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> restart
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Symbolic powers of functions defined by second-order linear ODEs

Maple Conference 2022: Algorithms and Software
MC2022, virtual, November 2-3, 2022

November 3, 2022

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1. Observation

For a fixed nonnegative integer k, the function $f_k(t)$ is the exponential generating function of $(\text{Stirling2(n,k)})_n$

```
> f[k]:=(exp(t)-1)^k/k!
```

$$f_k := \frac{\left(e^t - 1\right)^k}{k!} \quad \textbf{(1)}$$

```
> f[1]:=subs(k=1,f[k])
```

$$f_1 := \frac{e^t - 1}{1!} \quad \textbf{(2)}$$

```
> DEtools:-FindODE(f[1],y(t))
```

$$-\frac{d}{dt}\, y(t) + \frac{d^2}{dt^2}\, y(t) \quad \textbf{(3)}$$

```
> f[2]:=subs(k=2,f[k])
```

$$f_2 := \frac{\left(e^t - 1\right)^2}{2!} \quad \textbf{(4)}$$

```
> DEtools:-FindODE(f[2],y(t))
```

$$2\,\frac{d}{dt}\, y(t) - 3\,\frac{d^2}{dt^2}\, y(t) + \frac{d^3}{dt^3}\, y(t) \quad \textbf{(5)}$$

```
> for j to 5 do: DEtools:-FindODE(subs(k=j,f[k]),y(t)) end do
```

$$-\frac{d}{dt}\, y(t) + \frac{d^2}{dt^2}\, y(t)$$

$$2 \frac{d}{dt} y(t) - 3 \frac{d^2}{dt^2} y(t) + \frac{d^3}{dt^3} y(t)$$

$$-6 \frac{d}{dt} y(t) + 11 \frac{d^2}{dt^2} y(t) - 6 \frac{d^3}{dt^3} y(t) + \frac{d^4}{dt^4} y(t)$$

$$24 \frac{d}{dt} y(t) - 50 \frac{d^2}{dt^2} y(t) + 35 \frac{d^3}{dt^3} y(t) - 10 \frac{d^4}{dt^4} y(t) + \frac{d^5}{dt^5} y(t)$$

$$-120 \frac{d}{dt} y(t) + 274 \frac{d^2}{dt^2} y(t) - 225 \frac{d^3}{dt^3} y(t) + 85 \frac{d^4}{dt^4} y(t) - 15 \frac{d^5}{dt^5} y(t) + \frac{d^6}{dt^6} y(t) \quad (6)$$

```
> g[n]:=(sin(t)+cos(t))^n
```

$$g_n := (\sin(t) + \cos(t))^n \quad (7)$$

```
> for j to 5 do: DEtools:-FindODE(subs(n=j,g[n]),y(t)) end do
```

$$y(t) + \frac{d^2}{dt^2} y(t)$$

$$4 \frac{d}{dt} y(t) + \frac{d^3}{dt^3} y(t)$$

$$9 y(t) + 10 \frac{d^2}{dt^2} y(t) + \frac{d^4}{dt^4} y(t)$$

$$64 \frac{d}{dt} y(t) + 20 \frac{d^3}{dt^3} y(t) + \frac{d^5}{dt^5} y(t)$$

$$225 \frac{d}{dt} y(t) + 259 \frac{d^3}{dt^3} y(t) + 35 \frac{d^5}{dt^5} y(t) + \frac{d^7}{dt^7} y(t) \quad (8)$$

1.1. Quadratic differential equations in Maple 2022

```
> DEtools:-FindODE(f[k],y(t),method=quadratic)
```

$$-k \left(\frac{d}{dt} y(t)\right) y(t) + (-k + 1) \left(\frac{d}{dt} y(t)\right)^2 + k \left(\frac{d^2}{dt^2} y(t)\right) y(t) \quad (9)$$

```
> DEtools:-FindODE(g[n],y(t),method=quadratic)
```

$$n^2 y(t)^2 + (-n + 1) \left(\frac{d}{dt} y(t)\right)^2 + n \left(\frac{d^2}{dt^2} y(t)\right) y(t) \quad (10)$$

```
> DEtools:-FindODE(exp(exp(t)-1),y(t),method=quadratic)
```

$$-\left(\frac{d}{dt} y(t)\right) y(t) - \left(\frac{d}{dt} y(t)\right)^2 + \left(\frac{d^2}{dt^2} y(t)\right) y(t) \quad (11)$$

```
> DEtools:-FindODE((exp(exp(t)-1))^n,y(t),method=quadratic)
```

(12)

$$-\left(\frac{d}{dt}y(t)\right)y(t)-\left(\frac{d}{dt}y(t)\right)^2+\left(\frac{d^2}{dt^2}y(t)\right)y(t) \tag{12}$$

```
> simplify(eval(%,y(t)=(exp(exp(t)-1))^n))
```

$$0 \tag{13}$$

```
> simplify(eval((12),y(t)=(exp(exp(t)-1))^(1+I)))
```

$$0 \tag{14}$$

Theorem 1.

Theorem 1. *Let f be a function satisfying a second-order holonomic ODE. Then for all* <u>*symbol*</u> n, f^n *satisfies a second order quadratic differential equation with polynomial coefficients.*

Proof. Define $g=f^n$, and show that $\frac{g^{(1)}}{g}=n\frac{f^{(1)}}{f}$ and $\frac{g^{(2)}}{g}=n\frac{f^{(2)}}{f}+n(n-1)\cdot\left(\frac{f^{(1)}}{f}\right)^2$. Finally use a second-order (homogeneous) holonomic ODE to derive a quadratic differential equation satisfied by g.

The theorem can be extended to some nonlinear ODEs. This can be established by trying to generalize the proof to a more general class of ODEs.

```
> DEtools:-FindODE(LambertW(t),y(t),method=quadratic)
```

$$-y(t)+t\left(\frac{d}{dt}y(t)\right)+t\left(\frac{d}{dt}y(t)\right)y(t) \tag{15}$$

```
> DEtools:-FindODE(LambertW(t)^n,y(t),method=quadratic)
```

$$-n\left(2n^2-5n+1\right)\left(\frac{d}{dt}y(t)\right)y(t)+t\left(2n^3-7n^2+11n-8\right)\left(\frac{d}{dt}y(t)\right)^2-tn\left(2n^2-11n+3\right)\left(\frac{d^2}{dt^2}y(t)\right)y(t)-t^2\left(3n^2-9n+11\right)\left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dt}y(t)\right)-9t^3\left(\frac{d^2}{dt^2}y(t)\right)^2+t^2n(3n-1)\left(\frac{d^3}{dt^3}y(t)\right)y(t)+3t^3\left(\frac{d^3}{dt^3}y(t)\right)\left(\frac{d}{dt}y(t)\right) \tag{16}$$

```
> DEtools:-FindODE(sec(t)^n,y(t),method=quadratic)
```

$$-n^2y(t)^2+(-n-1)\left(\frac{d}{dt}y(t)\right)^2+n\left(\frac{d^2}{dt^2}y(t)\right)y(t) \tag{17}$$

2. Consequences for *guessing*

2.1. What is guessing?

A technique in experimental mathematics that consists of constructing a hypothetical recurrence or differential equation from finitely many of the first terms of an infinite sequence. The obtained recurrence equation should be valid for the nth term of the sequence, and the differential equation for its generating function.

2.2. Guessing in Maple

```
> C:=n-> binomial(2*n,n)/(n+1)
```

$$C := n \mapsto \frac{\binom{2\cdot n}{n}}{n+1} \quad \textbf{(18)}$$

```
> L:=[seq(C(j),j=0..8)]
```

$$L := [1, 1, 2, 5, 14, 42, 132, 429, 1430] \quad \textbf{(19)}$$

```
> gfun:-listtorec(L,a(n))
```

$$[\{(-4\,n-6)\,a(n+1)+(n+3)\,a(n+2), a(0)=1, a(1)=1\}, \mathit{ogf}] \quad \textbf{(20)}$$

```
> L:=PolynomialTools:-CoefficientList(convert(series(subs(n=1,g[n]
  ),t,9),polynom),t)
```

$$L := \left[1, 1, -\frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, \frac{1}{120}, -\frac{1}{720}, -\frac{1}{5040}, \frac{1}{40320}\right] \quad \textbf{(21)}$$

```
> gfun:-listtorec(L,a(n))
```

$$\left[\left\{a(n+1)+\left(n^2+5\,n+6\right)a(n+3), a(0)=1, a(1)=1, a(2)=-\frac{1}{2}\right\}, \mathit{ogf}\right] \quad \textbf{(22)}$$

```
> L:=PolynomialTools:-CoefficientList(convert(series(subs(n=2,g[n]
  ),t,12),polynom),t)
```

$$L := \left[1, 2, 0, -\frac{4}{3}, 0, \frac{4}{15}, 0, -\frac{8}{315}, 0, \frac{4}{2835}, 0, -\frac{8}{155925}\right] \quad \textbf{(23)}$$

```
> gfun:-listtorec(L,a(n))
```

$$[\{4\,a(n)+\left(n^2+3\,n+2\right)a(n+2), a(0)=1, a(1)=2, a(2)=0\}, \mathit{ogf}] \quad \textbf{(24)}$$

```
> L:=PolynomialTools:-CoefficientList(convert(series(subs(n=5,g[n]
  ),t,21),polynom),t)
```

$$L := \left[1, 5, \frac{15}{2}, -\frac{5}{6}, -\frac{85}{8}, -\frac{95}{24}, \frac{107}{16}, \frac{3359}{1008}, -\frac{7057}{2688}, -\frac{92735}{72576}, \frac{167681}{241920}, \frac{2397119}{7983360}, -\frac{457033}{3548160}, -\frac{12127315}{249080832}, \frac{102123841}{5811886080}, \frac{1522291679}{261534873600}, -\frac{2546718737}{1394852659200}, -\frac{7622937523}{14227497123840}, \frac{21203524267}{142274971238400}, \frac{953383751039}{24329020081766400}, -\frac{93514574081}{9540792188928000}\right] \quad \textbf{(25)}$$

```
> gfun:-listtorec(L,a(n))
```

$$FAIL \tag{26}$$

Theorem 2.

Theorem 2. *Let f be a function satisfying a second-order holonomic ODE. Then there exists a positive integer N such that the sequence of power series coefficients of f^n, for all positive integer n, can be recovered from its first N terms.*

With ONLY 21 coefficients of the series expension of $g_n(t) = (\sin(t) + \cos(t))^n$ for all positive interger n, one can recover the entire sequence of coefficients!

```
> L:=PolynomialTools:-CoefficientList(convert(series(subs(n=12,g[n]
  ),t,21),polynom),t):
> FPS:-delta2guess(L)
```

$$\left[\_C\left(\sum_{k=0}^{n-2}(k+1)(k+2)\,a(k+2)\,a(n-2-k)\right) - \frac{11\,\_C\left(\sum_{k=0}^{n-2}(k+1)\,a(k+1)\,(n-1-k)\,a(n-1-k)\right)}{12} + 12\,\_C\left(\sum_{k=0}^{n-2}a(k)\,a(n-2-k)\right) = 0, \_C\,z^2\left(\frac{d^2}{dz^2}y(z)\right)y(z) - \frac{11\,\_C\,z^2\left(\frac{d}{dz}y(z)\right)^2}{12} + 12\,\_C\,z^2\,y(z)^2 = 0\right] \tag{2.1}$$

http://www.mathematik.uni-kassel.de/~bteguia/FPS_webpage/FPS.htm

But that is not the whole story of Theorem 2. What if the constant term of the series expansion of f is zero? Then of course we cannot derive N from f^n directly.

Let us come back to the Stirling numbers of the second kind.

```
> f[k]
```

(27)

$$\frac{\left(e^{t}-1\right)^{k}}{k!} \quad \textbf{(27)}$$

```
> h[k]:=f[k]/t^k
```

$$h_{k} := \frac{\left(e^{t}-1\right)^{k}}{k!\, t^{k}} \quad \textbf{(28)}$$

```
> DEtools:-FindODE(h[k],y(t),method=quadratic)
```

$$-k^{2}\, y(t)^{2}-(t-2)\, k \left(\frac{d}{dt}\, y(t)\right) y(t)-t\,(k-1)\left(\frac{d}{dt}\, y(t)\right)^{2}+t\, k \left(\frac{d^{2}}{dt^{2}}\, y(t)\right) y(t) \quad \textbf{(29)}$$

Therefore when a sequence starts with several zeros, one can neglect them and try to guess a generating function of the form $\frac{f^{n}}{t^{n}}$

Thank you!

[1] Manuel Kauers. **The guess-and-prove paradigm in action**. *Internationale Mathematische Nachrichten*. 237. pp 1--15. 2018
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[4] Teguia Tabuguia, Bertrand and Koepf, Wolfram. **On the representation of non-holonomic univariate power series**. *Maple Trans.* 2, 1. Article 14315, 18 pages. August 2022.