Differentialgleichungen

Differentialgleichung von y/x

In[53]:= `DE = y'[x] == (y[x] + x)/(y[x] - x)`

$$\text{DE} = y'[x] == \frac{y[x] + x}{y[x] - x}$$

Out[53]= $$y'(x) = \frac{y(x) + x}{y(x) - x}$$

In[54]:= `DSolve[DE, y[x], x]`

Out[54]= $$\left\{\left\{y(x) \to x - \sqrt{e^{2 c_1} + 2 x^2}\right\}, \left\{y(x) \to \sqrt{e^{2 c_1} + 2 x^2} + x\right\}\right\}$$

schrittweise Lösung

In[55]:= $$\text{Gleichung} = \int \frac{1}{\frac{u+1}{u-1} - u} \, du == \int \frac{1}{x} \, dx$$

Out[55]= $$-\frac{1}{2} \log\left(u^2 - 2 u - 1\right) = \log(x)$$

In[56]:= `Solve[Gleichung, u]`

Out[56]= $$\left\{\left\{u \to \frac{x^2 - \sqrt{2 x^4 + x^2}}{x^2}\right\}, \left\{u \to \frac{x^2 + \sqrt{2 x^4 + x^2}}{x^2}\right\}\right\}$$

Hausaufgabe: Beispiel 1.23

In[57]:= $$\text{gleichung} = \frac{1}{2} \int \frac{2 y - 1}{y^2 - y - 2} \, dy == \int x \, dx$$

Out[57]= $$\frac{1}{2} \log\left(-y^2 + y + 2\right) = \frac{x^2}{2}$$

In[58]:= `Solve[gleichung, y]`

Out[58]= $$\left\{\left\{y \to \frac{1}{2}\left(1 - \sqrt{9 - 4 e^{x^2}}\right)\right\}, \left\{y \to \frac{1}{2}\left(\sqrt{9 - 4 e^{x^2}} + 1\right)\right\}\right\}$$

In[59]:= $$\text{DE} = y'[x] == x \frac{2 y[x]^2 - 2 y[x] - 4}{2 y[x] - 1}$$

Out[59]= $$y'(x) = \frac{x \left(2 y(x)^2 - 2 y(x) - 4\right)}{2 y(x) - 1}$$

In[60]:= `DSolve[DE, y[x], x]`

Out[60]= $$\left\{\left\{y(x) \to \frac{1}{2}\left(1 - \sqrt{9 - 4 e^{c_1 + x^2}}\right)\right\}, \left\{y(x) \to \frac{1}{2}\left(\sqrt{9 - 4 e^{c_1 + x^2}} + 1\right)\right\}\right\}$$

In[61]:=
```
DSolve[{y'[x] == x (2 y[x]^2 - 2 y[x] - 4)/(2 y[x] - 1), y[x0] == y0}, y[x], x]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

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Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[61]= $\left\{\left\{y(x) \to \frac{1}{2}\left(1 - \sqrt{9 - \left(-4\,\mathrm{y0}^2 + 4\,\mathrm{y0} + 8\right) e^{x^2 - \mathrm{x0}^2}}\right)\right\}, \left\{y(x) \to \frac{1}{2}\left(\sqrt{9 - \left(-4\,\mathrm{y0}^2 + 4\,\mathrm{y0} + 8\right) e^{x^2 - \mathrm{x0}^2}} + 1\right)\right\}\right\}$

In[62]:=
```
DSolve[{y'[x] == x (2 y[x]^2 - 2 y[x] - 4)/(2 y[x] - 1), y[0] == 3}, y[x], x]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

DSolve::bvnul : For some branches of the general solution, the given boundary conditions lead to an empty solution. ≫

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

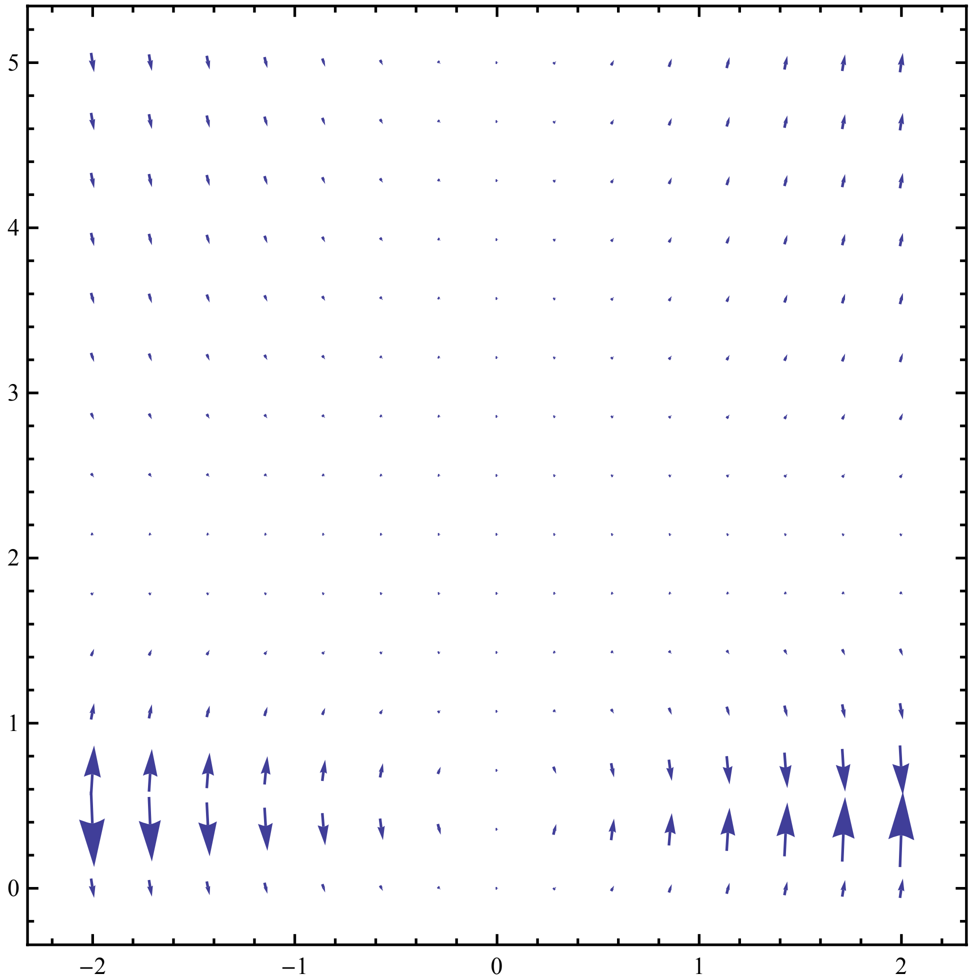
General::stop :
Further output of Solve::ifun will be suppressed during this calculation. ≫

Out[62]= $\left\{\left\{y(x) \to \frac{1}{2}\left(\sqrt{16\, e^{x^2} + 9} + 1\right)\right\}\right\}$

In[63]:=
```
DirectionField[DE_, y_[x_], {x_, a_, b_},
  {y_, c_, d_}, options___] := Module[{g},
  g = DE[[2]] /. y[x] → y;
  VectorPlot[{1, g}, {x, a, b}, {y, c, d}, options]
 ]
```

In[64]:=
```
plot1 = DirectionField[DE, y[x], {x, -2, 2}, {y, 0, 5}, Frame → True]
```

Out[64]=



In[65]:=
```
plot2 = Plot[
  Evaluate[Table[y[x] /. DSolve[{DE, y[0] == k/5}, y[x], x][[1]], {k, 20}]],
  {x, -2, 2},
  PlotStyle → Table[{Thickness[0.005], RGBColor[k/20, 0, 1 - k/20]}, {k, 20}]]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

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Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

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General::stop :
Further output of Solve::ifun will be suppressed during this calculation. ≫

DSolve::bvnul : For some branches of the general solution,
the given boundary conditions lead to an empty solution. ≫

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the given boundary conditions lead to an empty solution. ≫

General::stop :
Further output of DSolve::bvnul will be suppressed during this calculation. ≫

Part::partw : Part 1 of {} does not exist. ≫

ReplaceAll::reps :
{{}⟦1⟧} is neither a list of replacement rules nor a valid dispatch table, and
so cannot be used for replacing. ≫

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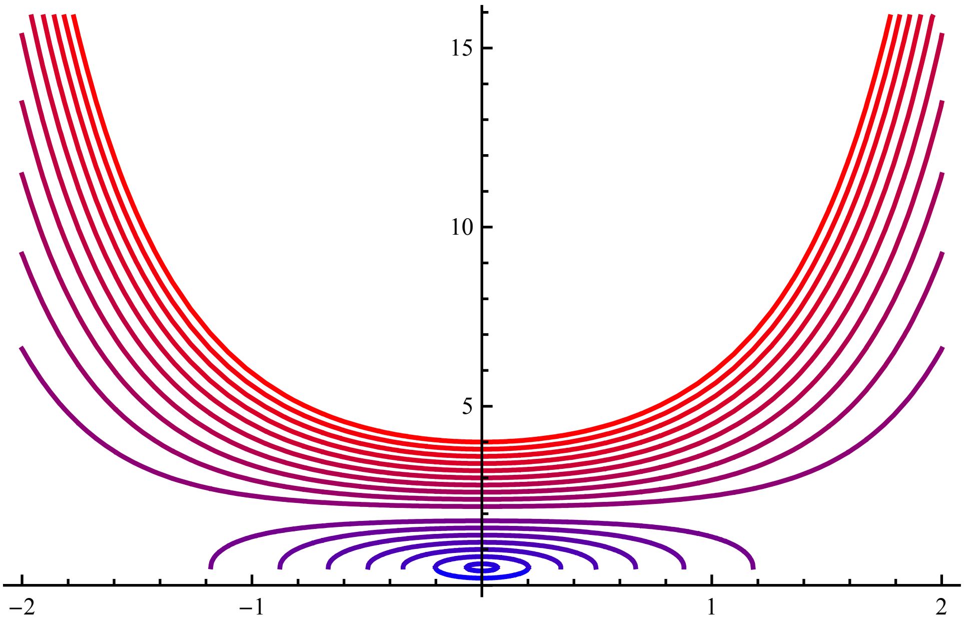
{{}⟦1⟧} is neither a list of replacement rules nor a valid dispatch table, and
so cannot be used for replacing. ≫

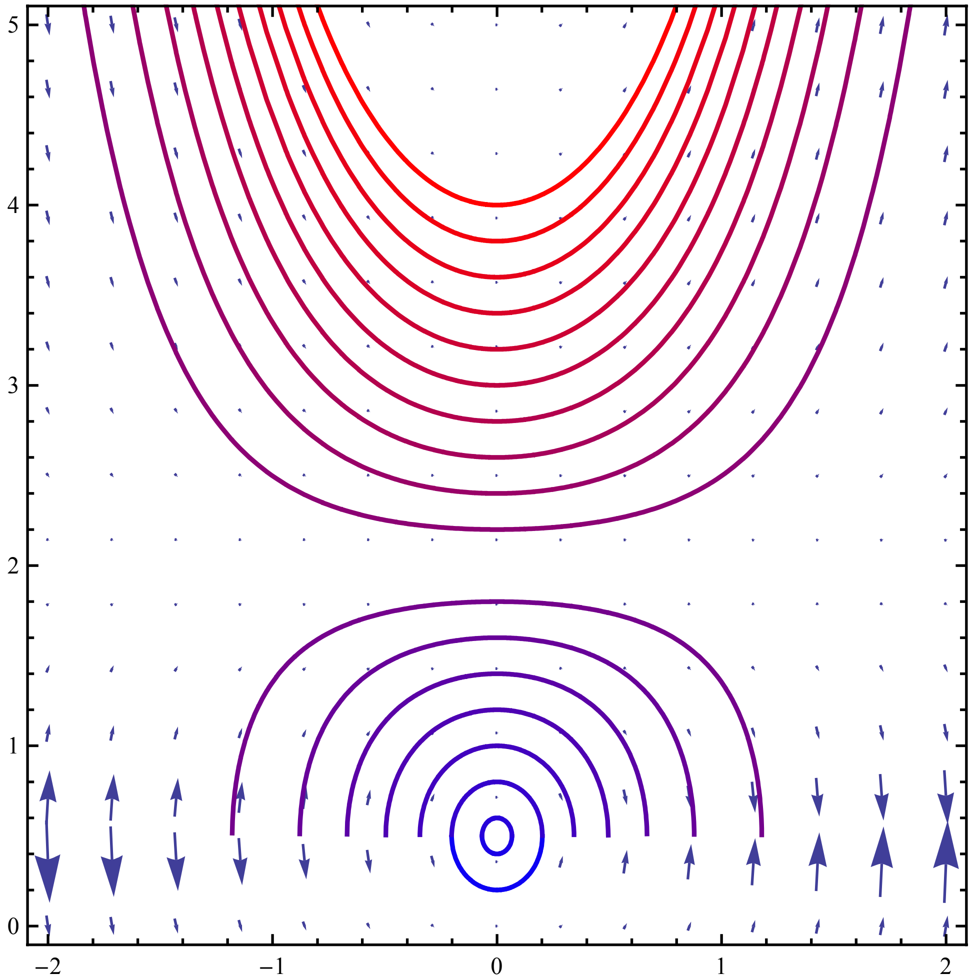
General::stop :
Further output of ReplaceAll::reps will be suppressed during this calculation. ≫

Out[65]=



In[66]:= `Show[plot1, plot2, PlotRange → {0, 5}]`

Out[66]=



Ein Beispiel, bei welchem *Mathematica* die Lösung (nach Auflösen nach y[x]) durch spezielle Funktionen ausdrückt.

In[67]:= `DE1 = y'[x] == (y[x] + 1)/(y[x] - 1)`

Out[67]= $y'(x) = \frac{y(x)+1}{y(x)-1}$

In[68]:= `DSolve[DE1, y[x], x]`

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found;
use Reduce for complete solution information. ≫

Out[68]= $\left\{\left\{y(x) \to -2\, W\left(-\frac{1}{2}\sqrt{e^{-c_1-x-1}}\right)-1\right\}, \left\{y(x) \to -2\, W\left(\frac{1}{2}\sqrt{e^{-c_1-x-1}}\right)-1\right\}\right\}$

In[69]:=
```
FullForm[%]
```

Out[69]//FullForm=

List[List[Rule[y[x], Plus[−1, Times[−2, ProductLog[Times[Rational[−1, 2],
Power[Power[E, Plus[−1, Times[−1, x], Times[−1, C[1]]]], Rational[1, 2]]]]]]]]],
List[Rule[y[x], Plus[−1, Times[−2, ProductLog[Times[Rational[1, 2],
Power[Power[E, Plus[−1, Times[−1, x], Times[−1, C[1]]]], Rational[1, 2]]]]]]]]]

In[70]:=
```
? ProductLog
```

ProductLog[z] gives the principal solution for w in $z = we^w$.
ProductLog[k, z] gives the k^{th} solution. ≫

Zum Schluss noch ein Beispiel, bei welchem *Mathematica* fälschlicherweise keine korrekte Fallunterscheidung vornimmt.

In[71]:=
```
DE = y'[x] == x √(1 + y[x])
```

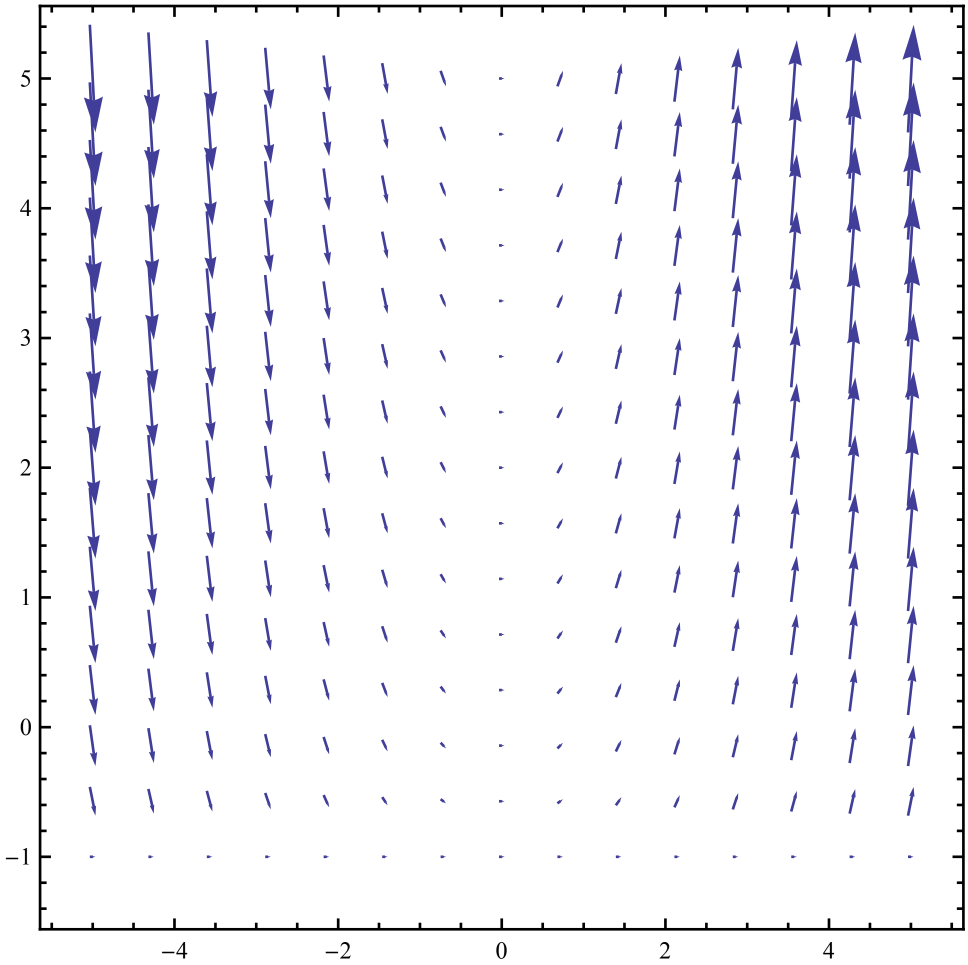
Out[71]= $y'(x) = x\sqrt{y(x) + 1}$

In[72]:=
```
DSolve[DE, y[x], x]
```

Out[72]= $\left\{\left\{y(x) \to \frac{1}{16}\left(4 c_1 x^2 + 4 c_1^2 + x^4 - 16\right)\right\}\right\}$

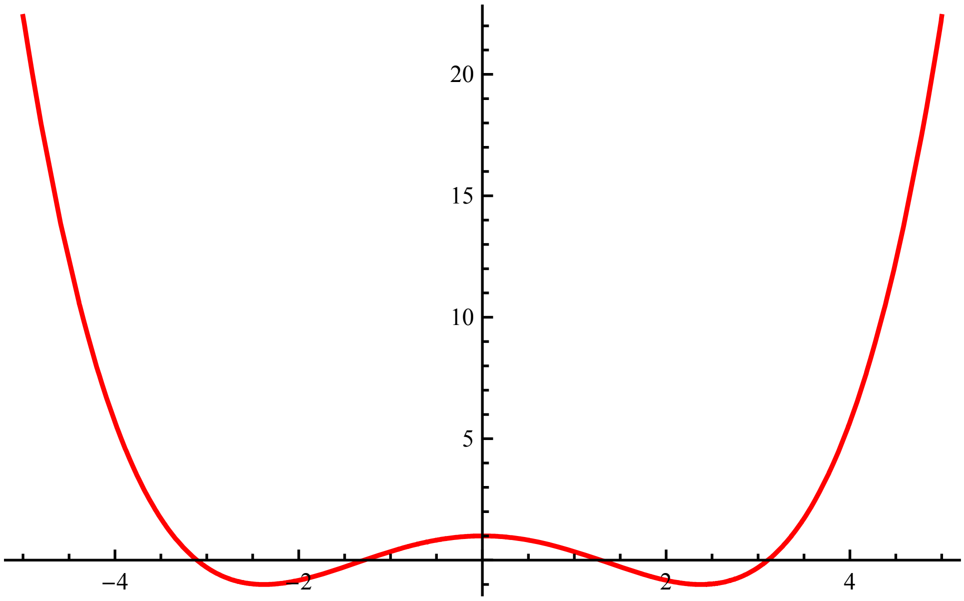
In[73]:= plot1 = DirectionField[DE, y[x], {x, -5, 5}, {y, -1, 5}, Frame → True]

Out[73]=



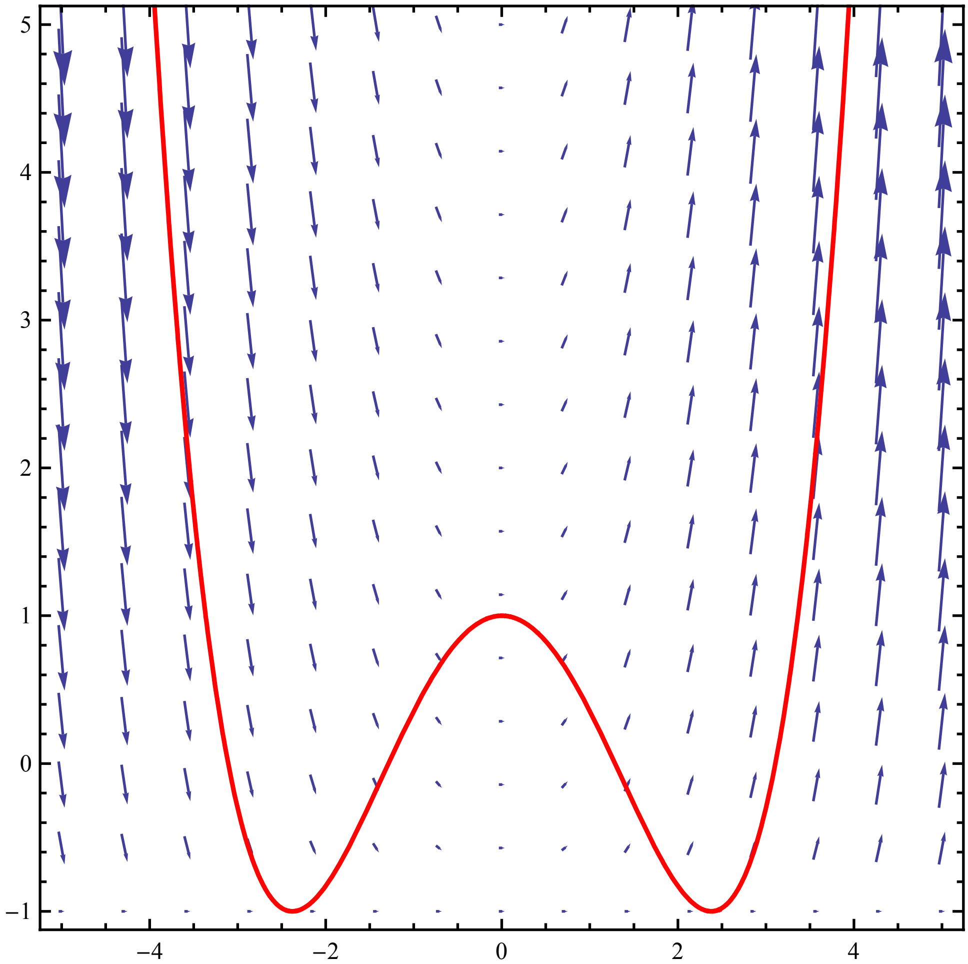
In[74]:= plot2 = Plot[Evaluate[y[x] /. DSolve[{DE, y[0] == 1}, y[x], x][[1]]],
{x, -5, 5}, PlotStyle → {Thickness[0.005], RGBColor[1, 0, 0]}]

Out[74]=



In[75]:= `Show[plot1, plot2, PlotRange → {-1, 5}]`

Out[75]=



Lineare Differentialgleichungen

■ Lineare Differentialgleichung erster Ordnung

In[76]:= `DSolve[y'[x] + a[x] * y[x] == b[x], y[x], x]`

Out[76]= $\left\{\left\{y(x) \to e^{\int_1^x -a(K[1])\, dK[1]} \int_1^x b(K[2])\, e^{-\int_1^{K[2]} -a(K[1])\, dK[1]}\, dK[2] + c_1\, e^{\int_1^x -a(K[1])\, dK[1]}\right\}\right\}$

In[77]:= `DSolve[y'[x] + a[x] * y[x] == 0, y[x], x]`

Out[77]= $\left\{\left\{y(x) \to c_1\, e^{\int_1^x -a(K[1])\, dK[1]}\right\}\right\}$

■ Beispiel 1.12

In[78]:=
```
DSolve[y'[x] == Sin[x] y[x], y[x], x]
```
Out[78]= $\{\{y(x) \to c_1 e^{-\cos(x)}\}\}$

In[79]:=
```
DSolve[{y'[x] == Sin[x] y[x], y [0] == 1}, y[x], x]
```
Out[79]= $\{\{y(x) \to e^{1-\cos(x)}\}\}$

■ Variation der Konstanten

Die homogene Gleichung ist separierbar

In[94]:=
```
DE = y'[x] + a[x] * y[x] == 0
```
Out[94]= $a(x)\, y(x) + y'(x) = 0$

und hat die Lösung

In[95]:=
```
homogeneLösung = y[x] → K * Exp[∫-a[x] ⅆx]
```
Out[95]= $y(x) \to K\, e^{-\int a(x)\, dx}$

Diese setzen wir ein und bekommen

In[96]:=
```
DE /. {homogeneLösung, D[homogeneLösung, x]}
```
Out[96]= True

Um eine Lösung der inhomogenen Differentialgleichung

In[97]:=
```
DE = y'[x] + a[x] * y[x] == b[x]
```
Out[97]= $a(x)\, y(x) + y'(x) = b(x)$

zu finden, machen wir den Ansatz (Variation der Konstanten)

In[98]:=
```
inhomogeneLösung = y[x] → K[x] * Exp[∫-a[x] ⅆx]
```
Out[98]= $y(x) \to K[x]\, e^{-\int a(x)\, dx}$

Diese setzen wir ein und bekommen

In[99]:=
```
newDE = DE /. {inhomogeneLösung, D[inhomogeneLösung, x]}
```
Out[99]= $K'(x)\, e^{-\int a(x)\, dx} = b(x)$

Diese einfache Differentialgleichung für K[x] können wir aber durch Integration lösen und wir erhalten

In[100]:= `spezielleLösung = y[x] → (Exp[∫-a[x] ⅆx] * ∫b[x] Exp[∫a[x] ⅆx] ⅆx)`

Out[100]= $y(x) \to e^{-\int a(x)\,dx} \int b(x)\, e^{\int a(x)\,dx}\, dx$

Test:

In[101]:= `test = DE /. {spezielleLösung, D[spezielleLösung, x]}`

Out[101]= True

DSolve kann dies auch alleine, liefert aber wieder eine kompliziert aussehende Lösung.

In[102]:= `DSolve[DE, y[x], x]`

Out[102]= $\left\{\left\{y(x) \to e^{\int_1^x -a(K[1])\,dK[1]} \int_1^x b(K[2])\, e^{-\int_1^{K[2]} -a(K[1])\,dK[1]}\, dK[2] + c_1\, e^{\int_1^x -a(K[1])\,dK[1]}\right\}\right\}$

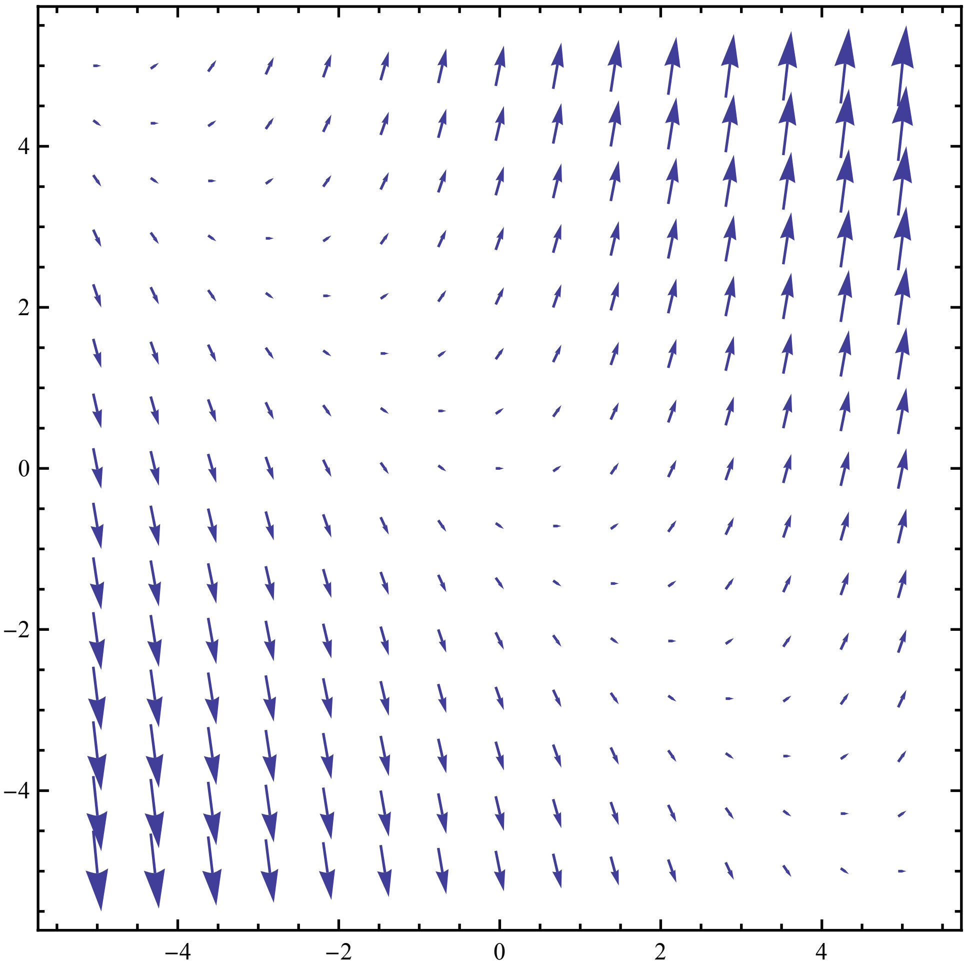
Beispiel 1.16

In[103]:= `DE = y'[x] == y[x] + x`

Out[103]= $y'(x) == y(x) + x$

In[104]:= `plot1 = DirectionField[DE, y[x ], {x, -5, 5}, {y, -5, 5}, Frame → True]`

Out[104]=

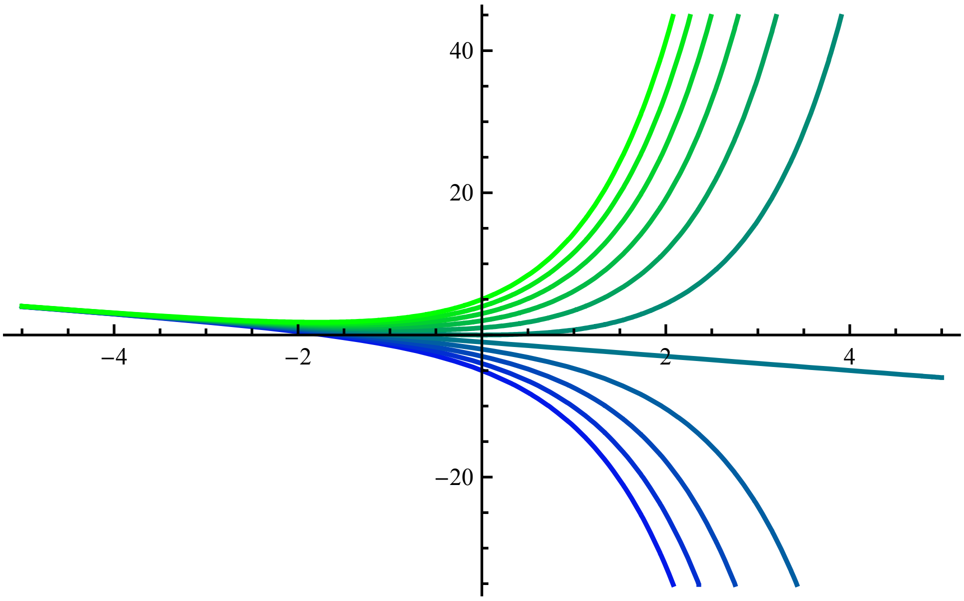


In[105]:= `DSolve[DE, y[x], x]`

Out[105]= $\{\{y(x) \to c_1\, e^x - x - 1\}\}$

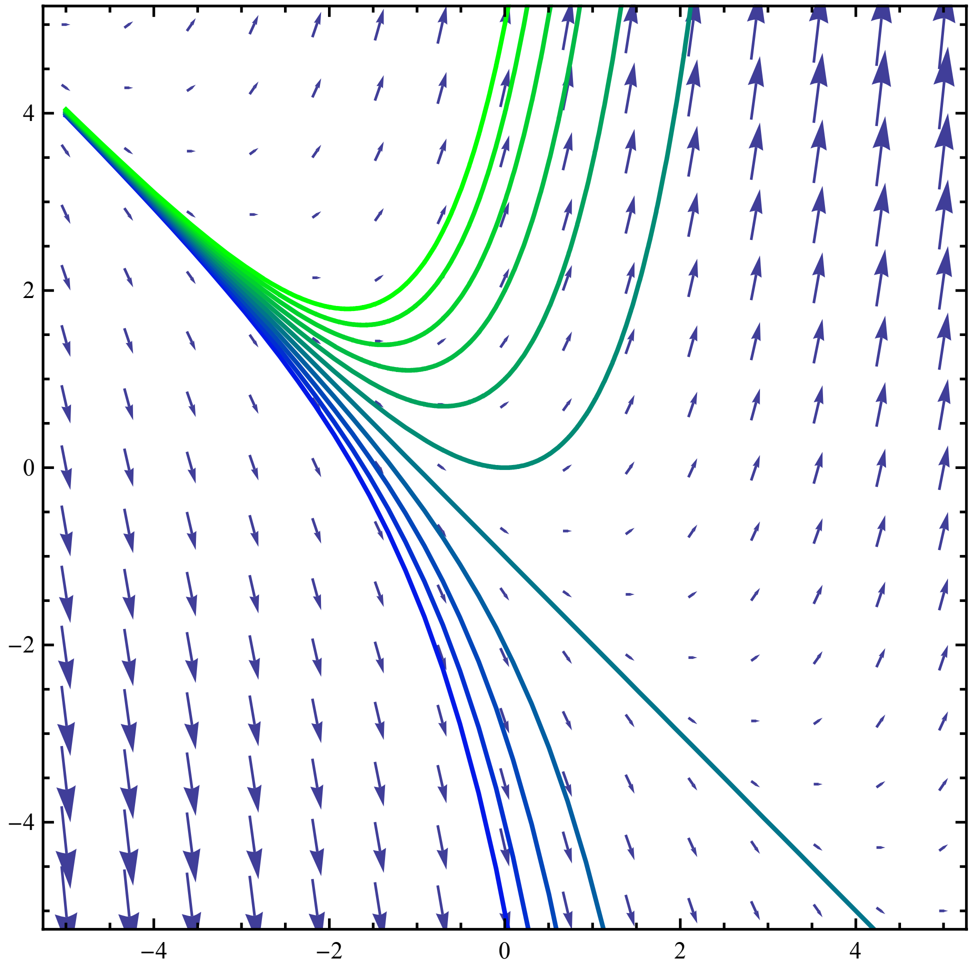
In[106]:=
```
plot2 = Plot[
  Evaluate[Table[y[x] /. DSolve[{DE, y[0] == k}, y[x], x][[1]], {k, -5, 5}]],
  {x, -5, 5},
  PlotStyle → Table[{Thickness[0.005], RGBColor[0, k/11, 1 - k/11]}, {k, 11}]]
```

Out[106]=



In[107]:= `Show[plot1, plot2, PlotRange → {-5, 5}]`

Out[107]=



nach unserer Formel:

In[108]:= `a = -1; b = x;`

Allgemeine Lösung der homogenen Differentialgleichung:

In[109]:= $\text{hom} = \text{y1} \to K * e^{\int -a\, dx}$

Out[109]= $\text{y1} \to K\, e^{x}$

Variation der Konstanten:

In[110]:= $\int b\, e^{\int a\, dx}\, dx$

Out[110]= $e^{-x}\,(-x-1)$

Spezielle Lösung der inhomogenen Differentialgleichung:

In[111]:= $\text{var} = \text{y2} \to e^{\int -a\, dx} * \int b\, e^{\int a\, dx}\, dx$

Out[111]= $\text{y2} \to -x-1$

Allgemeine Lösung der inhomogenen Differentialgleichung:

In[112]:=
```
lösung = y → y1 + y2 /. {hom, var}
```
Out[112]= $y \to K\, e^{x} - x - 1$

Wir lösen das Anfangswertproblem mit y(x0)=y0:

In[113]:=
```
Solve[(lösung[[2]] /. {x → x0}) == y0, K]
```
Out[113]= $\{\{K \to e^{-\text{x0}}\,(\text{x0} + \text{y0} + 1)\}\}$

oder mit DSolve

In[114]:=
```
lösung = DSolve[{DE, y[x0] == y0}, y[x], x]
```
Out[114]= $\{\{y(x) \to -e^{-\text{x0}}\left(x\, e^{\text{x0}} - e^{x}\,\text{x0} - e^{x}\,\text{y0} - e^{x} + e^{\text{x0}}\right)\}\}$

In[115]:=
```
Simplify[lösung]
```
Out[115]= $\{\{y(x) \to e^{-\text{x0}}\left(e^{x}\,(\text{x0} + \text{y0} + 1) - (x + 1)\, e^{\text{x0}}\right)\}\}$