Allgemeine
Variationsmethoden,
WS0809
Dr. Dorothee Knees
19. Januar 2009

Exercises 4

Exercise 13
Let X be a reflexive Banach space, $M \subset X$ and $u_* \in X \backslash M$. The functional $I : X \to \mathbb{R}_\infty$ is defined as

$$I(u) = \begin{cases} \|u_* - u\| & u \in M \\ \infty & \text{else} \end{cases}.$$

a) Show that I is strongly lower semicontinuous if M is closed.

b) Show that I is convex if M is convex.

c) Let M be convex and closed. Show that there exists a minimizer of I.

d) Find a closed set M and a point u_* such that I does not have a minimizer.

Exercise 14
Let $\Omega = B_R(0) \subset \mathbb{R}^3$ and assume that γ, μ are positive constants. For $u \in W^{1,2}(\Omega)$ we define

$$I(u) = \int_\Omega \mu \left|\nabla u\right|^2 \, \mathrm{d}x - \int_\Omega \frac{\gamma}{|x|} \left|u\right|^2 \, \mathrm{d}x.$$

Show that the functional I has a minimizer with respect to the set

$$M = \{\, u \in W^{1,2}(\Omega)\,;\ \|u\|_{L^2(\Omega)} = 1 \,\}.$$

Hint: Use and prove that $(x \mapsto \frac{1}{|x|}) \in L^2(\Omega)$ and that $\|u^2\|_{L^2(\Omega)} \leq C \left\|u\right\|_{L^2(\Omega)}^{\frac{1}{2}} \left\|u\right\|_{H^1(\Omega)}^{\frac{3}{2}}$. Moreover, if $u_n \rightharpoonup u_*$ weakly in $W^{1,2}(\Omega)$, then $u_n^2 \to u_*^2$ strongly in $L^2(\Omega)$. (Embedding theorems!)

Exercise 15
Let $\Omega = B_1(0) \subset \mathbb{R}^3$ and $s \in [1, \infty)$. For $u \in \mathbb{R}$ and $A \in \mathbb{R}^3$ let $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be defined as $f(u, A) = \frac{1}{2} \left|A\right|^2 - 1000 \left|u\right|^s$.

Show that $I : W^{1,2} \to \mathbb{R}_\infty$ with $I(u) = \int_\Omega f(u, \nabla u) \, \mathrm{d}x$ is sequentially weakly lower semicontinuous on $W^{1,2}(\Omega)$ if and only if $s \in [1, 6)$.

This example shows that although f is convex in ∇u, the functional is not lower semicontinuous if s is large enough.

Hint: The sequence $u_k(x) = k \max\{0, 1 - k^2 \left|x\right|\}$, $k \in \mathbb{N}$, is quite interesting.