# Exercises 2

**Exercise 4: Scheeffer's example**

Show that $u(x) \equiv 0$ is a critical point of the functional $I : M \to \mathbb{R}$ with

$$I(u) = \int_0^1 (u'(x))^2 + (u'(x))^3 \,\mathrm{d}x,$$

where $M = \{ u \in \mathrm{C}^1([0,1]) \mid u(0) = 0,\, u(1) = 0 \}$.

Show that $u$ satisfies the necessary condition for weak local minimizers and that the necessary condition for strong local minimizers is violated. (Hint in the latter case: find a suitable “zig-zag”-function). Is $u$ a weak local minimizer?

**Exercise 5: Legendre-Hadamard condition and convexity**

Let $m = d = 2$ and $f_c(A) = cA : A + \det A$ with $c \in \mathbb{R}$ and $A \in \mathbb{R}^{2\times 2}$.

(a) Find all $c \in \mathbb{R}$, for which $f_c$ satisfies the Legendre-Hadamard condition in 0.

(b) Show that for the numbers $c$ from (a) the Legendre-Hadamard condition is satisfied in every $A \in \mathbb{R}^{2\times 2}$.

(c) Find all $c \in \mathbb{R}$ for which $f_c : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is convex.

(d) Find $c > 0$ and $u \in C^1(\overline{\Omega}; \mathbb{R}^2)$ for which $\int_\Omega f_c(\nabla u(x))\mathrm{d}x < 0$, $\Omega = (-1,1)^2$.

**Exercise 6: Convexity**

Let $V = \mathbb{R}^{m\times d}$ and $f : V \to \mathbb{R}$.

(a) Show that $f$ is convex if and only if for every $A \in V$ there exists $\Sigma \in V$ such that for all $B \in V$ we have $f(B) \geq f(A) + \Sigma : (B - A)$.

(b) Assume that $f \in C^1(V)$. Show that (b1), (b2) and (b3) are equivalent.

(b1) $f$ is convex,

(b2) $\forall\, A, B \in V : \quad f(B) \geq f(A) + \partial_A f(A) : (B - A)$,

(b3) $\forall\, A_1, A_2 \in V$: $\quad \big(\partial_A f(A_1) - \partial_A f(A_2)\big) : (A_1 - A_2) \geq 0$.

(c) Let $f \in C^2(V)$. Show that $f$ is convex if and only if for all $A \in V$ the Hessian $\partial_A^2 f(A)$ is positive semi-definite, i.e. $\mathrm{D}_A^2 f(A)[B, B] \geq 0$ for all $B \in V$.

**Exercise 7: Convexity and continuity**

(a) Show that every convex function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz.

Hint: Choose an interval $B_R(x_0) = (x_0 - R, x_0 + R)$ and define $s_{2R} = \sup\{ f(x) \,|\, x \in B_{2R}(x_0) \}$ and $i_{2R} = \inf\{ f(x) \,|\, x \in B_{2R}(x_0) \}$. Using suitable tangents one can prove the following estimate for $x_1, x_2 \in B_R(x_0)$: $|f(x_1) - f(x_2)| \leq \frac{\|x_1 - x_2\|}{R}(s_{2R} - i_{2R})$.

(b) Let $V$ be a finite dimensional normed vector space. Show that every convex function $f : V \to \mathbb{R}$ is locally Lipschitz.

(c) Construct a convex function $f : \mathbb{R} \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ which is not continuous.

(d) Construct a convex function $f : \mathbb{R} \to \mathbb{R}_\infty$ which is continuous and which takes values in $\mathbb{R}$ and in $\{\infty\}$.

(e) Find a Banach space $X$ and a convex functional $I : X \to [0, \infty]$ which is nowhere continuous.

**Exercise 8: Jensen's inequality**

Let $V$ be a finite dimensional vector space and let $f : V \to \mathbb{R}$ be convex.

(a) Show that $f(q_1 u_1 + \ldots + q_n u_n) \leq \sum_{i=1}^n q_i f(u_i)$ for all $u_i \in V$ and all $q_i \geq 0$ with $q_1 + \ldots + q_n = 1$.

(b) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $V = \mathbb{R}$ and let $u : \Omega \to \mathbb{R}$ be a piecewise constant function (finitely many values). Show that

$$f\left(\frac{1}{\mathrm{vol}(\Omega)} \int_\Omega u(x)\mathrm{d}x\right) \leq \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega f(u(x))\mathrm{d}x.$$

(c) Show that (b) is also valid for $u \in C^0(\overline{\Omega})$ and for $u \in L^1(\Omega)$.