Multiplicity-free permutation representations of the sporadic groups

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1. Let G be a finite group, and let $H \leq G$ be of index $n = [G : H]$. Let $\Omega = H|G = \{\omega_1, \ldots, \omega_n\}$ be the set of the right cosets of H in G, where $\omega_1 := H \cdot 1$, and let $\mathbb{Z}\Omega$ be the corresponding $\mathbb{Z}G$-permutation module, with permutation character $\pi_\Omega \in \mathbb{Z}\mathrm{Irr}_{\mathbb{C}}(G)$. Let $\Omega = \coprod_{i=1}^r \Omega_i$ be the decomposition of Ω as a disjoint union of H-orbits, where $\Omega_1 := \{\omega_1\}$, and let $A_i := [a_{ijk}]_{jk} \in \{0,1\}^{n \times n}$, where $a_{ijk} = 1$ if and only if $(\omega_j, \omega_k) \in \Omega \times \Omega$ is in the G-orbit $(\omega_1 \times \Omega_i) \cdot G$. Then $\{A_i; 1 \leq i \leq r\}$ is a $\mathbb{Z}$-basis for the endomorphism ring $E := \mathrm{End}_{\mathbb{Z}G}(\mathbb{Z}\Omega)$.

The algebra $E^{\mathbb{C}} := E \otimes_{\mathbb{Z}} \mathbb{C}$ is a split semisimple $\mathbb{C}$-algebra, and by the *Fitting correspondence* there is a bijection $\{\varphi \in \mathrm{Irr}_{\mathbb{C}}(E^{\mathbb{C}})\} \longleftrightarrow \{\chi \in \mathrm{Irr}_{\mathbb{C}}(G); \langle \pi_\Omega, \chi \rangle_G \neq 0\}$. The ring E is commutative if and only if Ω is *multiplicity-free*, i. e. $\langle \pi_\Omega, \chi \rangle_G \leq 1$ for all $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$. In this case we have $|\mathrm{Irr}_{\mathbb{C}}(E^{\mathbb{C}})| = r = \dim_{\mathbb{C}}(E^{\mathbb{C}})$, and $\varphi(A_1) = 1$ for all $\varphi \in \mathrm{Irr}_{\mathbb{C}}(E^{\mathbb{C}})$; the matrix $\Phi := [\varphi(A_i); \varphi \in \mathrm{Irr}_{\mathbb{C}}(E^{\mathbb{C}}), 1 \leq i \leq r] \in \mathbb{C}^{r \times r}$ is called the *character table* of $E^{\mathbb{C}}$.

2. For the sporadic finite simple groups, their automorphism groups and their Schur covering groups there are 328 equivalence classes of multiplicity-free permutation actions Ω [T.Breuer, K.Lux, 1996; S.Linton, Z.Mpono, 2001]. For these we have compiled a database containing the character tables Φ of the corresponding endomorphism rings $E^{\mathbb{C}}$ (currently up to one exception). The data is available as a GAP-library [T.Breuer, J.M., 2002]:

http://www.math.rwth-aachen.de/~Juergen.Mueller/mferctbl/mferctbl.html.

As an example, the character table for the case $G = B$ and $H = 2^{1+22}.Co_2$ is shown below; the degrees m_φ of the Fitting correspondents are indicated as well.

The results are based on work of [S. Norton, 1985; C.Praeger, L.Soicher, 1997; I. Höhler, 2001; J.M., 2003], amongst others, and on computational techniques encompassing a bunch of the most powerful tools of computational representation theory and computational group theory [F.Lübeck, M.Neunhöffer, 2001; R.Parker, 2001; J.M., 2002].

3. As an application, we have completed the classification of the distance-transitive orbital graphs for the sporadic finite simple groups, their automorphism groups and their Schur covering groups: The primitive cases have been classified in [A.Ivanov, S.Linton, K. Lux, J. Saxl, L. Soicher, 1995] and the non-primitive cases in [J.M., 2003].

The non-primitive cases are shown below; the rank r, valency k, number of vertices n, diameter d, and the intersection array is given as well, and it is indicated whether the corresponding graph is bipartite or antipodal.

Charaktertafel f”ur $G = B$ und $H = 2^{1+22}.Co_2$

φ	m_φ	1	2	3	4	5	6	7	8	9	10
1	1	1	93150	7286400	262310400	4196966400	9646899200	470060236800	537211699200	4000762036224	6685301145600
2	96255	1	−2025	772200	−5702400	42768000	290816000	−2714342400	5474304000	8833204224	−11921817600
3	9458750	1	10287	215424	3777840	25974432	35514368	607533696	100362240	−42467328	−730920960
4	347643114	1	−2025	99000	356400	−5702400	8806400	.	45619200	−191102976	141926400
5	4275362520	1	495	48960	−334800	1631520	2769920	−9636480	−12441600	−2359296	20321280
6	9287037474	1	3375	28800	356400	1015200	−870400	−6652800	4147200	−14155776	16128000
7	536105794455	1	1095	1560	7200	−113280	81920	107520	−921600	2555904	−1720320
8	635966233056	1	−425	9400	−3600	−57600	−115200	358400	−76800	1409024	−1523200
9	4375623425250	1	135	−360	−12960	17280	−40960	138240	414720	−884736	368640
10	6145833622500	1	−153	−936	8640	1152	32768	−129024	−207360	294912	.

Imprimitive distanz-transitive Orbital-Graphen

G	H	i	r	k	n	d	intersection array	
$HS.2$	$M_{22} < M_{22}.2$	3	6	22	200	5	$\{22, 21, 16, 6, 1; 1, 6, 16, 21, 22\}$	ba
$M_{22}.2$	$2^4{:}\,A_6 < 2^4{:}\,S_6$	3	6	16	154	5	$\{16, 15, 12, 4, 1; 1, 4, 12, 15, 16\}$	ba
Co_3	$McL < McL.2$	3	4	275	552	3	$\{275, 112, 1; 1, 112, 275\}$	a
Co_3	$McL < McL.2$	4	4	275	552	3	$\{275, 162, 1; 1, 162, 275\}$	a
HS	$U_3(5) < U_3(5).2$	3	4	175	352	3	$\{175, 72, 1; 1, 72, 175\}$	a
HS	$U_3(5) < U_3(5).2$	4	4	175	352	3	$\{175, 102, 1; 1, 102, 175\}$	a
HS	$U_3(5) < U_3(5).2$	3	4	175	352	3	$\{175, 72, 1; 1, 72, 175\}$	a
HS	$U_3(5) < U_3(5).2$	4	4	175	352	3	$\{175, 102, 1; 1, 102, 175\}$	a
$HS.2$	$U_3(5).2$	3	4	126	352	3	$\{126, 125, 36; 1, 90, 126\}$	b
$HS.2$	$U_3(5).2$	2	4	50	352	3	$\{50, 49, 36; 1, 14, 50\}$	b
$M_{22}.2$	$L_3(4) < L_3(4).2_2$	4	4	21	44	3	$\{21, 20, 1; 1, 20, 21\}$	ba
$2.M_{12}$	M_{11}	3	3	22	24	2	$\{22, 1; 1, 22\}$	ba
$2.M_{12}$	M_{11}	3	3	22	24	2	$\{22, 1; 1, 22\}$	ba
M_{11}	$A_6 < A_6.2_3$	3	3	20	22	2	$\{20, 1; 1, 20\}$	ba
$M_{12}.2$	M_{11}	3	3	12	24	2	$\{12, 11; 1, 12\}$	ba