ON MULTIPLIER IDEAL SHEAVES AND FIBRE PRODUCTS

by

MARKUS WESSLER

Abstract

In this paper we show in how far multiplier ideal sheaves are compatible with fibre products. Unlike in the case of products, this compatibility is not optimal, but we manage to construct a map in one direction and examine under which conditions it is an isomorphism.

1 Introduction

Throughout this paper, we consider varieties which are defined over some algebraically closed field $k$ of characteristic zero. Let $f : X \to Y$ be a projective surjective morphism of smooth quasi-projective varieties. Let $Y_0 \subseteq Y$ be the open subset such that $f|_{f^{-1}(Y_0)}$ is smooth. Then by the Fujita Kawamata Positivity Theorem ([2], [4]), $f_* \omega_{X/Y}$ is weakly positive over $Y_0$. An algebraic proof of this theorem is given in [5] and uses, besides Kollár’s Vanishing Theorem, basically the compatibility of the relatively canonical sheaf $\omega_{X/Y}$ with flat base change: If $X'$ denotes the $r$-fold fibre product of $X$ over $Y$, then one has an isomorphism

$$\omega_{X'/Y} \to \omega_{X/Y} \boxtimes \cdots \boxtimes \omega_{X/Y}.$$

Here and later we use the symbol $\boxtimes$ to denote the tensor product of sheaves which have been pulled back from the factors to the product. We want to examine the more general situation of a normal variety $X$ with at most rational singularities together with an effective Cartier divisor $D$, considering multiplier ideal sheaves. It is well-known that there exists an analogue of the Fujita Kawamata Positivity Theorem for the sheaf $f_* \left( \omega_{X/Y} \{-D\} \right)$ (see [7], 3.5, for example). Since this follows easily from the original version by use of vanishing theorems, one does not need an analogue of the isomorphism above for this situation.

But as an application, we showed in [7] a geometric version of Dyson’s Lemma, which was essentially achieved by showing weak positivity of a certain sheaf over some product open subset. In order to describe this set, one would need to have control over the subset where multiplier ideal sheaves and fibre products are compatible. Moreover, the multiplier ideal sheaves themselves are still not understood very well. For these reasons it is interesting to observe their interaction with fibre products. Whereas multiplier ideal sheaves are compatible with products (see [7], 3.4, for example), we show that they are compatible with fibre products only under certain conditions.

The main result of this paper is the following:

**Theorem 1.1** Let $X$ be a normal variety with at most rational singularities and let $D$ be an effective Cartier divisor on $X$. Let $Y$ be a smooth variety and $f : X \to Y$ a surjective flat morphism. Let us assume that $\omega_{X/Y} \{-D\}$ is locally free. If $X'$ denotes the $r$-fold fibre product of $X$ over $Y$, then for every $N \in \mathbb{N} - \{0\}$ there exists a map

$$f_N : \omega_{X'/Y} \left\{-\frac{\Gamma}{N}\right\} \to \omega_{X/Y} \left\{-\frac{D}{N}\right\} \boxtimes \cdots \boxtimes \omega_{X/Y} \left\{-\frac{D}{N}\right\},$$

where $\Gamma$ is the induced divisor on $X'$.
Moreover, we wish to understand, under which conditions this map is an isomorphism. To this end we construct the map directly, based on the properties of multiplier ideal sheaves. We start with summarizing briefly the main facts about multiplier ideal sheaves we need. In the third section, we construct the required map \( f_N \) and give an upper bound for \( N \) such that \( f_N \) is an isomorphism. In the fourth section we give an example, showing that this bound is not optimal.

## 2 Multiplier Ideal Sheaves

We start with a statement about integral parts of \( \mathbb{Q} \)-divisors.

**Lemma 2.1** Let \( f : X \to Y \) be a surjective morphism of smooth varieties and let \( D \) and \( D' \) be normal crossing divisors on \( Y \) such that \( f^*D \) also has normal crossings. Then for every \( N \in \mathbb{N} - \{0\} \) we have injective maps

\[
\mathcal{O}_Y \left( - \left[ \frac{D + D'}{N} \right] \right) \to \mathcal{O}_Y \left( - \left[ \frac{D}{N} \right] \right) \otimes \mathcal{O}_Y \left( - \left[ \frac{D'}{N} \right] \right)
\]

and

\[
\mathcal{O}_X \left( - \left[ \frac{f^*D}{N} \right] \right) \to f^* \mathcal{O}_Y \left( - \left[ \frac{D}{N} \right] \right).
\]

**Proof:** Let us write \( D = \sum \alpha_i \cdot D_i \) and \( D' = \sum \beta_i \cdot D_i \), where some of the \( \alpha_i \) or \( \beta_i \) may vanish. The first map follows from the inequality

\[
\left[ \frac{D + D'}{N} \right] = \sum_i \left[ \frac{\alpha_i + \beta_i}{N} \right] \cdot D_i \geq \sum_i \left( \left[ \frac{\alpha_i}{N} \right] + \left[ \frac{\beta_i}{N} \right] \right) \cdot D_i = \left[ \frac{D}{N} \right] + \left[ \frac{D'}{N} \right].
\]

Writing \( f^*D = \sum \alpha_i \cdot f^*D_i = \sum \alpha_i \cdot \left( \sum_{i,j} \mu_{ij} \cdot B_{ij} \right) \), we have

\[
f^* \left[ \frac{D}{N} \right] = f^* \sum_i \left[ \frac{\alpha_i}{N} \right] \cdot D_i = \sum_{i,j} \frac{\alpha_i \cdot \mu_{ij}}{N} \cdot B_{ij} \leq \sum_{i,j} \left[ \frac{\alpha_i \cdot \mu_{ij}}{N} \right] \cdot B_{ij} = \left[ \frac{f^*D}{N} \right],
\]

and thus obtain the second map. \( \square \)

Now we come to the definition of multiplier ideal sheaves, following the line of [6].

**Definition 2.2** Let \( X \) be a normal variety with at most rational singularities and let \( \Gamma \) be an effective Cartier divisor on \( X \). Let \( \tau : X' \to X \) be a blowing up such that both \( X' \) is smooth and \( \Gamma' = \tau^* \Gamma \) has normal crossings. For every \( N \in \mathbb{N} - \{0\} \) let us define

\[
\omega_X \left\{ - \frac{\Gamma}{N} \right\} = \tau_* \omega_{X'} \left( - \left[ \frac{\Gamma'}{N} \right] \right).
\]

The sheaf \( \omega_X \left\{ - \frac{\Gamma}{N} \right\} \otimes \omega_X^{-1} \) (or sometimes the sheaf \( \omega_X \left\{ - \frac{\Gamma}{N} \right\} \) itself) is called the multiplier ideal sheaf associated to \( D \). Due to [6], 5.10, this definition does not depend on the chosen blowing up. Moreover, we define

\[
\mathcal{C}_X(\Gamma, N) = \text{coker} \left( \omega_X \left\{ - \frac{\Gamma}{N} \right\} \longrightarrow \omega_X \right).
\]
Since \( \omega_X = \tau_\omega X \), we have \( C_X(\Gamma, N) = 0 \) for \( N \) sufficiently large, so we define
\[
e(\Gamma) = \min \left\{ N \in \mathbb{N} \setminus \{0\} ; C_X(\Gamma, N) = 0 \right\}.
\]

**Remark 2.3** It is sometimes convenient to consider rational numbers as denominators instead of integers (and in the fourth section we shall in fact do this) and to replace minimum by infimum. Obviously, all the properties of \( e(\Gamma) \), given in [6], for example, then still hold.

**Lemma 2.4** We have the following properties.

(a) Given any birational morphism \( \delta : Z \to X \) of normal varieties with at most rational singularities and any effective Cartier divisor \( \Gamma \) on \( X \), we have
\[
\delta_* \omega_Z \left\{-\frac{\delta^* \Gamma}{N}\right\} = \omega_X \left\{-\frac{\Gamma}{N}\right\}.
\]

(b) If \( X \) is smooth and \( \Gamma \) is a normal crossing divisor on \( X \), then
\[
\omega_X \left\{-\frac{\Gamma}{N}\right\} = \omega_X \left(-\left[\frac{\Gamma}{N}\right]\right).
\]

Moreover, for \( \Gamma = \sum \nu_i \cdot \Gamma_i \) with smooth reduced divisors \( \Gamma_i \), we have \( e(\Gamma) = \max\{\nu_i\} + 1 \).

**Proof:** For (a) we may choose a blowing up \( \tau : X' \to X \) in such a way that \( X' \) is smooth and \( \tau^* \Gamma \) is a normal crossing divisor on \( X' \) and that in addition \( \tau \) factors over \( \delta \). For (b) see [6], 5.10.

## 3 Proof of 1.1

In this section we shall construct the map \( f_N \) from 1.1. In fact, we even show it for the more general case of different factors.

**Lemma 3.1** Let \( f : X \to Y \) be a surjective flat morphism of normal varieties with at most rational singularities. Let \( D \) be an effective Cartier divisor on \( Y \) and \( N \in \mathbb{N} \setminus \{0\} \). Then there exists a map
\[
\omega_X \left\{-\frac{f^* D}{N}\right\} \to f^* \omega_Y \left\{-\frac{D}{N}\right\} \otimes \omega_{X/Y}.
\]

**Proof:** If \( X \) and \( Y \) are smooth and if \( D \) and \( f^* D \) are normal crossing divisors, then by 2.1, there exists a map
\[
\mathcal{O}_X \left(-\left[\frac{f^* D}{N}\right]\right) \to f^* \mathcal{O}_Y \left(-\left[\frac{D}{N}\right]\right),
\]
inducing a map
\[
\omega_X \left(-\left[\frac{f^* D}{N}\right]\right) \to f^* \omega_Y \left(-\left[\frac{D}{N}\right]\right) \otimes \omega_{X/Y}.
\]
For the general case, let $\tau : Y' \to Y$ be a blowing up such that $Y'$ is smooth and $D' = \tau^*D$ is a normal crossing divisor. We consider the fibre product $X' = Y' \times_Y X$ together with the induced maps $f' : X' \to Y'$ and $\tau' : X' \to X$. Let $\delta : Z \to X'$ be a desingularization such that $\delta^*f'^*D'$ is a normal crossing divisor, too. Thus we have

$$\omega_X \left( - \frac{f^*D}{N} \right) = \tau'^* \delta_\omega Z \left( - \left[ \frac{\delta^* \tau'^* f'^* D}{N} \right] \right) = \tau'^* \delta_\omega Z \left( - \left[ \frac{\delta f'^* D'}{N} \right] \right),$$

and according to the first case there exists a map

$$\omega_Z \left( - \left[ \frac{\delta f'^* D'}{N} \right] \right) \to \delta^* f'^* \omega_{Y'}, \left( - \left[ \frac{D'}{N} \right] \right) \otimes \omega_{Z/Y'},$$

for $Z$ is smooth and $D'$ and $\delta f'^* D$ are normal crossing divisors. Thus we obtain a map

$$\omega_X \left( - \frac{f^*D}{N} \right) \to \tau'^* \delta_\omega \left( \frac{\delta f'^* \omega_{Y'}}{\frac{D'}{N}} \right) \otimes \omega_{Z/Y'} = \tau'^* \left( f'^* \omega_Y \left( - \left[ \frac{D'}{N} \right] \right) \otimes \delta_\omega \omega_{Z/Y'} \right).$$

Now by considering the trace map $\delta_\omega \omega_{Z/Y'} \to \omega_{X'/Y'}$, the compatibility of the relatively canonical sheaf with base change and the projection formula, we obtain maps

$$\omega_X \left( - \frac{f^*D}{N} \right) \to \tau'^* \left( f'^* \omega_{Y'} \left( - \left[ \frac{D'}{N} \right] \right) \otimes \omega_{X'/Y'} \right) = \tau'^* \left( f'^* \omega_{Y'} \left( - \left[ \frac{D'}{N} \right] \right) \otimes \tau'^* \omega_{X/Y} \right) = \tau'^* f'^* \omega_{Y'} \left( - \left[ \frac{D'}{N} \right] \right) \otimes \omega_{X/Y} = f'^* \omega_Y \left( - \left[ \frac{D'}{N} \right] \right) \otimes \omega_{X/Y} = f'^* \left( - \frac{D'}{N} \right) \otimes \omega_{X/Y},$$

where the last but one equality is implied by flat base change (see [3], III.9.3). \hfill \Box

Now we are able to prove 1.1. To this end, let us consider a slightly more general situation. Assume we are given normal varieties $X_1, \ldots, X_r$ with at most rational singularities. Assume in addition we are given, for $i = 1, \ldots, r$, effective Cartier divisors $D_i$ on $X_i$ and surjective flat morphisms $f_i : X_i \to Y$ onto a smooth variety $Y$. Let

$$X' = X_1 \times_Y \cdots \times_Y X_r$$

be the fibre product together with the projections $p_i : X' \to X_i$. Let

$$\Gamma = D_1 \boxplus \cdots \boxplus D_r = \sum p_i^{-1}(D_i)$$

be the induced divisor on $X'$.

**Claim 3.2** If all the sheaves $\omega_{X_i/Y} \left\{ - \frac{D_i}{N} \right\}$ (except possibly one) are locally free, then for every $N \in \mathbb{N} - \{0\}$ there exists a map

$$\omega_{X'/Y} \left\{ - \frac{\Gamma}{N} \right\} \to \omega_{X_1/Y} \left\{ - \frac{D_1}{N} \right\} \boxplus \cdots \boxplus \omega_{X_r/Y} \left\{ - \frac{D_r}{N} \right\}.$$

**Proof:** It suffices to consider the case $r = 2$. We choose a desingularization $\tau : Z \to X'$ of the fibre product and denote by $\pi_1 : Z \to X_1$ and $\pi_2 : Z \to X_2$ the induced projections. We have to consider three cases.
Let us start with the case where $X_1$ and $X_2$ are smooth and $D_1$ and $D_2$ are normal crossing divisors. We may assume $\tau : Z \to X'$ to be chosen in such a way that $\tau^* \Gamma$ and $\pi_1^*D_1$ and

$$
\omega_{X'/Y} \left\{ -\frac{\Gamma}{N} \right\} = \tau_* \omega_{Z/Y} \left( -\left[ \frac{\tau^* \Gamma}{N} \right] \right) = \tau_* \omega_{Z/Y} \left( -\left[ \frac{\pi_1^*D_1 + \pi_2^*D_2}{N} \right] \right),
$$

and by 2.1 there exists an injective map

$$
\mathcal{O}_Z \left( -\left[ \frac{\pi_1^*D_1 + \pi_2^*D_2}{N} \right] \right) \to \pi_1^* \mathcal{O}_{X_1} \left( -\left[ \frac{D_1}{N} \right] \right) \otimes \pi_2^* \mathcal{O}_{X_2} \left( -\left[ \frac{D_2}{N} \right] \right),
$$

such that we obtain a map

$$
\omega_{X'/Y} \left\{ -\frac{\Gamma}{N} \right\} \to \tau_* \left( \omega_{Z/Y} \otimes \pi_1^* \mathcal{O}_{X_1} \left( -\left[ \frac{D_1}{N} \right] \right) \otimes \pi_2^* \mathcal{O}_{X_2} \left( -\left[ \frac{D_2}{N} \right] \right) \right)
$$

By the trace map $\tau_* \omega_Z \to \omega_X$, we obtain

$$
\omega_{X'/Y} \left\{ -\frac{\Gamma}{N} \right\} \to \omega_{X'/Y} \otimes \left( \mathcal{O}_{X_1} \left( -\left[ \frac{D_1}{N} \right] \right) \boxtimes \mathcal{O}_{X_2} \left( -\left[ \frac{D_2}{N} \right] \right) \right)
$$

and, since $\omega_{X'/Y} = \omega_{X_1/Y} \boxtimes \omega_{X_2/Y}$, there exists a map

$$
\omega_{X'/Y} \left\{ -\frac{\Gamma}{N} \right\} \to \omega_{X_1/Y} \left( -\left[ \frac{D_1}{N} \right] \right) \boxtimes \omega_{X_2/Y} \left( -\left[ \frac{D_2}{N} \right] \right)
$$

Let us now consider the case where only one factor, say $X_2$, is smooth and only $D_2$ is a normal crossing divisor. In this case we only obtain a map

$$
\omega_{X'/Y} \left\{ -\frac{\Gamma}{N} \right\} \to \tau_* \left( \omega_{Z/Y} \otimes \mathcal{O}_Z \left( -\left[ \frac{\pi_1^*D_1}{N} \right] \right) \otimes \pi_2^* \mathcal{O}_{X_2} \left( -\left[ \frac{D_2}{N} \right] \right) \right)
$$

By applying 3.1 to $p_1 : X' \to X_1$ and $D_1$, we obtain a map

$$
\omega_{X'} \left\{ -\frac{p_1^*D_1}{N} \right\} \to p_1^* \omega_{X_1} \left\{ -\frac{D_1}{N} \right\} \otimes \omega_{X'/X_1},
$$

inducing a map

$$
\omega_{X'/Y} \left\{ -\frac{\Gamma}{N} \right\} \to p_1^* \omega_{X_1/Y} \left\{ -\frac{D_1}{N} \right\} \otimes \omega_{X'/X_1} \otimes p_2^* \mathcal{O}_{X_2} \left( -\left[ \frac{D_2}{N} \right] \right)
$$

$$
= \omega_{X_1/Y} \left\{ -\frac{D_1}{N} \right\} \boxtimes \omega_{X_2/Y} \left( -\left[ \frac{D_2}{N} \right] \right)
$$

$$
= \omega_{X_1/Y} \left\{ -\frac{D_1}{N} \right\} \boxtimes \omega_{X_2/Y} \left\{ -\frac{D_2}{N} \right\}.
$$
In the remaining case, none of the varieties is smooth and none of the divisors has normal crossings, but we may assume that, for example, $\omega_{X_1/Y}\left\{-\frac{D_1}{N}\right\}$ is locally free. We choose a blowing up $\sigma : X'_2 \to X_2$ such that $X'_2$ is smooth and $D'_2 = \sigma^*D_2$ is a normal crossing divisor. Moreover, we consider the fibre product $X'' = X' \times_{X_2} X'_2 = X_1 \times_Y X'_2$ together with the induced maps $p'_1 : X'' \to X'$ and $p'_2 : X'' \to X'_2$. Then the second case implies the existence of a map

$$\omega_{X''/Y} \left\{-\frac{D_1 \oplus D'_2}{N}\right\} \to \omega_{X_1/Y} \left\{-\frac{D_1}{N}\right\} \boxtimes \omega_{X'_2/Y} \left\{-\frac{D'_2}{N}\right\} = p'_1 \ast p'_1 \ast \omega_{X_1/Y} \left\{-\frac{D_1}{N}\right\} \boxtimes p'_2 \ast \omega_{X'_2/Y} \left( -\left\lceil \frac{D'_2}{N} \right\rceil \right).$$

By 2.4 we have

$$p'_1 \ast \omega_{X''/Y} \left\{-\frac{D_1 \oplus D'_2}{N}\right\} = \omega_{X'/Y} \left\{-\frac{\Gamma}{N}\right\},$$

therefore we obtain a map

$$\omega_{X'/Y} \left\{-\frac{\Gamma}{N}\right\} \to p'_1 \ast \left( p'_1 \ast \omega_{X_1/Y} \left\{-\frac{D_1}{N}\right\} \boxtimes p'_2 \ast \omega_{X'_2/Y} \left( -\left\lceil \frac{D'_2}{N} \right\rceil \right) \right).$$

Since $\omega_{X_1/Y} \left\{-\frac{D_1}{N}\right\}$ is locally free, we obtain, by projection formula, a map

$$\omega_{X'/Y} \left\{-\frac{\Gamma}{N}\right\} \to p'_1 \ast \omega_{X_1/Y} \left\{-\frac{D_1}{N}\right\} \boxtimes p'_2 \ast \omega_{X'_2/Y} \left( -\left\lceil \frac{D'_2}{N} \right\rceil \right).$$

Using flat base change ([3], III.9.3), this yields a map

$$\omega_{X'/Y} \left\{-\frac{\Gamma}{N}\right\} \to p'_1 \ast \omega_{X_1/Y} \left\{-\frac{D_1}{N}\right\} \boxtimes p'_2 \ast \sigma \ast \omega_{X'_2/Y} \left( -\left\lceil \frac{D'_2}{N} \right\rceil \right) = p'_1 \ast \omega_{X_1/Y} \left\{-\frac{D_1}{N}\right\} \boxtimes p'_2 \ast \omega_{X'_2/Y} \left\{-\frac{D'_2}{N}\right\} = \omega_{X_1/Y} \left\{-\frac{D_1}{N}\right\} \boxtimes \omega_{X_2/Y} \left\{-\frac{D'_2}{N}\right\}.$$ 

This proves 3.2, and considering the special case where all the factors are the same, we obtain the map $f_N$ from 1.1. \qed

As mentioned in the introduction, the case where the map $f_N$ from 1.1 is an isomorphism is of special interest. This case can be approximated from two directions. First one could try, having fixed $N \in \mathbb{N} - \{0\}$, to determine the open subset where $f_N$ is an isomorphism. On the other hand, an optimal choice for $N$ is interesting, which means the determination of a minimal $N \in \mathbb{N} - \{0\}$ such that $f_N$ is an isomorphism on $X'$. 

**Lemma 3.3** Under the assumptions from 1.1, let in addition all the fibres $X_y = f^{-1}(y)$ be normal varieties, not contained in $D$, and with at most rational singularities. Let for every $N \in \mathbb{N} - \{0\}$

$$U_N = \left\{ y \in Y ; e(D|_{X_y}) \leq N \right\},$$

which by the semicontinuity of $e$ on the fibres ([6], 5.14) is an open subset of $Y$. Then $f_N$ is an isomorphism over $(f')^{-1}(U_N)$, where $f' : X' \to Y$ is the structure map.
Proof: By [6], 5.14 we have \( e(D|_{f^{-1}(U_N)}) \leq N \). On the other hand, for \( y \in U_N \) the \( r \)-fold product of the fibre \( X_y \) is exactly the fibre of \( y \) under the structure map \( f^r : X^r \to Y \). By the compatibility of multiplier ideal sheaves with products (see [7], 3.4), we have for all \( y \in U_N \)
\[
e(\Gamma|_{(f^r)^{-1}(y)}) = e(D|_{X_y}) \leq N.
\]
Again by [6], 5.14 we have \( e(\Gamma|_{(f^r)^{-1}(U_N)}) \leq N \), and so, over \((f^r)^{-1}(U_N)\), \( f_N \) turns out to be the isomorphism \( \omega_{X^r/Y} \to \omega_{X/Y} \otimes \cdots \otimes \omega_{X/Y} \).

We should remark that the open subset where \( f_N \) is an isomorphism may be larger than \((f^r)^{-1}(U_N)\) and, in fact, may be difficult to describe. As an example, it is hardly possible to keep control over the product open subset in Dyson’s Lemma ([7], 1.1) where weak positivity holds.

On the other hand, for \( N \) sufficiently large, \( f_N \) will turn out to be the well-known isomorphism \( \omega_{X^r/Y} \to \omega_{X/Y} \otimes \cdots \otimes \omega_{X/Y} \). Therefore the question arises, whether one can determine a minimal \( N \) such that \( f_N \) is an isomorphism. One certainly has the following upper bound, given by 3.3.

**Corollary 3.4** For \( M = \max \{ e(D|_{X_y}) ; y \in Y \} \), the map \( f_M \) from 1.1 is an isomorphism.

There are cases where the bound given in 3.4 is not optimal. Sometimes one may take a smaller value for \( N \), since single fibres with a higher value for \( e \) do not disturb. Thus restricting \( D \) to the general fibre and taking the value of \( e \) may give a better upper bound for the minimal \( N \), which, however, still does not need to be optimal.

4 An Example

We shall now give an example where the bound given in 3.4 is not the optimal one. To this end we consider \( X = \mathbb{A}^2_k \) with coordinates \((x,y)\), \( Y = \mathbb{A}^1_k \) and we take \( f : X \to Y \) to be the projection onto the first factor. Let \( D \) be the smooth reduced divisor on \( X \) defined by \( y^2 - x \). For \( r \in \mathbb{N} - \{0\} \) the \( r \)-fold fibre product is given by \( X^r = \mathbb{A}^{r+1}_k \) with coordinates \((x,y_1,\ldots,y_r)\), where the induced divisor \( \Gamma \) is given by \((y_i^2 - x) \cdots (y_r^2 - x)\).

Since \( D \) is smooth, irreducible and reduced, we have \( e(D) = 2 \) by 2.4, and this holds if one restricts \( D \) to the general fibre of \( f \), since one obtains two separated points. But \( D|_{f^{-1}(0)} \) is a double point, which implies \( e(D|_{f^{-1}(0)}) = 3 \). Thus we obtain \( M = \max \{ e(D|_{X_y}) ; y \in Y \} = 3 \) here, and so \( f_3 \) is an isomorphism by 3.4.

Considering the situation on \( X^r \), we observe that all the components of \( \Gamma \) are smooth, yet \( \Gamma \) is not a normal crossing divisor. Therefore we have to construct a blowing up \( \tau : X^{r'} \to X^r \) such that \( \Gamma' = \tau^* \Gamma \) is a normal crossing divisor. We start with the blowing up \( \tau_1 : X_0 \to X^r \) of the zero point, which is contained in all the \( r \) components of \( \Gamma \). Since the equation of \( \Gamma \) is symmetric in the coordinates \( y_i \), we only have to consider two cards of the blowing up:

On the card \( x \neq 0 \), we find out that \( \tau^*_1 \Gamma \) is given by the equation
\[
x^{r'} \cdot (z_1^2 x - 1) \cdots (z_r^2 x - 1),
\]
where \( z_i x = y_i \) for \( i = 1, \ldots, r \). Here the exceptional divisor \( E_1 \), which is given by \( x^r = 0 \), does not intersect the strict transforms of the components.

Considering one of the other cards, say \( y_1 \neq 0 \), one obtains that \( \tau_1^* \Gamma \) is defined by
\[
y_1^r \cdot (y_1 - w_0) \cdot (w_2^2 y_1 - w_0) \cdots (w_r^2 y_1 - w_0),
\]
where \( w_0 y_1 = x \) and \( w_i y_1 = y_i \) for \( i = 2, \ldots, r \). Here we have
\[
\tau_1^* \Gamma = r \cdot E_1 + \Gamma_1 + \cdots + \Gamma_r.
\]
Thus \( E_1 \) does intersect the components \( \tilde{\Gamma} = \sum_{i=1}^r \tilde{\Gamma}_i \) in a copy of \( \mathbb{P}^{r-1}_k \), given by coordinates \( (w_2, \ldots, w_r) \). Obviously, \( \tau_1^* \Gamma \) is still no normal crossing divisor, and we have to continue blowing up. Let \( \tau_2 : X^{r'} \to X_0^r \) be the blowing up with this copy of \( \mathbb{P}^{r-1}_k \) as centre, and we consider the card \( w_0 \neq 0 \). Writing \( \beta_1 w_0 = y_1 \), the equation of \( \tau_1^* \tau_2^* \Gamma \) on this card has the form
\[
\beta_1^r \cdot w_0^2 \cdot (\beta_1 - 1) \cdot (w_2^2 \beta_1 - 1) \cdots (w_r^2 \beta_1 - 1).
\]
If \( \tau : X^{r'} \to X^r \) denotes the composed blowing up, we obtain
\[
\Gamma' = \tau^* \Gamma = 2r \cdot E_2 + r \cdot E'_1 + \tilde{\Gamma},
\]
where \( E_2 \) is the exceptional divisor of \( \tau_2 \), defined by \( w_0^{2r} = 0 \), \( E'_1 \) is the strict transform of \( E_1 \) under \( \tau_2 \), defined by \( \beta_1^r = 0 \), and \( \tilde{\Gamma} \) is the union of the strict transforms of the components of \( \tau_1^* \Gamma \) under \( \tau_2 \). We observe that \( E'_1 \) and \( \tilde{\Gamma} \) do no longer intersect and that the intersection of the component is given by \( \beta_1 = 1 \) and \( w_i^2 = 1 \). So \( \Gamma' \) is a normal crossing divisor on \( X^{r'} \), and we are done.

**Claim 4.1** For the canonical sheaf one has \( \omega_{X^{r'}} = \tau^* \omega_{X^r} \otimes \mathcal{O}_{X^{r'}}(r \cdot E'_1 + (r + 1) \cdot E_2). \)

**Proof:** The centre of \( \tau_1 \) has codimension \( r+1 \), thus we have \( \omega_{X_0^r} = \tau_1^* \omega_{X^r} \otimes \mathcal{O}_{X_0^r}(r \cdot E_1) \).
Since \( \tau_2 \) has centre in codimension 2, we have \( \omega_{X^{r'}} = \tau_2^* \omega_{X_0^r} \otimes \mathcal{O}_{X^{r'}}(E_2) \), thus we obtain
\[
\omega_{X^{r'}} = \tau_2^* (\tau_1^* \omega_{X^r} \otimes \mathcal{O}_{X_0^r}(r \cdot E_1)) \otimes \mathcal{O}_{X^{r'}}(E_2) = \tau^* \omega_{X^r} \otimes \mathcal{O}_{X^{r'}}(\tau_2^* (r \cdot E_1) + E_2) = \tau^* \omega_{X^r} \otimes \mathcal{O}_{X^{r'}}(r \cdot E'_1 + r \cdot E_2 + E_2).
\]

\[\Box\]

Now by 4.1 and by the description of \( \Gamma' \) we obtain
\[
\omega_{X^r} \left\{ -\frac{\Gamma}{N} \right\} = \omega_{X^r} \otimes \tau_* \mathcal{O}_{X^{r'}} \left( \left( r - \left[ \frac{r}{N} \right] \right) \cdot E'_1 + \left( r + 1 - \frac{2r}{N} \right) \cdot E_2 \right).
\]
This gives the following condition for \( N \). One has \( \omega_{X^r} \left\{ -\frac{\Gamma}{N} \right\} = \omega_{X^r} \) if and only if
\[
r \geq \left[ \frac{r}{N} \right] \quad \text{and} \quad r + 1 \geq \left[ \frac{2r}{N} \right].
\]
But this happens if and only if \( r + 2 > \frac{2r}{N} \), hence if \( N > \frac{2r}{r+2} \). Since the number \( r \) of factors is arbitrary, we finally obtain that \( \omega_{X^r} \left\{ -\frac{\Gamma}{N} \right\} = \omega_{X^r} \) if and only if \( N \geq 2 \).
Since $\omega_{X/Y} \left\{-\frac{D}{N}\right\} = \omega_{X/Y}$, we obtain that $f_N$ is an isomorphism for $N \geq 2$, which shows that $M = 3$ is here not an optimal bound. Considering rational denominators $(2,3)$, this equation still holds after replacing 2 by $1 + \varepsilon$ for some positive rational number $\varepsilon$. On the other hand, taking arbitrary $r$, the calculation above shows that $\omega_{X/r} \left\{-\frac{D}{N}\right\} = \omega_{X/r}$ does no longer hold for $N < 2$. Thus $N = 2$ is the minimal number such that $f_N$ is an isomorphism, and for this example, this coincides with the value of $e$ on the general fibre.

References


Markus Wessler, Universität Kassel, Fachbereich 17, Mathematik und Informatik
Heinrich-Plütt-Str. 40, D-34132 Kassel

e-Mail: wessler@mathematik.uni-kassel.de