In[1]:= `Needs["SpecialFunctions`"]`

SpecialFunctions, (C) Wolfram Koepf, version 2.03, 2011

Fast Zeilberger, (C) Peter Paule and Markus Schorn (V 2.2) loaded

■ Übungsaufgabe

In[2]:= $f = \frac{1}{z^2 - 3z}$

Out[2]= $\frac{1}{z^2 - 3z}$

In[3]:= `Apart[f]`

Out[3]= $\frac{1}{3(z-3)} - \frac{1}{3z}$

In[4]:= `FPS[f, {z, 3}]`

Out[4]= $\text{sum}\left((-1)^k\, 3^{-k-1}\,(z-3)^{k-1},\ \{k, 0, \infty\}\right)$

In[5]:= `FPS[f, {z, -1}]`

Out[5]= $\text{sum}\left(\frac{1}{12}\left(4 - 4^{-k}\right)(z+1)^k,\ \{k, 0, \infty\}\right)$

In[6]:= `Series[f, {z, 1, 10}]`

Out[6]= $$-\frac{1}{2} + \frac{z-1}{4} - \frac{3}{8}(z-1)^2 + \frac{5}{16}(z-1)^3 - \frac{11}{32}(z-1)^4 + \frac{21}{64}(z-1)^5 - \frac{43}{128}(z-1)^6 + \frac{85}{256}(z-1)^7 - \frac{171}{512}(z-1)^8 + \frac{341(z-1)^9}{1024} - \frac{683(z-1)^{10}}{2048} + O\left((z-1)^{11}\right)$$

In[7]:= `FPS[f, {z, 1}]`

Out[7]= $\text{sum}\left(\frac{1}{6}\left(-2(-1)^k - 2^{-k}\right)(z-1)^k,\ \{k, 0, \infty\}\right)$

In[8]:= `Series[f, {z, 2 i, 10}]`

Out[8]= $$-\left(\frac{1}{13} - \frac{3i}{26}\right) - \left(\frac{63}{676} + \frac{4i}{169}\right)(z-2i) + \left(\frac{3}{2197} - \frac{855i}{17576}\right)(z-2i)^2 + \left(\frac{10155}{456976} - \frac{40i}{28561}\right)(z-2i)^3 + \left(\frac{199}{371293} + \frac{122463i}{11881376}\right)(z-2i)^4 - \left(\frac{1565523}{308915776} - \frac{276i}{4826809}\right)(z-2i)^5 + \left(\frac{1483}{62748517} - \frac{20636535i}{8031810176}\right)(z-2i)^6 + \left(\frac{271930635}{208827064576} + \frac{9520i}{815730721}\right)(z-2i)^7 - \left(\frac{18801}{10604499373} - \frac{3549537423i}{5429503678976}\right)(z-2i)^8 - \left(\frac{46069404483}{141167095653376} - \frac{48556i}{137858491849}\right)(z-2i)^9 - \left(\frac{438637}{1792160394037} + \frac{597554765415i}{3670344486987776}\right)(z-2i)^{10} + O\left((z-2i)^{11}\right)$$

In[9]:= `FPS[f, {z, 2 i}]`

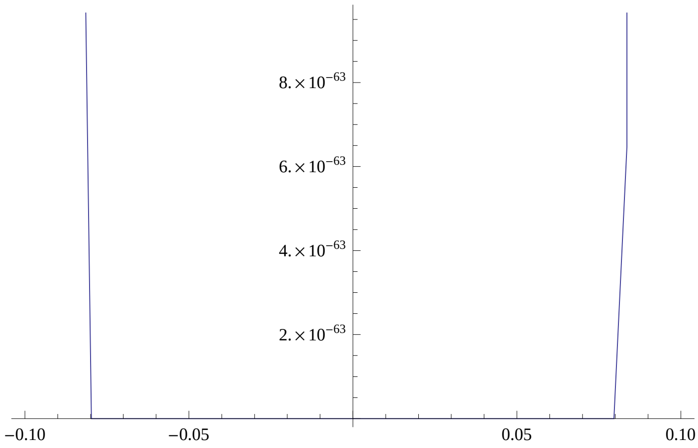
Out[9]= $\text{sum}\left(\left(\frac{1}{26} + \frac{i}{39}\right)\left((2+3i)\left(\frac{i}{2}\right)^k - 2\left(\frac{3}{13} + \frac{2i}{13}\right)^k\right)(z-2i)^k,\ \{k, 0, \infty\}\right)$

Beispiel

In[10]:= `Plot[Exp[-1/x^2], {x, -0.1, 0.1}]`

$$\text{Plot}\left[\text{Exp}\left[-\frac{1}{x^2}\right], \{x, -0.1, 0.1\}\right]$$

Out[10]=



Beispiel (4)

In[11]:= $$\text{Series}\left[\frac{\text{Sin}[z]}{z}, \{z, 0, 5\}\right]$$

Out[11]= $$1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)$$

In[12]:= $$\text{FPS}\left[\frac{\text{Sin}[z]}{z}, \{z, 0\}\right]$$

Out[12]= $$\text{sum}\left(\frac{(-1)^k z^{2k}}{(2k+1)!}, \{k, 0, \infty\}\right)$$

In[13]:= $$\text{Series}\left[\frac{\text{Sin}[z]}{z^2}, \{z, 0, 5\}\right]$$

Out[13]= $$\frac{1}{z} - \frac{z}{6} + \frac{z^3}{120} - \frac{z^5}{5040} + O(z^6)$$

In[14]:= $$\text{FPS}\left[\frac{\text{Sin}[z]}{z^2}, \{z, 0\}\right]$$

Out[14]= $$\text{sum}\left(\frac{(-1)^k z^{2k-1}}{(2k+1)!}, \{k, 0, \infty\}\right)$$

In[15]:= $$\text{Series}\left[\frac{z}{\text{Sin}[z]}, \{z, 0, 5\}\right]$$

Out[15]= $$1 + \frac{z^2}{6} + \frac{7z^4}{360} + O(z^6)$$

In[16]:= $$\frac{1}{36} - \frac{1}{120}$$

Out[16]= $$\frac{7}{360}$$

Residuen

In[17]:= $$f = \frac{1}{z^2 - 3z}$$

Out[17]= $$\frac{1}{z^2 - 3z}$$

In[18]:= `Apart[f]`

Out[18]= $\frac{1}{3(z-3)} - \frac{1}{3z}$

In[19]:= `Series[f, {z, 0, 5}]`

Out[19]= $-\frac{1}{3z} - \frac{1}{9} - \frac{z}{27} - \frac{z^2}{81} - \frac{z^3}{243} - \frac{z^4}{729} - \frac{z^5}{2187} + O(z^6)$

In[20]:= `Residue[f, {z, 0}]`

Out[20]= $-\frac{1}{3}$

In[21]:= `Residue[f, {z, 3}]`

Out[21]= $\frac{1}{3}$

In[22]:= `Residue[f, {z, 5}]`

Out[22]= 0

In[23]:= Residue$\left[e^{\frac{1}{z}}, \{z, 0\}\right]$

Out[23]= $\text{res}\left(e^{\frac{1}{z}}, \{z, 0\}\right)$

In[24]:= Series$\left[\frac{\text{Sin}[z]}{z^4}, \{z, 0, 2\}\right]$

Out[24]= $\frac{1}{z^3} - \frac{1}{6z} + \frac{z}{120} + O(z^3)$

In[25]:= Residue$\left[\frac{\text{Sin}[z]}{z^6}, \{z, 0\}\right]$

Out[25]= $\frac{1}{120}$

In[26]:= Series$\left[\frac{1+z}{z-z^3}, \{z, 0, 5\}\right]$

Out[26]= $\frac{1}{z} + 1 + z + z^2 + z^3 + z^4 + z^5 + O(z^6)$

In[27]:= Residue$\left[\frac{1+z}{z-z^3}, \{z, 0\}\right]$

Out[27]= 1

In[28]:= Residue$\left[\frac{1+z}{z-z^3}, \{z, 1\}\right]$

Out[28]= -1

In[29]:= Residue$\left[\frac{1+z}{z-z^3}, \{z, -1\}\right]$

Out[29]= 0

In[30]:= Apart$\left[\frac{1+z}{z-z^3}\right]$

Out[30]= $\frac{1}{z} - \frac{1}{z-1}$

■ Beispiel 4.13

In[31]:=
$$f = \frac{2 z^2 + z + 1}{(z^2 + 1)\ (z - 2\, i)^2}$$

Out[31]=
$$\frac{2 z^2 + z + 1}{(z - 2\, i)^2 (z^2 + 1)}$$

In[32]:= `Apart[f]`

Out[32]=
$$-\frac{\frac{1}{18} - \frac{i}{18}}{z + i} + \frac{\frac{5}{9} + \frac{4 i}{9}}{z - 2 i} + \frac{\frac{7}{3} - \frac{2 i}{3}}{(z - 2 i)^2} - \frac{\frac{1}{2} + \frac{i}{2}}{z - i}$$

In[33]:= `r1 = Residue[f, {z, i}]`

Out[33]=
$$-\frac{1}{2} - \frac{i}{2}$$

In[34]:= `r2 = Residue[f, {z, -i}]`

Out[34]=
$$-\frac{1}{18} + \frac{i}{18}$$

In[35]:= `r3 = Residue[f, {z, 2 i}]`

Out[35]=
$$\frac{5}{9} + \frac{4 i}{9}$$

In[36]:= `2 π i (r1 + r2 + r3) // Simplify`

Out[36]= 0

In[37]:= `2 π i (r1 + r2) // Simplify`

Out[37]=
$$\left(\frac{8}{9} - \frac{10 i}{9}\right)\pi$$

■ Beispiel 4.14

In[38]:=
$$f = \frac{\text{Sin}[z]}{(z^2 + 1)^4\ (z + 3\, i)^3}$$

Out[38]=
$$\frac{\sin(z)}{(z + 3 i)^3 (z^2 + 1)^4}$$

In[39]:=
$$\text{Apart}\left[\frac{f}{\text{Sin}[z]}\right]$$

Out[39]=
$$-\frac{1}{256 (z + i)} - \frac{41}{32\,768 (z + 3 i)} - \frac{i}{128 (z + i)^2} - \frac{3 i}{4096 (z + 3 i)^2} + \frac{1}{256 (z + i)^3} +$$
$$\frac{1}{4096 (z + 3 i)^3} + \frac{i}{128 (z + i)^4} + \frac{169}{32\,768 (z - i)} - \frac{35 i}{8192 (z - i)^2} - \frac{11}{4096 (z - i)^3} + \frac{i}{1024 (z - i)^4}$$

In[40]:= `Series[f, {z, i, 2}]`

Out[40]=
$$-\frac{\sinh(1)}{1024 (z - i)^4} + \frac{i (4 \cosh(1) - 11 \sinh(1))}{4096 (z - i)^3} + \frac{\frac{39 \sinh(1)}{8192} - \frac{11 \cosh(1)}{4096}}{(z - i)^2} -$$
$$\frac{i (436 \cosh(1) - 639 \sinh(1))}{98\,304 (z - i)} + \left(\frac{551 \cosh(1)}{98\,304} - \frac{5837 \sinh(1)}{786\,432}\right) +$$
$$(i (z - i) (93\,828 \cosh(1) - 116\,915 \sinh(1)))/15\,728\,640 + (z - i)^2 \left(\frac{317\,929 \sinh(1)}{47\,185\,920} - \frac{29\,489 \cosh(1)}{5\,242\,880}\right) + O((z - i)^3)$$

In[41]:= `r1 = Residue[f, {z, ⅈ}]`

Out[41]= $-\frac{i\,(436\cosh(1)-639\sinh(1))}{98\,304}$

In[42]:= `r2 = Residue[f, {z, -ⅈ}]`

Out[42]= $-\frac{i\,(14\cosh(1)-9\sinh(1))}{1536}$

In[43]:= `r3 = Residue[f, {z, -3 ⅈ}]`

Out[43]= $-\frac{3\,i\,(8\cosh(3)-15\sinh(3))}{32\,768}$

In[44]:= `res = 2 π ⅈ (r1 + r2 + r3) // Simplify`

Out[44]= $\frac{1}{16\,384}3\,\pi\,(-15\,(9\sinh(1)+\sinh(3))+148\cosh(1)+8\cosh(3))$

In[45]:= `res /. {Cosh[x_] → (ⅇ^x + ⅇ^-x)/2, Sinh[x_] → (ⅇ^x - ⅇ^-x)/2} // Simplify`

Out[45]= $-\frac{3\,(-23-283\,e^2-13\,e^4+7\,e^6)\,\pi}{32\,768\,e^3}$

■ Beispiel 4.15

In[46]:= `f = 1/(1 + z^2 + z^4)`

Out[46]= $\frac{1}{z^4+z^2+1}$

In[47]:= `Apart[f]`

Out[47]= $\frac{1-z}{2\,(z^2-z+1)}+\frac{z+1}{2\,(z^2+z+1)}$

In[48]:= `MapAll[ComplexExpand, Solve[1/f == 0, z]]`

Out[48]= $\left\{\left\{z\to-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right\},\left\{z\to\frac{1}{2}+\frac{i\sqrt{3}}{2}\right\},\left\{z\to\frac{1}{2}-\frac{i\sqrt{3}}{2}\right\},\left\{z\to-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right\}\right\}$

In[49]:= `r1 = Residue[f, {z, -1/2 + ⅈ √3/2}]`

Out[49]= $-\frac{i}{\sqrt{3}-3\,i}$

In[50]:= `r2 = Residue[f, {z, 1/2 + ⅈ √3/2}]`

Out[50]= $-\frac{i}{\sqrt{3}+3\,i}$

In[51]:= `2 π ⅈ (r1 + r2) // Simplify`

Out[51]= $\frac{\pi}{\sqrt{3}}$

In[52]:= `N [%]`

Out[52]= 1.8138

■ Beispiel 4.16

In[53]:= $f = \frac{e^{i z}}{(1 + z^2 + z^4)}$

Out[53]= $\frac{e^{i z}}{z^4 + z^2 + 1}$

In[54]:= `Apart[f]`

Out[54]= $\frac{e^{i z}}{z^4 + z^2 + 1}$

In[55]:= `MapAll[ComplexExpand, Solve[1/f == 0, z]]`

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[55]= $\left\{\left\{z \to -\frac{1}{2} - \frac{i\sqrt{3}}{2}\right\}, \left\{z \to \frac{1}{2} + \frac{i\sqrt{3}}{2}\right\}, \left\{z \to \frac{1}{2} - \frac{i\sqrt{3}}{2}\right\}, \left\{z \to -\frac{1}{2} + \frac{i\sqrt{3}}{2}\right\}\right\}$

In[56]:= $r1 = \text{Residue}\left[f, \left\{z, -\frac{1}{2} + \frac{i\sqrt{3}}{2}\right\}\right]$

Out[56]= $-\frac{i\, e^{-\frac{\sqrt{3}}{2} - \frac{i}{2}}}{\sqrt{3} - 3 i}$

In[57]:= $r2 = \text{Residue}\left[f, \left\{z, \frac{1}{2} + \frac{i\sqrt{3}}{2}\right\}\right]$

Out[57]= $\frac{e^{-\frac{\sqrt{3}}{2} + \frac{i}{2}}}{-3 + i\sqrt{3}}$

In[58]:= `2 π i (r1 + r2) // ComplexExpand // Simplify`

Out[58]= $\frac{1}{3} e^{-\frac{\sqrt{3}}{2}} \pi \left(3 \sin\left(\frac{1}{2}\right) + \sqrt{3} \cos\left(\frac{1}{2}\right)\right)$

In[59]:= `N [%]`

Out[59]= 1.30305