Explicit formulas for concatenations of arithmetic progressions

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Abstract

The sequence $(Sm(n))_{n\geqslant 0}$: 1, 12, 123, $\ldots$ formed by concatenating the first $n+1$ positive integers is often called Smarandache consecutive numbers. We consider the more general case of concatenating arithmetic progressions and establish formulas to compute them. Three types of concatenation are taken into account: the right-concatenation like $(Sm(n))_{n\geqslant 0}$ or the concatenation of odd integers: 1, 13, 135, $\ldots$; the left-concatenation like the reverse of Smarandache consecutive numbers $(Smr(n))_{n\geqslant 0}$: 1, 21, 321, $\ldots$; and the concatenation of right-concatenation and left-concatenation like 1, 121, 12321, 1234321,$\ldots$, formed by $Sm(n)$ and $Smr(n-1)$ for $n\geqslant 1$, with the initial term $Sm(0)$. The resulting formulas enable fast computations of asymptotic terms of these sequences. In particular, we use our implementation in the Computer Algebra System Maple to compute billionth terms of $(Sm(n))_{n\geqslant 0}$ and $(Smr(n))_{n\geqslant 0}$.

Keywords: Smarandache sequences, generating functions, hypergeometric terms, holonomic recurrence equations
AMS subject Classification: 11B83, 11Y55, 68W30, 11-04.

1. Introduction

"*The scientist does not study nature because it is useful to do so. He studies it because he takes pleasure in it, and he takes pleasure in it because it is beautiful*"– Henri Poincaré

Suppose we want to find the function $f(z)$ whose series expansion about the origin starts

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as follows:

$$f(z)=\sum_{n=0}^{8}\overline{12\ldots(n+1)}z^n+\cdots=1+12z+123z^2+\cdots+123456789z^8+\cdots. \quad (1)$$

A common technique used in Computer Algebra is the so-called Guessing that tries to compute f_1 as a solution of a differential equation (mostly linear) built from the first coefficients in (1). There are many existing packages to perform Guessing. For instance, one can use the Maple `gfun` package of Salvy and Zimmermann [5]; the Mathematica `GeneratingFunctions` package of Mallinger [3]; or the Sage `ore_algebra` package [2] of Kauers, Jaroschek, and Johansson. In this paper, we use the Computer Algebra System (CAS) Maple and the package `gfun` [1] . Using the command `listodiffeq` of `gfun` we find the differential equation

$$\left(-10z^2+11z-1\right)\left(\frac{d}{dz}y(z)\right)+(-30z+12)\,y(z)=0, \quad (2)$$

which with the initial value $y(0)=1$, leads to the rational function

$$f_1(z):=-\frac{1}{(10z-1)(z-1)^2}. \quad (3)$$

The latter has a Maclaurin series that starts as in (1), but continues in an *unexpected* manner. Indeed, we have

$$\frac{\left(\frac{d^9}{dz^9}f_1(z)\right)_{|(z=0)}}{9!}=1234567900,\quad \frac{\left(\frac{d^{10}}{dz^{10}}f_1(z)\right)_{|(z=0)}}{10!}=12345679011, \quad (4)$$

preventing f_1 from continuing with the concatenation of positive integers. This is another example that shows the risk of generalizing with only few constraints. However, we observe that the series coefficients of f_1 diverge from the sequence of concatenation of positive integers at the first 2-digit concatenation, indicating an impossibility to find the generating function of these coefficients by using linear equations[2] . Alternatively, one can explore the 2-digit concatenation with the same strategy and look for similarities with the 1-digit concatenation.

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[1]Latest version available at `https://perso.ens-lyon.fr/bruno.salvy/software/the-gfun-package/`

[2]The `gfun` package gives no result with more than 9 initial coefficients in (1).

Let $g(z)$ be the function with the following Maclaurin truncated expansion

$$\begin{aligned} g(z) &= \sum_{n=0}^{10} \overline{12\ldots(n+10)} z^n + \cdots \\ &= 12345678910 + \cdots + 12345678910111213141516171819 z^9 + \cdots. \end{aligned} \tag{5}$$

This time, we proceed with Guessing approach based on the search of a quadratic differential equation (see [9]), though a valid first-order differential equation (containing some larger integers) can be computed with `gfun`. We find

$$\left(150z^2 - 201z + 51\right) y(z)^2 + \left(-\frac{24691357809}{2} + 12345678900z\right) y(z) + \left(-6172839455 + \frac{24691357809}{2} z - 6172839450z^2\right) \left(\frac{d}{dz} y(z)\right) = 0, \tag{6}$$

with the initial condition $y(0) = 12345678910$. Therefore we get the function

$$g_1(z) := -\frac{12345678900z^2 - 24691357809z + 12345678910}{(100z - 1)(z - 1)^2}. \tag{7}$$

As expected, $g_1(z)$ continues all 2-digit concatenations accurately and fails at the first 3-digit concatenation. Following our experiment, it remains to see whether g_1 and f_1 present a similarity that may enable us to see a pattern that generalizes the concatenation of any digit length. For that purpose, we compute the power series representations of f_1 and g_1 using the symbolic method from [12, 11, 8][3] . This yields

$$\begin{aligned} f_1(z) &:= \sum_{n=0}^{\infty} \left(\frac{100\,10^n}{81} - \frac{n}{9} - \frac{19}{81} \right) z^n, \\ g_1(z) &:= \sum_{n=0}^{\infty} \left(\frac{120999998998000\,100^n}{9801} - \frac{n}{99} - \frac{1090}{9801} \right) z^n. \end{aligned} \tag{8}$$

It may seem not straightforward to see the pattern from (8), however, this becomes clearer with the result obtained for the 3-digit concatenation. Our investigation comes from observing some intermediate steps of the power series computation used above. In this paper, we give precise positive answers to the questions:

[3]Download latest Maple and Maxima implementations at `http://www.mathematik.uni-kassel.de/~bteguia/FPS_webpage/FPS.htm`.

1. Given a positive integer l, what is the recurrence equation for the l-digit concatenation?
2. What is the solution of that recurrence equation and how does it relate to the l-digit concatenation?
3. Does this lead to a formula to compute any concatenation of natural numbers?

We answer these questions for natural numbers as a particular case of the concatenation of arithmetic progressions.
Throughout this paper, the sequence $(Sm(n))_{n\geqslant 0}$: $1, 12, 123, \ldots$, formed by concatenating the first $n+1$ positive integers denotes Smarandache numbers, and the sequence $(Smr(n))_{n\geqslant 0}$: 1, 21, 321, ..., denotes reverse Smarandache numbers (see [7, 6]). We use the word 'right-concatenation' for concatenations like that of $(Sm(n))_{n\geqslant 0}$, and 'left-concatenation' for those like that of $(Smr(n))_{n\geqslant 0}$. Depending on the context, 'right' and 'left' could be omitted. Our aim is to find explicit formulas for three types of concatenation of arithmetic progressions: the right-concatenation, the left-concatenation, and the concatenation of right-concatenations and left-concatenations that combines $(Sm(n))_{n\geqslant 0}$ and $(Smr(n))_{n\geqslant 0}$ to build the sequence 1, 121, 12321, 1234321,... formed by $Sm(n)$ and $Smr(n-1)$ for $n \geqslant 1$, with the initial term $Sm(0)$. Thus a formula to compute $Sm(n)$ or $Smr(n)$ will be deduced as a particular case. The most challenging question about Smarandache numbers concerns the primality of $Sm(n)$. It is verified that all Smarandache numbers $Sm(n)$ for $n \leqslant 344869$ are composite. The web page `https://www.mersenneforum.org/showthread.php?t=20527&page=9` has a comment from Serge Balatov that seems to say that the search reached 10^6 without finding a prime. Our results simplify the step of computing $Sm(n)$ for large n and help theoretical approaches attempting to prove or disprove the primality of $Sm(n)$ by explaining its formula. For recent history about Smarandache numbers visit OEIS A007908, which at the time of writing this article, illustrates the lack of an explicit general formula to efficiently compute Smarandache concatenated type sequences. The only reference we found with a connection to Computer Algebra is the work in [13] that essentially explores these sequences with naive implementations.
In (6), we used a quadratic differential equation for 2-digit concatenation. One might wonder if the same strategy can lead to a generating function that combines 1-digit and 2-digit concatenations. This appears to be possible when the degree of polynomial coefficients sought is raised to five. Unfortunately, the resulting second-order differential equation contains very large integers that dwarf every chance to find a suitable formula for $Sm(n)$ in this manner.
We shall focus here on finding formulas for fixed-length concatenation and deduce an explicit formula for $Sm(n)$. In Section 2, we establish recurrence equations encoding

fixed-length concatenations. In Section 3, we solve those recurrence equations and deduce formulas associated with any fixed-length concatenation. This will lead us to formulas for $Sm(n)$ and $Smr(n)$ and some other Smarandache sequences (see [7, 6]). Section 4 presents an algorithm to compute $Sm(n)$. The one to compute $Smr(n)$ can be deduced easily. Some computations in the CAS Maple are also presented.

2. Recurrence equations for concatenations of arithmetic progressions

A holonomic recurrence equation (RE) is a linear homogeneous recurrence equation with polynomial coefficients. Although all the recurrence equations of this paper have constant coefficients, we use this general setting as an introduction to the solver used in Section 3, which still effectively applies if polynomial coefficients appear in such a case study. As observed in the introduction, from the use of the author's algorithm in [12], the formulas to compute Smarandache numbers of a fixed digit length right-concatenation are obtained as solutions of holonomic REs. Throughout this section, the index variable $a(n)$ will be used for arbitrary holonomic REs. Using the `gfun` package [5] (procedure `listtorec`) with enough initial coefficients, we observe the following:

1. The holonomic RE corresponding to the 1-digit right-concatenation is

$$a(n+3) - 12 \cdot a(n+2) + 21 \cdot a(n+1) - 10 \cdot a(n) = 0 \tag{9}$$

2. The holonomic RE for the 2-digit right-concatenation is

$$a(n+3) - 102 \cdot a(n+2) + 201 \cdot a(n+1) - 100 \cdot a(n) = 0 \tag{10}$$

3. The holonomic RE for the 3-digit right-concatenation is

$$a(n+3) - 1002 \cdot a(n+2) + 2001 \cdot a(n+1) - 1000 \cdot a(n) = 0 \tag{11}$$

Therefore we are led to prove the following lemma.

Lemma 1. *Let l be a positive integer. Smarandache numbers of l-digit right-concatenation satisfy the holonomic recurrence equation*

$$a(n+3) - (10^l + 2) \cdot a(n+2) + (2 \cdot 10^l + 1) \cdot a(n+1) - 10^l \cdot a(n) = 0. \tag{12}$$

Proof. Assume $10^{l-1} - 1 \leqslant n \leqslant 10^l - 2$. Let N be the number occupying the last l digits of $Sm(n+1)$. We have the relationships

$$\begin{aligned} Sm(n+1) &= 10^l \cdot Sm(n) + N, & (13)\\ Sm(n+2) &= 10^l \cdot Sm(n+1) + N + 1, & (14)\\ Sm(n+3) &= 10^l \cdot Sm(n+2) + N + 2. & (15) \end{aligned}$$

From 13 we get $N = Sm(n+1) - 10^l Sm(n)$, and from 14, $1 = Sm(n+2) - 10^l Sm(n+1) - N$. Therefore we can write

$$2 \times 1 = 2 \cdot \left(Sm(n+2) - (10^l + 1) \cdot Sm(n+1) + 10^l \cdot Sm(n) \right). \qquad (16)$$

Finally, we substitute 2 in 15 by the right-hand side of 16 and obtain

$$Sm(n+3) = (10^l + 2) \cdot Sm(n+2) - (2 \cdot 10^l + 1) \cdot Sm(n+1) + 10^l \cdot Sm(n), \qquad (17)$$

which shows that Smarandache numbers of the l-digit right-concatenation satisfy 12. □

The above proof can be used as a template to prove that any sequence $(S(n))_{n \geqslant 0}$ formed by the right-concatenation of an arithmetic progression satisfies holonomic REs depending on the digit lengths of the concatenation. The following lemma gives the recurrence equations for right-concatenations and left-concatenations of arithmetic progressions. Note that we ignore the case where the common difference disqualifies l-digit concatenations. Eg: there is no 1-digit concatenation with 10 as the common difference. We assume that the l-digit concatenation involves at least 4 terms in $(S(n))_{n \geqslant 0}$.

Lemma 2. *Let l be a positive integer.*

i. Any l-digit concatenation of an arithmetic progression satisfy the recurrence equation 12.

ii. The recurrence equation for the l-digit left-concatenation of an arithmetic progression is

$$a(n+3) - (2 \cdot 10^l + 1) \cdot a(n+2) + (10^{2l} + 2 \cdot 10^l) \cdot a(n+1) - 10^{2l} \cdot a(n) = 0. \qquad (18)$$

Proof. For i., it suffices to observe that 14 becomes $S(n+2) = 10^l S(n+1) + N + d$, where d denotes the common difference of the arithmetic progression considered. The proof is then straightforward by substituting d and $2d$ similarly as we did for 1 and 2 in the proof of Lemma 1.
For *ii.*, assume $(S(n))_{n \geqslant 0}$ denotes the left-concatenation of an arithmetic progression of common difference d and initial term U_0. Suppose $10^{l-1} - U_0 \leqslant n \cdot d \leqslant 10^l - U_0$. Let N be the first l digit of $S(n+1)$, and p the digit length of $S(n)$. We have the relationships

$$S(n+1) = N \cdot 10^p + S(n), \qquad (19)$$
$$S(n+2) = (N+d) \cdot 10^{l+p} + S(n+1) = 10^l \left(N 10^p \right) + d 10^{l+p} + S(n+1), \qquad (20)$$
$$S(n+3) = (N+2d) \cdot 10^{2l+p} + S(n+2). \qquad (21)$$

From 19, we get $N10^p = S(n+1) - S(n)$, and from 20, $d10^{l+p} = S(n+2) - 10^l \cdot (N10^p) - S(n+1)$. Therefore

$$2 \cdot d \cdot 10^{2l+p} = 2 \cdot 10^l \left(S(n+2) - (10^l+1) \cdot S(n+1) + 10^l \cdot S(n)\right). \tag{22}$$

Finally, by substitution in 21 we get

$$\begin{aligned} S(n+3) &= 10^{2l}(S(n+1) - S(n)) \\ &\quad +2 \cdot 10^l \left(S(n+2) - (10^l+1)S(n+1) + 10^l S(n)\right) + S(n+2) \\ &= (2 \cdot 10^l + 1) \cdot S(n+2) - (10^{2l} + 2 \cdot 10^l) \cdot S(n+1) + 10^{2l} \cdot S(n), \end{aligned} \tag{23}$$

which concludes the proof. □

We can similarly define recurrence equations for sequences formed by concatenating right-concatenations and left-concatenations of arithmetic progressions. A typical example is the sequence $\overline{123\ldots n(n+1)n\ldots 321}$, $n \geqslant 0$: 1, 121, 12321, …, whose 10th term is a prime number.

Lemma 3. *Let l be a positive integer. The recurrence equation for the l-digit concatenation of right-concatenations and left-concatenations of an arithmetic progression is encoded by the recurrence equation*

$$a(n+3) - \left(1 + 10^l + 10^{2l}\right) \cdot a(n+2) + \left(10^l + 10^{2l} + 10^{3l}\right) \cdot a(n+1) - 10^{3l} \cdot a(n) = 0. \tag{24}$$

A technique to prove Lemma 3 easily followed from the proofs of Lemma 2.
The recurrence equations given in the lemmas 2 and 3 encode the main common concatenations used to define sequences from arithmetic progressions. Obviously, several other concatenations can similarly be considered. In particular, one can also define the recurrence equation for sequences like $\overline{123\ldots n(n+1)(n+1)n\ldots 321}$, $n \geqslant 0$: 11, 1221, 123321, …, for which the 10th is also a prime number. However, in this paper we only concentrate on the concatenations of Lemma 2 and Lemma 3.

3. Formulas for concatenations of arithmetic progressions

All recurrence equations of the previous section are third-order linear recurrences. Therefore each of them has three linearly independent solutions. In this section, we solve these recurrence equations by using the algorithm in [10] which implements a variant of van Hoeij's algorithm (see [14, 4])[4]. The formulas for concatenations of arithmetic progressions are deduced as linear combinations of the bases of solutions computed.
As usual, we start with the l-digit right-concatenation that includes $(Sm(n))_{n \geqslant 0}$.

[4] [10] is more adapted in this context because it handles the symbolic coefficients better

Theorem 4. *Let $(S(n))_{n\geqslant 0}$ be a sequence formed by the right-concatenation of an arithmetic progression $(U(n))_{n\geqslant 0}$ of common difference d. Then $S(n)$ can be computed as follows:*

$$S(n) = \alpha_l + \mu_l(n - t_l) + \theta_l 10^{l(n-t_l)} \tag{25}$$

$$l = \lceil \log_{10}(nd + S(0) + 1) \rceil,\ t_l = \left\lfloor \frac{10^{l-1} - S(0)}{d} \right\rfloor \tag{26}$$

$$\alpha_l = -\frac{\left(10^l - 1\right) \cdot U(t_l) + d \cdot 10^l}{\left(10^l - 1\right)^2} \tag{27}$$

$$\mu_l = -\frac{d}{10^l - 1} \tag{28}$$

$$\theta_l = \frac{s_2 - 2 \cdot s_1 + s_0}{\left(10^l - 1\right)^2} \tag{29}$$

$$s_0 = S(t_l),\ s_1 = S(t_l + 1),\ s_2 = S(t_l + 2) \tag{30}$$

.

Proof. The recurrence equation for l-digit right-concatenation is (see Lemma 2)

$$a(n+3) - (2 \cdot 10^l + 1) \cdot a(n+2) + (10^{2l} + 2 \cdot 10^l) \cdot a(n+1) - 10^{2l} \cdot a(n) = 0.$$

Using the algorithm in [10], we find the basis of solutions

$$\left\{1,\ n,\ 10^{ln}\right\}, \tag{31}$$

which can be easily verified. Therefore there exist $\alpha_l, \mu_l, \theta_l$ to compute terms of the l-digit concatenation in $(S(n))_{n\geqslant 0}$ as follows:

$$\alpha_l + \mu_l n + \theta_l 10^{ln}. \tag{32}$$

The constants $\alpha_l, \mu_l, \theta_l$ can be computed by solving the linear system

$$\begin{cases} \alpha_l + \theta_l = s_0 \\ \alpha_l + \mu_l + \theta_l 10^l = s_1 \\ \alpha_l + 2\mu_l + \theta_l 10^{2l} = s_2 \end{cases}, \tag{33}$$

where s_0, s_1, and s_2 correspond to the first three l-digit concatenations in $(S(n))_{n\geqslant 0}$; these are, respectively, $S(t_l)$, $S(t_l + 1)$, and $S(t_l + 2)$, where $t_l = \left\lfloor \left(10^{l-1} - S(0)\right)/d \right\rfloor$, and $l = \lceil \log_{10}(nd + S(0) + 1) \rceil$. Solving (28) yields

$$\alpha_l = \frac{2 \cdot \left(s_1 - 10^l \cdot s_0\right) - \left(s_2 - 10^{2l} \cdot s_0\right)}{\left(10^l - 1\right)^2},\ \mu_l = \frac{s_2 - 10^l \cdot s_1 - \left(s_1 - 10^l \cdot s_0\right)}{10^l - 1}, \tag{34}$$

and θ_l as in (29). The coefficients α_l and μ_l can be further simplified by using properties of the arithmetic progression $(U_n)_{n\geqslant 0}$. It is easy to see that

$$S(t_l+1) - 10^l \cdot S(t_l) = U(t_l) + d \tag{35}$$

$$S(t_l+2) - 10^{2l} \cdot S(t_l) = (U(t_l)+d) \cdot 10^l + U(t_l) + 2 \cdot d \tag{36}$$

$$S(t_l+2) - 10^l \cdot S(t_l+1) = U(t_l) + 2 \cdot d \tag{37}$$

After substitution in (34) we find α_l and θ_l as expected.
Finally, to use 32 as the formula to compute $S(n)$ for all non-negative integers n, we shift the variable index n in the range $[\![0, t_{l+1} - t_l - 1]\!]$ by substituting n by $n - t_l$. □

From Theorem 4, we see that an efficient computation of l-digit concatenations need only compute α_l, μ_l, and θ_l once. For example, for $(Sm(n))_{n\geqslant 0}$, this yields effective formulas for indices in ranges like $[\![10^6-1, 10^7-2]\!]$, $[\![10^7-1, 10^8-2]\!]$, etc.
We mention that the formula of the coefficient θ_l in Theorem 4 can be written in terms of s_0, $U(t_l)$, and d. Nevertheless, it seems more efficient to compute θ_l with s_0, s_1 and s_2, which are respectively computed by the l-digit concatenation of $U(t_l)$ to $S(t_l-1)$, $U(t_l)+d$ to s_0, and $U(t_l)+2d$ to s_1.

Corollary 5. *[Formula for Smarandache numbers] An explicit formula to compute terms of the sequence $(Sm(n))_{n\geqslant 0}$ formed by the concatenation of positive integers is given by*

$$Sm(n) = \alpha_l + \mu_l(n-t_l) + \theta_l 10^{l(n-t_l)} \tag{38}$$

$$l = \lceil \log_{10}(n+2) \rceil, \; t_l = 10^{l-1} - 1 \tag{39}$$

$$\alpha_l = -\frac{10^{2l-1} + 9 \cdot 10^{l-1}}{(10^l-1)^2} \tag{40}$$

$$\mu_l = -\frac{1}{10^l - 1} \tag{41}$$

$$\theta_l = \frac{s_2 - 2 \cdot s_1 + s_0}{(10^l-1)^2} \tag{42}$$

$$s_0 = Sm(t_l), \; s_1 = Sm(t_l+1), \; s_2 = Sm(t_l+2) \tag{43}$$

Proof. Immediate application of Theorem 4 for the concatenation of natural numbers $(n+1)_{n\geqslant 0}$ of common difference 1. □

Let us now give the formula for left-concatenations.

Theorem 6. *Let $(S(n))_{n\geqslant 0}$ be a sequence formed by the left-concatenation of an arithmetic progression $(U(n))_{n\geqslant 0}$ of common difference d. Then $S(n)$ can be computed as follows:*

$$S(n) = \alpha_l + \mu_l \cdot 10^{l(n-t_l)} + \theta_l \cdot (n - t_l) \cdot 10^{l(n-t_l)} \tag{44}$$

$$l = \lceil \log_{10}(nd + S(0) + 1) \rceil,\ t_l = \left\lfloor \frac{10^{l-1} - S(0)}{d} \right\rfloor,\ p_l \equiv \textit{ digit length of } S(t_l) \tag{45}$$

$$\alpha_l = \frac{s_2 - 2 \cdot 10^l \cdot s_1 + 10^{2l} \cdot s_0}{(10^l - 1)^2} \tag{46}$$

$$\mu_l = \frac{\big((10^l - 1) \cdot U(t_l) - d\big) \cdot 10^{p_l}}{\left(10^l - 1\right)^2} \tag{47}$$

$$\theta_l = \frac{d \cdot 10^{p_l}}{10^l - 1} \tag{48}$$

$$s_0 = S(t_l),\ s_1 = S(t_l + 1),\ s_2 = S(t_l + 2) \tag{49}$$

Proof. We solve (see [10]) the corresponding recurrence equation

$$a(n+3) - (2 \cdot 10^l + 1) \cdot a(n+2) + (10^{2l} + 2 \cdot 10^l) \cdot a(n+1) - 10^{2l} \cdot a(n) = 0,$$

and get the basis of solutions

$$\left\{1,\ 10^{l \cdot n},\ n \cdot 10^{l \cdot n}\right\}. \tag{50}$$

We proceed similarly as in the proof of Theorem 4 to find the expected formulas. One should notice that

$$S(t_l + 2) - S(t_l + 1) = 10^{p_l + l} \cdot (U(t_l) + 2 \cdot d) \tag{51}$$

$$S(t_l + 1) - S(t_l) = 10^{p_l} \cdot (U(t_l) + d) \tag{52}$$

$$S(t_l + 2) - S(t_l) = 10^{p_l} \cdot \big((U(t_l) + 2 \cdot d) \cdot 10^l + U(t_l) + d\big) \tag{53}$$

to simplify the coefficients μ_l and θ_l after solving the linear system of initial conditions. □

Corollary 7. *[Formula for reverse Smarandache numbers] An explicit formula to compute terms of the sequence $(Smr(n))_{n\geqslant 0}$ formed by the left-concatenation of positive integers is given by*

$$Smr(n) = \alpha_l + \mu_l \cdot 10^{l(n-t_l)} + \theta_l \cdot (n - t_l) \cdot 10^{l(n-t_l)} \tag{54}$$

$$l = \lceil \log_{10}(n + 2) \rceil,\ t_l = 10^{l-1} - 1,\ p_l = 10^{l-1} \cdot \left(l - \frac{10}{9}\right) + l + \frac{1}{9} \tag{55}$$

$$\alpha_l = \frac{s_2 - 2 \cdot 10^l \cdot s_1 + 10^{2l} \cdot s_0}{(10^l - 1)^2} \tag{56}$$

$$\mu_l = \frac{10^{p_l} \cdot \left(10^{2l-1} - 10^{l-1} - 1\right)}{\left(10^l - 1\right)^2} \tag{57}$$

$$\theta_l = \frac{10^{p_l}}{10^l - 1} \tag{58}$$

$$s_0 = Smr(t_l),\ s_1 = Smr(t_l + 1),\ s_2 = Smr(t_l + 2) \tag{59}$$

Proof. Immediate application of Theorem 6 for the concatenation of natural numbers $(n+1)_{n\geqslant 0}$ of common difference 1. The formula for p_l is deduced by the sum

$$1 + \sum_{k=1}^{l-1} \left(10^k - 1 - 10^{k-1}\right) \cdot k + k + 1 = 10^{l-1} \cdot \left(l - \frac{10}{9}\right) + l + \frac{1}{9}, \tag{60}$$

where $\left(10^k - 1 - 10^{k-1}\right) \cdot k$ counts all the digits of the k-digit left-concatenation, and $k+1$ stands for the first $(k+1)$-digit left-concatenation. The extra 1 before the sum is to compensate the 1-digit concatenation as the sequence starts at index 0. □

We end this section with the formula for concatenations of right-concatenations and reverse concatenations.

Theorem 8. *Let $(S(n))_{n\geqslant 0}$ be a sequence formed by the concatenation of the right-concatenation and the left-concatenation of an arithmetic progression of common difference d as in Lemma 3. Then $S(n)$ can be computed as follows:*

$$S(n) = \alpha_l + \mu_l \cdot 10^{l(n-t_l)} + \theta_l \cdot 10^{2l(n-t_l)} \tag{61}$$

$$l = \lceil \log_{10}(nd + S(0) + 1) \rceil,\ t_l = \left\lfloor \frac{10^{l-1} - S(0)}{d} \right\rfloor \tag{62}$$

$$\alpha_l = \frac{10^{3l} \cdot s_0 - 10^l \cdot \left(10^l + 1\right) \cdot s_1 + \cdot s_2}{\left(10^l + 1\right) \cdot \left(10^l - 1\right)^2} \tag{63}$$

$$\mu_l = -\frac{10^{2l} \cdot s_0 - \left(10^{2l} + 1\right) \cdot s_1 + s_2}{10^l \cdot \left(10^l - 1\right)^2} \tag{64}$$

$$\theta_l = \frac{10^l \cdot s_0 - \left(10^l + 1\right) \cdot s_1 + s_2}{10^l \cdot \left(10^l + 1\right) \cdot \left(10^l - 1\right)^2} \tag{65}$$

$$s_0 = S(t_l),\ s_1 = S(t_l + 1),\ s_2 = S(t_l + 2) \tag{66}$$

4. Algorithms and implementations

Using the theorems 4, 6, and 8, one can define algorithms for efficient computation of concatenations of arithmetic progressions. We implemented the particular cases of Smarandache numbers $(Smr(n))_{n\geqslant 0}$ and reverse Smarandache numbers $(Smr(n))_{n\geqslant 0}$ in the CAS

Maple. The resulting package can be downloaded from the link Smarandache.mla. The code for concatenations of arbitrary arithmetic progressions is currently under development.

Algorithm 1 $Sm(n)$

Input: A non-negative integer n.
Output: $Sm(n)$: the $(n+1)^{\text{st}}$ Smarandache number.

1. if $Sm(n)$ is defined then return $Sm(n)$
2. $l = \lceil \log_{10}(n+2) \rceil$
3. if α_l is not defined then
   (a) $d = (10^l - 1)^2$
   (b) $t_l = 10^{l-1} - 1$ and save t_l
   (c) $\alpha_l = -\left(10^{2l-1} + 9 \cdot 10^{l-1}\right)/d$ and save α_l
   (d) $\mu_l = -1/\left(10^l - 1\right)$ and save μ_l
   (e) $\theta_l =$ `numtheaSm(l)` $/d$ and save θ_l
4. Return and save $Sm(n) = \alpha_l + \mu_l \cdot (n - t_l) + \theta_l \cdot 10^{l(n-t_l)}$

The numerator of θ_l is computed by the algorithm `numthetaSm(l)` given below.

Algorithm 2 `numthetaSm(l)`

Input: A positive integer l.
Output: The numerator of θ_l in Algorithm 1.

1. if $l = 1$ then return 100
2. $s_0 = \texttt{conc}_l(Sm(t_l - 1), t_l + 1)$
3. $s_1 = \texttt{conc}_l(s_0, t_l + 2)$
4. $s_2 = \texttt{conc}_l(s_1, t_l + 3)$
5. return $s_2 - 2 \cdot s_1 + s_0$

The concatenation $\texttt{conc}_l$ is defined as: $(a, b) \mapsto a \cdot 10^l + b$. Notice that Algorithm 1 uses a remembering effect for t_l α_l, μ_l, θ_l, and the returned values. This helps avoiding computing the same values several times, and is really needed for the coefficients α_l, μ_l, and θ_l.
The algorithm to compute $Smr(n)$ is similar to Algorithm 1. The formulas are the only changes, and p_l is also computed with the remembering effect.

Let us now compare the efficiency of our implementations with the naive implementations of $Sm(n)$ and $Smr(n)$ in Maple (see the codes from [13], OEIS A007908, and OEIS A000422). The most efficient among the naive implementations of $Sm(n)$ seems to be following code.

```
> sm := n-> parse(cat($(n+1))):
```

We do not consider recursive implementations because we will do evaluations on very distant integers. The recursive approach is more suitable for the calculation of the "triangle of the gods": printing consecutive terms of $(Sm(n))_{n\geqslant 0}$, one per line. We use the command `CPUtime` of the Maple package `CodeTools` to compute the CPU times.

```
> sm(9)
```

12345678910

```
> Sm(9)
```

12345678910

```
> t1:=[CPUTime(sm(10^5-1))]:t1[1]
```

0.094

```
> t2:=[CPUTime(Smarandache:-Sm(10^5-1))]:t2[1]
```

0.015

```
> t2[2]-t1[2]
```

0

```
> t3:=[CPUTime(sm(10^6-1))]:t3[1]
```

0.953

```
> t4:=[CPUTime(Smarandache:-Sm(10^6-1))]:t4[1]
```

0.141

```
> t4[2]-t3[2]
```

0

```
> t5:=[CPUTime(sm(10^7-1))]:t5[1]
```

$$12.703$$

```
> t6:=[CPUTime(Smarandache:-Sm(10^7-1))]:t6[1]
```

$$2.047$$

```
> t6[2]-t5[2]
```

$$0$$

```
> t7:=[CPUTime(sm(10^8-1))]:t7[1]
```

$$195.359$$

```
> t8:=[CPUTime(Smarandache:-Sm(10^8-1))]:t8[1]
```

$$33.188$$

```
> t8[2]-t7[2]
```

$$0$$

As one can see, our code `Smarandache:-Sm` is faster than `sm` for asymptotic computations. Note, however, that these two codes can be combined so that closer terms are computed using `sm` and distant terms using `Smarandache:-Sm`.
For reverse Smarandache numbers, we define `smr` similarly.

```
> smr := n-> local i; parse(cat(n+1-i$i=0..n)):
```

The computations are summarized in the following table.

Table 1: `Smarandache:-Smr` vs `smr`

l	5	6	7	8
CPUTime(Smarandache:-Smr(10^l-1))	0.047	0.437	7.610	110
CPUTime(smr(10^l-1))	0.094	1	13	191.296

One can also note that Smarandache numbers are faster to compute than their reverses. This could be explained by p_l (see Corollary 7 and Theorem 6), which is also the reason why the coefficients α_l, μ_l, and θ_l have more decimal digits in left-concatenations.

5. Conclusion

The main results of this article are the three theorems 4, 6, and 8. These give explicit formulas for the most used concatenations of arithmetic progressions. This includes many Smarandache sequences, among which the most popular might be OEIS A007908 and OEIS A000422, and their concatenation OEIS A173426. These are all listed on the Online Encyclopedia of Integer Sequences (OEIS). Another interesting example is the concatenation of odd integer which is a particular case of Theorem 4 with common difference $d = 2$. One can see a few similarities between Corollary 5 and the work in [1, Section 4] which deals with right-concatenation in an arbitrary base. We implemented the formulas for Smarandache numbers and their reverses (see the corollaries 5 and 7) in the CAS Maple. The resulting package is available at Smarandache.mla. In Section 4 we have also shown that these implementations are important for efficient computations of asymptotic Smarandache numbers. Our approach to finding these formulas can be used to establish formulas for other concatenations of arithmetic progressions. The computation of the basis of hypergeometric term solutions in Section 3 suggests that the same strategy might still apply for concatenation of hypergeometric sequences. Thus the method can be further extended to concatenations of more general sequences, including geometric progressions.

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