Exercises 3

Exercise 9
Let X be a Banach space and $I : X \to \mathbb{R}_\infty$ a functional on X. Assume that the sublevel sets $S_\alpha = \{u \in X;\, I(u) \le \alpha\}$ are sequentially compact for every $\alpha \in \mathbb{R}$.
Show that the direct problem has a solution.

Exercise 10: Criterium for weak convergence
Let X be a Banach space, $u^* \in X$ and $(u_n)_{n\in\mathbb{N}} \subset X$. Assume that the following holds true:

(a) $(u_n)_{n\in\mathbb{N}}$ is bounded in X: $\sup_n \|u_n\|_X \le c$.

(b) $\ell(u_n) \to \ell(u^*)$ for $\ell \in S$, where $S \subset X'$ and the linear hull of S (finite linear combinations of elements from S) is dense in X'.

Show that $u_n \rightharpoonup u^*$ in X.

Exercise 11: Weak convergence of oscillating functions
Let $X = L^p((0,1))$ with $p \in (1,\infty)$ and let $u \in X$. The function $\widetilde{u}$ is defined by the periodic extension of u to $\mathbb{R}$. Consider the sequence $(u_n)_{n\in\mathbb{N}} \subset X$, which is defined via

$$u_n(x) = \widetilde{u}(nx), \quad x \in (0,1).$$

Show that $u_n \rightharpoonup u^*$ in X with $u^*(x) = \int_0^1 u(\xi)\,\mathrm{d}\xi$ for all $x \in (0,1)$.
Hint: Use exercise 10 with $S = \{1_{[0,\alpha]}, 0 \le \alpha \le 1\}$, where $1_{[0,\alpha]}$ is the indicator function of the set $[0,\alpha]$, i.e. $1_{[0,\alpha]}(x) = 1$ if $x \in [0,\alpha]$ and 0 otherwise.

Exercise 12: Weakly continuous integral functionals in L^2 are affine
Let $g : \mathbb{R} \to \mathbb{R}$ be continuous with $|g(u)| \le C(1+u^2)$ and let the functional $I : L^2((0,1)) \to \mathbb{R}$ be defined through $I(u) = \int_0^1 g(u(x))\,\mathrm{d}x$. Show that the following statements are equivalent:

(i) I is weakly continuous.

(ii) I is affine, i.e. $I(u) = I(0) + \langle y, u\rangle$ with a suitable $y \in L^2((0,1),\mathbb{R})$.

Hint: Use exercise 11 and proove first that $I(u) = g(\int_0^1 u\,\mathrm{d}x)$ for all u. Next consider sequences u_n with $u_n(x) \in \{\alpha,\beta\}$ and show that g is convex and concave.