x\, y) - y \sin(2\, x\, y))$$

```
D[u, x] - D[v, y] // Together
```

$$0$$

```
D[u, y]
```

$$-2\, x\, e^{x^2-y^2} \sin(2\, x\, y) - 2\, y\, e^{x^2-y^2} \cos(2\, x\, y)$$

```
D[v, x]
```

$$2\, x\, e^{x^2-y^2} \sin(2\, x\, y) + 2\, y\, e^{x^2-y^2} \cos(2\, x\, y)$$

```
D[u, y] + D[v, x] // Together
```

0

```
ableitung = D[u + i v, x]
```

$2\,i\,x\,e^{x^2-y^2}\sin(2\,x\,y) - 2\,y\,e^{x^2-y^2}\sin(2\,x\,y) + 2\,x\,e^{x^2-y^2}\cos(2\,x\,y) + 2\,i\,y\,e^{x^2-y^2}\cos(2\,x\,y)$

```
Together[ableitung]
```

$2\,e^{x^2-y^2}\,(x + i\,y)\,(\cos(2\,x\,y) + i\sin(2\,x\,y))$

u,v sind harmonisch:

```
Delta[u_, {x_, y_}] := Together[D[u, {x, 2}] + D[u, {y, 2}]]
```

```
u
```

$e^{x^2-y^2}\cos(2\,x\,y)$

```
Delta[u, {x, y}]
```

0

```
v
```

$e^{x^2-y^2}\sin(2\,x\,y)$

```
Delta[v, {x, y}]
```

0