A GEOMETRIC VERSION OF DYSON’S LEMMA
FOR FACTORS OF ARBITRARY DIMENSION

by

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Abstract

This paper generalizes the geometric part of the Esnault-Viehweg paper on Dyson’s Lemma for a product of projective lines. Using the method of weak positivity from algebraic geometry, we are able to study products of smooth projective varieties of arbitrary dimension and to prove a geometric analogue of Dyson’s Lemma for this case. Our main result is in fact a quantitative version of Faltings’ product theorem.

1 Introduction

Let us start with a brief glance at the history of Dyson’s Lemma. By Liouville’s theorem, complex numbers which can be approximated by rational numbers very well are necessarily transcendental. In other words, if $\alpha$ is algebraic of degree $d \geq 2$, then for all $\varepsilon > 0$ there are only finitely many rational numbers $p/q$ such that $|p/q - \alpha| \leq q^{-(d+\varepsilon)}$. However, the existence of infinite sequences of rational numbers $(p_n/q_n)_{n \in \mathbb{N}}$, satisfying $|p_n/q_n - \alpha| \leq q_n^{-2}$ for all $n \in \mathbb{N}$, implied that there had to be some lower bound for the exponent. It took about one hundred years until Roth in 1955 could prove that replacing $d$ by 2 was in fact the optimal bound.

In most approaches, auxiliary polynomials in two or more variables have been constructed. In his paper [1], Dyson describes explicitly which properties these polynomials should have. In order to replace the exponent $d+\varepsilon$ by $\sqrt{2d}+\varepsilon$, he proves a statement about the existence of certain polynomials, which is known as Dyson’s Lemma today.

Fixing a certain number of points in the complex plane, Dyson considers polynomials in two variables of multidegree $d = (d_1, d_2)$ with respect to a special kind of zero conditions in these points, called the index. Clearly there exists some polynomial satisfying them, provided the number of conditions is bounded by $d_1 \cdot d_2$. Now Dyson finds out that, increasing the number of conditions too much, it becomes impossible to find such a polynomial at all. In other words, assuming the existence of such a polynomial, he shows that the number of conditions is necessarily bounded by $C \cdot d_1 \cdot d_2$ for some constant $C$ depending on $d_1$ and $d_2$. Moreover, this constant tends to one when increasing the ratio $d_2/d_1$, which means that the conditions are asymptotically independent.

We want to reformulate this in terms of algebraic geometry. The question should be the following. If one compactifies the situation by considering $(\mathbb{P}^1)^n$ over $\mathbb{C}$ (or over any algebraically closed field) and if one identifies polynomials with sections of some special sheaf and encodes the zero conditions into some ideal sheaf, how can the existence of a special section be interpreted then? This immediately leads to the notion of weak positivity, and it is exactly what Esnault-Viehweg stated and proved in [3], using positivity statements and vanishing theorems.

The proof in [3] splits in two parts. The first part is a statement about the weak positivity of a certain sheaf over some product open subset and can be regarded as a geometric version
of Dyson’s Lemma. We should mention here, too, that there are very close relations with Faltings’ product theorem (see [5]), since in both cases the proof is based on induction on the dimension of product subvarieties. The second part in [3] then gives, by combinatorial methods, a proof for an equivalent of Dyson’s Lemma itself.

Whereas the whole result can be generalized for a product of arbitrary curves (see [11] or [13], for example), the proof of the second part does no longer hold if one considers a product of smooth projective varieties of arbitrary dimension. Yet, a geometric statement, corresponding to the one in the first part of [3], can be made in this situation. It is the aim of this paper to state and prove this geometric version of Dyson’s Lemma or, if you want, of Faltings’ product theorem. As the most powerful tool to achieve this, we consider multiplier ideal sheaves. Their theory had not been developed in its full strength by the time [3] was published.

Let us formulate the main theorem of this paper. We consider the following situation. Let $X = X_1 \times \cdots \times X_n$ be a product of smooth projective varieties, defined over an algebraically closed field $k$ of characteristic zero. For $\nu = 1, \ldots, n$, let us denote by $m_\nu$ the dimension of $X_\nu$ and by $p_\nu : X \to X_\nu$ the projection onto the $\nu$-th factor. Let us fix very ample sheaves $L_\nu$ on $X_\nu$ and let us write $e_\nu = c_1(L_\nu)^{m_\nu}$. For every $n$-tuple $\delta = (\delta_1, \ldots, \delta_n)$ of non-negative integers let us denote by $L_\delta = L_1^{\delta_1} \otimes \cdots \otimes L_n^{\delta_n}$ the induced sheaf on $X$.

Let $S = \{\xi_1, \ldots, \xi_M\}$ be a finite set of $M \geq 2$ points on $X$. Let $\xi_{\mu, \nu} = p_\nu(\xi_\mu)$ denote the projections onto the factors and let us write $m_{\xi_{\mu, \nu}}$ for the maximal ideal sheaf of the point $\xi_{\mu, \nu}$. We fix positive rational numbers $t_1, \ldots, t_M$ and an $n$-tuple $\underline{a} = (a_1, \ldots, a_n)$ of non-negative integers and we assume that $d_1 \cdot e_1 \geq \cdots \geq d_n \cdot e_n$. For $\mu = 1, \ldots, M$ we define $I_{\xi_{\mu}}$ to be the ideal sheaf generated by $m_{\xi_{\mu, 1}}^{a_1} \otimes \cdots \otimes m_{\xi_{\mu, n}}^{a_n}$ for $\frac{a_1}{d_1} + \cdots + \frac{a_n}{d_n} \geq t_\mu$, and we let $I = \bigcap_{\mu=1}^M I_{\xi_{\mu}, t_\mu}$. We fix non-negative integers $\gamma_\nu$ such that $L_\delta^{\nu} \otimes \omega_{X_\nu}^{-1}$ is globally generated. Finally, let us define $M'_\nu = \min\{2, [p_\nu(S)]\}$ and $M_\nu = M'_\nu \cdot m_\nu + \gamma_\nu$ for $\nu = 1, \ldots, n$. In this situation, we are going to prove:

**Theorem 1.1** If $\underline{c} \otimes I$ is effective, then $\underline{c} \otimes I$ is weakly positive over some product open set, where $\underline{d} = (d_1, \ldots, d_n)$ with

$$d'_\nu = d_\nu + M_\nu \cdot \sum_{j=\nu+1}^n d_j \cdot e_j.$$ 

From this, we recover the classical situation as follows. The ideal sheaf $I$ carries the index conditions, the effectivity of $\underline{c} \otimes I$ corresponds to the existence of a polynomial of multidegree $\underline{a}$ satisfying these conditions, and the maximal number of possible conditions is encoded in $\underline{b}$.

## 2 Weak Positivity

Let us recall the following positivity notions:

**Definition 2.1** Let $X$ be a quasi-projective variety and let $U \subseteq X$ be an open subset.

1. A locally free sheaf $G$ on $X$ is called weakly positive over $U$, if there exists an ample invertible sheaf $\mathcal{H}$ on $X$ such that for all $\alpha > 0$ the sheaf $S^\alpha(G) \otimes \mathcal{H}$ is semi-ample.
over $U$, which means that for some $\beta > 0$ the sheaf $S^{\alpha \beta}(\mathcal{G}) \otimes \mathcal{H}^\beta$ is globally generated over $U$. (Here $S^\alpha(\mathcal{G})$ denotes the symmetric power of $\mathcal{G}$, see [8], II. Exercise 5.16, for example.) If $\mathcal{G}$ is weakly positive over $X$, we call $\mathcal{G}$ weakly positive. Obviously, we may replace $\beta$ by arbitrary multiples.

2. An invertible sheaf $\mathcal{L}$ on $X$ is called very ample with respect to $U$, if $\mathcal{L}$ is globally generated over $U$ by sections of a finite dimensional subspace $V \subseteq H^n(X, \mathcal{L})$ and the natural map $U \to \mathbb{P}(V)$ defined by these sections is an embedding. $\mathcal{L}$ is called ample with respect to $U$, if some power of $\mathcal{L}$ is very ample with respect to $U$.

**Lemma 2.2** Let $X$ be a quasi-projective variety, let $\mathcal{L}$ be an invertible sheaf on $X$ and let $U \subseteq X$ be an open subset. Then $\mathcal{L}$ is ample with respect to $U$ if and only if there exists a blowing up $\tau : X' \to X$ with centre outside $U$ such that for some ample invertible sheaf $\mathcal{L}'$ on $X'$ and some $\mu > 0$ we have an inclusion $\mathcal{L}' \to \tau^* \mathcal{L}^\mu$, which is an isomorphism over $\tau^{-1}(U)$.

**Proof:** We only have to show the condition is necessary and to this end we may assume that $\mathcal{L}$ is very ample with respect to $U$. Let $V$ be the space of sections generating $\mathcal{L}$ over $U$. Thus we have a rational map $\varphi : X \to \mathbb{P}(V)$, given by the map $V \otimes \mathcal{O}_X \to \mathcal{L}$, surjective over $U$. If $\mathcal{G}$ is the image sheaf of this map, then we consider the blowing up $\tau' : X'' \to X$ with respect to the ideal sheaf $\mathcal{G} \otimes \mathcal{L}^{-1}$. By [8], II.7.17.3 there exists a morphism $\varphi'' : X'' \to \mathbb{P}(V)$ and an inclusion $\varphi''*\mathcal{O}_{\mathbb{P}(V)}(1) \to \tau'^* \mathcal{L}$. Since $\varphi''*\mathcal{O}_{\mathbb{P}(V)}(1)$ is not necessarily ample, we have to continue blowing up. Let $Y$ be the image of $X''$ under $\varphi''$ and let $M \in \mathbb{N} - \{0\}$ such that $X'' \subseteq \mathbb{P}^M$. Then $\varphi''$ factorizes over $Y \times \mathbb{P}^M = \mathbb{P}(\mathcal{E})$, where $\mathbb{P}(\mathcal{E})$ is the projective bundle of $\mathcal{E} = p_2^* \left( \bigoplus_{M+1} \mathcal{O}_{\mathbb{P}^M} \right)$ (see [8], II.7). We have natural maps

$$\mathcal{E} = p_1*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to \varphi''*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{X''} \to \mathcal{O}_Y(1).$$

Let $\mathcal{B}$ be the image sheaf of this composed map. We consider the blowing up $\varphi' : X' \to Y$ of $\mathcal{B} \otimes \mathcal{O}_Y(-1)$, which is an ideal sheaf, since $\varphi''$ is birational. The sheaf

$$\mathcal{B}' = \varphi'^{-1}(\mathcal{B} \otimes \mathcal{O}_Y(-1)) \cdot \mathcal{O}_{X'},$$

is therefore invertible on $X'$, and we obtain $\mathcal{B}' \otimes \mathcal{O}_{X'}(1)$ as an invertible quotient of $\tau'^* \mathcal{E}$. By [8], II.7.12 this corresponds to a morphism $X' \to \mathbb{P}(\mathcal{E})$, factoring canonically over some morphism $\eta : X' \to X''$. Then $\tau = \tau' \circ \eta$ is the blowing up we need.

By construction we obtain an exceptional divisor $E$ for $\tau$ such that $\mathcal{O}_{X'}(-E)$ is relatively ample. Then there exists some $\mu > 0$ such that

$$\mathcal{L}' = \eta^* \varphi''*\mathcal{O}_Y(\mu) \otimes \mathcal{O}_{X'}(-E)$$

is ample. This sheaf is contained in $\eta^* \tau'^* \mathcal{L}^\mu = \tau^* \mathcal{L}^\mu$, and this inclusion is an isomorphism over $\tau^{-1}(U)$. \hfill $\square$

This immediately implies the compatibility of locally ample sheaves with finite morphisms:

**Corollary 2.3** Let $\sigma : Y \to X$ be a morphism of normal quasi-projective varieties, let $U \subseteq X$ be an open subset such that $\sigma|_{\sigma^{-1}(U)}$ is finite, and let $\mathcal{L}$ be an invertible sheaf on $X$. If $\mathcal{L}$ is ample with respect to $U$, then $\sigma^* \mathcal{L}$ is ample with respect to $\sigma^{-1}(U)$.  

3
Proof: If \( \sigma \) is finite, then by 2.2 we find a blowing up \( \tau : X' \to X \) with centre outside \( U \), an ample sheaf \( \mathcal{L}' \) on \( X' \), some \( \mu > 0 \) and an inclusion \( \mathcal{L}' \to \tau^* \mathcal{L}^\mu \), being an isomorphism over \( \tau^{-1}(U) \). Let
\[
\begin{align*}
Y' & \overset{\sigma'}{\to} X' \\
\tau' & \downarrow \tau \\
Y & \overset{\sigma}{\to} X
\end{align*}
\]
be the fibre product. Since \( \sigma' \) is finite, \( \sigma'^* \mathcal{L}' \) is ample on \( Y' \). Moreover, we have an inclusion \( \sigma'^* \mathcal{L}' \to \sigma'^* \tau^* \mathcal{L}^\mu = \tau^* \sigma^* \mathcal{L}^\mu \), being an isomorphism over \( \sigma^{-1}(U) \), which by 2.2 implies that \( \sigma^* \mathcal{L} \) is ample with respect to \( \sigma^{-1}(U) \).

For the general case we may, considering the Stein factorization of \( \sigma \), assume that \( \sigma \) is birational and an isomorphism over \( U \). But then we are done, since \( \sigma_* \sigma^* \mathcal{L} = \mathcal{L} \) and hence the sections of \( \mathcal{L} \) correspond to the sections of \( \sigma^* \mathcal{L} \).

The definition of weak positivity is independent of the choice of the ample invertible sheaf \( \mathcal{H} \). Moreover, we have:

**Theorem 2.4** Let \( X \) a quasi-projective variety, \( \mathcal{G} \) a locally free sheaf on \( X \) and \( U \subseteq X \) an open subset. Then the following statements are equivalent:

(a) \( \mathcal{G} \) is weakly positive over \( U \).

(b) There exists an ample invertible sheaf \( \mathcal{H} \) on \( X \) such that for all \( \alpha > 0 \) the sheaf \( S^{\alpha \delta}(\mathcal{G}) \otimes \mathcal{H}^\beta \) is globally generated over \( U \) for some \( \beta > 0 \).

(c) For every ample invertible sheaf \( \mathcal{H} \) on \( X \) and for all \( \alpha > 0 \), the sheaf \( S^{\alpha \delta}(\mathcal{G}) \otimes \mathcal{H}^\beta \) is globally generated over \( U \) for some \( \beta > 0 \).

(d) There exists an invertible sheaf \( \mathcal{H} \) on \( X \), ample with respect to \( U \), such that for all \( \alpha > 0 \) the sheaf \( S^{\alpha \delta}(\mathcal{G}) \otimes \mathcal{H}^\beta \) is globally generated over \( U \) for some \( \beta > 0 \).

(e) For every invertible sheaf \( \mathcal{H} \) on \( X \), ample with respect to \( U \), and for all \( \alpha > 0 \), the sheaf \( S^{\alpha \delta}(\mathcal{G}) \otimes \mathcal{H}^\beta \) is globally generated over \( U \) for some \( \beta > 0 \).

(f) There exists an invertible sheaf \( \mathcal{L} \) on \( X \), such that for all \( \alpha > 0 \) the sheaf \( S^{\alpha \delta}(\mathcal{G}) \otimes \mathcal{L}^\beta \) is globally generated over \( U \) for some \( \beta > 0 \).

Proof: (a) \( \Leftrightarrow \) (b) is just the definition of weak positivity, and we obviously have \( (e) \Rightarrow (c) \Rightarrow (b) \Rightarrow (d) \Rightarrow (f) \). So it remains to show \( (f) \Rightarrow (e) \). Let \( \mathcal{H} \) be ample with respect to \( U \) and let \( \alpha > 0 \). By 2.2 we find a blowing up \( \tau : X' \to X \), an ample sheaf \( \mathcal{H}' \) on \( X' \) and some \( \mu > 0 \) with an inclusion \( \mathcal{H}' \to \tau^* \mathcal{H}^\mu \), being an isomorphism over \( \tau^{-1}(U) \). Let \( \mathcal{L}' = \tau^* \mathcal{L} \) and let us choose \( \gamma > 0 \) such that \( \mathcal{L}'^{-1} \otimes \mathcal{H}'^\gamma \) is globally generated. Thus, for some \( r > 0 \), we have a surjective map \( \bigoplus r \mathcal{L}' \to \mathcal{H}'^\gamma \). By assumption, there exists some \( \beta' > 0 \) such that, for some \( s > 0 \), we find a map
\[
\bigoplus s \mathcal{O}_{X'} \to \tau^* \left( S^{\alpha \gamma \mu \beta'}(\mathcal{G}) \otimes \mathcal{L}^\beta \right) = \tau^* S^{\alpha \gamma \mu \beta'}(\mathcal{G}) \otimes \mathcal{L}^\beta,
\]
surjective over \( \tau^{-1}(U) \). Thus we obtain maps
\[
\bigoplus \bigoplus s \mathcal{O}_{X'} \to \bigoplus \left( \tau^* S^{\alpha \gamma \mu \beta'}(\mathcal{G}) \otimes \mathcal{L}^\beta \right) = \tau^* S^{\alpha \gamma \mu \beta'}(\mathcal{G}) \otimes \bigoplus r \mathcal{L}^\beta.
\]
and hence
\[
\bigoplus_{i} \mathcal{O}_{X'} \to \tau^* S^{2\alpha \gamma \mu \beta'} (\mathcal{G}) \otimes \mathcal{H}^{\gamma \beta'} \to \tau^* \left( S^{2\alpha \gamma \mu \beta'} (\mathcal{G}) \otimes \mathcal{H}^{\mu \gamma \beta'} \right),
\]
surjective over \( \tau^{-1}(U) \). This induces a map
\[
\bigoplus_{i} \tau_* \mathcal{O}_{X'} \otimes \mathcal{H}^{\mu \gamma \beta'} \to S^{2\alpha \gamma \mu \beta'} (\mathcal{G}) \otimes \mathcal{H}^{2\mu \gamma \beta'},
\]
surjective over \( U \). We may assume that the sheaf on the left hand side is globally generated, and, taking \( \beta = 2\gamma \mu \beta' \), this implies (e). \( \Box \)

We consider the following class of ideal sheaves:

**Definition 2.5** Let \( X \) be a normal quasi-projective variety and let \( \mathcal{I} \) be an ideal sheaf on \( X \). Let \( \tau : X' \to X \) be a birational morphism such that \( X' \) is normal and \( \mathcal{I}' = \tau^{-1} \mathcal{I} \cdot \mathcal{O}_{X'} \) is invertible.

1. We call \( \mathcal{I} \) **full**, if the natural map \( \mathcal{I} \to \tau_* \mathcal{I}' \) is an isomorphism.

2. If \( \mathcal{L} \) is an invertible sheaf on \( X \) and \( U \subseteq X \) an open subset, then we call \( \mathcal{L} \otimes \mathcal{I} \) weakly positive over \( U \), if \( \tau^* \mathcal{L} \otimes \mathcal{I}' \) is weakly positive over \( \tau^{-1}(U) \).

3. If \( X \) is projective, then we call \( \mathcal{L} \otimes \mathcal{I} \) nef, if \( \tau^* \mathcal{L} \otimes \mathcal{I}' \) is nef on \( X' \).

**Lemma 2.6** Let \( X \) be a normal quasi-projective variety, let \( \mathcal{L} \) be an invertible sheaf on \( X \), let \( \mathcal{I} \) be a full ideal sheaf on \( X \) and let \( U \subseteq X \) be an open subset.

(a) \( \mathcal{L} \otimes \mathcal{I} \) is weakly positive over \( U \) if and only if for every sheaf \( \mathcal{H} \), ample with respect to \( U \), and for all \( \alpha > 0 \) the sheaf \( \mathcal{L}^{\alpha \beta} \otimes \mathcal{I}^{\alpha \beta} \otimes \mathcal{H}^{\beta} \) is globally generated over \( U \) for some \( \beta > 0 \).

(b) If \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are invertible sheaves on \( X \) and if for every \( \mu > 0 \) there is some \( \nu > 0 \) such that \( \mathcal{L}^{\mu \nu} \otimes \mathcal{I}^{\mu \nu} \otimes \mathcal{L}_1^{\nu} \otimes \mathcal{L}_2 \) is weakly positive over \( U \), then so is \( \mathcal{L} \otimes \mathcal{I} \).

**Proof:** (b) follows immediately from (a) and from [3], 4.3. In order to show (a), let us assume \( \mathcal{L} \otimes \mathcal{I} \) is weakly positive over \( U \). We choose some birational morphism \( \tau : X' \to X \) such that \( X' \) is normal, \( \mathcal{I}' = \tau^{-1} \mathcal{I} \cdot \mathcal{O}_{X'} \) is invertible on \( X' \) and \( \tau|_{\tau^{-1}(U)} \) is an isomorphism. If \( \mathcal{H} \) is ample with respect to \( U \), then \( \tau^* \mathcal{H} \) is ample with respect to \( \tau^{-1}(U) \) by 2.3. If we choose \( \alpha > 0 \), then by 2.4 there exists some \( \beta > 0 \) such that \( (\tau^* \mathcal{L} \otimes \mathcal{I}')^{2\alpha \beta} \otimes \tau^* \mathcal{H}^{\beta} \) is globally generated over \( \tau^{-1}(U) \). Thus there exists a map
\[
\bigoplus_{i} \mathcal{O}_{X'} \to (\tau^* \mathcal{L} \otimes \mathcal{I}')^{2\alpha \beta} \otimes \tau^* \mathcal{H}^{\beta},
\]
surjective over \( \tau^{-1}(U) \), and hence we obtain a map
\[
\bigoplus_{i} \tau_* \mathcal{O}_{X'} \otimes \mathcal{H}^{\beta} \to \tau_* \left( (\tau^* \mathcal{L} \otimes \mathcal{I}')^{2\alpha \beta} \otimes \tau^* \mathcal{H}^{\beta} \right) \otimes \mathcal{H}^{\beta}
\]
\[
= \mathcal{L}^{2\alpha \beta} \otimes \tau_* \mathcal{I}^{2\alpha \beta} \otimes \mathcal{H}^{2\beta} = \mathcal{L}^{2\alpha \beta} \otimes \mathcal{I}^{2\alpha \beta} \otimes \mathcal{H}^{2\beta},
\]
surjective over \( U \), where the last equality holds because \( \mathcal{I} \) is full. Now we may assume that the sheaf on the left hand side is globally generated, which proves the necessity of the condition.
In order to show it is sufficient, too, let us choose again some birational morphism \( \tau : X' \to X \) with the properties above. Let \( \alpha > 0 \) and let \( \mathcal{H} \) be ample with respect to \( U \), hence \( \tau^* \mathcal{H} \) is ample with respect to \( \tau^{-1}(U) \) by 2.3. Then \( \tau^* \left( \mathcal{L} \alpha^\beta \otimes \mathcal{I}^\alpha \otimes \mathcal{H}^\beta \right) \) is globally generated over \( \tau^{-1}(U) \). By definition there exists a map
\[
\tau^* \left( \mathcal{L} \alpha^\beta \otimes \mathcal{I}^\alpha \otimes \mathcal{H}^\beta \right) \to (\tau^* \mathcal{L} \otimes \mathcal{I})^\alpha \otimes \tau^* \mathcal{H}^\beta,
\]
surjective over \( \tau^{-1}(U) \). This implies the weak positivity of \( \tau^* \mathcal{L} \otimes \mathcal{I} \otimes \tau^{-1}(U) \) and so by definition the weak positivity of \( \mathcal{L} \otimes \mathcal{I} \) over \( U \), which completes the proof of (a). □

We shall need two more properties:

**Lemma 2.7** Let \( \varphi : Y \to X \) be a surjective morphism of normal quasi-projective varieties and let \( U \subseteq X \) be an open subset such that \( \varphi(\varphi^{-1}(U)) \) is finite. Let \( \mathcal{G} \) be a locally free sheaf on \( Y \), let \( \mathcal{L} \) be an invertible sheaf and let \( \mathcal{J} \) be an ideal sheaf on \( X \) such that for every \( l \in \mathbb{N} - \{0\} \) the sheaf \( \mathcal{J} \) is full and there exists a map
\[
\varphi_* \mathcal{S}^l(\mathcal{G}) \to \mathcal{L}^l \otimes \mathcal{J},
\]
surjective over \( U \). If in this situation \( \mathcal{G} \) is weakly positive over \( \varphi^{-1}(U) \), then \( \mathcal{L} \otimes \mathcal{J} \) is weakly positive over \( U \).

**Proof:** Let \( \alpha > 0 \) and let \( \mathcal{H} \) be an ample invertible sheaf on \( X \). Since \( \varphi \) is finite over \( U \), \( \varphi^* \mathcal{H} \) is ample over \( \varphi^{-1}(U) \) by 2.3. If \( \mathcal{G} \) is weakly positive over \( \varphi^{-1}(U) \), then by 2.4 there exists some \( \beta > 0 \) such that the sheaf \( \mathcal{S}^2 \alpha^\beta \mathcal{G} \otimes \varphi^* \mathcal{H}^\beta \) is globally generated over \( \varphi^{-1}(U) \). Thus we obtain a map
\[
\bigoplus \mathcal{O}_Y \to \mathcal{S}^2 \alpha^\beta \mathcal{G} \otimes \varphi^* \mathcal{H}^\beta,
\]
surjective over \( \varphi^{-1}(U) \) and hence a map
\[
\bigoplus \varphi_* \mathcal{O}_Y \otimes \mathcal{H}^\beta \to \varphi_* \left( \mathcal{S}^2 \alpha^\beta \mathcal{G} \otimes \varphi^* \mathcal{H}^\beta \right) \otimes \mathcal{H}^\beta = \varphi_* \mathcal{S}^2 \alpha^\beta \mathcal{G} \otimes \mathcal{H}^{2\beta},
\]
surjective over \( U \). By assumption and since \( \mathcal{J} \) is full for every \( l \in \mathbb{N} - \{0\} \), there exists a map
\[
\bigoplus \varphi_* \mathcal{O}_Y \otimes \mathcal{H}^\beta \to \mathcal{L}^{2\alpha^\beta} \otimes \mathcal{J}^{2\alpha^\beta} \otimes \mathcal{H}^{2\beta},
\]
surjective over \( U \). We may assume that the sheaf on the left hand side is globally generated, therefore \( \mathcal{L}^{2\alpha^\beta} \otimes \mathcal{J}^{2\alpha^\beta} \otimes \mathcal{H}^{2\beta} \) is globally generated over \( U \) which by 2.6 yields the weak positivity of \( \mathcal{L} \otimes \mathcal{J} \) over \( U \). □

**Lemma 2.8** Let \( X \) be a normal quasi-projective variety, let \( \mathcal{G} \) be a locally free sheaf on \( X \) and let \( U \subseteq X \) be an open subset. Then the following statements are equivalent:

(a) \( \mathcal{G} \) is weakly positive over \( U \).

(b) There exists some \( \mu > 0 \) and some sheaf \( \mathcal{H} \) on \( X \), ample with respect to \( U \), such that for every morphism \( \tau : X' \to X \), finite over \( U \), with \( \tau^* \mathcal{H} = \mathcal{H}^\delta \) for some \( \delta > 0 \) and some invertible sheaf \( \mathcal{H}' \) on \( X' \), the sheaf \( \tau^* \mathcal{G} \otimes \mathcal{H}^\mu \) is weakly positive over \( \tau^{-1}(U) \).

(c) There exists some \( \mu > 0 \), some invertible sheaf \( \mathcal{L} \) on \( X \) and for every \( \delta \in \mathbb{N} - \{0\} \) a morphism \( \tau_\delta : X^\delta \to X \), finite over \( U \), such that \( \tau_\delta^* \mathcal{G} = \mathcal{L}^\delta \) for some invertible sheaf \( \mathcal{L}' \) on \( X^\delta \) and such that \( \tau_\delta^* \mathcal{G} \otimes \mathcal{L}'^\mu \) is weakly positive over \( \tau_\delta^{-1}(U) \).
PROOF: We only have to show (c) ⇒ (a). Let α > 0 and let \( \mathcal{H} \) be an ample invertible sheaf on \( X \) such that \( \mathcal{L} \otimes \mathcal{H} \) is ample, too. By assumption there exists a morphism \( \tau : X' \to X \), finite over \( U \), such that \( \tau^* \mathcal{L} = \mathcal{L}'^{1+2 \alpha \mu} \) for some invertible sheaf \( \mathcal{L}' \) on \( X' \). By [12], 2.1 we may assume that in addition \( \tau^* \mathcal{H} = \mathcal{H}'^{1+2 \alpha \mu} \) for some invertible sheaf \( \mathcal{H}' \) on \( X' \). Since \( \tau^* \mathcal{G} \otimes \mathcal{L}'^\beta \) is weakly positive over \( \tau^{-1}(U) \), so is \( \tau^* \mathcal{G} \otimes (\mathcal{L}' \otimes \mathcal{H}')^\beta \). Hence, for some \( \beta > 0 \) we obtain a map
\[
\bigoplus \mathcal{O}_{X'} \to S^{2 \alpha \beta'} (\tau^* \mathcal{G} \otimes (\mathcal{L}' \otimes \mathcal{H}')^\beta) \cong \tau^* \left( S^{2 \alpha \beta'} (\mathcal{G}) \otimes (\mathcal{L}' \otimes \mathcal{H}')^{(1+2 \alpha \mu)\beta'} \right),
\]
surjective over \( \tau^{-1}(U) \) and thus a map
\[
\bigoplus \tau_* \mathcal{O}_{X'} \otimes (\mathcal{L} \otimes \mathcal{H})^\beta \to S^{2 \alpha \beta'} (\mathcal{G}) \otimes (\mathcal{L} \otimes \mathcal{H})^{2\beta'},
\]
surjective over \( U \). We may assume that the sheaf on the left hand side is globally generated and we find that \( S^{2 \alpha \beta'} (\mathcal{G}) \otimes (\mathcal{L} \otimes \mathcal{H})^{2\beta'} \) is globally generated over \( U \). Choosing \( \beta = 2 \beta' \), we are done. \( \square \)

Finally, let us recall the Fujita-Kawamata Positivity Theorem in a slightly modified version. A proof due to Kollár can be found in [12], 2.41.

**Theorem 2.9 (Fujita [6], Kawamata [9])** Let \( f : X \to Y \) be a projective surjective morphism of smooth quasi-projective varieties and let \( Y_0 \subseteq Y \) be an open subset such that \( f^{-1}(Y_0) \) is smooth and \( f_* \omega_{X/Y} \) is locally free over \( Y_0 \). Then \( f_* \omega_{X/Y} \) is weakly positive over \( Y_0 \).

**Corollary 2.10** In the situation of 2.9, let \( \mathcal{L} \) be an invertible sheaf on \( X \) and let \( D \) be an effective normal crossing divisor on \( X \) such that \( \mathcal{L}^N (-D) \) is semi-ample. Then \( f_* (\mathcal{L} \otimes \omega_{X/Y} (- [D/N])) \) is weakly positive over \( Y_0 \).

**Proof:** For \( \beta > 0 \) sufficiently large, we have \( \mathcal{L}^{\beta N} = \mathcal{O}_X (\Gamma + \beta \cdot D) \), where \( \Gamma \) is a smooth divisor and \( \Gamma + \beta \cdot D \) is a normal crossing divisor, and on the other hand we have \( \omega_{X/Y} (- \left[ \Gamma + \beta \cdot D \right]/\beta N) = \omega_{X/Y} (- [D/N]) \). We thus may assume that \( \mathcal{L}^N = \mathcal{O}_X (D) \), and the statement follows from considering the cyclic covering according to this situation (see [12], 2.43). \( \square \)

### 3 Some Positivity Statements

**Definition 3.1** Let \( X \) be a normal variety with at most rational singularities and let \( \Gamma \) be an effective Cartier divisor on \( X \). Let \( \tau : X' \to X \) be a blowing up such that both \( X' \) is smooth and \( \tau^* \Gamma \) has normal crossings. For every \( N \in \mathbb{N} - \{0\} \) let us define
\[
\omega_X \left( - \left[ \frac{\Gamma}{N} \right] \right) = \tau_* \omega_{X'} \left( - \left[ \frac{\Gamma'}{N} \right] \right).
\]
Due to [12], 5.10, this definition does not depend on the chosen blowing up. Moreover, we define
\[ \mathcal{C}_X(\Gamma, N) = \text{coker} \left( \omega_X \left\{ -\frac{\Gamma}{N} \right\} \rightarrow \omega_X \right). \]
Since \( \omega_X = \tau_* \omega_{X'} \), we have \( \mathcal{C}_X(\Gamma, N) = 0 \) for \( N \) sufficiently large, so we define
\[ e(\Gamma) = \min \left\{ N \in \mathbb{N} - \{0\}; \mathcal{C}_X(\Gamma, N) = 0 \right\}. \]
If in addition \( X \) is projective and \( \mathcal{L} \) is an effective invertible sheaf on \( X \), then let
\[ e(\mathcal{L}) = \max \left\{ e(\Gamma); \Gamma \text{ effective Cartier divisor with } \mathcal{L} = \mathcal{O}_X(\Gamma) \right\}. \]

**Remark 3.2** Immediately from the definition we see that, given any birational morphism \( \delta : Z \rightarrow X \) of normal varieties with at most rational singularities and any effective Cartier divisor \( \Gamma \) on \( X \), we may choose a blowing up \( \tau : X' \rightarrow X \) in such a way that \( X' \) is smooth and \( \Gamma' = \tau^* \Gamma \) is a normal crossing divisor on \( X' \) and that in addition \( \tau \) factors over \( \delta \), hence
\[ \delta_* \omega_Z \left\{ -\frac{\delta^* \Gamma}{N} \right\} = \omega_X \left\{ -\frac{\Gamma}{N} \right\}. \]

**Lemma 3.3** If \( \sigma : Y \rightarrow X \) is a finite surjective morphism of smooth varieties, \( \Gamma \) an effective Cartier divisor on \( X \) and \( N \in \mathbb{N} - \{0\} \), then there exists a map
\[ \omega_Y \left\{ -\frac{\sigma^* \Gamma}{N} \right\} \rightarrow \sigma^* \omega_X \left\{ -\frac{\Gamma}{N} \right\}. \]
If \( U \subseteq X \) is the open subset such that \( \sigma|_{\sigma^{-1}(U)} \) is étale, then this map is an isomorphism over \( \sigma^{-1}(U) \).

**Proof:** Let \( \tau : X' \rightarrow X \) be a blowing up such that \( X' \) is smooth and \( \Gamma' = \tau^* \Gamma \) is a normal crossing divisor. Let \( Y' \) be a desingularization of the fibre product \( Y \times_X X' \) such that we have the following diagram:

\[
\begin{array}{ccc}
Y' & \xrightarrow{\sigma'} & X' \\
\downarrow{\tau'} & & \downarrow{\tau} \\
Y & \xrightarrow{\sigma} & X.
\end{array}
\]

We may assume that \( \sigma'^* \Gamma' = \tau'^* \sigma^* \Gamma \) is a normal crossing divisor on \( Y' \). We have
\[ \omega_Y \left\{ -\frac{\sigma^* \Gamma}{N} \right\} \overset{\tau'_* \omega_{Y'}}{\longrightarrow} \tau'_* \omega_{Y'} \left( -\frac{\tau'^* \sigma^* \Gamma}{N} \right) = \tau'_* \omega_{Y'} \left( -\frac{\sigma'^* \Gamma'}{N} \right). \]

Now let us write \( \Gamma' = \sum_i \alpha_i \Gamma'_i \) and
\[ \sigma'^* \Gamma' = \sum_i \alpha_i \cdot \sigma'^* \Gamma'_i = \sum_i \alpha_i \cdot \left( \sum_j \beta_{ij} \cdot \Gamma'_{ij} \right). \]
Then we have
\[
\sigma'' \left[ \frac{\Gamma'}{N} \right] = \sigma'' \sum_i \left[ \frac{\alpha_i}{N} \right] \cdot \Gamma_i' = \sum_{i,j} \left[ \frac{\alpha_i \beta_{ij}}{N} \right] \cdot \Gamma_{ij}' \leq \sum_{i,j} \left[ \frac{\alpha_i \beta_{ij}}{N} \right] \cdot \Gamma_{ij}' = \left[ \frac{\sigma'' \Gamma'}{N} \right],
\]

hence we obtain an injective map
\[
\omega_{Y'} \left( - \left[ \frac{\sigma'' \Gamma'}{N} \right] \right) \rightarrow \sigma'' \omega_{X'}, \quad \left( - \left[ \frac{\Gamma'}{N} \right] \right),
\]

which is an isomorphism over \( \sigma^{-1}(U) \). Thus we obtain a map
\[
\omega_Y \left\{ - \frac{\sigma^* \Gamma}{N} \right\} \rightarrow \tau' \sigma^* \omega_{X'}, \quad \left( - \left[ \frac{\Gamma}{N} \right] \right).
\]

Since \( \sigma \) is flat by [8], III. Exercise 9.3, we obtain by flat base change ([8], III.9.3) a map
\[
\omega_Y \left\{ - \frac{\sigma^* \Gamma}{N} \right\} \rightarrow \tau' \sigma^* \omega_{X'}, \quad \left( - \left[ \frac{\Gamma}{N} \right] \right) = \sigma^* \tau_* \omega_{X'}, \quad \left( - \left[ \frac{\Gamma}{N} \right] \right) = \sigma^* \omega_X \left\{ - \frac{\Gamma}{N} \right\},
\]

being an isomorphism over \( \sigma^{-1}(U) \). \( \square \)

We shall need the following fact which generalizes [12], 5.21:

**Proposition 3.4** Let \( X_1, \ldots, X_r \) be smooth projective varieties and let \( X = X_1 \times \cdots \times X_r \) denote their product. For \( i = 1, \ldots, r \) let us consider effective divisors \( D_i \) and effective invertible sheaves \( L_i \) on \( X_i \). Let \( \Gamma \) denote the induced divisor on \( X \) and let \( \mathcal{L} \) denote the induced invertible sheaf on \( X \). Then we have

(a) \( e(\Gamma) = \max \{ e(D_1), \ldots, e(D_r) \} \),

(b) \( e(\mathcal{L}) = \max \{ e(L_1), \ldots, e(L_r) \} \).

**Proof:** We may restrict ourselves to the case of two factors. In order to show (a), we may assume \( e = e(D_2) = e(D_1) \). Let us start with the case where \( D_2 \) is a normal crossing divisor on \( X_2 \). Let \( p_2 : X_1 \times X_2 \rightarrow X_2 \) be the second projection. For \( \Gamma = D_1 \times X_2 \) one has \( e(\Gamma_{p_2^{-1}(x)}) = e(D_1) \) for all fibres of the second projection, and by [12], 5.18 one has
\[
\omega_{X_1 \times X_2} \left\{ - \frac{D_1 \oplus D_2}{N} \right\} = \omega_{X_1 \times X_2} \left( - \left[ \frac{D_2}{N} \right] \right)
\]

for \( N \geq e(D_1) \). In particular this holds for \( N = e \), and since in this case the sheaf on the right hand side is equal to \( \omega_{X_1 \times X_2} \), we have \( e(D_1 \oplus D_2) \leq e \).

Let now \( D_2 \) be an arbitrary effective divisor on \( X_2 \) and \( N \geq e(D_1) \). We consider a blowing up \( \tau : X_2' \rightarrow X_2 \) such that \( X_2' \) is smooth and \( D_2' = \tau^* D_2 \) is a normal crossing divisor on \( X_2' \):

\[
\begin{align*}
X_1 \times X_2' & \xrightarrow{\tau'} X_1 \times X_2 \\
\downarrow p_2' & \quad \downarrow p_2 \\
X_2' & \xrightarrow{\tau} X_2.
\end{align*}
\]
For $D_1 oxplus D_2'$ on $X_1 \times X_2'$ we use the first case and the compatibility of relatively canonical sheaf with base change to obtain

$$
\omega_{X_1 \times X_2'} \left\{ \frac{D_1 \boxplus D_2'}{N} \right\} = \omega_{X_1 \times X_2'} \left( -p_2^* \left[ \frac{D_2'}{N} \right] \right)
$$

$$
= \omega_{X_1 \times X_2'} / X_2' \otimes p_2^* \omega_{X_2'} \left( -\left[ \frac{D_2'}{N} \right] \right)
$$

$$
= \tau^* \omega_{X_1 \times X_2} / X_2 \otimes p_2^* \omega_{X_2} \left( -\left[ \frac{D_2'}{N} \right] \right).
$$

Using 3.2 and flat base change ([8], III.9.3), this yields

$$
\omega_{X_1 \times X_2} \left\{ \frac{D_1 \boxplus D_2}{e \cdot N} \right\} = \omega_{X_1 \times X_2} / X_2 \otimes p_2^* \tau^* \omega_{X_2} \left( -\left[ \frac{D_2'}{N} \right] \right).
$$

For $N = e$ the right hand side is nothing but $\omega_{X_1 \times X_2} / X_2 \otimes p_2^* \omega_{X_2} = \omega_{X_1 \times X_2}$, and we obtain $e(D_1 \boxplus D_2) \leq e$ for this case, too.

To show that $e$ is a lower bound, too, it suffices to show that for an open subset $U \subseteq X_1 \times X_2$ one has $e((D_1 \boxplus D_2)|_U) \geq e$. We thus may assume that $D_1 = 0$, and in the above calculation may choose $N = e - 1$. Then, since $e(D_2) = e$, we obtain that

$$
\omega_{X_1 \times X_2} \left\{ \frac{D_1 \boxplus D_2}{e - 1} \right\} \longrightarrow \omega_{X_1 \times X_2}
$$

is not an isomorphism, so $e - 1 < e(D_1 \boxplus D_2)$.

For (b) we may assume $e(L_2) \geq e(L_1)$. Let us choose effective divisors $D_1$ on $X_1$ and $D_2$ on $X_2$ with $e(L_1) = e(D_1)$ and $e(L_2) = e(D_2)$. Then $D_1 \boxplus D_2$ is a section of $L$, and by (a) we obtain

$$
e(L) \geq e(D_1 \boxplus D_2) = e(D_2) = e(L_2).
$$

For the other direction we consider the projections $p_1 : X \rightarrow X_1$ and $p_2 : X \rightarrow X_2$. One has $e(L|_{p_1^{-1}(x_1)}) = e(L_2)$ for every fibre $p_1^{-1}(x_1)$ and $e(L|_{p_2^{-1}(x_2)}) = e(L_1)$ for every fibre $p_2^{-1}(x_2)$. Let $\Gamma$ be an effective divisor with $L = O_X(\Gamma)$. By [12], 5.19, the support of $\mathcal{C}_X(\Gamma, e(L_2))$ is of the form $p_1^{-1}(S_1) = S_1 \times X_2$ for some closed subvariety $S_1 \subseteq X_1$ and the support of $\mathcal{C}_X(\Gamma, e(L_1))$ is of the form $p_2^{-1}(S_2) = X_1 \times S_2$ for some closed subvariety $S_2 \subseteq X_2$. But the support of $\mathcal{C}_X(\Gamma, e(L_2))$ is contained in the support of $\mathcal{C}_X(\Gamma, e(L_1))$, which yields the vanishing of $\mathcal{C}_X(\Gamma, e(L_2))$ or in other words $e(L) \leq e(L_2)$.

The usual vanishing and positivity theorems (see [12], for example) can be extended to the situation described above:

**Theorem 3.5** Let $X$ be a normal projective variety with at most rational singularities, let $L$ be an invertible sheaf on $X$, and let $D$ be an effective Cartier divisor on $X$ and $N \in \mathbb{N} - \{0\}$.

(a) If $L^N(-D)$ is nef and big, then for all $i > 0$ one has

$$
H^i\left(X, L \otimes \omega_X \left\{ \frac{D}{N} \right\} \right) = 0.
$$

10
(b) If $L^N(-D)$ is semi-ample and $(L^N(-D))^\nu(-B)$ is effective for some effective Cartier divisor $B$ on $X$ and some $\nu \in \mathbb{N} - \{0\}$, then for all $i > 0$ the map

$$H^i \left( X, L(B) \otimes \omega_X \left\{ -\frac{D}{N} \right\} \right) \to H^i \left( B, \left( L(B) \otimes \omega_X \left\{ -\frac{D}{N} \right\} \right) |_B \right)$$

is surjective.

(c) Let $f : X \to Y$ be a projective surjective morphism on a smooth quasi-projective variety $Y$ and $Y_0 \subseteq Y$ the open subset such that $f_* (L \otimes \omega_{X/Y} \left\{ -\frac{D}{N} \right\})$ is locally free over $Y_0$ and $f|_{f^{-1}(Y_0)}$ is smooth. If $L^N(-D)$ is semi-ample, then $f_* (L \otimes \omega_{X/Y} \left\{ -\frac{D}{N} \right\})$ is weakly positive over $Y_0$.

PROOF: Let $\tau : X' \to X$ a blowing up such that $D' = \tau^* D$ is a normal crossing divisor and $X'$ is smooth. Writing $L' = \tau^* L$, the sheaf $L'^N(-D')$ is nef and big on $X'$, and writing $B' = \tau^* B$, the sheaf $(L'^N(-D'))^\nu(-B')$ is effective. We obtain (a) and (b) from the corresponding vanishing theorems for integral parts of $\mathbb{Q}$-divisors which hold on $X'$ (see [12], 2.28 and 2.33, respectively). Finally, (c) follows from 2.10: If $f' : X' \to Y$ is the induced map, then

$$f'_* \left( L' \otimes \omega_{X'/Y} \left\{ -\frac{D'}{N} \right\} \right) = f_* \left( L \otimes \omega_{X/Y} \left\{ -\frac{D}{N} \right\} \right)$$

is weakly positive over $Y_0$. \hfill $\Box$

For the following two statements, which correspond to [3], 6.5 and 6.4, respectively, we have to recall briefly the notion of Seshadri index (see [2], for example). If $X$ is a smooth projective variety and $\mathcal{L}$ a nef invertible sheaf on $X$, then for every $x \in X$ we define $\varepsilon(\mathcal{L}, x)$ to be the supremum over all the rational numbers $\varepsilon$ such that $\tau^* \mathcal{L} \otimes \mathcal{O}_X(-\varepsilon E)$ is nef, where $\tau : X' \to X$ denotes the blowing up of $X$ at the point $x$ with exceptional divisor $E$.

The global Seshadri index $\varepsilon(\mathcal{L})$ is then nothing but the infimum over all the local ones.

**Lemma 3.6** Let $Z$ and $B$ be smooth quasi-projective varieties and let $p : Z \to B$ be a projective surjective morphism of relative dimension $k$. Let us consider the open subset $W' \subseteq B$ such that $p|_{p^{-1}(W')} : p^{-1}(W') \to W'$ is smooth. Let $\mathcal{M}$ be an invertible sheaf on $Z$, let $D$ be an effective divisor on $Z$ and let $U \subseteq Z$ be an open subset such that $\mathcal{M}^N(-D)$ is relatively nef over $U$ and $e(D|_{Z_b}) \leq N$ for all $b \in W'$. Let $K$ be an invertible sheaf on $Z$ such that $K|_{Z_b}$ is ample and such that $\varepsilon = \varepsilon(K|_{Z_b}) \geq 1$ for all $b \in W'$. Then there exists a nonempty open subset $W \subseteq B$ such that

$$p^* p_* \left( \mathcal{M} \otimes \mathcal{K}^{k+1} \otimes \omega_{Z/B} \left\{ -\frac{D}{N} \right\} \right) \to \mathcal{M} \otimes \mathcal{K}^{k+1} \otimes \omega_{Z/B} \left\{ -\frac{D}{N} \right\}$$

over $U \cap p^{-1}(W)$ is surjective.

**Proof:** We shall follow the line of the proof of [3], 6.5. By [12], 5.10. we may choose an open subset $W \subseteq W'$ where $\omega_{Z/B} \left\{ -\frac{D}{N} \right\} |_{Z_b} = \omega_{Z_b} \left\{ -\frac{D_{Z_b}}{N} \right\}$ for all $b \in W$. (In fact, considering some blowing up $\tau' : Z' \to Z$ such that $D' = \tau^* D$ is a normal crossing divisor, we may take $W$ as the set of all $b \in W$ such that $D'|_{Z_b}$ is a normal crossing divisor.)
Let $b \in W$ be in general position and let us fix some $z \in Z_b \cap U$. From now on, the index $b$ means restriction to the fibre $Z_b$. We consider the blowing up $\tau : \check{Z}_b \to Z_b$ of the fibre $Z_b$ in $z$ and we denote by $E_z$ the reduced exceptional divisor of $\tau$. Then, by [8], II. Exercise 8.5, we obtain that $\omega_{\check{Z}_b} = \tau^* \omega_{Z_b} \otimes \mathcal{O}_{\check{Z}_b}((k-1) \cdot E_z)$. Moreover, let $\varrho : Z_b' \to Z_b$ be a blowing up such that $\varrho^* D_b$ is a normal crossing divisor and $Z_b'$ is smooth. We may assume that $\varrho$ factors as $\varrho = \tau \circ \eta$ for some morphism $\eta : Z_b' \to \check{Z}_b$. We write $\omega_{Z_b'} = \varrho^* \omega_{Z_b} \otimes \mathcal{O}_{Z_b'}(E + F)$, where $E$ is the part of the exceptional divisor with $\varrho(E) = z$.

As in [3], 6.5, after making $W$ smaller if necessary and using flat base change, it suffices to show that the sheaf $\mathcal{M}_b \otimes K_b^{k+1} \otimes \omega_{Z_b}\{ - \frac{D_b}{N} \}$ is globally generated over $Z_b \cap U$.

By assumption, $\tau^* \mathcal{M}_b^N(-D_b)$ and $\tau^* K_b^N \otimes \mathcal{O}_{Z_b}(-N \cdot k \cdot E_z)$ are nef, and $\tau^* K_b^N$ is nef and big, hence $\tau^*(\mathcal{M}_b \otimes K_b^{k+1})^N \otimes \mathcal{O}_{Z_b}(-N \cdot k \cdot E_z - \tau^* D_b)$ is nef and big. By [12], 5.22 we obtain a surjection

$$H^0 \left( Z_b, \mathcal{M}_b \otimes K_b^{k+1} \otimes \omega_{Z_b} \right) \to H^0 \left( Z_b, \mathcal{C}_b \right),$$

where $\mathcal{C}_b$ denotes the cokernel of

$$\mathcal{M}_b \otimes K_b^{k+1} \otimes \varrho_* \omega_{Z_b}' \left( - \left[ \frac{\varrho^* D_b}{N} \right] - k \cdot \eta^* E_z \right) \to \mathcal{M}_b \otimes K_b^{k+1} \otimes \omega_{Z_b}.$$

Since $e(D|_{Z_b}) \leq N$, the above map is an isomorphism outside the centre of the blowing up $\rho$, hence the support of $\mathcal{C}_b|_{U \cap Z_b}$ is contained in $\{ z \}$. Assuming $\mathcal{C}_b|_{U \cap Z_b} = 0$, we obtain

$$\varrho_* \omega_{Z_b}' \left( - \left[ \frac{\varrho^* D_b}{N} \right] - k \cdot \eta^* E_z \right) = \omega_{Z_b} \otimes \varrho_* \mathcal{O}_{Z_b} \left( E - \left[ \frac{\varrho^* D_b}{N} \right] - k \cdot \eta^* E_z \right) = \omega_{Z_b}.$$

But this means

$$E \geq \left[ \frac{\varrho^* D_b}{N} \right] + k \cdot \eta^* E_z,$$

so the relative canonical divisor contains $\eta^* E_z$ with a multiplicity of at least $k$, which contradicts its description from above. So $\mathcal{C}_b$ is concentrated in the point $\{ z \}$. So far, we have repeated the argument from [3], 6.5, but now we are faced with the problem that the sheaves we consider are not locally free. Yet, since $K_b$ is ample, we may apply [10], 2.11 to obtain that $\mathcal{M}_b \otimes K_b^{k+1} \otimes \omega_{Z_b}\{ - \frac{D_b}{N} \}$ is globally generated over $Z_b \cap U$. \quad \square

**Proposition 3.7** Keeping the assumptions from 3.6, let $\mathcal{L}$ and $\mathcal{B}$ be invertible sheaves on $Z$ such that $\mathcal{L}(-D)$ is semi-ample over $U$, $\mathcal{L}$ is relatively semi-ample over $U$ and $\mathcal{B}$ is relatively nef. Let us assume moreover that $e(D|_U) \leq N$ and that, for some $r > 0$, there exists a map $\bigoplus \mathcal{O}_{W/B} \to \mathcal{B}$, surjective over $U$. Then there exists a nonempty open subset $W \subseteq B$ such that $\mathcal{L} \otimes \mathcal{B}^N \otimes K^{k+1}$ is weakly positive over $U \cap p^{-1}(W)$.

**Proof:** Let us start with the case where $p : Z \to B$ is flat. Taking $\mathcal{M} = \mathcal{L} \otimes \mathcal{B}^{N-1}$, we see that $\mathcal{M}^N(-D)$ is relatively nef over $U$. Using 3.6 and since $e(D|_U) \leq N$, we obtain maps

$$p^* p_* \left( \mathcal{L} \otimes \mathcal{B}^{N-1} \otimes \omega_{Z/B} \left\{ - \frac{D}{N} \right\} \right) \to \mathcal{L} \otimes \mathcal{B}^{N-1} \otimes K^{k+1} \otimes \omega_{Z/B} \left\{ - \frac{D}{N} \right\},$$

$$\to \mathcal{L} \otimes \mathcal{B}^{N-1} \otimes K^{k+1} \otimes \omega_{Z/B},$$

12
surjective over \( U \cap p^{-1}(W) \) for some nonempty open subset \( W' \subseteq B \). If \( \mathcal{H} \) is an ample invertible sheaf on \( B \), then the sheaf
\[
p_* \left( \mathcal{L} \otimes \mathcal{B}^{N-1} \otimes \mathcal{K}^{k+1} \otimes \omega_{Z/B} \left\{ -\frac{D}{N} \right\} \right) \otimes \mathcal{H}^\mu
\]
is globally generated for \( \mu \) sufficiently large, and with \( \mathcal{H} = p^*\mathcal{H} \) we obtain that
\[
p^*p_* \left( \mathcal{L} \otimes \mathcal{B}^{N-1} \otimes \mathcal{K}^{k+1} \otimes \omega_{Z/B} \left\{ -\frac{D}{N} \right\} \right) \otimes \mathcal{H}^\mu
\]
is globally generated, too. Since by [12], 2.16 quotient sheaves inherit weak positivity, we obtain that
\[
\mathcal{L} \otimes \mathcal{B}^{N-1} \otimes \mathcal{K}^{k+1} \otimes \omega_{Z/B} \otimes \mathcal{H}^\mu
\]
is weakly positive over \( U \cap p^{-1}(W') \). By assumption we have a map
\[
\bigoplus \left( \mathcal{L} \otimes \mathcal{B}^{N-1} \otimes \mathcal{K}^{k+1} \otimes \omega_{Z/B} \otimes \mathcal{H}^\mu \right) = \mathcal{L} \otimes \mathcal{B}^{N-1} \otimes \mathcal{K}^{k+1} \otimes \bigoplus \omega_{Z/B} \otimes \mathcal{H}^\mu
\]
\[
\to \mathcal{L} \otimes \mathcal{B}^N \otimes \mathcal{K}^{k+1} \otimes \mathcal{H}^\mu,
\]
surjective over \( U \), which yields, using [12], 2.16 again, the weak positivity of \( \mathcal{L} \otimes \mathcal{B}^N \otimes \mathcal{K}^{k+1} \otimes \mathcal{H}^\mu \) over \( U \cap p^{-1}(W') \), where \( \mu \) is sufficiently large. This gives sense to the following definition:
\[
\gamma = \min \left\{ \mu \in N \cdot \mathbb{N}; \mathcal{L} \otimes \mathcal{B}^N \otimes \mathcal{K}^{k+1} \otimes \mathcal{H}^\mu \text{ weakly positive over } U \cap p^{-1}(W) \text{ for a nonempty open subset } W \subseteq B \right\}.
\]

**Claim 3.8** We have \( \gamma \leq N^2 \), hence there exists a nonempty open subset \( W \subseteq B \) such that \( \mathcal{L} \otimes \mathcal{B}^N \otimes \mathcal{K}^{k+1} \otimes \mathcal{H}^{N^2} \) is weakly positive over \( U \cap p^{-1}(W) \).

**Proof of 3.8:** By 2.4 there exists some \( \beta > 0 \) such that
\[
\left( \mathcal{L} \otimes \mathcal{B}^N \otimes \mathcal{K}^{k+1} \otimes \mathcal{H}^\gamma \right)^{(N-1) \cdot \beta} \otimes \left( \mathcal{H}^N \otimes \mathcal{K}^{k+1} \right)^\beta,
\]
and hence
\[
\left( \mathcal{L} \otimes \mathcal{B}^{N-1} \otimes \mathcal{K}^{k+1} \otimes \mathcal{H}^{\frac{(N-1) \cdot \beta + N}{N}} \right)^{N \cdot \beta} \otimes \mathcal{L}^{-\beta}
\]
is globally generated over \( U \cap p^{-1}(W) \). We may assume that \( \mathcal{L}^\beta (-\beta \cdot D) \) is globally generated over \( U \). Taking
\[
\mathcal{M} = \mathcal{L} \otimes \mathcal{B}^{N-1} \otimes \mathcal{H}^{\frac{(N-1) \cdot \beta + N}{N}},
\]
we obtain that \( (\mathcal{M} \otimes \mathcal{K}^{k+1})^{N \cdot \beta} = \mathcal{O}_Z (\Gamma + \beta \cdot D) \) for some general section \( \Gamma \). Since \( p \) is flat, we may assume that
\[
p_* \left( \mathcal{M} \otimes \mathcal{K}^{k+1} \otimes \omega_{Z/B} \left\{ -\frac{\Gamma + \beta \cdot D}{\beta \cdot N} \right\} \right)
\]
is locally free over $W$, and hence, by 3.5, weakly positive over $W$. By [3], 4.3 we obtain that
\[ p^*p_* \left( \mathcal{M} \otimes K^{k+1} \otimes \omega_{Z/B} \left\{ -\frac{\Gamma + \beta \cdot D}{\beta \cdot N} \right\} \right) \]
is weakly positive over $p^{-1}(W)$, too.

The sheaves $\mathcal{M}$ and $\mathcal{K}$ satisfy the conditions of 3.6. Moreover, making $\beta$ larger if required, we may assume
\[ \omega_{Z/B} \left\{ -\frac{\Gamma + \beta \cdot D}{\beta \cdot N} \right\} = \omega_{Z/B} \left\{ -\frac{D}{N} \right\}. \]

Thus we obtain that the composed map
\[ p^*p_* \left( \mathcal{M} \otimes K^{k+1} \otimes \omega_{Z/B} \left\{ -\frac{\Gamma + \beta \cdot D}{\beta \cdot N} \right\} \right) \rightarrow \mathcal{M} \otimes K^{k+1} \otimes \omega_{Z/B} \rightarrow \mathcal{M} \otimes K^{k+1} \otimes \omega_{Z/B} \]
is surjective over $U \cap p^{-1}(W)$, since $\epsilon(D|_U) \leq N$. Using [12], 2.16, we obtain that $\mathcal{M} \otimes K^{k+1} \otimes \omega_{Z/B}$ is weakly positive over $U \cap p^{-1}(W)$, and so is
\[ (\mathcal{M} \otimes K^{k+1} \otimes \omega_{Z/B}) = \mathcal{M} \otimes K^{k+1} \otimes \bigoplus \omega_{Z/B}. \]

Again by [12], 2.16 we obtain the weak positivity of the quotient sheaf
\[ \mathcal{M} \otimes K^{k+1} \otimes \mathcal{B} = \mathcal{L} \otimes \mathcal{B}^N \otimes K^{k+1} \otimes \mathcal{H}^{(N-1)+k} \]
over $U \cap p^{-1}(W)$. But by definition of $\gamma$ we obtain
\[ \gamma \cdot \frac{(N-1)+N}{N} > \gamma - N, \]
and so $\gamma \leq N^2$. \hfill \Box

Now we have to get rid of the twist $\mathcal{H}^{N^2}$, which we manage to do by 2.8 and the following fact:

**Lemma 3.9** In the situation above, let $\tau : B' \rightarrow B$ be a finite morphism, ramified over some divisor $\Delta \subseteq B$, where $D$ intersects the divisor $p^{-1}(\Delta)$ in codimension $\geq 2$. Let $Z' = B' \times_B Z$ denote the fibre product. Then, after replacing $B'$ by the complement of a closed subvariety of codimension $\geq 2$ if necessary, the assumptions made in 3.7 hold true for the induced morphism $\tau' : Z' \rightarrow B'$ as well.

**Proof:** By assumption we may assume that $D \cap p^{-1}(\Delta)$ is even empty and that the induced morphism $\tau' : Z' \rightarrow Z$ is smooth over $Z - p^{-1}(\Delta)$. Since moreover $\tau' : Z' \rightarrow B'$ is smooth over $\tau^{-1}(B - \Delta)$, we obtain that $Z'$ is smooth and so in particular $\omega_{Z'/B'} = \tau'^* \omega_{Z/B}$. The compatibility of the relation $\epsilon \geq 1$ with the covering holds by [7], 4.3. It remains to show that the relation $\epsilon(D|_U) \leq N$ still holds. But taking $U' = \tau'^{-1}(U)$ and
Δ' = τ*Δ, we may assume, after making W' smaller if necessary, that U' ∩ p'^-1(Δ') is empty and the relation then follows from 3.3. □

We now choose for every δ ∈ N \ {0} a finite morphism τδ : Bδ → B, which satisfies the properties of 3.9. In addition we may, by [12], 2.1, assume that τδ*H = Hδ for some invertible sheaf H on Bδ. This induces, for every δ ∈ N \ {0}, a morphism τδ : Zδ → Z such that τδ*H = Hδ for some invertible sheaf H on Zδ. By 3.9, the bound γ ≤ N² holds for pδ : Zδ → Bδ as well. By 2.8 this implies, taking μ = N², the weak positivity of L ⊗ B^N ⊗ K^{k+1} over U ∩ p^{-1}(W), which proves 3.7 for the flat case.

Now let us sketch how to reduce to the flat case if p : Z → B is not flat. Let H denote the Hilbert scheme parametrizing the flat subvarieties of Z (see [12], p. 42, for example) and inducing, by its universal property, a rational map B₀ → H. Extending this by [8], II.7.17.3 to a morphism B' → H, we obtain a factorization of the inclusion B₀ → B over some birational map σ : B' → B. By [8], II.9.8.1, the morphism B' → H corresponds to a flat morphism p₀ : Z₀ → B₀, where Z₀ turns out to be a component of the fibre product Z' = Z ×_B B'.

**Claim 3.10** Let δ : Z₀'' → Z₀' be a desingularization such that the preimage of the singular locus of Z₀' is a divisor and let σ₀'' : Z₀'' → Z and p₀'' : Z₀'' → B' denote the induced maps. Then, replacing the morphism p : Z → B in 3.7 by

\[ p₀'' : Z₀'' - p₀''^{-1}(p₀''(B)) \to B' - p₀''(B), \]

where B is the maximal divisor in Z₀'' such that \text{codim}(p₀''(B)) ≥ 2, the assumptions from 3.7 hold true for the induced sheaves and divisors as well.

To prove 3.10, the essential fact is the existence of a morphism ω_{Z₀''/Z} → p₀''*ω'_{B'/B} or, correspondingly, ω_{Z₀''/B'} → σ₀'"*ω_{Z/\text{B}}. But this follows from duality of finite morphisms (see [8], III. Exercise 6.10) and from the fact that p₀''*ω'_{B'/B} = ω'_{Z/\text{B}} or, correspondingly, σ₀'"*ω_{Z/\text{B}} = ω'_{Z/\text{B}'}, which can be proved using methods from [8], III.6 and 7.

By [8], III.10.2, we find that p₀'' : Z₀'' → B', being an equidimensional morphism of smooth varieties outside B, is flat outside B, which by 3.10 and 3.7 for the flat case implies the weak positivity of σ₀'"* (L ⊗ B^N ⊗ K^{k+1}) |_{Z₀'' - B} over σ₀'"^{-1}(U) ∩ p₀''^{-1}(W') for an open subset W' ⊆ B'. We may assume that σ₀'"^{-1}(U) ∩ p₀''^{-1}(W') is of the form σ₀'"^{-1}(U ∩ p^{-1}(W)) for an open subset W ⊆ B. But then L ⊗ B^N ⊗ K^{k+1} is weakly positive over U ∩ p^{-1}(W) by [3], 4.3, which completes the proof of 3.7. □

### 4 A Covering Construction

Following the notations from the introduction, we shall construct a covering which simplifies the index conditions defining the ideal sheaf \( \mathcal{I} \).

**Construction 4.1** Let \( \nu \in \{1, \ldots, n\} \). For every point \( \xi_{\mu, \nu} \in p_{\nu}(S) \) we choose exactly \( m_\nu \) smooth hyperplane sections \( H_\nu^{\mu_1}, \ldots, H_\nu^{\mu_m} \) of \( L_\nu \) in general position through \( \xi_{\mu, \nu} \). (If
\[ [p_{\nu}(S)] = 1 \text{ we just add another arbitrary point.} \text{ So } \xi_{\mu,\nu} \in \cap_{k=1}^{m_{\nu}} H_{\mu,k}^{\nu} \text{ is an isolated point, where the hyperplane sections } H_{\mu,k}^{\nu} \text{ intersect transversally. We denote by} \\
\Delta_{\nu} = \sum H_{\mu,k}^{\nu} \\
\text{the corresponding normal crossing divisor on } X_{\nu}, \text{ where the sum runs over } \mu = 1, \ldots, M_{\nu} \text{ and } k = 1, \ldots, m_{\nu}. \text{ So we have } \mathcal{O}_{X_{\nu}}(\Delta_{\nu}) = \mathcal{L}_{\nu}^{M_{\nu} - m_{\nu}}. \\
\text{Now let us consider} \\
N = \min \{ n \in \mathbb{N} - \{0\}; \frac{n}{d_{\nu}} \cdot t_{\mu} \in \mathbb{N} - \{0\} \text{ for } \mu = 1, \ldots, M \text{ and } \nu = 1, \ldots, n \}. \\
\text{By [12], 2.1, we may assume that } \Delta_{\nu} \text{ is a section of the } \frac{N}{d_{\nu}} \text{-th tensor power of some invertible sheaf. Thus we may apply [12], 2.3 to find out that there exists a smooth projective variety } Y_{\nu} \text{ and a finite surjective morphism } \sigma_{\nu} : Y_{\nu} \to X_{\nu} \text{ such that } \sigma_{\nu}^{*}\Delta_{\nu} \text{ is a normal crossing divisor on } Y_{\nu}. \text{ Moreover, } \sigma_{\nu} \text{ ramifies exactly over } \Delta_{\nu} \text{ and, since } \Delta_{\nu} \text{ has reduced components, we obtain} \\
\sigma_{\nu}^{*}H_{\mu,k}^{\nu} = \frac{N}{d_{\nu}} \cdot (\sigma_{\nu}^{*}H_{\mu,k}^{\nu})_{\text{red}} \\
\text{for } \mu = 1, \ldots, M \text{ and } k = 1, \ldots, m_{\nu}. \\
\text{The coverings } \sigma_{\nu} : Y_{\nu} \to X_{\nu} \text{ constructed in this way for } \nu = 1, \ldots, n \text{ induce a covering} \\
\sigma : Y = Y_{1} \times \cdots \times Y_{n} \to X, \\
\text{which is étale over } X_{0} = (X_{1} - \Delta_{1}) \times \cdots \times (X_{n} - \Delta_{n}). \\
\textbf{Lemma 4.2} \text{ In the situation of 4.1 we consider the sheaf} \\
\mathcal{M} = \bigcap_{\mu=1}^{M} \bigcap_{\eta=\sigma^{-1}(\xi_{\nu})}^{\mathbb{N} \cdot \mathbb{N}_{t_{\mu}}^{\nu}} \\
\text{which is full. Let } \tau' : Z \to Y \text{ be a birational morphism such that } Z \text{ is smooth and} \\
\mathcal{M}' = \tau'^{-1}\mathcal{M} \cdot \mathcal{O}_{Z} \text{ is invertible on } Z \text{ and let us fix a blowing up } \tau : X' \to X \text{ such that } \mathcal{I}' = \tau^{-1}\mathcal{I} \cdot \mathcal{O}_{X'} \text{ is invertible. Let us assume moreover that there exists a morphism} \\
\sigma' : Z \to X' \text{ making the diagram} \\
\begin{array}{ccc}
Z & \xrightarrow{\tau'} & Y \\
\sigma' \downarrow & & \downarrow \sigma \\
X' & \xrightarrow{\tau} & X.
\end{array} \\
\text{commutative. Let us denote by } q : Z \to X \text{ the induced morphism. Then for all } l \in \mathbb{N} - \{0\} \\
\text{the trace map induces a surjective map } q_{*}\mathcal{M}'^{l} \to \mathcal{I}'^{l}, \text{ and moreover, the ideal sheaf } \mathcal{I}'^{l} \text{ is full.} \\
\textbf{Proof:} \text{ Keeping notations simple, we restrict ourselves to the case } l = 1. \text{ For the general case one has to consider the } l\text{-th powers. We shall first show that the image of}
$\sigma_*\mathcal{M}$ under the trace map $\sigma_*\mathcal{O}_Y \to \mathcal{O}_X$ is contained in $\mathcal{I}$. This statement is local, and we may, after fixing a point $\xi_\mu \in S$ and some point $\eta \in Y$ mapping to $\xi_\mu$ and keeping all the notations, replace $X$ and $Y$ by the corresponding local rings.

Let $h_{\mu,k}^\nu$ and $g_{\mu,k}^\nu$ denote the local equations of $H_{\mu,k}^\nu$ at the point $\xi_\mu,\eta$ and of $(\sigma_*H_{\mu,k}^\nu)_{\text{red}}$ at a point $\eta$ mapping to $\xi_\mu,\eta$, respectively. Thus for $\mu = 1, \ldots, M$, the ideal sheaves $\mathcal{I}_{\xi_\mu,\eta}$ and $\sigma^*\mathcal{I}_{\xi_\mu,\eta}$ are generated by expressions of the form

$$\prod_{k=1}^{m_1} (h_{\mu,k}^1)^{\alpha_1} \cdots \prod_{k=1}^{m_n} (h_{\mu,k}^n)^{\alpha_n}$$

where $\frac{\alpha_1}{d_1} + \cdots + \frac{\alpha_n}{d_n} \geq t_\mu$ and

$$\prod_{k=1}^{m_1} (g_{\mu,k}^1)^{\alpha_1,N} \cdots \prod_{k=1}^{m_n} (g_{\mu,k}^n)^{\alpha_n,N}$$

where $\frac{\alpha_1,N}{d_1} + \cdots + \frac{\alpha_n,N}{d_n} \geq N \cdot t_\mu$, respectively. It remains to show that the image of $\sigma_*m_{\eta,\eta}^{N,t_\mu}$ under the trace map is contained in $\mathcal{I}_{\xi_\mu,\eta}$. The coverings $\sigma_\nu : Y_\nu \to X_\nu$ are determined by the ramifications $\beta_\nu = \frac{N}{d_\nu} \in \mathbb{N} - \{0\}$ in $\Delta_\nu$, and we may now assume that $\sigma : Y \to X$ is a Galois covering with Galois group $G = \mathbb{Z}/\beta_1 \times \cdots \times \mathbb{Z}/\beta_n$. The trace map then is nothing but the sum over all conjugates of $G$. Now $m_{\eta,\eta}^{N,t_\mu}$ is invariant under $G$, so the image of $\sigma_*m_{\eta,\eta}^{N,t_\mu}$ under the trace map in $\mathcal{O}_X$ is generated by the images of the $G$-invariant elements

$$\sum_{\mathcal{L} \in \{l_1, \ldots, l_n\}} f_{\mathcal{L}} \cdot \left( \prod_{k=1}^{m_1} g_{\mu,k}^1 \right)^{l_1} \cdots \left( \prod_{k=1}^{m_n} g_{\mu,k}^n \right)^{l_n},$$

where $f_{\mathcal{L}}$ are units for $\mathcal{L} \in \mathbb{N}^n$. But the $G$-invariance of such expressions just means the $G$-invariance of the single summands, and for such a summand, given by $\mathcal{L} = (l_1, \ldots, l_n)$, this just means that $l_\nu = \beta_\nu \cdot s_\nu$ for $\nu = 1, \ldots, n$ and certain $s_\nu \in \mathbb{N} - \{0\}$. So the image of $\sigma_*m_{\eta,\eta}^{N,t_\mu}$ under the trace map in $\mathcal{O}_X$ is generated by expressions of the form

$$\left( \prod_{k=1}^{m_1} h_{\mu,k}^1 \right)^{s_1} \cdots \left( \prod_{k=1}^{m_n} h_{\mu,k}^n \right)^{s_n},$$

where $\sum_{\nu=1}^{N} l_\nu = \sum_{\nu=1}^{N} \beta_\nu \cdot s_\nu \geq N \cdot t_\mu$. This means that $\sum_{\nu=1}^{N} \frac{s_\nu}{d_\nu} \geq t_\mu$, so the image of $\sigma_*m_{\eta,\eta}^{N,t_\mu}$ under the trace map is contained in $\mathcal{I}_{\xi_\mu,\eta}$.

So the trace map induces a surjective map $\sigma_*\mathcal{M} \to \mathcal{I}$. Since $\mathcal{M}$ is full, we have $\tau'_*\mathcal{M}' = \mathcal{M}$, and we obtain a surjective map $\varrho_*\mathcal{M}' \to \mathcal{I}$.

Finally, we observe that the inclusion $\tau_*\mathcal{I}' \to \sigma_*\tau'_*\mathcal{M}' = \sigma_*\mathcal{M}$ splits, and we obtain a surjective map $\sigma_*\mathcal{M} \to \tau_*\mathcal{I}'$, which factors over $\mathcal{I}$, as we have just seen. So $\mathcal{I} \to \tau_*\mathcal{I}'$ is surjective and $\mathcal{I}$ is full.

5 The Proof of 1.1

We shall now give the proof for 1.1. First we need some more notations.
Definition 5.1 An open subset $U \subseteq X$ is called an open subset of type $k$, if for $\nu = 1, \ldots, k$ there exist open subsets $U_{\nu} \subseteq X_{\nu}$ and an open subset $W \subseteq X_{k+1} \times \cdots \times X_{n}$ such that $U$ is of the form $U = U_{1} \times \cdots \times U_{k} \times W$. A subset of type $n-1$ is called a product open set.

As one immediately sees, 1.1 follows from inductively applying the following

Theorem 5.2 If in the situation of 1.1 the sheaf $\mathcal{L}^{d} \otimes \mathcal{I}$ is weakly positive over some open set $U_{k-1}$ of type $k-1$, then the sheaf $\mathcal{L}^{d'} \otimes \mathcal{I}$ is weakly positive over some open set $U^{k}$ of type $k$, where $d' = (d_{1}', \ldots, d_{n}')$ with

$$d_{\nu}' = \begin{cases} d_{\nu} + M_{\nu} \cdot d_{k+1} + e_{k+1} & (\nu \leq k) \\ d_{\nu} & (\nu \geq k + 1) \end{cases}$$

and $M_{\nu} = M_{\nu}' \cdot m_{\nu} + \gamma_{\nu}$ for $\nu = 1, \ldots, n$.

So we shall prove this. To this end let $\alpha > 0$. By 2.6 there exists some $\beta > 0$ such that $\mathcal{L}^{d} \otimes \mathcal{J}$ is globally generated over $U_{k-1} = U_{1} \times \cdots \times U_{k-1} \times W$, where $d = (\delta_{1}, \ldots, \delta_{n})$ with $\delta_{\nu} = \alpha \beta d_{\nu} + \beta$ for $\nu = 1, \ldots, n$ and $\mathcal{J} = \mathcal{I}^{\alpha \beta}$ and where $U_{\nu} \subseteq X_{\nu}$ for $\nu = 1, \ldots, k-1$ and $W \subseteq U_{k} \times \cdots \times U_{n}$ are open subsets. We notice that the ordering of the $d_{\nu}, e_{\nu}$ is preserved: $\delta_{1} \cdot e_{1} \geq \cdots \geq \delta_{n} \cdot e_{n}$. We choose a general section $\Gamma$ of $\mathcal{L}^{d} \otimes \mathcal{J}$ and observe:

Claim 5.3 There exists an open subset $U^{i} \subseteq X$ of type $k$ such that

$$e(\Gamma|_{U^{i}}) \leq \delta_{k+1} \cdot e_{k+1} + 1.$$ 

Moreover, by 4.1 we may assume that $\mathcal{L}_{|_{U^{i}}}$ is étale, and so by 3.3 we have $e(\mathcal{L}_{|_{U^{i}}}) \leq 1$.

Proof: We consider the projection $p_{1, \ldots, k}: X \to X_{1} \times \cdots \times X_{k}$ onto the first $k$ factors of the product, whose fibres are isomorphic to $X_{k+1} \times \cdots \times X_{n}$. One has $p_{1, \ldots, k}(U_{k-1}) = U_{1} \times \cdots \times U_{k-1} \times U_{k}$ for an open subset $U_{k} \subseteq X_{k}$. Let $x \in p_{1, \ldots, k}(U_{k-1})$, in other words, the fibre $p_{1, \ldots, k}^{-1}(x)$ intersects the open subset $U_{k-1}$. Then we obtain

$$e(\Gamma|_{p_{1, \ldots, k}^{-1}(x)}) \leq e(\mathcal{L}_{p_{1, \ldots, k}^{-1}(x)}^{d}) = e(\mathcal{L}_{k+1}^{\delta_{k+1}} \boxtimes \cdots \boxtimes \mathcal{L}_{n}^{\delta_{n}}).$$

By [12], 5.11 we have

$$e(\mathcal{L}_{\nu}^{d_{\nu}}) \leq \delta_{\nu} \cdot c_{1}(\mathcal{L}_{\nu})^{m_{\nu}} + 1 = \delta_{\nu} \cdot e_{\nu} + 1$$

for $\nu = k + 1, \ldots, n$. Now by 3.4 and since the $\delta_{\nu}, e_{\nu}$ are ordered, we obtain

$$e(\Gamma|_{p_{1, \ldots, k}^{-1}(x)}) \leq \delta_{k+1} \cdot e_{k+1} + 1,$$

which, by [12], 5.14, yields $e(\Gamma|_{U^{i}}) \leq \delta_{k+1} \cdot e_{k+1} + 1$, where $U^{i}$ is a neighbourhood of the fibre $p_{1, \ldots, k}^{-1}(x)$. We can say more precisely that $U^{i} = p_{1, \ldots, k}(U_{k-1}) \times X_{k+1} \times \cdots \times X_{n} \cap X_{0}$, where we recall $X_{0}$ to be the étale locus of $\sigma$ (see 4.1). $\square$

In order to apply 3.7 to our situation, we shall now check whether the assumptions are fulfilled. For $\nu = 1, \ldots, n$ let $\sigma_{\nu} : Y_{\nu} \to X_{\nu}$ be the coverings constructed in 4.1, which ramify exactly in $\Delta_{\nu}$, and let $\Delta_{\nu}' = p_{\nu}^{-1}(\Delta_{\nu})$. Let $\Delta = \sum_{\nu=1}^{k} \Delta_{\nu}'$ and let us write $\sigma_{1, \ldots, k} = \sigma_{1} \times \cdots \times \sigma_{k}$ and $\sigma_{k+1, \ldots, n} = \sigma_{k+1} \times \cdots \times \sigma_{n}$.

18
Claim 5.4 In the situation described above, we have

$$\omega_Y/(Y_{k+1} \times \cdots \times Y_n) = \sigma^* \omega_X/(X_{k+1} \times \cdots \times X_n) \otimes \mathcal{O}_Y \left(-\sigma^* \Delta \right)\sigma^* \Delta.$$  

Proof of 5.4: We consider the fibre product

$$X_1 \times \cdots \times X_k \times Y_{k+1} \times \cdots \times Y_n \xrightarrow{\sigma_{k+1,\ldots,n}'} X$$

and notice that $\sigma_{k+1,\ldots,n}'$ ramifies exactly in $\sum_{\nu=1}^k \Delta_{\nu}'$. Compatibility of relatively canonical sheaves with base change yields

$$\omega[X_1 \times \cdots \times X_k \times Y_{k+1} \times \cdots \times Y_n]/(Y_{k+1} \times \cdots \times Y_n) = \sigma_{k+1,\ldots,n}'^* \omega X/(X_{k+1} \times \cdots \times X_n),$$

and this holds true after adding divisors, which are not contained in the ramification locus, hence we have

$$\omega[X_1 \times \cdots \times X_k \times Y_{k+1} \times \cdots \times Y_n]/(Y_{k+1} \times \cdots \times Y_n) \left(\log \sigma_{k+1,\ldots,n}' \left(\sum_{\nu=1}^k \Delta_{\nu}'\right)\right)$$

$$= \sigma_{k+1,\ldots,n}'^* \left(\omega X/(X_{k+1} \times \cdots \times X_n) \left(\log \left(\sum_{\nu=1}^k \Delta_{\nu}'\right)\right)\right).$$

Considering on the other hand the fibre product

$$Y \xrightarrow{\sigma_{1,\ldots,k}'} X_1 \times \cdots \times X_k \times Y_{k+1} \times \cdots \times Y_n$$

$$\downarrow \quad \downarrow$$

$$Y_1 \times \cdots \times Y_k \xrightarrow{\sigma_{1,\ldots,k}'} X_1 \times \cdots \times X_k,$$

then $\sigma_{1,\ldots,k}'$ ramifies exactly in $\sum_{\nu=1}^k \sigma_{k+1,\ldots,n}'^* \Delta_{\nu}'$, and the Riemann Hurwitz Formula implies

$$\omega_Y/(Y_{k+1} \times \cdots \times Y_n) \left(\log \sigma_{1,\ldots,k}' \left(\sum_{\nu=1}^k \sigma_{k+1,\ldots,n}'^* \Delta_{\nu}'\right)\right)$$

$$= \sigma_{1,\ldots,k}'^* \left(\omega X_1 \times \cdots \times X_k \times Y_{k+1} \times \cdots \times Y_n)/(Y_{k+1} \times \cdots \times Y_n) \left(\log \left(\sum_{\nu=1}^k \sigma_{k+1,\ldots,n}'^* \Delta_{\nu}'\right)\right)\right).$$

$$= \sigma_{1,\ldots,k}'^* \left(\sigma_{k+1,\ldots,n}'^* \left(\omega X_1 \times \cdots \times X_n) \left(\log \left(\sum_{\nu=1}^k \Delta_{\nu}'\right)\right)\right)\right).$$

Thus we get

$$\omega_Y/(Y_{k+1} \times \cdots \times Y_n) \left(\log \sigma^* \left(\sum_{\nu=1}^k \Delta_{\nu}'\right)\right)$$

$$= \sigma^* \left(\omega X_1 \times \cdots \times X_n) \left(\log \left(\sum_{\nu=1}^k \Delta_{\nu}'\right)\right)\right).$$
and so
\[
\omega_Y/(Y_{k+1} \times \cdots \times Y_n) \otimes \mathcal{O}_Y \left( \left( \sigma^* \left( \sum_{\nu=1}^k \Delta_{\nu} \right) \right)_{red} \right) = \sigma^* \left( \omega_X/(X_{k+1} \times \cdots \times X_n) \otimes \mathcal{O}_X \left( \left( \sum_{\nu=1}^k \Delta_{\nu} \right)_{red} \right) \right).
\]

This proves 5.4. □

Now \( \tau' : Z \to Y \) is the blowing up of the ideal sheaf \( \mathcal{M} \), given by powers of maximal ideal sheaves. This, together with 5.4, implies
\[
\omega_Z/(Y_{k+1} \times \cdots \times Y_n) = \tau'^* \left( \omega_Y/(Y_{k+1} \times \cdots \times Y_n) \otimes \mathcal{O}_Z(E) \right) \]
\[
= \varrho^* \omega_X/(X_{k+1} \times \cdots \times X_n) \otimes \varrho^* \mathcal{O}_X(\Delta) \otimes \tau'^* \mathcal{O}_Y(- (\sigma^* \Delta)_{red}) \otimes \mathcal{O}_Z(E),
\]
where \( E \) is the exceptional divisor of \( \tau' \). By choice of \( \gamma_\nu \) there exists a surjective map
\[
\bigoplus r \omega_X/(X_{k+1} \times \cdots \times X_n) \longrightarrow \mathcal{L}^{(\gamma_1, \ldots, \gamma_k, 0, \ldots, 0)},
\]
so we obtain a surjective map
\[
\bigoplus r \omega_Z/(Y_{k+1} \times \cdots \times Y_n) \rightarrow \varrho^* \mathcal{L}^{(\gamma_1, \ldots, \gamma_k, 0, \ldots, 0)} \otimes \tau'^* \mathcal{O}_Y(- (\sigma^* \Delta)_{red}) \otimes \mathcal{O}_Z(E)
\]
\[
= \varrho^* \mathcal{L}^{(M_1, \ldots, M_k, 0, \ldots, 0)} \otimes \tau'^* \mathcal{O}_Y(- (\sigma^* \Delta)_{red}) \otimes \mathcal{O}_Z(E),
\]
where the last sheaf coincides with \( \varrho^* \mathcal{L}^{(M_1, \ldots, M_k, 0, \ldots, 0)} \) over \( \varrho^{-1}(U') \). Hence, there exists a map
\[
\bigoplus r \omega_Z/(Y_{k+1} \times \cdots \times Y_n) \longrightarrow \varrho^* \mathcal{L}^{(M_1, \ldots, M_k, 0, \ldots, 0)},
\]
surjective over \( \varrho^{-1}(U') \).

Now let us take \( B = Y_{k+1} \times \cdots \times Y_n, D = \varrho^* \Gamma, U = \varrho^{-1}(U') \) and \( N = \delta_{k+1} \cdot \epsilon_{k+1} + 1 \). Considering the sheaves \( \mathcal{L} = \varrho^* \mathcal{L}^{d'}, \mathcal{B} = \varrho^* \mathcal{L}^{(M_1, \ldots, M_k, 0, \ldots, 0)} \) and \( \mathcal{K} = \varrho^* \mathcal{L}^{(1, \ldots, 1)} \), we have justified above that we may apply 3.7 to the induced morphism \( p : Z \to B \). Thus we obtain that \( \varrho^* \mathcal{L}^{\delta'} \otimes \mathcal{M}^{(a, b)} \) is weakly positive over
\[
\varrho^{-1}(U') \cap \varrho^{-1}(W) = \varrho^{-1}(U') \cap \tau'^{-1}(Y_1 \times \cdots \times Y_k \times W),
\]
for some non-empty subset \( W \subseteq Y_{k+1} \times \cdots \times Y_n \) and for \( \delta' = (\delta_1', \ldots, \delta_n') \) with
\[
\delta_\nu' = \begin{cases} 
\delta_\nu + M_\nu \cdot (\delta_{k+1} \cdot \epsilon_{k+1} + 1) + k + 1 & (\nu \leq k) \\
\delta_\nu + k + 1 & (\nu \geq k + 1).
\end{cases}
\]

Now we may assume that \( \varrho^{-1}(U') \cap \tau'^{-1}(Y_1 \times \cdots \times Y_k \times W) \) is of the form \( \varrho^{-1}(U^k) \), where \( U^k \subseteq X \) is an open subset of type \( k \). By 4.2, for every \( l \in \mathbb{N} - \{0\} \) there exists a surjective map
\[
\varrho_* \left( \varrho^* \mathcal{L}^{\delta'} \otimes \mathcal{M}^{(a, b)} \right)^l = (\mathcal{L}_d^{\delta'})^l \otimes \varrho_* \mathcal{M}^{(a, b)} \longrightarrow (\mathcal{L}_d^{\delta'})^l \otimes \mathcal{J}^l,
\]

20
and $\mathcal{J}$ is full for every $l \in \mathbb{N} - \{0\}$. So taking

$$\mathcal{G} = q^* \mathcal{L}^g \otimes \mathcal{M}^{[\alpha \beta]}$$

and $\mathcal{L} = \mathcal{L}^g$, and since $q_{|q^{-1}(U_k)}$ is finite, the assumptions of 2.7 are fulfilled, and we obtain the weak positivity of $\mathcal{L}^g \otimes \mathcal{J}$ over $U_k$. Now we have

$$\mathcal{L}^g \otimes \mathcal{J} = \left( \mathcal{L}^{(d_1 + M_1, d_k + 1, e_{h+1}, \ldots, d_k + M_k, d_k + 1, d_h + 1, \ldots, d_n)} \right)^{[\alpha \beta]} \otimes \left( \mathcal{L}^{[M_1, e_h + 1, 1, \ldots, M_k, e_{k+1}, 1, \ldots, 1]} \right)^{[\beta]} \otimes \left( \mathcal{L}^{[M_1 + k + 1, \ldots, M_k + k + 1, k + 1, \ldots, k + 1]} \right).
$$

Hence by 2.6 we obtain the weak positivity of

$$\mathcal{L}^{(d_1 + M_1, d_k + 1, e_{h+1}, \ldots, d_k + M_k, d_k + 1, d_h + 1, \ldots, d_n)} \otimes \mathcal{I} = \mathcal{L}^g \otimes \mathcal{I}$$

over the open set $U_k \subseteq X$ of type $k$, which completes the proof of 5.2.

\[ \square \]

\textbf{Remark 5.5} As we mentioned in the introduction, in the case of a product of smooth projective curves, one can do slightly better. Making a very mild assumption on the position of the points and modifying the ideal sheaves a little, one can replace the notion of weak positivity over some product open set by the notion of numerical effectivity, hence by a global positivity statement. This finally yields an analogue of Dyson’s Lemma, using basically the same arguments as in the combinatorial part of the proof in [3], 5.

\textbf{References}


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